Differential and Integral Equations Volume 11, Number 6, November 1998, pp. 895–915.

KDV AND BO EQUATIONS WITH BORE-LIKE DATA

R. Iorio* and F. Linares
IMPA, Estrada Dona Castorina 110, 22640-320 Rio de Janeiro, Brazil

M. Scialom
IMECC-UNICAMP, CP 6065, 13083-970, Campinas SP, Brazil

(Submitted by: J.L. Bona)

Abstract. The global well-posedness of the initial-value problem associated to the Korteweg-de Vries equation with bore-like data is studied. In particular, we show that in the Sobolev space H^s, $s \geq 2$, the solutions of this problem remain bounded for any time. We also establish similar results for solutions of the initial-value problem associated to the Benjamin-Ono equation with this kind of data.

1. Introduction. In this work we are interested in the initial-value problem associated to the Korteweg-de Vries (KdV) and Benjamin-Ono (BO) equations with the so-called bore-like initial data, that is,

$$\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u = 0, \quad x \in \mathbb{R},\ t > 0 \\ u(x,0) = g(x) \end{cases} \tag{1.1}$$

and

$$\begin{cases} \partial_t u + \sigma\partial_x^2 u + u\partial_x u = 0, \quad x \in \mathbb{R},\ t > 0 \\ u(x,0) = g(x) \end{cases} \tag{1.2}$$

where σ denotes the Hilbert transform and g satisfies

$$\begin{cases} \quad i)\ g(x) \to C_{\pm} \ \text{ as } \quad x \to \pm\infty, \\ \ ii)\ g' \in H^s,\ \text{for some } s \geq 0, \\ iii)\ (g - C_+) \in L^2([0,\infty)) \quad \text{and} \quad (g - C_-) \in L^2((-\infty, 0])\,. \end{cases} \tag{1.3}$$

Received for publication October 1997.

*Instituto Politécnico UERJ, Parque da Cascata s/n°, 28630-050 Nova Friburgo RJ, Brazil.

AMS Subject Classifications: 35G25, 35Q53, 35Q30, 76B15.

Notice that a function g with these properties is necessarily bounded. We begin by considering the initial-value problem (IVP) (1.1). The (IVP) (1.1) models the evolution of bores on the surface of a channel, incorporating non-linear and dispersive effects intrinsic to such propagation. Some dissipative effects should also be included as noted in [1], where Bona, Rajopadhye and Schonbek established the existence of a unique global solution u of (1.1) such that $u(x,t)-\psi(x)\in C([0,T];\, H^s(\mathbb{R}))$, for all $T>0$ and $s\geq 3$and some $\psi\in C^\infty$ with $\psi'\in H^\infty$. This solution depends continuously on the initial data g. In what follows we refrain from discussing the physics of the problem at hand. The interested reader should consult [1] and references therein. Our purpose here consists of improving and simplifying the previous results obtained for (1.1). We also point out some smoothing properties associated to an auxiliary model which we use to establish our results. It is constructed as follows: according to Lemma 2.1 below, there exists a function $\psi\in C^\infty$ with $\psi'\in H^\infty$ satisfying (1.3) and such that $g-\psi\in H^{s+1},\ s\geq 0$. It is then natural to define $u(x,t)=w(x,t)+\psi(x)$ and study the (IVP) associated with $w(x,t)$, namely,

$$\begin{cases} \partial_t w+\partial_x^3 w+w\partial_x w+\partial_x(w\psi)+(\psi\psi'+\psi^{(3)})=0,\\ w(x,0)=g(x)-\psi(x)\equiv\phi(x), \end{cases} \tag{1.4}$$

where ϕ is a function in H^r, for some $r\geq 1$. With these comments in mind, our main results can be stated as follows:

Theorem 1.1 (Local well-posedness)**.** *Let $\phi\in H^s, s>3/2$. Then there exist $T=T(s,\|\phi\|_s)$and a unique solution w of the IVP (1.4) such that $w\in C([0,T]:H^s)\cap C^1([0,T]:H^{s-3})$. Moreover, suppose that $\phi^n\to\phi\in H^s$ and w^n is the solution of (1.4) with data $w^n(0)=\phi^n$. Then given $T'\in(0,T)$, there exists $N_0=N_0(T')$ such that for $n\geq N_0$, w^n is defined in $[0,T']$ with*

$$\lim_{n\to\infty}\sup_{[0,T]}\|w^n(t)-w(t)\|_s=0\,.$$

Theorem 1.2 (Global well-posedness)**.** *If $\phi\in H^s,\ s\geq 2$, then the solution w obtained in Theorem 1.1 can be extended for any $T>0$.*

The estimates obtained in the proof of Theorem 1.2 imply that $w\in L^\infty([0,\infty);\, H^2(\mathbb{R}))$. In fact, it can be shown that $w\in L^\infty([0,\infty);H^s(\mathbb{R}))$, $s\geq 2$. This can be done, using the remaining conserved quantities associated to KdV and non-linear interpolation.

Corollary 1.3. *Let w be the solution of IVP* (1.4) *given in Theorem* 1.1. *Then*

i) $u = w + \psi$ *is the unique solution of IVP* (1.1) *satisfying* $u - g \in C([0,T];\, H^s(\mathbb{R}))$, $s > 3/2$.
ii) *If* $s \geq 2$, *then u is a bounded solution of IVP* (1.1) *for any* $T > 0$.

Before proceeding, some remarks are in order. Note that the PDE in (1.4) is just KdV perturbed by two terms. The first one is linear in w and has a x -dependent coefficient, while the second does not depend on w and is rather smooth. In view of this, it is to be expected that (1.4) can be treated very much in the same way as KdV. This is indeed true, up to a point. If we are below the "Sobolev barrier"; i.e., in the range $s > 3/2$ (in order to easily make sense of the non-linear term) we can employ parabolic regularization and the Bona-Smith approximations to obtain Theorem 1.1 and then prove the *a priori* estimates needed to establish Theorem 1.2 combining the conserved quantities associated to the KdV equation (and in both cases a technical lemma due to T. Kato; see Lemma 2.3 below). It should be noted that local well-posedness can also be obtained using Kato's theory of quasi-linear equations ([6]). In case of KdV however, the sharpest results to date are those obtained by Kenig, Ponce and Vega [7] who have shown that the IVP

$$\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u = 0, & x \in \mathbb{R},\ t > 0 \\ u(x,0) = u_0(x) \end{cases}$$

is locally well-posed in H^s, $s > -3/4$ and globally well-posed in case $s \geq 0$. Unfortunately, their method does not seem to be applicable here because ψ is merely a bounded function and does not belong to L^2. It is interesting to remark that solutions of IVP (1.4) satisfy similar smoothing properties as those satisfied by solutions of the KdV equation, namely,

Theorem 1.4. *Let* $\phi \in H^s$, $s > 3/2$ *and* $0 < T < \infty$. *Then the solution* $w(t)$ *of the IVP* (1.4) *given in Theorem* 1.1 *satisfies* $w(t) \in L^2([0,T] : H^{s+1}(-R,\, R))$ *for any* $R < \infty$.

We will omit the proof of this result since it is almost identical to that given by Kato [5] in the usual KdV case. Finally, it is not difficult to verify that the methods employed in Section 3 can be used to establish the local well-posedness of

$$\begin{cases} \partial_t u + \partial_x^3 u + u^p\partial_x u = 0, & x \in \mathbb{R},\ t > 0 \\ u(x,0) = g(x) \end{cases}$$

for any integer $p > 1$. Global existence is, as far as we know, an open problem in this case. Concerning the IVP (1.2), we obtain results similar to those described for the IVP (1.4) related to local and global well-posedness. As in the previous case we use an auxiliary problem, that is, we look for solutions of the form $u(x,t) = w(x,t) + \psi(x)$ where w satisfies

$$\begin{cases} \partial_t w + \sigma\,\partial_x^2 w + w\partial_x w + \partial_x(w\psi) + (\psi\psi' + \sigma\psi^{(2)}) = 0, \\ w(x,0) = g(x) - \psi(x) \equiv \phi(x). \end{cases} \tag{1.5}$$

with $\phi = g - \psi$ in H^{s+1}, $s \geq 0$ and ψ is such that $\psi \in C^\infty$, and $\psi' \in H^\infty$. The plan of this paper is as follows. In Section 2, we present some preliminary results that we will use in the remainder of this work. Local well-posedness for the IVP (1.3) will be studied in Section 3. Section 4 is devoted to establishing *a priori* estimates and proving the global results for the IVP(1.3). Finally, in Section 5 we show the local and global well-posedness results concerning the IVP (1.2) and (1.5). Before leaving this section we present some standard notation which will be used throughout this work.

Notation. We denote the inner product in H^s by $(\cdot,\cdot)_s$ and write $\|\cdot\|_s$ for the corresponding norm. The Bessel and Riesz potentials of order - s are denoted by $J^s = (1-\partial_x^2)^{s/2}$ and $I^s = (-\partial_x^2)^{s/2}$, respectively.

2. Preliminary results. In this section we will state some results which will be used several times in our development. Some of them are well known and will be presented just for the sake of completeness. The first one concerns the construction of data which allows us to implement the scheme proposed above to study local and global well-posedness of the IVP (1.4).

Lemma 2.1. *Let g satisfy conditions* (i) *and* (ii) *in* (1.3), *then for each $\theta \in (0,\infty)$ there exists a $\psi_\theta \in C^\infty$ such that $\psi'_\theta \in H^\infty$, and $\phi_\theta = g - \psi_\theta \in H^{s+1}$, $s \geq 0$. Moreover,*

$$\|g - \psi_\theta\|_0 \leq \sqrt{2\theta}\, e^{-1/2}\, \|g'\|_0, \quad \|\psi'_\theta\|_s \leq C\|g'\|_0,\ C = C(s,\theta). \tag{2.1}$$

Finally, if g satisfies (iii) *in* (1.3), *then ψ_θ also has the property*

$$\lim_{x\to\pm\infty} \psi_\theta(x) = C_\pm.$$

Proof. Define $\psi_\theta = k_\theta * g$, where $k_\theta(x) = (4\pi\theta)^{-1/2}\exp(-x^2/4\theta)$, and use the Fourier transform to obtain the estimates in (2.1). $\square$

Since we will use the parabolic regularization method to establish existence and uniqueness of the IVP (1.4) we need to study the linear problem

$$\begin{cases} \partial_t u + \partial_x^3 u - \mu \partial_x^2 u = 0, \quad \mu > 0, \ x \in \mathbb{R}, \quad t > 0 \\ u(x,0) = f(x). \end{cases} \tag{2.2}$$

For this problem, we have the following result.

Lemma 2.2. *Let $\mu \geq 0$, $\lambda \geq 0$ and $s \in \mathbb{R}$ and define $W_\mu(t) = \exp\{(-\mu\partial_x^2 - i\partial_x^3)t\}$. Then*

$$\|W_\mu(t)f\|_{s+\lambda} \leq K_\lambda \big(1 + \big(\frac{1}{2\mu t}\big)^\lambda\big)^{1/2} \|f\|_s \, ,$$

for all $t, \mu \in (0,\infty)$, $\lambda \geq 0$ and $f \in H^s$. The map $t \in (0,\infty) \to W_\mu(t)f$ is continuous with respect to the topology of $H^{s+\lambda}(\mathbb{R})$. Moreover, $W_\mu(t)$ defines a C^0 -semigroup in H^s, $s \in \mathbb{R}$ (which can be extended to an unitary group if $\mu = 0$), and $t \in (0,\infty) \to W_\mu(t)f$, $\mu \geq 0$ is the unique solution of (2.2).

Proof. See Lemma 1.1 in [3].

Remark. A similar result is satisfied by solution of the regularized linear problem associated with the Benjamin-Ono equation, i.e.,

$$\begin{cases} \partial_t u + \sigma \partial_x^2 u - \mu \partial_x^2 u = 0, \quad \mu > 0, \ x \in \mathbb{R}, \quad t > 0 \\ u(x,0) = f(x). \end{cases}$$

(see [3]).

Next we will introduce a series of estimates. We begin by presenting a useful result due to Kato (see Lemma A5 in [5]).

Lemma 2.3. *Let $s > \frac{n}{2} + 1$, $t > 1$ and $f, g \in \mathcal{S}(\mathbb{R}^n)$. Then there exists $C = C(s,n,t) > 0$ such that*

$$|(fDg, g)_t| \leq C\{\|\nabla f\|_{s-1}\|g\|_t^2 + \|\nabla f\|_{t-1}\|g\|_t\|g\|_s\},$$

where $D = \partial^\alpha$, $|\alpha| = 1$.

The commutator estimate below is due to Kato and Ponce [6].

Lemma 2.4. *Let $s \geq 0$, $1 < p < \infty$, and $f, g \in \mathcal{S}(\mathbb{R}^n)$. Then*

$$\|[J^s, f]g\|_{L^p} \leq c\{\|\nabla f\|_{L^{p_1}} \|J^{s-1}g\|_{L^{p_2}} + \|J^s f\|_{L^{p_3}} \|g\|_{L^{p_4}}\},$$

where p_2, $p_3 \in (1, \infty)$ are such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$.

The next result, due to Saut and Temam [9], will be used to prove some estimates necessary to show the continuous dependence of solutions on the initial data for the IVP (1.4).

Lemma 2.5. *If $s > 1$, $r > 1/2$ and $f, g \in \mathcal{S}(\mathbb{R}^n)$. Then there exists a constant $C = C(r, s, n)$ such that*

$$\|[D^s, f]g\| \leq C\{\|f\|_s \|g\|_r + \|f\|_{r+1} \|g\|_{s-1}\}.$$

In the next lemma we introduce the so-called Bona-Smith approximations and present their basic properties (see Lemma 5 in [2]).

Lemma 2.6. *Let $\varphi \in C_0^\infty(\mathbb{R})$, $0 \leq \varphi \leq 1$ be such that $supp\, \varphi \in [-1, 1]$, $\varphi(0) = 1$ and if $\psi(x) = 1 - \varphi(x)$, then $\psi^k(0) = 0$, for all $k \in \mathbb{N}$. For $\epsilon > 0$ and $\phi \in H^s$, $s > 0$, define $\phi_\epsilon(x) = (\varphi(\epsilon\xi)\hat{\phi})\check{}(x)$, $x \in \mathbb{R}$. Then, $\phi_\epsilon \in H^\infty(\mathbb{R})$ and for $0 < \epsilon \leq 1$, $r \geq 0$ there exists $C = C(s, r, \varphi) > 0$ such that*

$$\|\phi_\epsilon\|_{s+r} \leq C\epsilon^{-r}\|\phi\|_s, \quad \|\phi_\epsilon - \phi\|_{s-r} \leq C\epsilon^r\|\phi\|_s. \tag{2.3}$$

$\phi_\epsilon \to 0$ in H^s as $\epsilon \to 0$. Moreover, if $\phi^n \to \phi$ in H^s as $n \to \infty$, then $\|\phi_\epsilon^n - \phi^n\|_s \to 0$ as $\epsilon \downarrow 0$, uniformly with respect to n.

3. Local theory.

To establish existence and uniqueness of solutions of the IVP (1.4) we use the parabolic regularization method; that is, we consider the following IVP

$$\begin{cases} \partial_t w + \partial_x^3 w + w\partial_x w + \partial_x(w\psi) + (\psi\psi' + \psi^{(3)}) - \mu\partial_x^2 w = 0, \\ w(x, 0) = g(x) - \psi(x) \equiv \phi. \end{cases} \tag{3.1}$$

The first step is to show the existence and uniqueness of solutions to the IVP (3.1) for any $\mu > 0$.

Proposition 3.1. *Let $\mu > 0$ be fixed and $\phi \in H^s,\ s > 3/2$. Then there exists $T_s = T(s, \|\phi\|_s, \mu) > 0$ and a unique solution w_μ of IVP (3.1) satisfying*

$$w_\mu \in C([0, T_s], H^s(\mathbb{R})) \cap C((0, T_s], H^\infty(\mathbb{R})),$$

where $H^\infty(\mathbb{R})) = \bigcap_r H^r(\mathbb{R})$ endowed with its natural Fréchet-space topology.

Proof. Using Lemma 2.2, it is not difficult to prove that IVP (3.1) is equivalent to the integral equation

$$w_\mu(t) = W_\mu(t)\phi - \int_0^t W_\mu(t-t')(w_\mu \partial_x w_\mu(t') + \partial_x(w_\mu(t')\psi) + (\psi\psi' + \psi^{(3)}))\, dt'$$

in $C([0, T_s], H^s),\ s > 3/2$. Let

$$X_s(T) = \{v \in C([0, T], H^s) \mid \|v(t) - W_\mu(t)\phi\|_s \leq M,\ t \in [0, T]\}$$

and $d_s(u, v) = \sup_{[0,T]} \|u - v\|_s$. Then a standard argument combined with Lemma 2.2 shows that there exists T such that the map

$$\Phi(v)(t) = W_\mu(t)\phi - \int_0^t W_\mu(t-t')(v\partial_x v(t') + \partial_x(v(t')\psi) + (\psi\psi' + \psi^{(3)}))\}\, dt'$$

has a unique fixed point in the complete metric space $(X_s(T),\ d_s)$. This gives us T and $w_\mu \in C([0, T], H^s(\mathbb{R}))$. The fact that $w_\mu \in C((0, T], H^\infty(\mathbb{R}))$ now follows from the integral equation using a simple bootstrapping argument. $\square$

The next step is to study the limit $w = \lim_{\mu \downarrow 0} w_\mu$. First we must show that w_μ can be extended to an interval of time independent of μ.

Lemma 3.2. *Let $s > 3/2$ and $w_\mu \in C([0, T_s], H^s(\mathbb{R}))$ be the solution of the IVP (3.1) with $\mu > 0$. Then w_μ can be extended to an interval $[0, T_s']$, where $T_s' = T_s'(s, \|\phi\|_s)$ is independent of μ. Moreover, there exists a $\rho \in C([0, T_s'], \mathbb{R})$ such that*

$$\|w_\mu(t)\|_s^2 \leq \rho(t), \qquad \rho(0) = \|\phi\|_s^2, \quad t \in [0, T_s']. \tag{3.2}$$

Proof. Since $w_\mu \in C((0, T], H^\infty(\mathbb{R}))$ we may differentiate $\|w_\mu(t)\|_s^2$ with respect to t, and use integration by parts to obtain

$$\begin{aligned}\partial_t \|w_\mu(t)\|_s^2 \leq &-2(w_\mu, w_\mu \partial_x w_\mu)_s - 2(w_\mu, \psi\partial_x w_\mu)_s \\ &- 2(w_\mu, \psi' w_\mu)_s - 2(w_\mu, \psi\psi' + \psi^{(3)})_s\,.\end{aligned} \tag{3.3}$$

Using Lemma 2.3, Lemma 2.4, the Cauchy-Schwartz inequality and the properties of the function ψ, the expressions on the right-hand side of (3.3) can be estimated as follows:

$$|(w_\mu, w_\mu \partial_x w_\mu)_s| \le C_s \|w_\mu\|_s^3, \quad |(w_\mu, \psi \partial_x w_\mu)_s| \le \|\psi'\|_{s_0} \|w_\mu\|_s^2,$$
$$|(w_\mu, \psi' w_\mu)_s| \le \|\psi'\|_{s_0} \|w_\mu\|_s^2,$$
$$|(w_\mu, \psi\psi' + \psi^{(3)})_s| \le C\{(\|\psi'\|_k + \|\psi\|_{L^\infty})^2 + \|\psi'\|_{s+2}\}\|w_\mu\|_s,$$

where, to obtain the last inequality, we combined the Cauchy-Schwartz inequality, interpolation ($\|J^s(\psi\psi')\| \le c\|J^k\psi\|^\theta \|u\|^{1-\theta}$, $s = \theta k$, $k \ge 1$, an integer) and the Leibniz rule. Above, s_0 denotes a positive number sufficiently large. Combining these inequalities with (3.3) we get

$$\partial_t \|w_\mu(t)\|_s^2 \le C'(\|w_\mu\|_s^3 + \|w_\mu\|_s^2 + \|w_\mu\|_s),$$

where $C' = \max(C_s, C\|\psi'\|_{s_0}, C\{(\|\psi'\|_k + \|\psi\|_{L^\infty})^2 + \|\psi'\|_{s+2}\})$. Hence, $\|w_\mu\|_s \le \rho^{1/2}(t)$ in $[0,T]$ where $\rho(t)$ satisfies

$$\partial_t \rho = C(\rho^{3/2} + \rho + \rho^{1/2}), \quad \rho(0) = \|\phi\|_s^2\,.$$

The solution to this Cauchy problem is

$$\rho^{1/2}(t) = \frac{\sqrt{3}}{2}\tan(Ct + \arctan(\frac{2\|\phi\|_s + 1}{\sqrt{3}})) - \frac{1}{2}.$$

It is well defined for any $[0,T]$ such that $0 < T$ and

$$CT + \arctan(\frac{2\|\phi\|_s + 1}{\sqrt{3}}) < \frac{\pi}{2}\,.$$

This proves the lemma. □

Now we are ready to establish the first part of Theorem 1.1.

Proof of Theorem 1.1 (Existence and uniqueness). Let $\mu > 0$ and $w = w_\mu$ and $v = w_\nu$. It is easy to check that

$$\partial_t \|w - v\|_0^2 \le 4M^2|\mu - \nu| + (C\|\psi'\|_{s_0} + 2C_s M)\|w - v\|_0^2$$

for all $t \in [0, T_s']$, where $M = \sup_{[0,T_s']} \rho(t)$ and $C_s = C(s)$, where ρ is defined as in the previous lemma and T_s' is independent of μ. A straightforward application of Gronwall's inequality implies then that $w = \lim_{\mu\downarrow 0} w_\mu$ exists in L^2. Therefore, $t \in [0, T'_s] \to w_\mu(t)$ is continuous and uniformly bounded in L^2 so that $w_\mu(t) \to w$ weakly in H^s uniformly on $[0, T]$. In particular, $w(t)$ is weakly continuous and uniformly bounded by $(\rho(t))^{1/2}$. These observations and the weak continuity of the map $t \in [0, T'_s] \to \partial_x^3 w + w\partial_x w + \partial_x(\psi w) + (\psi\psi' + \psi^{(3)}) \in H^{s-3}$ imply

$$w(t) - w(t') = -\int_{t'}^{t} (\partial_x^3 w + w\partial_x w + \partial_x(\psi w) + (\psi\psi' + \psi^{(3)}) - \mu\partial_x^2 w)(s)\, ds.$$

Thus, $w \in AC([0, T]; H^{s-3}) \cap L^\infty([0, T]; H^s)$ ($AC(I; X)$ denotes the space of absolutely continuous functions from I to X) and satisfies (3.1) almost everywhere in $[0, T]$. Next, we show that w is unique. In pursuit of this result, we take w and v satisfying the above conditions. It follows easily that

$$\partial_t \|w - v\|_0^2 \leq 4\tilde{C}M\, \|w - v\|_0^2.$$

Therefore, Gronwall's inequality implies the uniqueness. Since $\|w(t)\|_s \leq \rho^{1/2}(t)$, it is easy to see that $\limsup_{t\to 0^+} \|w(t)\|_s = \|\phi\|_s$. Thus, continuity at zero is a consequence of weak continuity. If $\tau \in (0, T]$ we get the right continuity from continuity at zero and uniqueness. Left continuity follows because (1.4) is invariant under the transformation $(x, t) = (-x, \tau - t)$. This completes the proof of the first part of Theorem 1.1. □

To prove the second part of Theorem 1.1, that is, the continuous dependence, we will use the Bona-Smith approximations [2, §2]. First we have the following result.

Lemma 3.3. *Let $0 < \delta \leq \epsilon \leq \epsilon_0 < 1$. There exists $C = C(s, T', \|\phi\|_s)$ such that*

$$\sup_{0\leq t\leq T'} \|w_\delta(t) - w_\epsilon(t)\|_s \leq C\big\{\epsilon^{\gamma(s)} + \|\phi_\delta - \phi_\epsilon\|_s\big\}, \tag{3.4}$$

where $\gamma(s) = \frac{\mu s}{\mu+1}$, $0 \leq \mu < s - 3/2$, w_δ and w_ϵ are solutions of problem (3.1) *given by Proposition* 3.1 *with initial data ϕ_δ and ϕ_ϵ, respectively.*

Proof. Let $u = w_\delta - w_\epsilon$ so that

$$\partial_t u + \partial_x^3 u + u\partial_x u - \partial_x(w_\epsilon u) + \partial_x(\psi u) = 0, \qquad u(0) = \phi_\delta - \phi_\epsilon. \tag{3.5}$$

Since ϕ_δ and $\phi_\epsilon \in H^\infty$ we have $w_\delta,\ w_\epsilon \in H^\infty$. Thus, $u(t) \in H^\infty$ for $0 \leq t \leq T'$. Notice that an argument similar to the one used in Lemma 3.2 guarantees that w_δ and w_ϵ can be defined in the whole interval $[0,T']$. Moreover, it can be proved that $\|w_\epsilon\|_s \le C(s,T',\|\phi\|_s)$. Combining integration by parts, Sobolev embedding, the properties of w_μ and ψ with the partial differential equation in (3.5) we obtain,

$$\begin{aligned}\partial_t\|u(t)\|_0^2 &= 2(u,\partial_t u)_0 = -(u^2,\partial_x w_\epsilon)_0 - (u^2,\psi')_0 \\ &\le C(s,\|\phi\|_s,\,\|\psi'\|_{s_0})\|u(t)\|_0^2\,.\end{aligned} \tag{3.6}$$

Integration over $(0,t)$, Gronwall's inequality and Lemma 2.6 with $\delta<\epsilon$ give

$$\sup_{0\le t\le T'}\|w_\delta(t)-w_\epsilon(t)\|_0 \le C\{\epsilon^{2s}+\delta^{2s}\}\|\phi\|_s \le C(s,T',\|\phi\|_s)\,\epsilon^{2s}. \tag{3.7}$$

Next we turn to $\|I^s u(t)\|_0$. We again use the equation in (3.5) to obtain the following:

$$\begin{aligned}&\partial_t\|I^s u(t)\|_0^2 = 2(I^s u(t),\partial_t I^s u(t))_0 \\ &= -2(I^s u, I^s(u\partial_x u))_0 + 2(I^s u, I^s\partial_x(w_\epsilon u))_0 - 2(I^s u, I^s\partial_x(\psi u))_0 \\ &\le 2|(I^s u, I^s(u\partial_x u))_0| + 2|(I^s u, I^s(w_\epsilon\partial_x u))_0| + 2|(I^s u, I^s(\partial_x w_\epsilon u)_0| \\ &\quad + 2|(I^s u, I^s(\psi\partial_x u))_0| + 2|(I^s u, I^s(\psi' u))_0|.\end{aligned} \tag{3.8}$$

Use of the Cauchy-Schwartz inequality, Lemma 2.5 and the properties of w_δ and w_ϵ leads to

$$\begin{aligned}|(I^s u, I^s(u\partial_x u))_0| &\le \|I^s u\|_0\|[I^s,u]\partial_x u\|_0 + \tfrac{1}{2}\,|(\partial_x u I^s u, I^s u)_0| \\ &\le C_s\|w_\delta - w_\epsilon\|_s\|u\|_s^2 \le C\|u(t)\|_s^2.\end{aligned} \tag{3.9}$$

A similar argument shows that

$$|(I^s u, I^s(w_\epsilon\partial_x u))_0| \le C(s,T',\|\phi\|_s)\|u(t)\|_s^2. \tag{3.10}$$

Lemma 2.3 implies

$$\begin{aligned}|(I^s u, I^s(\psi\partial_x u))_0| &\le |(u,\psi\partial_x u)_s| + \|\psi\|_{L^\infty}\|u\|_s^2 \\ &\le C(\|\psi'\|_s + \|\psi\|_{L^\infty})\|u\|_s^2\,.\end{aligned} \tag{3.11}$$

Since $s>3/2$, it follows easily that

$$|(I^s u, I^s(\psi' u))_0| \le C\|\psi'\|_s\|u\|_s^2\,. \tag{3.12}$$

Finally, using the Cauchy-Schwartz inequality, Lemma 2.5 with $\frac{1}{2} \le \gamma \le s-1$ and Sobolev embedding, we obtain

$$|(I^s u(t), I^s(u\partial_x w_\epsilon)_0| \le \|u\|_s\{C(s,\gamma)[\|w_\epsilon\|_{s+1}\|u\|_\gamma + \|w_\epsilon\|_{\gamma+2}\|u\|_{s-1}]\} \\ + \|w_\epsilon\|_s\|u\|_s \,.$$

It is not difficult to show that $\|w_\epsilon\|_{s+1} \le C(s,T',\|\phi\|_s)\|\phi_\epsilon\|_{s+1}$ (see Lemma 2.6). Hence, this inequality, interpolation, Lemma 2.6 and (3.7)imply that

$$\begin{aligned} \|w_\epsilon\|_{s+1}\|u\|_\gamma\|u\|_s &\le C\epsilon^{-1}\|u\|_s^{1+\gamma/s}\|u\|_0^{1-\gamma/s} \\ &\le C\{2\nu s\epsilon^{-(1+\nu)} + \|u\|_s^2\}. \end{aligned} \tag{3.13}$$

If $\gamma \in (s-2, s-1)$ the above remark and Lemma 2.6, with $r = \gamma + 2$, combine to yield

$$\|w_\epsilon\|_{\gamma+2} \le C(s,T',\|\phi\|_s)\|\phi_\epsilon\|_{\gamma+2} \le C\epsilon^{s-2-\gamma}\|\phi\|_s \,.$$

From interpolation, the previous inequality and (3.8), it follows that

$$\|w_\epsilon\|_{\gamma+2}\|u\|_{s-1}\|u\|_s \le C'\{\epsilon^{2\nu s} + \|u\|_s^2\}\,. \tag{3.14}$$

Combining (3.13) and (3.14) we have

$$|(I^s u, I^s(\partial_x w_\epsilon\, u))_0| \le C\{\epsilon^{\frac{2\nu s}{1+\nu}} + \|u\|_s^2\}. \tag{3.15}$$

Combining (3.9), (3.10), (3.11), (3.12) and (3.15) it follows that

$$\partial_t\|I^s u(t)\|_0^2 \le C\{\epsilon^{\frac{2\nu s}{1+\nu}} + \|u(t)\|_s^2\}.$$

Integration from 0 to t yields

$$\|I^s u(t)\|_0^2 \le \|u(0)\|_s^2 + CT'\epsilon^{\frac{2\nu s}{1+\nu}} + C\int_0^t \|u(t')\|_s^2\, dt', \text{ for } t \in (0,T'). \tag{3.16}$$

From inequalities (3.8) and (3.16) we obtain

$$\|u(t)\|_s^2 \le \|u(t)\|_0^2 + \|I^s u(t)\|_0^2 \le C\epsilon^{2s} + \|u(0)\|_s^2 + C\epsilon^{\frac{2\nu s}{1+\nu}} + C\int_0^t \|u(t')\|_s^2\, dt' \,.$$

An application of Gronwall's inequality finishes the proof. □

It is not difficult to verify that $w_\epsilon \underset{\epsilon\downarrow 0}{\to} v(t)$ in H^s uniformly in $t \in [0, T']$ and $v(t) = w(t)$ is a solution of (3.1). We are now in position to prove the second part of Theorem 1.1.

Proof of Theorem 1.1 (Continuous dependence). Let $\phi_\epsilon,\ \phi_\epsilon^n$ be the Bona-Smith approximations associated with ϕ, ϕ^n, respectively. Consider $w_\epsilon = w_\epsilon(t)$ and $w_\epsilon^n = w_\epsilon^n(t)$ the solutions of (3.1) with initial data $\phi,\ \phi^n$.Using the observations in the beginning of the previous lemma it is easy to see that $w_\epsilon(t)$ and $w_\epsilon^n(t)$ can be extended to $[0, T']$ if $n \geq \mathbb{N}_0$ and $\epsilon_0 \geq \epsilon \geq 0$. So for $t \in [0, T']$, we have

$$\|w^n(t) - w(t)\|_s \leq \|w^n(t) - w_\epsilon^n(t)\|_s + \|w_\epsilon(t) - w_\epsilon^n(t)\|_s + \|w_\epsilon(t) - w(t)\|_s$$

Letting $\delta \downarrow 0$ in (3.4) and using the remarks on the convergence, we obtain

$$\sup_{0\leq t\leq T'} \|w_\epsilon(t) - w(t)\|_s \leq C\big\{\epsilon^{\gamma(s)} + \|\phi_\epsilon - \phi\|_s\big\}, \tag{3.17}$$

$$\sup_{0\leq t\leq T'} \|w^n(t) - w_\epsilon^n(t)\|_s \leq C\big\{\epsilon^{\gamma(s)} + \|\phi^n - \phi_\epsilon^n\|_s\big\}, \tag{3.18}$$

From Lemma 2.6, (3.17) and (3.18) it follows that $\|w^n(t) - w_\epsilon^n(t)\|_s$ and $\|w_\epsilon(t) - w_\epsilon^n(t)\|_s \to 0$ as $\epsilon \downarrow 0$ uniformly on $[0, T']$ and n. It remains to show that $\|w_\epsilon(t) - w_\epsilon^n(t)\|_s \to 0$ as $n \to \infty$ and $\epsilon \downarrow 0$. So to do this we consider $v = w_\epsilon^n - w_\epsilon$ which satisfies

$$\partial_t v + \partial_x^3 v + v\partial_x v - \partial_x(w_\epsilon v) + \partial_x(\psi v) = 0, \qquad v(0) = \phi_\epsilon^n - \phi_\epsilon,$$

for $0 \leq t \leq T'$. The argument used in (3.6) and (3.7) gives us

$$\|v(t)\|_0^2 \leq C\|v(0)\|_0^2. \tag{3.20}$$

The same argument used to obtain (3.16) shows that

$$\|I^s v(t)\|_0^2 \leq \|v(0)\|_s^2 + CT'\epsilon^{2\gamma(s)} + c\int_0^t \|v(t')\|_s^2\, dt', \text{ for } t \in (0, T'). \tag{3.21}$$

Combining (3.20), (3.21) and Gronwall's inequality, we obtain

$$\|w_\epsilon(t) - w_\epsilon^n(t)\|_s \leq \big\{\|\phi_\epsilon^n - \phi_\epsilon\|_s + \epsilon^{2\gamma(s)}\big\} \to 0$$

as $n \to 0$ and $\epsilon \downarrow 0$. This completes the proof of Theorem 1.1. □

4. Global theory. In this section we will prove Theorem 1.2 concerning the global well-posedness of the IVP (1.4). This will be a consequence of global well-posedness of the IVP

$$\begin{cases} \partial_t w + \partial_x^3 w + w\partial_x w + \partial_x(w\psi) + (\psi\psi' + \psi^{(3)}) - \mu\partial_x^2 w = 0, \\ w(x,0) = \phi(x). \end{cases} \tag{4.1}$$

with $\mu > 0$. First we establish a series of *a priori* estimates.

Lemma 4.1. *If w satisfies* (4.1), *then for* $0 \le t \le T$

$$\begin{aligned} \|w\|_0 &\le \left\{\|\phi\|_0 + C\|\psi'\|_{s_0}\,(1 + \|\psi\|_{L^\infty})\,t\right\} \exp\left(t\|\psi'\|_{s_0}\right) \\ &\equiv h(\|\psi'\|_{s_0}, \|\psi\|_{L^\infty}, \|\phi\|_0, t). \end{aligned} \tag{4.2}$$

Proof. Multiplying by w and integrating with respect to x the equation in (4.1), we have

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_0^2 = -\int w\partial_x(\psi w)\,dx - \int w(\psi\psi' + \psi^{(3)})\,dx + \int w\partial_x^2 w\,dx\,.$$

Integration by parts, the Cauchy-Schwartz inequality and the hypotheses on ψ yield

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_0^2 \le \frac{1}{2}\|\psi'\|_{s_0}\|w(t)\|_0^2 + (1 + \|\psi\|_{L^\infty}\|\psi'\|_{s_0})\|w(t)\|_0\,.$$

An application of Gronwall's inequality gives (4.2). □

To find an *a priori* estimate for the H^2-norm of the solutions of (4.1), we will make use of the following quantities which are conserved by the KdV flow.

$$\begin{aligned} \Phi_3(u) &= \int ((\partial_x u)^2 - \frac{1}{3}u^3)\,dx, \\ \Phi_4(u) &= \int ((\partial_x^2 u)^2 - \frac{5}{3}u(\partial_x u)^2 + \frac{5}{36}u^4)\,dx. \end{aligned}$$

Notice that the equation in (4.1) can be written as

$$\partial_t w + \partial_x\Phi_3'(w) + \partial_x(w\psi) + (\psi\psi' + \psi^{(3)}) - \mu\partial_x^2 w = 0, \tag{4.3}$$

where Φ_3' denotes the Gateaux derivative of the real valued functional Φ_3. It is also easy to check that

$$\partial_t\Phi_4(w(t)) = (\Phi_4' w(t), \partial_t w(t)) \quad \text{and} \quad (\Phi_4'(w(t)), \partial_x \Phi_3'(w(t))) = 0,$$

where

$$\Phi_3'(u) = \partial_x^2 u + \tfrac{1}{2}u^2 \quad \text{and} \quad \Phi_4'(u) = 2\partial_x^4 u + \tfrac{5}{3}(\partial_x u)^2 + \tfrac{10}{3} u\, \partial_x^2 u + \tfrac{5}{9}u^3. \tag{4.4}$$

Note also that $\Phi_4(\phi) \le C(\|\phi\|_2^2 + \|\phi\|_2^3 + \|\phi\|_2^4)$. Next we have the following estimate involving $\Phi_4(w)$.

Lemma 4.2. *If w satisfies* (4.1), *then*

$$\begin{aligned} \Phi_4(w(t)) \le \Phi_4(\phi) + C\int_0^t (\|\psi\|_{L^\infty} + 8)\|\psi'\|_{s_0}\|w(s)\|_2^2 ds \\ + G(\|\psi\|_{L^\infty}, \|\psi'\|_{s_0}, \|\phi\|_0, t, \mu), \end{aligned} \tag{4.5}$$

where G is defined in (4.9) *below.*

Proof. From the above formulas (4.3)–(4.4), we have

$$\partial_t\Phi_4(w(t)) = (\Phi_4'(w), \partial_x(w\psi)) - (\Phi_4'(w), (\psi\psi' + \psi^{(3)})) + \mu(\Phi_4'(w), \partial_x^2 w). \tag{4.6}$$

We first show that

$$\mu(\Phi_4'(w), \partial_x^2 w) \le \mu C\|w\|_0^6. \tag{4.7}$$

In fact, integration by parts, the Cauchy-Schwartz, Gagliardo-Nirenberg and Young inequalities give us

$$\begin{aligned} \mu(\Phi_4'(w), \partial_x^2 w) &= \mu\{-2\|\partial_x^3 w\|_2^2 + \tfrac{10}{3}(w\partial_x^2 w, \partial_x^2 w) + \tfrac{5}{9}(w^3, \partial_x^2 w)\} \\ &\le \mu\{(\eta_0 + \eta_1 - 2)\|\partial_x^3 w\|_0^2 + (C(\eta_0) + C(\eta_1))\|w\|_0^6\} \end{aligned}$$

and (4.7) follows choosing η_0 and $\eta_1 \in (0,1)$. It is not difficult to show that

$$\begin{aligned} (\Phi_4'(w), (\psi\psi' + \psi^{(3)})) \le 2\|w\|_2^2 + C(\|\psi\|_{L^\infty} + 1)^2\|\psi'\|_{s_0}^2\|w\|_2^2 \\ + C(\|\psi\|_{L^\infty} + 1)^2\|\psi'\|_{s_0}^2 + C_2(\{\|\psi\|_{L^\infty} + 1)^2\|\psi'\|_{s_0}^2\}^{8/7}\|w\|_0^{22/7}. \end{aligned} \tag{4.8}$$

Finally, using integration by parts, the Cauchy-Schwartz inequality, the assumptions on ψ and Gagliardo-Nirenberg interpolation, we obtain the estimate

$$-(\Phi_4'(w), \partial_x(w\psi)) \leq (5 + c\|\psi\|_\infty)\|w\|_2^2 + C_1\|w\|_0^{10}\|\psi'\|_{s_0}^8 \\ + C_2(1 + \|\psi'\|_{s_0}^{1/2})\|w\|_0^{14/3}\|\psi'\|_{s_0}^{8/3}.$$

Hence, it follows that

$$\partial_t\Phi_4(w(t)) \leq C(8 + \|\psi\|_{L^\infty})\|\psi'\|_{s_0}\|w\|_2^2 + \mu c\|w\|_0^6 \\ + c\|\psi'\|_{s_0}(1 + \|\psi\|_{L^\infty})^{8/7}\|w\|_0^{22/7} + C_1\|\psi\|_{L^\infty}^8\|w\|_0^{10} + C_2\|w\|_0^{14/3} \\ + \|\psi'\|_{s_0}(1 + \|\psi\|_{L^\infty})^2 .$$

Integrating from 0 to t and applying Lemma 4.1, we have

$$\Phi_4(w(t)) \leq \Phi_4(\phi) + C(8 + \|\psi\|_{L^\infty})\|\psi'\|_{s_0}\int_0^t \|w(s)\|_2^2 ds + \mu c\int_0^t h^6(s)\,ds \\ + c\|\psi'\|_{s_0}(1 + \|\psi\|_{L^\infty})^{8/7}\int_0^t h^{22/7}(s)\,ds + C_1(\|\psi'\|_{s_0})\|\psi\|_{L^\infty}^8\int_0^t h^{10}(s)\,ds \\ + C_2(\|\psi'\|_{s_0})\int_0^t h^{14/3}(s)\,ds + \|\psi'\|_{s_0}(1 + \|\psi\|_{L^\infty})^2 t \tag{4.9}$$

$$\equiv \Phi_4(\phi) + C(8 + \|\psi\|_{L^\infty})\|\psi'\|_{s_0}\int_0^t\|w(s)\|_2^2 ds + G(\|\phi\|, \|\psi'\|_{s_0}, \|\psi\|_{L^\infty}, t, \mu),$$

where h is defined as in (4.2). This completes the proof of (4.5). □

Next we have an H^2 -*a priori* estimate.

Lemma 4.3. *If w satisfies* (4.1) *with $\phi \in H^2$, then*

$$\|w(t)\|_2^2 \leq \left\{\|\phi\|_2^2 + \int_0^t (\Phi_4(\phi) + h^2(s) + h^{14/3}(s) + G(\cdot,\, s,\, \mu))\,ds\right\} \\ \times \exp\{C(8 + \|\psi\|_{L^\infty})\|\psi'\|_{s_0} t\} \tag{4.10}$$

where h and G are defined in (4.2) *and* (4.9), *respectively.*

Proof. We have the following inequalities:

$$\|w(t)\|_2^2 \leq \|w(t)\|_0^2 + \|\partial_x^2 w(t)\|_0^2 \\ = \|w(t)\|_0^2 + \Phi_4(w(t)) + \frac{5}{3}\int w(\partial_x w)^2\,dx - \frac{5}{36}\int w^4\,dx \tag{4.11} \\ \leq \|w(t)\|_0^2 + \Phi_4(w(t)) + \eta\|w\|_2^2 + c_2(\eta)\|w\|_0^{14/3} + \eta\|w\|_2^2 + c_2(\eta)\|w\|_0^{14/3},$$

where, in the last step, we have made use of Sobolev's embedding, the Cauchy-Schwartz, Gagliardo-Nirenberg and Young inequalities. Choosing $\eta = 1/4$ we get from (4.11)

$$\|w(t)\|_2^2 \le c(\|w\|_0^2 + \|w\|_0^{14/3}) + \Phi_4(w(t)).$$

The desired result then follows from Lemma 4.1, Lemma 4.2 and Gronwall's inequality. □

The preceding estimates show that the limiting process used to establish local existence can be repeated on any interval $[0, T]$, with $\rho(t)$ replaced by the right-hand side of (4.10), which is bounded with respect to $\mu \in [0, 1]$. Therefore we have:

Theorem 4.4. *Let $\phi \in H^2(\mathbb{R})$. Then for each $\mu \ge 0$ there exists a unique w_μ satisfying* (4.1) *and such that*

$$w_\mu \in C([0,\infty); H^2(\mathbb{R})) \cap C^1([0,\infty); L^2(\mathbb{R})).$$

Proof. This follows from Lemma 4.3 and Theorem 1.1.

Theorem 4.5. *Let $\phi \in H^s(\mathbb{R})$, $s \ge 2$. Then for each $\mu \ge 0$ there exists a unique w_μ satisfying* (4.1) *and such that*

$$w_\mu \in C([0,\infty); H^s(\mathbb{R})) \cap C^1([0,\infty); H^{s-2}(\mathbb{R})).$$

Proof. This follows using a standard argument (see for instance [3]). □

Notice that Theorem 1.2 is a consequence of the result just proved.

5. Benjamin-Ono equation. In this section we will establish the local and global well-posedness results concerning the IVP (1.2). As we mentioned in the introduction, we first do so for the IVP (1.5), that is,

$$\begin{cases} \partial_t w + \sigma\partial_x^2 w + w\partial_x w + \partial_x(w\psi) + (\psi\psi' + \sigma\psi^{(2)}) = 0, \\ w(x,0) = g(x) - \psi(x) \equiv \phi(x), \end{cases}$$

where ϕ is a function in H^r, $r \ge 1$.

Theorem 5.1 (Local well-posedness)**.** *Let $\phi \in H^s$, $s > 3/2$. Then there exist $T = T(s, \|\phi\|_s)$ and a unique solution w of the IVP* (1.2) *such that $w \in C([0,T] : H^s) \cap C^1([0,T] : H^{s-3})$. Moreover, suppose that $\phi^n \to \phi \in H^s$ and w^n is the solution of* (1.4) *with data $w^n(0) = \phi^n$, then given $T' \in (0,T)$, there exists $N_0 = N_0(T')$ such that for $n \geq N_0$, w^n is defined in $[0,T']$ with*

$$\lim_{n\to\infty} \sup_{[0,T]} \|w^n(t) - w(t)\|_s = 0.$$

Proof. This result follows from the remark after Lemma 2.2 and the same argument employed in the proof of Theorem 1.1.

Theorem 5.2 (Global well-posedness)**.** *If $\phi \in H^s$, $s \geq 2$, then the solution w obtained in Theorem* 5.1 *can be extended for any $T > 0$.*

To show this, we first establish the result in H^2. As in the case of the KdV equation, it will be a consequence of proving a global estimate for solutions of the IVP

$$\begin{cases} \partial_t w + \sigma\partial_x^2 w + w\partial_x w + \partial_x(w\psi) + (\psi\psi' + \sigma\psi^{(2)}) - \mu\partial_x^2 w = 0, \\ w(x,0) = \phi(x). \end{cases} \tag{5.1}$$

We begin by establishing an *a priori* estimate for solutions of the IVP (5.1) in the L^2 norm.

Lemma 5.3. *If w satisfies* (5.1), *then for $0 \leq t \leq T$*

$$\|w\|_0 \leq \big\{\|\phi\|_0 + C\|\psi'\|_{s_0}\,(1 + \|\psi\|_{L^\infty})\,t\big\} \exp\left(t\|\psi'\|_{s_0}\right) \equiv h(\psi, \|\phi\|_0, t). \tag{5.2}$$

Proof. The argument is the same as that used in the proof of Lemma 4.1 and will therefore be omitted. □

To find *a priori* estimates for the H^1 - and H^2 -norms of the solutions of (5.1), we will make use of the following quantities which are conserved by the BO flow (see [4] and the references therein):

$$\Phi_3(u) = \int \Big\{\frac{u^3}{3} + u\sigma\partial_x u\Big\}\,dx, \quad \Phi_4(u) = \int \Big\{\frac{u^4}{4} + \frac{3}{2}u^2\sigma\partial_x u + 2(\partial_x u)^2\Big\}\,dx,$$

$$\begin{aligned} \Phi_6(u) = &\int \Big\{\frac{u^6}{6} + \frac{5}{4}u^4\sigma\partial_x u + \frac{5}{3}u^3\sigma(u\partial_x u)\Big\}\,dx \\ &- \int 10\big\{(\partial_x u)^2\sigma\partial_x u + 2u\partial_x^2 u(\sigma\partial_x u)\big\} + 8(\partial_x^2 u)^2\,dx \\ &+ \frac{5}{2}\int \big\{5u^2(\partial_x u)^2 + u^2(\sigma\partial_x u)^2 + 2u(\sigma\partial_x u)(\sigma u\partial_x u)\big\}\,dx\,. \end{aligned}$$

Lemma 5.4. *If w satisfies* (5.1), *then*

$$\Phi_4(w(t)) \leq \Phi_4(\phi) + C(\psi)t + \int_0^t G_0(\psi, \|\phi\|_0, s, \mu)\|w(s)\|_1^2\, ds.$$

Proof. We first make some observations similar to those preceding Lemma 4.2. Let

$$\Phi_3'(u) = u^2 + 2\sigma\partial_x u$$

and

$$\Phi_4'(u) = u^3 + 3u\sigma\partial_x u + 3\sigma(u\partial_x u) - 4\partial_x^2 u$$

denote the Gateaux derivatives of the functionals Φ_3 and Φ_4, respectively. Equation (5.1) can be written as

$$\partial_t w + \partial_x \Phi_3'(w) + \partial_x(w\psi) + (\psi\psi' + \sigma\psi^{(2)}) - \mu\partial_x^2 w = 0.$$

Also note that

$$\partial_t \Phi_4(w(t)) = (\Phi_4'(w(t)), \partial_t w(t))$$

and

$$(\Phi_4'(w(t)), \partial_x \Phi_3'(w(t))) = 0.$$

With the previous observations in mind, we have

$$\begin{aligned}\partial_t \Phi_4(w(t)) = &-(\Phi_4'(w(t)), \partial_x(\psi w)) - (\Phi_4'(w(t)), \psi\psi' + \sigma\psi^{(2)}) \\ &+ \mu(\Phi_4'(w(t)), \partial_x^2 w(t)) \,. \end{aligned} \tag{5.3}$$

Using arguments similar to those of the previous section it follows that

$$\begin{aligned} -(\Phi_4' w(t), \psi\psi' + \sigma\psi^{(2)}) &\leq c(\psi) + C(\psi)(1 + h(t, \psi, \|\phi\|_0))\|w\|_1^2 \,, \\ \mu(\Phi_4'(w(t)), \partial_x^2 w(t)) &\leq -\mu c\tilde{h}(t, \psi, \|\phi\|_0)\|w\|_1^2 \,, \\ -(\Phi_4'(w(t)), \partial_x(\psi w)) &\leq C(\psi)(1 + h(t, \psi, \|\phi\|_0) + h^2(t, \psi, \|\phi\|_0))\|w(t)\|_1^2 \,. \end{aligned} \tag{5.4}$$

In the last inequality we made use of Calderon's commutator theorem (see [10]) to control the terms $(w\sigma\partial_x w, \partial_x(\psi w))$ and $(\sigma(w\partial_x w), \partial_x(\psi w))$. Combining the inequalities in (5.4) and (5.3) we have

$$\partial_t \Phi_4(w(t)) \leq C(\psi) + G_0(t, \psi, \|\phi\|_1)\|w(t)\|_1^2, \tag{5.5}$$

where

$$G_0(t, \psi, \|\phi\|_1) = 1 + (1 + \mu)h(t, \psi, \|\phi\|_1) + h^2(t, \psi, \|\phi\|_1).$$

The result then follows upon integrating (5.5) with respect to time. $\square$

Lemma 5.5. *If w is a solution of* (5.1), *then*

$$\|w(t)\|_1^2 \le \{\int_0^t G_1(\psi, s, \|\phi\|_1)\, ds\} \exp\{c \int_0^t G_0(\psi, \|\phi\|_0, s, \mu)\, ds\}.$$

Proof. Using the conservation law Φ_4, Lemma 5.4, the Cauchy-Schwartz inequality, Lemma 2.3 and Young's inequality, it follows that

$$2\|w(t)\|_1^2 \le 2\|w(t)\|_0^2 + C(\|\phi\|_1) + C(\psi)t + \int_0^t G_0(\psi, \|\phi\|_0, s, \mu)\|w(s)\|_1^2\, ds \\ + C(h^2(t, \psi, \|\phi\|_0) + h^6(t, \psi, \|\phi\|_0)) + \|w(t)\|_1^2\,. \tag{5.6}$$

Combining (5.6) and Gronwall's inequality yields the desired result.

Lemma 5.6. *If w satisfies* (5.1), *then*

$$-\Phi_6(w(t)) \le -\Phi_6(\phi) + C(\psi)t + \int_0^t F_0(\psi, \|\phi\|_1, s, \mu)\|w(s)\|_1^2\, ds. \tag{5.7}$$

Proof. To show inequality (5.7) we use observations similar to those at the beginning of the proof of Lemma 5.4. So,

$$-\partial_t \Phi_6(w(t)) = (\Phi_6'(w(t)), \partial_x(\psi w)) + (\Phi_6'(w(t)), \psi\psi' + \sigma\psi^{(2)}) \\ - \mu(\Phi_6'(w(t)), \partial_x^2 w(t)), \tag{5.8}$$

where $\Phi_6'(w)$ is the Gateaux derivative of the functional Φ_6. Using the Cauchy-Schwartz inequality, the Sobolev embedding theorem, Lemma 5.3, Lemma 5.4, Young's inequality and Lemma B.3 in [4], it is not difficult to show that

$$(\Phi_6'(w(t)), \partial_x(\psi w)) \le f_1(t, \psi, \|\phi\|_2, \mu)\|w(t)\|_2^2, \tag{5.9}$$

$$(\Phi_6'(w(t)), \psi\psi' + \sigma\psi^{(2)}) \le f_2(t, \psi, \|\phi\|_2, \mu)\|w(t)\|_2^2 + C(\psi), \tag{5.10}$$

$$-\mu(\Phi_6'(w(t)), \partial_x^2 w(t)) \le \mu\, f_3(t, \psi, \|\phi\|_2, \mu)\|w(t)\|_2^2\,. \tag{5.11}$$

Combining (5.9), (5.10), (5.11) and (5.8) and integrating in time, inequality(5.7) follows by defining

$$F_0(t, \psi, \|\phi\|_2, \mu) = \sum_{i=1}^{2} f_i(t, \psi, \|\phi\|_2, \mu) + \mu\, f_3(t, \psi, \|\phi\|_2, \mu).$$

Lemma 5.7. *If w is solution of* (5.1), *then*

$$\|w(t)\|_2^2 \leq \{c\int_0^t F_1(\psi, s, \|\phi\|_2)\, ds\} \exp\{c\int_0^t F_0(\psi, \|\phi\|_2, s, \mu)\, ds\}. \quad (5.12)$$

Proof. From the conservation law $\Phi_6(u)$, Lemma 5.6, inequality (4.10) in [4] and Young's inequality, it follows that

$$8\|w(t)\|_2^2 \leq F_1(\psi, \|\phi\|_1, t, \mu) + \int_0^t F_0(\psi, \|\phi\|_1, s, \mu)\|w(s)\|_2^2\, ds + 4\|w(t)\|_2^2\,.$$

So we have

$$\|w(t)\|_2^2 \leq c\, F_1(\psi, \|\phi\|_1, t, \mu) + c\int_0^t F_0(\psi, \|\phi\|_1, s, \mu)\|w(s)\|_2^2\, ds\,.$$

The result then follows by applying Gronwall's inequality. □

The preceding estimates show that the limiting process used to establish local existence can be repeated on any interval $[0, T]$.

Theorem 5.8. *Let $\phi \in H^2(\mathbb{R})$. Then for each $\mu \geq 0$, there exists a unique w_μ satisfying* (5.1) *and such that*

$$w_\mu \in C([0, \infty); H^2(\mathbb{R})) \cap C^1([0, \infty); L^2(\mathbb{R})).$$

Proof. This follows from Lemma 5.7 and Theorem 5.1.

This result can be extended to include all $s \geq 2$ using the arguments employed in the proof of Theorem 4.4.

Corollary 5.8. *Let w be the solution of IVP* (1.2) *given in Theorem* 5.1. *Then*

i) *$u = w + \psi$ is the unique solution of IVP* (1.2) *satisfying*

$$u - g \in C([0, T]; H^s(\mathbb{R})),\ s > 3/2.$$

ii) *If $s \geq 2$, then u is a bounded solution of IVP* (1.2) *for any $T > 0$.*

REFERENCES

[1] J. L. Bona, S. V. Rajopadhye, and M. E. Schonbek, *Models for propagation of Bores I. Two dimensional Theory*, Diff. Int. Eqns **7** (1994), 699–734.

[2] J. L. Bona, and R. Smith, *The initial value problem for the Korteweg-de Vries equation*, Philos Trans. Roy. Soc. London Ser A **278** (1975), 555–604.

[3] R. Iorio, *KdV, BO and friends in weighted Sobolev Spaces*, Functional-Analytic Methods for PDE, Lectures Notes in Math. **1450** (1990), 104–121.

[4] R. Iorio, *On the Cauchy problem for the Benjamin Ono equation*, Comm. PDE **11** (1986), 1031–1081.

[5] T. Kato, *On the Cauchy problem for the (generalized) Korteweg-de Vries equation*, Studies In Appl. Math., Adv. in Math. Suppl. Studies **8** (1983), 93–128.

[6] T. Kato, *Quasilinear equations of evolutions, with applications to Partial Differential Equations*, Lecture Notes in Math. **448** (1975), 25–70.

[7] T. Kato, and G. Ponce, *Commutator estimates and Euler and Navier-Stokes equation*, Comm. Pure Appl. Math **41** (1988), 891–907.

[8] C. E. Kenig, G. Ponce, and L. Vega, *A bilinear form with application to the Korteweg-de Vries equation*, Journal AMS, 9 (1996), 573–603.

[9] J-C. Saut, and R. Temam, *Remarks on the Korteweg-de Vries equation*, Israel J. of Math **24** (1976), 78–87.

[10] A. Torchinsky, *Real-variable methods in Harmonic Analysis*, Academic Press, New York, 1986.