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SIGNED SOLUTIONS FOR A SEMILINEAR ELLIPTIC PROBLEM

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Abstract. We show existence of signed solutions with positive energy of the problem $\Delta u + u_+^p - u_-^q = 0$ in Ω, $u = 0$ on $\partial\Omega$, where $0 < q < 1 < p$, $p < (N+2)/(N-2)$ if $N > 2$ and the domain $\Omega \subset \mathbb{R}^N$ is bounded and "sufficiently large." Our proof is based on the study of the dynamical system associated with the corresponding parabolic problem and it can be easily extended to more general problems. In particular, it does not rely on the uniqueness of the negative solution in contrast to the variational proof in [2] where the authors obtained signed solutions with negative energy.

1. Introduction. In this paper we study existence of signed solutions of the problem

$$\begin{aligned} \Delta u + u_+^p - u_-^q &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{S}$$

where Ω is a smoothly bounded domain in $\mathbb{R}^N$, $0 < q < 1 < p$, $p < \frac{N+2}{N-2}$ if $N > 2$, $u_+ = \max(u, 0)$ and $u_- = \max(-u, 0)$. It is known that this problem admits at least one positive solution and a unique negative solution; see [1] for the latter fact. On the other hand, signed solutions exist for this problem if and only if Ω is "sufficiently large." This fact was proved in [2] and [3]. More precisely, Cortázar, Elgueta and Felmer have shown in [2] that for Ω large there exists a signed solution for (S) and del Pino has shown nonexistence for Ω small in [3].

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The existence proof of a signed solution in [2] relies on the uniqueness of the negative solution u^- of (S). Denote by

$$J(u)=\int_\Omega\Big(\frac12|\nabla u|^2-\frac{1}{p+1}u_+^{p+1}-\frac{1}{q+1}u_-^{q+1}\Big)\,dx \tag{1}$$

the energy functional corresponding to the problem (S). The authors of [2] construct a path joining the local minimum u^- of the energy J with a function having the energy less than $J(u^-)$ in the region where $J<0$. Then they use the Mountain Pass Theorem to get existence of a solution $\tilde u\neq u^-$ with $J(u^-)\leq J(\tilde u)<0$. This solution can be neither negative (due to the uniqueness of the negative solution) nor positive (due to negativity of its energy).

In this paper we show that for Ω large there exists a signed solution with (strictly) positive energy. Moreover, in our proof we do not use the uniqueness of the negative solution u^- of (S) so that our proof can be generalized also to problems without this property. The method of our proof is based on the study of the dynamical system associated with the corresponding parabolic problem. More precisely, we deal with the problem

$$\begin{aligned} u_t&=\Delta u+u_+^p-g_M(u_-) && x\in\Omega,\ t>0,\\ u&=0 && x\in\partial\Omega,\ t>0,\\ u(x,0)&=u_o(x) && x\in\bar\Omega, \end{aligned}\tag{P$_M$}$$

where g_M is a Lipschitz-continuous function, $g_M(t)\to t^q$ as $M\to\infty$. We choose an appropriate two-dimensional subspace X_2 of the Sobolev space $H_0^1(\Omega)$ and then we consider a component D_M of the set of all initial functions $u_o\in X_2$ such that the corresponding solution of (P$_M$) becomes positive for t large enough and tends to zero as $t\to\infty$. We prove that D_M is a bounded open set in X_2 and then we use a result of Giga ([4]) to show that the trajectories of (P$_M$) starting on the boundary ∂D_M are global and bounded. Finally we use a topological degree argument to show that there exists an initial condition $u_o\in\partial D_M$ such that its ω-limit set contains a signed equilibrium u_{MS} with positive energy. Our signed solution of (S) is then obtained as a limit of the solutions u_{MS} as $M\to\infty$.

The method of our proof is similar to that used in the existence proof of signed solutions for the problem $\Delta u+|u|^{p-1}u+g(x,u)=0$ in Ω, $u=0$ on $\partial\Omega$ (where $|g(x,u)|\leq C(1+|u|^{p-\varepsilon})$) in [5, Example 1]. However, the proof in [5] contains a wrong argument and we correct it in the appendix to this paper.

In Remark 3 we consider the particular case $N = 1$ and we explicitly compute the energy for all solutions of (S).

2. Main results. For $M \geq 1$ and $t \geq 0$ define

$$g_M(t) = \begin{cases} Mt & \text{if } 0 \leq t < M^{-1/(1-q)}, \\ t^q & \text{otherwise} \end{cases}$$

and put $G_M(t) = \int_0^t g_M(s)\,ds$. Then g_M is a Lipschitz-continuous function, $g_M(t) \to t^q$ as $M \to \infty$ and

$$G_M(t) \begin{cases} \geq \frac{t^2}{2} & \text{if } t \leq 1, \\ \leq \frac{t^{q+1}}{q+1} & \text{for any } t\ . \end{cases} \tag{2}$$

Moreover, denoting $\delta(t) = \frac{1}{q+1}t^{q+1} - G_M(t)$ we get

$$0 \leq \delta(t) \leq \delta(M^{-1/(1-q)}) = c_M := \frac{1-q}{2(1+q)}M^{-(1+q)/(1-q)} \to 0 \text{ as } M \to \infty.$$

For $u \in H_0^1(\Omega)$ put

$$P(u) = \frac{1}{p+1}\int_\Omega |u|^{p+1}\,dx, \quad Q(u) = \frac{1}{q+1}\int_\Omega |u|^{q+1}\,dx,$$
$$Q_M(u) = \int_\Omega G_M(|u|)\,dx, \quad J_M(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - P(u_+) - Q_M(u_-)$$

and recall the definition of J in (1). Then

$$0 \leq J_M(u) - J(u) = \int_\Omega \big(\frac{1}{q+1}u_-^{q+1} - G_M(u_-)\big)\,dx \leq c_M|\Omega| \to 0 \text{ as } M \to \infty.$$

Lemma 1. *Let* $v_1, v_2 \in C^1(\bar{\Omega}) \cap H_0^1(\Omega)$ *be such that* $\operatorname{supp} v_1 \cap \operatorname{supp} v_2 = \emptyset$, $1 \geq v_i \geq 0$,

$$\int_\Omega |\nabla v_i|^2\,dx = 1, \qquad \int_\Omega v_i^2\,dx > 2$$

and $P(v_i) > 1$ *for* $i = 1, 2$. *Put* $u_\alpha = \alpha(v_2 - v_1)$ *for* $\alpha \in \mathbb{R}$. *Then* $J_M(u_\alpha) < 0$ *for any* $\alpha \neq 0$ *and* $M \geq 1$.

Remark 1. Lemma 1 is an analogue to [2, Lemma 3.2]. It can be easily seen that functions v_1, v_2 with the properties stated in Lemma 1 exist if Ω is "large enough," e.g., if $\Omega = \lambda\Omega_o$ where Ω_o is a fixed domain and $\lambda > 0$ is large enough; cf. also [2] and [3].

Proof of Lemma 1. Let $\alpha > 0$. We have

$$\frac{J_M(u_\alpha)}{\alpha^2} = 1 - \frac{P(u_{\alpha+})}{\alpha^2} - \frac{Q_M(u_{\alpha-})}{\alpha^2},$$

where $P(u_{\alpha+}), Q_M(u_{\alpha-}) \geq 0$.

If $\alpha \geq 1$ we get

$$\frac{P(u_{\alpha+})}{\alpha^2} = \frac{1}{\alpha^2}\frac{1}{p+1}\int_\Omega (\alpha v_2)^{p+1}\,dx = \alpha^{p-1}P(v_2) > 1,$$

so that $J_M(u_\alpha) < 0$.

If $0 < \alpha < 1$ we have

$$\frac{Q_M(u_{\alpha-})}{\alpha^2} = \frac{1}{\alpha^2}\int_\Omega G_M(\alpha v_1)\,dx \geq \frac{1}{\alpha^2}\int_\Omega \frac{(\alpha v_1)^2}{2}\,dx > 1,$$

so that $J_M(u_\alpha) < 0$.

The case $\alpha < 0$ is analogous. $\square$

In what follows we assume that Ω is large enough in the sense of Remark 1 and we fix functions v_1, v_2 with the properties stated in Lemma 1. We put $X_2 = \operatorname{span}\{v_1, v_2\}$, $K^+ = \{u \in H_0^1(\Omega)\,;\, u \geq 0\}$, $K^- = \{u \in H_0^1(\Omega)\,;\, u \leq 0\}$, $\langle u, v\rangle = \int_\Omega \nabla u \cdot \nabla v\,dx$ and $\|u\| = \sqrt{\langle u, u\rangle}$. By $u_M(t, u_o)$ we denote the solution of the problem (P_M) at time t and by $\mathcal{D}_M$ we denote the set of all initial functions $u_o \in X_2$ such that the solutions $u_M(\cdot, u_o)$ exist globally, tend to zero as $t \to \infty$ and $u_M(T, u_o) \in K^+ \setminus \{0\}$ for $T = T(u_o)$ large enough. Note that if $u_M(T, u_o) \in K^+ \setminus \{0\}$ and $t > T$ then, due to the maximum principle, $u_M(t, u_o)$ lies in the C^1-interior of the positive cone K^+ and solves the problem

$$\begin{aligned} u_t &= \Delta u + |u|^{p-1}u && x \in \Omega,\ t > T,\\ u &= 0 && x \in \partial\Omega,\ t > T. \end{aligned} \qquad (\mathrm{P}^+)$$

Due to the continuous dependence in $C^1(\bar\Omega)$ of solutions of (P_M) on the initial values in $H^1_0(\Omega)$ (i.e., $\|u_M(t,u_1) - u_M(t,u_2)\|_{C^1(\bar\Omega)} \to 0$ if $t > 0$, $u_1 \to u_2$ in $H^1_0(\Omega)$ and $u_M(t,u_1), u_M(t,u_2)$ exist) and the stability of the zero solution of (P^+) in $H^1_0(\Omega)$ we get that the set $\mathcal{D}_M$ is open in X_2 and $\varepsilon(v_1+v_2) \in \mathcal{D}_M$ for $0 < \varepsilon \le \varepsilon_o$ where $\varepsilon_o \in (0,1)$ does not depend on M. By D_M we denote the connected component of $\mathcal{D}_M$ containing the function $\varepsilon_o(v_1+v_2)$.

Lemma 2. *The set D_M is bounded* (*independently of M*).

Proof. Since J_M is the Lyapunov functional for (P_M) and $u_M(t,u_o) \to 0$ as $t \to \infty$ for any $u_o \in D_M$, we have $J_M > 0$ on D_M. Due to Lemma 1, we have also $J_M(u) \le 0$ for $u \in \{\alpha(v_2 - v_1)\,;\, \alpha \in \mathbb{R}\} = \{u \in X_2\,;\, \langle u, v_1+v_2\rangle = 0\}$. Consequently, $D_M \subset \{u \in X_2\,;\, \langle u, v_1+v_2\rangle > 0\}$.

Let $u = \alpha_1 v_1 + \alpha_2 v_2 \in D_M$. Then $\langle u, v_1+v_2\rangle > 0$, hence $\alpha_1+\alpha_2 > 0$. Let e.g. $\alpha_1 \ge \alpha_2$ (the case $\alpha_1 \le \alpha_2$ is analogous). Then $\alpha_1 > 0$ and

$$0 < J_M(u) \le \frac{1}{2}(\alpha_1^2 + \alpha_2^2) - \alpha_1^{p+1} P(v_1) \le \alpha_1^2 - \alpha_1^{p+1},$$

so that $\alpha_1 < 1$. $\square$

Denote by ∂D_M the boundary of D_M in X_2. A straightforward generalization of the result of Giga ([4]) shows that the trajectory $u_M(\cdot,u_o)$ is global and bounded in $H^1_0(\Omega)$ for any $u_o \in \partial D_M$ (the paper [4] deals with positive solutions; nevertheless, the negative part of $u_M(\cdot,u_o)$ can be uniformly bounded in an elementary way and this is sufficient for the proof in [4]). Moreover, the proof in [4] shows that even the set $\{u_M(t,u_o)\,;\, u_o \in \partial D_M,\ t \ge 0,\ M \ge 1\}$ is bounded in $H^1_0(\Omega)$ by some constant c_G. Consequently, if $u_o \in \partial D_M$ then its ω-limit set $\omega_M(u_o)$ (for the problem (P_M)) consists of stationary solutions u_{MS} with $J_M(u_{MS}) \ge 0$ and $\|u_{MS}\| \le c_G$.

Lemma 3. *Denote by E^+ the set of all stationary solutions of the problem* (P^+) *lying in the set $\{u \in K^+\,;\, 0 < \|u\| \le c_G\}$ and put $J^+ = \inf_{u\in E^+} J(u)$. Then $J^+ > 0$.*

Proof. If $u \in E^+$ then multiplying the equation in (P^+) by u and integrating by parts we get

$$\int_\Omega |\nabla u|^2\,dx = \int_\Omega u^{p+1}\,dx, \tag{3}$$

hence $J(u) > 0$ and $J^+ \ge 0$. Assume that $J^+ = 0$. Then there exist $u_k \in E^+$ such that $J(u_k) \to 0$. Passing to a subsequence we may assume that u_k converges weakly in $H^1_0(\Omega)$ to some function $\tilde u \in K^+$. Since any u_k is a stationary solution of the problem (P^+), i.e., a solution of the problem $F(u_k) = 0$ where F has the form (identity-compact), it follows that u_k converges to $\tilde u$ strongly in $H^1_0(\Omega)$. Consequently, $\tilde u \in K^+$ is a stationary solution of (P^+) with $J(\tilde u) = 0$. The equality (3) for $u = \tilde u$ together with $J(\tilde u) = 0$ imply $\tilde u = 0$. This means that $u_k \to 0$ in $H^1_0(\Omega)$ which is a contradiction since 0 is an isolated stationary solution of (P^+). □

In what follows put

$$N^+ = N^+_M := \{u_o \in \partial D_M\,;\, \omega_M(u_o) \subset K^+ \setminus \{0\}\,\},$$
$$N^- = N^-_M := \{u_o \in \partial D_M\,;\, \omega_M(u_o) \subset \{u\,;\, J_M(u) < J^+\}\,\}.$$

Using the continuous dependence of solutions of (P_M) in $C^1(\bar\Omega)$ on the initial values in $H^1_0(\Omega)$, the maximum principle for the solutions of (P^+) and the fact that zero is an isolated stationary solution of (P^+) it is easy to see that both N^+ and N^- are open in ∂D_M. Lemma 3 shows that $N^+ \cap N^- = \emptyset$. Since zero is stable from above and unstable from below for the problem (P_M), we have $0 \in \partial D_M$, hence $0 \in N^-$. On the other hand, any function $u_o \in \partial D_M \cap (K^+ \setminus \{0\})$ has to belong to the set N^+. This follows from the following two facts: $u_M(t, u_o) \in K^+$ for any $t \ge 0$ due to the maximum principle and the convergence $u_M(t, u_o) \to 0$ as $t \to \infty$ is ruled out by the definition of the open set D_M. Consequently, both N^+ and N^- are nonempty.

Lemma 4. $\partial D_M \ne N^+ \cup N^-$.

Proof. Assume the contrary. Then $\partial D_M = N^+ \cup N^-$, where both N^+ and N^- are open in ∂D_M, ∂D_M is compact and $N^+ \cap N^- = \emptyset$. Consequently, both N^+ and N^- are compact and have a positive distance so that the following homotopy $H : [0,1] \times \partial D_M \to X_2$ is continuous:

$$H(t,u) = \begin{cases} (1-2t)u, & u \in N^-,\ t \in [0,1/2], \\ (1-2t)u + 2t(v_1+v_2), & u \in N^+,\ t \in [0,1/2], \\ (2t-1)(v_2-v_1), & u \in N^-,\ t \in [1/2,1], \\ (2t-1)(v_2-v_1) + (2-2t)(v_1+v_2) & u \in N^+,\ t \in [1/2,1]. \end{cases}$$

Put $u_f = \varepsilon_o(v_1 + v_2)$. We shall show that $H(t,u) \ne u_f$ for any $t \in [0,1]$ and $u \in \partial D_M$ so that the homotopy invariance property of the Brouwer degree (with respect to D_M and u_f) yields

$$\begin{aligned}1 &= \deg(Id, D_M, u_f) = \deg(H(0,\cdot), D_M, u_f)\\ &= \deg(H(1,\cdot), D_M, u_f) = \deg(v_2 - v_1, D_M, u_f) = 0\end{aligned}$$

which is a contradiction.

The assertion $H(t,u) \neq u_f$ is obvious for $t = 0$ and $t \geq 1/2$ hence we may suppose $t \in (0, 1/2)$. Assume $H(t,u) = u_f$.

If $u \in N^-$ we have $u = \frac{\varepsilon_o}{1-2t}(v_1 + v_2)$ so that $u \in \partial D_M \cap (K^+ \setminus \{0\})$. However, we have shown $\partial D_M \cap (K^+ \setminus \{0\}) \subset N^+$ which is a contradiction.

If $u \in N^+$ we get $u = \frac{\varepsilon_o - 2t}{1-2t}(v_1 + v_2)$. The case $\varepsilon_o < 2t$ is impossible since $D_M \subset \{u\,;\, \langle u, v_1 + v_2\rangle > 0\}$. Since $0 \in N^-$ we have also $\varepsilon \neq 2t$ so that $\varepsilon_o > 2t$. However, since $0 < \frac{\varepsilon_o - 2t}{1-2t} < \varepsilon_o$, we have $u \in D_M$ which is again a contradiction. This concludes the proof. □

Now we are able to prove the following theorem.

Theorem. *Let Ω be large enough* (*so that the functions v_1, v_2 in Lemma* 1 *exist*). *Then the problem* (S) *admits a signed solution u_S with $J(u_S) \geq J^+$, where $J^+ > 0$ is defined in Lemma* 3*.*

Proof. Let $M \geq 1$ be fixed. By Lemma 4 there exists $u_o \in \partial D_M \setminus (N^+ \cup N^-)$. Choose $u_{MS} \in \omega_M(u_o)$. Then u_{MS} is a stationary solution of (P_M), $\|u_{MS}\| \leq c_G$, $u_{MS} \notin K^+ \setminus \{0\}$ and $J_M(u_{MS}) \geq J^+$. Consequently, $u_{MS} \notin K^+$.

Now choose a sequence $M_k \to \infty$ and let $u_k = u_{M_k S}$ be the corresponding sequence of stationary solutions with the properties stated above. Passing to a subsequence we may assume that u_k converges weakly in $H^1_0(\Omega)$ to some limit u_S. Since u_k are stationary solutions of the problems (P_{M_k}) with compact nonlinearities converging (in a suitable way) to the nonlinearity of the limiting problem (S), we obtain (similarly as in the proof of Lemma 3) that u_k converges to u_S strongly, $J(u_S) \geq J^+$, $\|u_S\| \leq c_G$ and u_S solves (S). A simple bootstrap argument shows that $u_k \to u_S$ in $C^1(\bar{\Omega})$.

If $u_S \in K^+$ then the maximum principle implies that u_S lies in the C^1-interior of K^+ and since $u_k \to u_S$ in $C^1(\bar{\Omega})$ we get a contradiction with $u_k \notin K^+$.

If $u_S \in K^-$ then similarly as in the proof of Lemma 3 we obtain

$$\int_\Omega |\nabla u|^2\, dx = \int_\Omega |u|^{q+1}\, dx,$$

so that $J(u_S) \le 0$, a contradiction.

Consequently, $u_S \notin K^+ \cup K^-$ which concludes the proof. $\square$

Remark 2. The fact that the stationary solution u_S obtained in the proof of Theorem cannot lie in the C^1-interior of the cone K^- can be excluded also by using the instability of zero from below for the problem (P_M) (without the information $J(u) \le 0$ for any stationary solution $u \in K^-$).

Remark 3. Let us consider the case $N = 1$. Then the problem (S) can be easily solved by the shooting method (cf. also the introduction in [2]). The solutions of (S) are composed of solutions u_r of the initial value problem

$$u'' + |u|^{r-1}u = 0,\ u(0) = 0,\ u'(0) = a_r,$$

where $r = p$ or $r = q$ and $a_p = -a_q \neq 0$. The first zero of u_r equals $T_r(a_r) = c_r|a_r|^{(1-r)/(1+r)}$, where $c_r = 2(\frac{r+1}{2})^{1/(r+1)} \int_0^1 (1 - s^{r+1})^{-1/2}\, ds$ and the corresponding energy equals

$$\begin{aligned} J_r(u_r) &= \Big(\frac{1}{2} - \frac{1}{r+1}\Big) \int_0^{T_r(a_r)} |u_r|^{r+1}\, dx \\ &= \frac{r-1}{2(r+3)} a_r^2 T_r(a_r) = \hat{c}_r |a_r|^{(r+3)/(r+1)} \end{aligned}$$

for some constants $\hat{c}_p > 0 > \hat{c}_q$ (where we used the following identity:

$$\begin{aligned} \frac{a_r^2}{2} T_r(a_r) &= \int_0^{T_r(a_r)} \big(\frac{1}{2}(u')^2 + \frac{1}{r+1}|u|^{r+1}\big)\, dx \\ &= \int_0^{T_r(a_r)} \big(-\frac{1}{2}u''u + \frac{1}{r+1}|u|^{r+1}\big)\, dx \\ &= \int_0^{T_r(a_r)} \big(\frac{1}{2} + \frac{1}{r+1}\big)|u|^{r+1}\, dx). \end{aligned}$$

Hence one can easily see that the resulting energy of any composed solution will be positive or negative if $|a_r|$ is small or large enough, respectively.

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Appendix. The proof of existence of a sign-changing equilibrium in [5, Example 1] (page 1553) is not correct since the initial conditions $v(\alpha)$ should lie in the set $M = \partial_Y(D_A \cap Y)$. To prove the corresponding result, assume $\mathcal{E} = \mathcal{E}^+ \cup \mathcal{E}^-$, $\mathcal{E}$ finite, and put $M^\pm = \{u \in M\,;\, \omega(u) \subset \mathcal{E}^\pm\}$, respectively. Then $M = M^+ \cup M^-$ and the sets M^+, M^- are disjoint and open, hence compact (the proof is the same as the corresponding proof for the sets $\Lambda^\pm$ in the paper). Put

$$H(t,u) = \begin{cases} (1-2t)u + 2tv_1, & u \in M^+,\ t \in [0,1/2] \\ (1-2t)u - 2tv_1, & u \in M^-,\ t \in [0,1/2] \\ (2-2t)v_1 + (2t-1)v_2, & u \in M^+,\ t \in [1/2,1] \\ (2-2t)(-v_1) + (2t-1)v_2, & u \in M^-,\ t \in [1/2,1]. \end{cases}$$

Then the homotopy invariance property of the topological degree in Y yields

$$1 = \deg\left(H(0,\cdot), 0, D_A \cap Y\right) = \deg\left(H(1,\cdot), 0, D_A \cap Y\right) = 0,$$

which is a contradiction (the admissibility of the homotopy H follows from the facts that $\operatorname{dist}(M^+, M^-) > 0$ and $M^\pm \cap \{\lambda v_1\,;\, \lambda \lesseqgtr 0\} = \emptyset$, respectively).

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