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EXISTENCE AND NONEXISTENCE FOR THE EXTERIOR DIRICHLET PROBLEM FOR THE MINIMAL SURFACE EQUATION IN THE PLANE

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1. Introduction and statement of main results. The aim of this paper is to investigate the classical solvability of the Dirichlet problem for the minimal surface equation in the plane

$$Mu := (1+|Du|^2)Du - D_iuD_juD_{ij}u = 0 \quad \text{in} \quad \Omega \tag{1}$$

$$u = f \quad \text{on} \quad \partial\Omega, \tag{2}$$

where $\Omega \subset \mathbb{R}^2$ is a domain of class C^2 with bounded complement $\mathbb{R}^2\backslash\Omega$, and the boundary data f are continuous. As is well known from classical results (see [11], Chapter VII) the Dirichlet problem (1), (2) in bounded domains $\Omega \subset \mathbb{R}^2$ is solvable for all data f if Ω is convex and in the multidimensional case if Ω is mean-convex ([3]). Already at the beginning of this century, Korn ([4]) and Müntz ([10]) proved by the method of successive approximations that, in the case of arbitrary bounded domains of class $C^{2,\alpha}, 0 < \alpha < 1$, the smallness of the $C^{2,\alpha}$-norm of the data guarantees the solvability of (1), (2) (see also [11], § 412–414.) Using Perron's method, Nitsche showed for bounded domains of class $C^{1,1}$ and data $f \in C^{1,1}$ that (1), (2) is solvable provided that $\operatorname{osc} f := \max f - \min f$ is small enough depending on Ω and the $C^{1,1}$-norm of f ([11], §649). A similar result (replacing $C^{1,1}$ by C^2)

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was established by Jenkins and Serrin for the multidimensional case ([3]). These results were considerably sharpened by Williams ([14]) who proved that for bounded C^2-domains Ω and $C^{0,1}$-data f with Lipschitz constant $K \in [0, 1/\sqrt{n-1}\,)$ problem (1), (2) has a classical solution if $osc\, f$ is small enough depending only on K, n and Ω. Moreover, for any $K > 1/\sqrt{n-1}$, Williams constructed Lipschitz data f with Lipschitz constant K for which the Dirichlet problem has no classical solutions. A similar result was proved in the two-dimensional case by Lau ([8]). If the domain Ω is unbounded, then problem (1), (2) is no more uniquely solvable because the asymptotic behavior of the solutions at infinity can be different. In particular, when Ω is an exterior domain, i.e., the complement of a compact subset in $\mathbb{R}^2$, then every solution of the minimal surface equation has the following C^1-convergent asymptotic development:

$$u(x) = c_1 x_1 + c_2 x_2 + c\,\ell n\,|x| + O(1) \tag{3}$$

when $|x| \to \infty$, with some appropriate constants c_i, c. For exterior domains Krust ([5]) extended the result of Jenkins and Serrin ([3]) under the corresponding hypotheses. We also note that existence results for unbounded but convex domains in arbitrary dimensions were proved by Massari and Miranda ([9]) excluding the case of the half space, and in the case of the half plane by Collin and Krust ([2]). As for more general surfaces than graphs, the exterior Plateau problem was investigated in [12], [13], [7].

In our preceding paper [6] we constructed smooth boundary data for any exterior domain, which has large enough $C^{0,1/2}$-norm and arbitrary small oscillation but for which the exterior Dirichlet problem has no solution, irrespective of its asymptotic behavior. In the present paper we extend this result as well as the main existence and nonexistence results of Williams ([14]) to exterior domains. In particular, concerning nonexistence, we sharpen our previous result by constructing boundary data with bounded $C^{0,1}$-norm and with arbitrarily small oscillation which do not extend as a solution of (1) on an exterior domain. Concerning existence, we confine ourselves to solutions with logarithmic growth, i.e., $c_1 = c_2 = 0$ in (3). The general case could also be treated but leads to unpleasant conditions.

We suppose that Ω is a domain of class C^2 and let $\partial^-\Omega$ denote the concave part of $\partial\Omega$; i.e., $\partial^-\Omega = \text{ clos } \{x \in \partial\Omega : \kappa(x) < 0\}$, where κ is the curvature of $\partial\Omega$ with respect to the interior unit normal. For $x \in \partial^-\Omega$ we denote by $R(x)$ the maximal radius of discs contained in $\mathbb{R}^2 \backslash \Omega$ and touching $\partial\Omega$ at x. With the definition $R_* = \text{ inf } \{R(x) : x \in \partial^-\Omega\}$, we can now state our main existence result.

Theorem E. *For any* $\mu \in (-R_*, R_*), \delta > 0,$ *and* $K \in \big[0, \sqrt{1-|\mu|/R_*}\,\big)$ *there exists* $\varepsilon = \varepsilon(\mu, \delta, K, \Omega) > 0$ *such that, whenever* $f \in C^0(\partial\Omega)$*, with*

$$|f(y) - f(x)| \le K|y - x|, \quad x \in \partial^-\Omega, \quad y \in \partial\Omega, \quad |y - x| \le \delta \tag{4}$$

and

$$\underset{\partial\Omega}{osc}\, f < \varepsilon, \tag{5}$$

the Dirichlet problem (1), (2) *possesses a solution* $u \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ *satisfying*

$$u(x) = \mu\ \ell n\,|x| + O(1) \text{ as } |x| \to \infty. \tag{6}$$

Moreover, u *is the unique solution with these properties.*

The condition $|\mu| < R_*$ is obviously sharp, as can be seen in the case $\Omega_R = \{x \in \mathbb{R}^2 : |x| > R\}$ where $R_* = R$; indeed, by a well-known and simple comparison argument using the half catenoids $\varphi_{\lambda,0}$ below it follows immediately that there are no solutions of (1) on Ω_R (with arbitrary boundary data) which satisfy (6) with $|\mu| > R_*$. Let us observe that for bounded solutions ($\mu = 0$) our condition on K agrees with Williams' condition in the two-dimensional case, namely $K < 1$. Finally, we extend Williams' nonexistence theorem which demonstrates the sharpness of the condition $K < 1$ for bounded domains to the case of exterior domains and solutions with arbitrary behavior at infinity by proving

Theorem N. *Let* $\Omega \subset \mathbb{R}^2$ *be a domain such that* $\mathbb{R}^2 \backslash \Omega$ *is bounded and* $\partial\Omega$ *is of class* $C^{2,\alpha}$, $0 < \alpha < 1$. *Then for any* $\varepsilon > 0$ *and* $K > 1$ *there are boundary data* f *which are the restriction of a polynomial, such that* $osc\, f < \varepsilon$, $1 < Lip(f) < K$, *and the Dirichlet problem* (1), (2) *has no solution in* Ω.

Theorem N is deduced from a purely local nonexistence theorem; cf. Theorem 3.2 below. Williams' nonexistence result is local as well, but he admits only solutions u with $\inf_{\partial\Omega} f \le u \le \sup_{\partial\Omega} \mathrm{f}$, a condition which is satisfied on bounded but not necessarily on unbounded domains.

2. Proof of the Existence Theorem. In this section we give the proof of Theorem E. Our method will be that of Perron, based on the notion of sub- and supersolutions; cf. [11], 7.2. By definition, a subsolution of (1) in a domain Ω is a continuous function v in Ω such that whenever Ω_0 is a bounded subdomain of Ω, $\bar{\Omega}_0 \subset \Omega$, and $u \in C^2(\Omega_0) \cap C^0(\bar{\Omega}_0)$ is a solution to (1) in Ω_0 with $v \leq u$ on $\partial\Omega_0$, then it follows that $v \leq u$ in all of Ω_0. Correspondingly, w is called a supersolution if $-w$ is a subsolution.

Given boundary data f on $\partial\Omega$ and some supersolution w in Ω such that

$$\liminf_{y \to x} w(y) \geq f(x) \qquad \text{for all } x \in \partial\Omega,$$

one considers the family $F_{f,w}$ of all subsolutions v in Ω with $v \leq w$ in Ω and $\limsup_{y \to x} v(y) \leq f(x)$ for all $x \in \partial\Omega$. Provided that $F_{f,w}$ is nonempty one then defines the Perron solution of the Dirichlet problem with boundary data f, and below w, as

$$u(x) = \sup\{v(x) : v \in F_{f,w}\}.$$

Whereas u is always (i.e., for any Ω) a solution of (1) in Ω (the same holds for a large class of second-order elliptic partial differential equations), the assumption of the assigned boundary data is of course not always certain, but in fact depends on the nature of the data and Ω (and, in the general case, on the equation replacing (1)). In order to show that the boundary data is actually assumed, suitable sub- and supersolutions must be constructed. For points $x \in \partial^-\Omega$ we use half catenoids

$$\varphi_{\lambda,x}(y) = \lambda\,(\cos\mathrm{h})^{-1}\Big(\frac{|y-z|}{|\lambda|}\Big), \quad z = x + |\lambda|\nu(x),$$

where $\nu(x)$ is the outer unit normal at x.

We continue to use the notation already introduced in Section 1 above. We may clearly assume that the growth constant μ in (6) is nonnegative. Choosing an arbitrary but fixed point $p \in \partial^-\Omega$, we have the inequality

$$m + \varphi_{\mu,p}(y) \leq f(y) \leq M + \varphi_{\mu,p}(y), \qquad y \in \partial\Omega, \tag{7}$$

with $m = \inf_{\partial\Omega} f - \sup_{\partial\Omega} \varphi_{\mu,p}$ and $M = \sup_{\partial\Omega} f$. Since $w = M + \varphi_{\mu,p}$ is a supersolution in Ω and, by (7), $m + \varphi_{\mu,p} \in F_{f,w}$, the Perron solution u corresponding to this family $F_{f,w}$ is well defined. From the definition of u we immediately have

$$m + \varphi_{\mu,p} \leq u \leq M + \varphi_{\mu,p},$$

and hence $u(x) = \mu\,\ell n\,|x| + O(1)$ for $|x| \to \infty$. We need therefore only worry about the assumption of the boundary data. Let us first suppose that f satisfies the condition

$$|f(y) - f(x)| \le \varphi_{\bar\mu,x}(y), \qquad x \in \partial^-\Omega, \qquad y \in \partial\Omega \tag{8}$$

for some $\bar\mu \in (\,\mu, R_*\,)$. Since $\bar\mu > \mu$ we have

$$f(x) + \varphi_{\bar\mu,y}(y) > M + \varphi_{\mu,p}(y) = w(y)$$

for all $x \in \partial^-\Omega$ and all y on a circle of sufficiently large radius. It therefore follows from (8) and the maximum principle that all subsolutions $v \in F_{f,w}$ satisfy

$$v(y) \le f(x) + \varphi_{\bar\mu,x}(y), \qquad x \in \partial^-\Omega, \qquad y \in \Omega,$$

and hence

$$u(y) \le f(x) + \varphi_{\bar\mu,x}(y), \qquad x \in \partial^-\Omega, \qquad y \in \Omega. \tag{9}$$

On the other hand, (8) implies that $f(x) - \varphi_{\bar\mu,x} \in F_{f,w}$, and hence

$$f(x) - \varphi_{\bar\mu,x}(y) \le u(y), \qquad x \in \partial^-\Omega, \qquad y \in \Omega. \tag{10}$$

Inequalities (9) and (10) immediately show that $u \in C^0(\Omega \cup \partial^-\Omega)$ and $u(x) = f(x)$ at all points $x \in \partial^-\Omega$. Since $\varphi_{\bar\mu,x}$ is Hölder continuous with exponent 1/2, it follows, moreover, that u satisfies the same local Hölder condition on $\Omega \cup \partial^-\Omega$. On the convex part of the boundary, i.e., $\partial\Omega \setminus \partial^-\Omega$, no further conditions on the data f besides continuity is needed. Given $x \in \partial\Omega \setminus \partial^-\Omega$, we may choose $\rho > 0$ such that $\Omega \cap U_\rho(x)$ is convex, where $U_\rho(x)$ denotes the open disc of radius ρ and center x. Hence the Dirichlet problem for (1) is solvable on $\Omega \cap U_\rho(x)$ with arbitrary continuous boundary data. Therefore, taking (7) into account, we can find solutions $w_x^\pm$ to (1) on $\Omega \cap U_\rho(x)$, with $w_x^\pm \in C^0(\overline{\Omega \cap U_\rho(x)})$, such that

$$\begin{aligned}
&\text{(i)}\ \ w_x^\pm(x) = f(x)\\
&\text{(ii)}\ w_x^+(y) \ge f(y), w_x^-(y) \le f(y), \quad y \in \partial\Omega \cap U_\rho(x)\\
&\text{(iii)} w_x^+(y) > M + \varphi_{\mu,p}(y),\ w_x^-(y) < m + \varphi_{\mu,p}(y), \quad y \in \Omega \cap \partial U_\rho(x).
\end{aligned} \tag{11}$$

From the maximum principle we immediately obtain

$$v(y) \le w_x^+(y), \quad y \in \Omega \cap U_\rho(x)$$

for any subsolution $v \in F_{f,w}$. Hence

$$u(y) \le w_x^+(y), \qquad y \in \Omega \cap U_\rho(x). \tag{12}$$

On the other hand, setting

$$W_x^-(y) = \begin{cases} \max\{w_x^-(y), m + \varphi_{\mu,p}(y)\} & \text{if } y \in \Omega \cap U_\rho(x) \\ m + \varphi_{\mu,p}(y) & \text{if } y \in \Omega \backslash U_\rho(x), \end{cases}$$

we see from (11, iii) that $W_x^- \in C^0(\bar{\Omega}) \cap C^{0,1}(\Omega)$ and that W_x^- is a subsolution of (1) in Ω. Moreover, it follows from (6), (7), and (11, ii) that $W_x^- \in F_{f,w}$. We therefore obtain $W_x^-(y) \le u(y)$ for $y \in \Omega$, and in particular,

$$W_x^-(y) \le u(y), \qquad y \in \Omega \cap U_\rho(x). \tag{13}$$

Inequalities (12), (13) immediately imply that $u(x) = f(x)$.

So far we have shown that the validity of (8) guarantees the solvability of the Dirichlet problem under the asymptotic condition (6). It is therefore enough to exhibit a positive quantity $\varepsilon = \varepsilon(\mu, \delta, K, \Omega)$ such that (4) and (5) imply (8). By elementary calculation one finds that, for any $x \in \partial^- \Omega$ and $0 < \lambda \le R_* \le R(x)$,

$$\varphi_{\lambda,x}(y) = \sqrt{1 + \lambda\kappa(x)}\,|y - x| + o\,(|y - x|), \qquad y \in \partial\Omega.$$

Since $|\kappa(x)| \le 1/R(x) \le 1/R_*$, we obtain

$$\varphi_{\lambda,x}(y) \ge \sqrt{1 - \tfrac{\lambda}{R_*}}\;|y - x| + o\,(|y - x|), \qquad y \in \partial\Omega. \tag{14}$$

Given $\mu \in [0, R_*)$ and $K \in [0, \sqrt{1 - \mu/R_*}\,)$ we may choose $\bar{\mu} \in (\,\mu, R_*)$ such that $K < \sqrt{1 - \bar{\mu}/R_*}$. Hence by (14)

$$\varphi_{\bar{\mu},x}(y) \ge K|y - x| \quad \text{for all} \quad x \in \partial^-\Omega, \qquad y \in \partial\Omega, \qquad |y - x| \le \delta \tag{15}$$

for a suitable $\delta > 0$ (not exceeding the earlier value for δ). Then we define

$$\varepsilon = \inf\{\varphi_{\bar{\mu},x}(y) : x \in \partial^-\Omega, y \in \partial\Omega, |y - x| \ge \delta\}.$$

Since $\bar{\mu} < R_*$, the exterior discs of radius $\bar{\mu}$ which touch at some $x \in \partial^-\Omega$ have no further point (besides x) in common with $\partial\Omega$, and hence $\varepsilon > 0$.

It is immediate from (15) and the definition of ε that the conditions (4) and (5) of Theorem E imply (8), which proves the existence part of the theorem. The uniqueness of the solution follows from a result of Collin-Krust ([2, Theorem 2]).

3. Nonexistence. In this section we give an extension of Williams' nonexistence theorem ([14, Theorem 4]). From this extension, which is a purely local result, we immediately obtain nonexistence for the Dirichlet problem (1), (2) on arbitrary unbounded domains, irrespective of the asymptotic behavior of solutions. For the convenience of the reader we restate the two-dimensional version of Williams' theorem as Theorem 3.1 below. We remark that his theorem holds in n dimensions (replacing curvature by mean curvature and the condition $K > 1$ by $K > 1/\sqrt{n-1}$) whereas the proof of our Theorem 3.2 below is valid only for $n = 2$.

Theorem 3.1 ([14])**.** *Let Γ be an arc of class $C^{2,\alpha}$ in $\mathbb{R}^2$ with nonzero curvature, $x^0 \in \Gamma$ and T^0 a unit tangent vector of Γ at x^0. For sufficiently small $\delta > 0$, denote by Ω_δ the component of $U_\delta(x^0)\backslash\Gamma$ which is not convex. Then there is no solution u of* (1) *which is defined and positive on Ω_δ, continuous up to $\Gamma_\delta = \Gamma \cap U_\delta(x^0)$ and which takes on the boundary data*

$$u(x) = K|\langle T^0, x - x^0\rangle| \qquad x \in \Gamma_\delta, \tag{16}$$

where K is any number greater than 1 *and $\langle , \rangle$ denotes the Euclidean scalar product.*

It is essential for the proof of the above theorem that the solution is assumed to be positive. In the case of bounded domains this is enough to construct boundary data which do not admit any solution of the Dirichlet problem, since the positivity of solutions follows from the positivity of the boundary data. This is however not the case for unbounded domains. Therefore we find it necessary to extend Williams' theorem, by dropping any additional positivity assumption on solutions. Our extension, however, requires a more complicated construction for boundary data, namely an infinite sequence of corners as in (16).

By a topological argument which invokes the Jordan curve theorem, and therefore limits the validity of our proof to two dimensions, we are able to reduce the situation to Theorem 3.1.

Theorem 3.2. *Let Γ, Ω_δ, and Γ_δ be as in Theorem* 3.1*. Then for any $\varepsilon > 0$ and any $K > 1$ there exist boundary data $f \in C^{0,1}(\Gamma_\delta)$, with osc $f < \varepsilon$,*

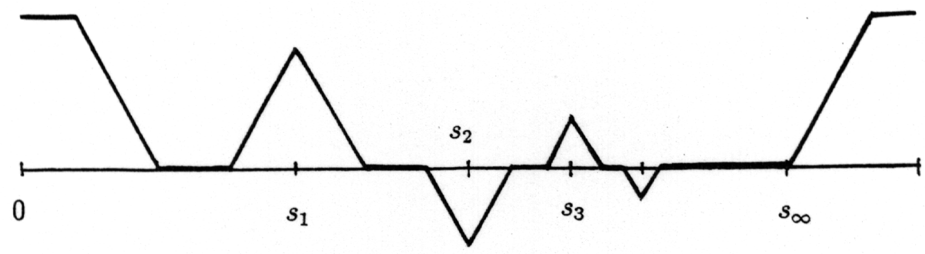

Figure 1. Graph of f.

Lip$(f) = K$, *such that no solution of* (1) *in* Ω_δ *exists which is continuous on* $\Omega_\delta \cup \Gamma_\delta$ *and takes on the boundary data* f *on* Γ_δ.

Proof. Let $\gamma : [0, l] \to \bar{\Gamma}_\delta$ be a parametrization by arc-length and choose a strictly increasing sequence $0 = s_0 < s_1 < \cdots$ with $s_\infty = \lim s_j < l$. Given K, we may choose numbers $\varepsilon_j, 0 < \varepsilon_j < \varepsilon$, so small that the connected component $\Gamma_{\delta,j}$ of the set $\{x \in \Gamma_\delta| : \langle \gamma'(s_j), x - \gamma(s_j)\rangle| < \varepsilon_j/K\}$ which contains $\gamma(s_j)$ is contained in $\gamma(\,(\frac{1}{2}(s_{j-1} + s_j), \frac{1}{2}(s_j + s_{j+1}))\,)$. It follows that the boundary segments $\Gamma_{\delta,1}$, $\Gamma_{\delta,2}, \ldots$ are pairwise disjoint and that the linear functions

$$\varphi_j(x) = K\langle \gamma'(s_j),\, x - \gamma(s_j)\rangle$$

take on the values ε_j and $-\varepsilon_j$ at the two endpoints of $\Gamma_{\delta,j}$, respectively. Next define f on Γ_δ by

$$f(x) = \begin{cases} -|\varphi_j(x)| + \varepsilon_j & \text{if } x \in \Gamma_{\delta,j},\ j \quad \text{odd} \\ |\varphi_j(x)| - \varepsilon_j & \text{if } x \in \Gamma_{\delta,j},\ j \quad \text{even} \\ \min\{|x - \gamma(\frac{1}{2}s_1)|, \varepsilon\} & \text{if } x \in \gamma([0, \frac{1}{2}s_1]) \\ \min\{|x - \gamma(s_\infty)|, \varepsilon\} & \text{if } x \in \gamma([s_\infty, l]) \\ \varepsilon & \text{elsewhere.} \end{cases}$$

See Figure 1.

Now assume for the sake of contradiction that a solution u of (1) exists in Ω_δ which is continuous on $\Omega_\delta \cup \Gamma_\delta$ and equals f on Γ_δ. Denote by $\Omega_{\delta,j}$ the connected components of $(\Omega_\delta \cup \Gamma_\delta)\backslash u^{-1}(0)$ which contain the boundary segment $\Gamma_{\delta,j}$, $j = 1, 2, \ldots$. Of course, the sets $\Omega_{\delta,j}$ can be identical for different j's. Let us now join a point $x^1 \in \gamma\,(\,(0, \frac{1}{2}s_1)\,)$ with a point $x^2 \in \gamma\,(\,(s_\infty, l)\,)$ by an embedded analytic arc σ contained in $\Omega_\delta \cup \{x^1, x^2\}$; e.g., we may choose σ as a circular arc. Since $u(x^i) = f(x^i) > 0$ $(i = 1, 2)$ and u is analytic in

Ω_δ, only finitely many different nodal domains $\Omega_{\delta,j}$ can intersect σ. If there are infinitely many different nodal domains, then infinitely many of them must therefore be contained in the subdomain $\tilde{\Omega}_\delta$ of Ω_δ which is enclosed by σ and the segment of Γ between x^1 and x^2. In case there exist only finitely many different domains $\Omega_{\delta,j,}$ there must be at least one Ω_{δ,j_0} which contains several of the boundary segments $\Gamma_{\delta,j}$, say for $j = j_0 < j_1 < \cdots$. Since f is positive (negative) on $\Gamma_{\delta,j}$ for j odd (even), the numbers j_0 and j_1 must be both odd or both even, and hence, by the Jordan curve theorem, the domain Ω_{δ,j_0} encloses a domain $\Omega_{\delta,k}$ where k has parity different from j_0. In both cases we obtain the existence of some $\Omega_{\delta,k}$ such that $\overline{\Omega_{\delta,k}} \subset \Omega_\delta \cup \Gamma_\delta$. Thus

$$u|\,(\partial\Omega_{\delta,k} \backslash \Gamma_\delta) = 0.$$

We may clearly assume that k is the least number j with $\Gamma_{\delta,j} \subset \Omega_{\delta,k}$ and, since we may also assume that the sequence ε_j is decreasing, it follows therefore from the maximum principle that $0 \le u \le \varepsilon_k$ or $-\varepsilon_k \le u \le 0$ on $\Omega_{\delta,k}$, depending on whether k is odd or even. Therefore we may apply Theorem 3.1 to either $\varepsilon_k - u$ or $u + \varepsilon_k$ in a neighborhood of the point $\gamma(s_k) \in \Gamma$ to obtain a contradiction to the assumed existence of u.

Corollary 3.3. *Let $\Omega \subset \mathbb{R}^2$ be any domain such that $\partial\Omega$ contains a boundary segment Γ of class $C^{2,\alpha}$ which has negative curvature with respect to the interior normal. Then, for any $\varepsilon > 0$ and $K > 1$ there exist boundary data $f \in \mathrm{Lip}\,(\Gamma)$, $\mathrm{Lip}(f) = K$, $\operatorname{osc} f < \varepsilon$ such that the Dirichlet problem* (1), (2) *has no solution on $\bar{\Omega}$ for any boundary data g on $\partial\Omega$ such that $g|\Gamma = f$.*

Remark 3.4. By interpolation, it follows that the C^β-norm of the boundary data constructed in Theorem 3.2 can be made arbitrarily small for any $\beta \in [0, 1)$.

Corollary 3.5. (*Instability of solutions of the Dirichlet problem*) *Let $\Omega \subset \mathbb{R}^2$ be a domain of class $C^{2,\alpha}$, and let $\varphi : \partial\Omega \to \mathbb{R}$ be such that φ is Lipschitz continuous near some point $x^0 \in \partial\Omega$ where $\partial\Omega$ is strictly concave. Then for any $\beta \in [0, 1)$ there are perturbations $\tilde{\varphi}$ of φ such that the support of $\tilde{\varphi} - \varphi$ is contained in an arbitrarily small neighborhood of x^0, the C^β-norm of $\tilde{\varphi} - \varphi$ is arbitrarily small, but the Dirichlet problem with boundary data $\tilde{\varphi}$ has no classical solution.*

Proof. We may assume that $\varphi(x^0) = 0$. For all sufficiently small δ such that the interior curvature of $\partial\Omega \cap U_{2\delta}(x^0)$ is negative, we define $\tilde{\varphi}$ on $\partial\Omega \cap U_\delta(x^0)$ as the function f constructed in Theorem 3.2 with $\operatorname{osc} f < \delta$ and

$\mathrm{Lip}\,(f) = K < 2$. Then we may find a Lipschitz extension $\tilde{\varphi}$ of f to all of $\partial\Omega$ such that $\tilde{\varphi} = \varphi$ outside of $U_{2\delta}(x^0) \cap \partial\Omega$ and $\mathrm{Lip}\,(\tilde{\varphi}|\,\partial\Omega \cap U_{2\delta}(x^0)) \leq 2(1 + \mathrm{Lip}\,(\varphi|\,\partial\Omega \cap U_{2\delta}(x^0))$. The conclusion then follows by the argument in Remark 3.4.

Our final result is the nonexistence Theorem N. For exterior domains we show that it is not the lack of smoothness of the data f which prevents the solvability of the Dirichlet problem, but rather the size of the first derivatives.

Proof of Theorem N. For simplicity of exposition we shall assume that $\mathbb{R}^2\backslash\Omega$ is homeomorphic to the disc, though the following arguments extend easily to the general case.

Since Ω is an exterior domain it contains some strictly concave boundary segment Γ. According to Theorem 3.2 we may choose boundary data $f \in \mathrm{Lip}\,(\partial\Omega)$ with $\mathrm{osc}\, f < \varepsilon$ and $1 < \mathrm{Lip}(f) < K$, which do not extend as a solution of (1) to any neighborhood of $\partial\Omega$ in $\bar{\Omega}$. Let us now construct a sequence (f_l) of polynomials such that $f_l \to f$ uniformly on $\partial\Omega$ and $\mathrm{osc}\, f_l|_{\partial\Omega} < \varepsilon$, $\mathrm{Lip}\, f_l|\partial\Omega < K$, and let us assume that the exterior Dirichlet problem with data f_l has solutions u_l, $l = 1, 2, \ldots$ Let $B_R = \{x \in \mathbb{R}^3 : |x| \leq R\}$ and choose R so large that the boundary graphs of $f_l|\partial\Omega$ are contained in $B_{R/2}$. The minimal surfaces $M_l = \mathrm{graph}\,(u_l) \cap B_R$ are of annulus type, $M_l \subset B_R \cap (\Omega \times \mathbb{R})$. Since any minimal graph on Ω is area-minimizing in $\Omega \times \mathbb{R}$, we have a uniform area estimate

$$A(M_l) \leq \mathrm{const}\,(R), \qquad l = 1, 2, \ldots. \tag{17}$$

Now choose conformal harmonic representations $X_l : \{z \in \mathbb{R}^2 : \rho_l \leq |z| \leq 1\} \to M_l$, $0 < \rho_l < 1$, $X_l(\{|z| = \rho_2\}) = \mathrm{graph}\,(f_l|\partial\Omega)$, where the X_l are incompressible as maps into $B_R \cap (\bar{\Omega} \times \mathbb{R})$; i.e. the image of any homotopically nontrivial loop in the annulus $A(\rho_l) = \{\rho_l \leq |z| \leq 1\}$ under X_l is homotopically nontrivial in $B_R \cap (\bar{\Omega} \times \mathbb{R})$. Then, (17) becomes an energy bound

$$E(X_l) = \frac{1}{2} \int |\nabla X_l|^2 \,\mathrm{dz} \leq \mathrm{const}\,(R). \tag{18}$$

By standard arguments (cf. [1, IV.2]) it follows from (18) and the incompressibility of X_l that $\rho_l \in [\delta, 1 - \delta]$ for some $\delta > 0$ and that the maps $\theta \longmapsto X_l(\rho_l \cos\theta, \rho_l \sin\theta)$ are equicontinuous. We may therefore find a subsequence of the X_l, again denoted by X_l, such that $\rho_l \to \rho, X_l$ converges uniformly on any annulus $\rho_l \leq |z| \leq r < 1$ towards a map $X \in C^0(\{\rho \leq |z| < 1\}, B_R \cap (\bar{\Omega} \times \mathbb{R}))$, which is harmonic and conformal on the open annulus

$\{\rho < |z| < 1\}$. Clearly, by the uniform convergence of the boundary values $X_l|\{|z| = \rho_l\}$ the map $X|\{|z| = \rho\}$ parametrizes the boundary graph $(f|\partial\Omega)$ and, in particular, X is not constant. Therefore, X is a conformal (possibly branched) minimal immersion. Since X is the limit of embeddings, the image of X, denoted by M, must be an embedded minimal surface in $B_R \cap (\bar{\Omega} \times \mathbb{R})$. The Gauss map of a minimal surface being either open or locally constant, it follows moreover that M, as a limit of graphs, must again be a graph or part of a vertical plane. The latter is impossible since M contains the boundary graph $(f|\partial\Omega)$. Therefore M is the graph of some solution u of (1) which is defined in some neighborhood of $\partial\Omega$ in $\bar{\Omega}$ and assumes boundary data f on $\partial\Omega$. This solution however cannot exist due to Theorem 3.2, and hence the u_l cannot exist for $l \geq l_0$, proving the theorem.

Corollary 3.6. *In the case of exterior domains the instability result corresponding to Corollary* 3.5 *holds in the class of* C^∞ *boundary data.*

REFERENCES

[1] R. Courant, "Dirichlet's Principle, Conformal Mapping, and Minimal Surfaces," Interscience, New York, 1950.

[2] P. Collin, R. Krust, *Le problème de Dirichlet pour l'équation des surfaces minimales sur des domaines non borns,* Bull. Soc. Math., France, 119 (1991), 443–462.

[3] H. Jenkins and J. Serrin, *The Dirichlet problem for the minimal surface equation in higher dimensions,* J. Reine Angew. Math., 229 (1968), 170–187.

[4] A. Korn, *Über Minimalflächen, deren Randkurven wenig von ebenen Kurven abweichen,* Abhand. König. Preuss. Akad. Berlin, Phys.-Math. Cl. II (1909), 1–37.

[5] R. Krust, *Remarques sur le problème exterieur de Plateau,* Duke Math. J., 59 (1989), 161–173.

[6] N. Kutev and F. Tomi, *Nonexistence and instability in the exterior Dirichlet problem for the minimal surface equation in the plane,* Pac. J. Math., 170 (1995), 535–542.

[7] E. Kuwert, *Embedded solutions for exterior minimal surface problems,* Manuscripta Math., 70 (1990), 51–65.

[8] C.-P. Lau, *The existence and nonexistence of a non-parametric solution to equations of minimal surface type,* Analysis, 4 (1984), 227–241.

[9] U. Massari and M. Miranda, "Minimal Surfaces of Codimension One," Math. Studies 91, North Holland, Amsterdam-New York-Oxford, 1984.

[10] C.H. Müntz, *Zum Randwertproblem der partiellen Differentialgleichung der Minimalflächen,* J. Reine Angew. Math., 139 (1911), 52–79.

[11] J.C.C. Nitsche, "Lectures on Minimal Surfaces," vol. I, Cambridge University Press, 1989.

[12] F. Tomi, *Plateau's problem for minimal surfaces with a catenoidal end,* Arch. Math., 59 (1990), 165–172.

[13] F. Tomi and R. Ye, *The exterior Plateau problem,* Math. Z., 205 (1990), 233–245.

[14] G.H. Williams, *The Dirichlet problem for the minimal surface equation with Lipschitz continuous boundary data,* J. Reine Angew. Math., 354 (1984), 123–140.