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FINITE-ENERGY SOLUTIONS, QUANTIZATION EFFECTS AND LIOUVILLE-TYPE RESULTS FOR A VARIANT OF THE GINZBURG-LANDAU SYSTEMS IN $\mathbb{R}^K$

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Abstract. We study the finite-energy solutions of a generalization of the classical Ginzburg-Landau system (problem (0.1)). In the first part of this paper we consider the solutions of (0.1) satisfying $K = M = 2$ and $\int_{\mathbb{R}^2} P_n(|u|^2) < \infty$. We establish a phenomenon of quantization for the "mass" $\int_{\mathbb{R}^2} P_n(|u|^2)$ that generalizes a well-known result of Brezis, Merle and Rivière for the classic Ginzburg-Landau system of equations in $\mathbb{R}^2$. The second part is devoted to solutions of (0.1) satisfying $\int_{\mathbb{R}^K} |\nabla u|^2 < \infty$. We establish some Liouville-type results and in particular we prove that any locally L^3 solution of the Ginzburg-Landau system $-\Delta u = u(1 - |u|^2)$ in $\mathbb{R}^K$ (K>1) satisfying $\int_{\mathbb{R}^K} |\nabla u|^2 < \infty$, is a constant function. We also obtain that any solution of (0.1) with finite energy (0.4) is a constant. In the last section we prove that any solution u of (0.1) is smooth and satisfies $|u|^2 \le k_n$ on $\mathbb{R}^K$.

0. Introduction. Let P_n be a polynomial of the form

$$P_n(t) = \frac{1}{2}\prod_{j=1}^{n}(t - k_j)^2,$$

where $n \in \mathbf{N}^+$ and $0 < k_1 < \cdots < k_n < \infty$. In this paper we study the finite-energy solutions of the problem

$$\begin{cases} u \in L^{4n-1}_{loc}(\mathbb{R}^K, \mathbb{R}^M) & (K, M) \in \mathbf{N}^+ \times \mathbf{N}^+ \\ \Delta u = u P_n'(|u|^2) & \text{in } \mathcal{D}'(\mathbb{R}^K, \mathbb{R}^M), \end{cases} \tag{0.1}$$

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that is, the solutions u of (0.1) satisfying

$$E_2(u) := \int_{\mathbb{R}^K} P_n(|u|^2) < \infty \tag{0.2}$$

or satisfying

$$E_3(u) := \int_{\mathbb{R}^K} |\nabla u|^2 < \infty \tag{0.3}$$

or

$$E_4(u) := E_2(u) + E_3(u) < \infty. \tag{0.4}$$

The diagonal system (0.1) is related to models introduced in classical field theory [6]; e.g. when $n = 1$ we get the well-known Ginzburg-Landau system used in the study of phase transition problems occurring in superfluidity, superconductivity and XY-magnetism [8, 9]. Our first result is a phenomenon of quantization for the "mass" $\int_{\mathbb{R}^2} P_n(|u|^2)$ when $M = 2$, namely,

$$\int_{\mathbb{R}^2} P_n(|u|^2) = \pi k_j d^2$$

for some integer $d \in \mathbf{Z}$ and $j \in \{1, \dots, n\}$. This result generalizes that of Brezis, Merle and Rivière for the classic Ginzburg-Landau system of equations in $\mathbb{R}^2$ (see [2], [5]). In the second part of this paper some Liouville-type results and Pohozaev-type identities are proven for solutions of (0.1) satisfying $(K, M) \in \mathbf{N}^+ \times \mathbf{N}^+$ and

$$\int_{\mathbb{R}^K} |\nabla u|^2 < \infty.$$

In particular we prove that any locally L^3 solution of the Ginzburg-Landau system

$$-\Delta u = u(1 - |u|^2) \quad \text{in} \quad \mathbb{R}^K \quad (K > 1)$$

satisfying (0.3) is a constant function. We also obtain that any solution of (0.1) with finite energy (0.4) and K>1 is a constant. In the third part we study the finite-energy solutions of (0.1) in the one-dimensional case. This is a special case since we always have non-constant finite-energy solutions. To conclude we prove that any solution of (0.1) is smooth and satisfies $|u|^2 \le k_n$ on $\mathbb{R}^K$.

Remark 0.1. In the sequel we shall use the following properties of polynomials P_n.

$$P_n^{'}(t) = \prod_{j=1}^{n}(t-k_j)Q_n(t),$$

where

$$Q_n(t) = n\prod_{l=1}^{n-1}(t-r_l)$$

and $r_l \in (k_l, k_{l+1})$, for any $l = 1, \dots, n-1$;

$$Q_n^{'}(t) = \sum_{\substack{i,j=1\\ i\neq j}}^{n}(\prod_{\substack{s=1\\ s\neq i,j}}^{n}(t-k_s)) = n\sum_{s=1}^{n-1}(\prod_{\substack{l=1\\ l\neq s}}^{n-1}(t-r_l)),$$

with the standard convention

$$\sum_{\alpha\in\emptyset} = 0 \qquad \text{and} \qquad \prod_{\alpha\in\emptyset} = 1.$$

1. Statement of results. Our first result is

Theorem 1.1. *Let u be a solution of the problem (0.1)–(0.2) with $K = M = 2$, then*

(i) *u is smooth and $|u|^2 \le k_n$*

(ii) *there is $j \in \{1, \dots, n\}$ such that*

$$\lim_{|x|\to\infty}|u(x)|^2 = k_j \quad \textit{and} \quad \int_{\mathbb{R}^2}(|u|^2 - k_j)^2 < \infty$$

(iii) *there is $d \in Z$ such that*

$$\int_{\mathbb{R}^2} P_n(|u|^2) = \pi k_j d^2.$$

The second result of this paper is stated as follows.

Theorem 1.2. *For $K>1$, let u be any solution of* (0.1) *and* (0.3). *Then one and only one of the following holds* (*where $P_{n,\alpha}:=P_n-P_n(\alpha)$*) :

(i) *u is a constant function and there is an integer $j\in\{1,\dots,n\}$ such that $|u|^2=k_j$*

(ii) *$u\in C^\infty\cap W^{1,\infty}\cap H^1$, $|u|^2\le k_n$ and u satisfies*:

$$\left(\tfrac{K}{2}-1\right)\int_{\mathbb{R}^K}|\nabla u|^2=-\tfrac{K}{2}\int_{\mathbb{R}^K}P_{n,0}(|u|^2) \tag{1.1}$$

(iii) *$u\in C^\infty\cap W^{1,\infty}$, $|u|^2\le k_n$ and there exists an integer $s\in\{1,\cdots,n-1\}$ such that $(|u|^2-r_s)\in H^1\cap W^{1,\infty}$; in addition u satisfies*:

$$\left(\frac{K}{2}-1\right)\int_{\mathbb{R}^K}|\nabla u|^2=-\frac{K}{2}\int_{\mathbb{R}^K}P_{n,r_s}(|u|^2). \tag{1.2}$$

A direct consequence of the previous Theorem is:

Corollary 1.3. (i) *Every $u\in L^3_{loc}(\mathbb{R}^K,\mathbb{R}^M)$, solution of the Ginzburg-Landau system of equations $-\Delta u=u(1-|u|^2)$ in $\mathbb{R}^K$ satisfying* (0.3) *with $K>1$, is a constant function* (*either $u\equiv\underline{0}$ or $u\equiv\underline{c}$, where $|\underline{c}|=1$*).

(ii) *Every solution u of* (0.1) *and* (0.4) *with $K>1$ is a constant function* (*and there exists $\gamma\in\{k_1,\dots,k_n\}$ such that $|u|^2=\gamma$*).

Theorem 1.2 and Corollary 1.3 do not hold for $K=1$. Note that

$$u_{k_1}=\sqrt{k_1}\tanh\left(\sqrt{\tfrac{k_1}{2}}x\right),$$

$x\in\mathbb{R}$ is a finite-energy solution of (0.1) with $n=K=M=1$. The one-dimensional counterpart of Theorem 1.2 is:

Theorem 1.4. *For every solution u of* (0.1) *and* (0.3) *with $K=1$, one and only one of the following holds*:

(i) *u is a constant function and there exists $\gamma\in\{0,k_1,\dots,k_n,r_1,\dots,r_{n-1}\}$ such that $|u|^2=\gamma$.*

(ii) *$u\in C^\infty$, $|u'|^2=P_n(|u|^2)>0$ and there exists $j\in\{1,\dots,n\}$ such that*:
(*if $j=1$*) $|u|^2<k_1$ and $\lim_{x\to\pm\infty}|u(x)|^2=k_1$,
(*if $j>1$*) $k_{j-1}<|u|^2<k_j$ and $\lim_{x\to\pm\infty}|u(x)|^2=l^\pm\in\{k_{j-1},k_j\}$.

Remark 1.5. Given $n\ge 1$ and $M>1$ there exists a solution u satisfying (ii) of Theorem 1.4. For example the function αv where $\alpha\in\mathbb{R}^M$, $|\alpha|=1$ and v is a solution of (0.1) and (0.3) with $K=M=1$. The following result proves the existence of such a function v.

Theorem 1.6. *For every solution u of* (0.1) *and* (0.3) *with $K = M = 1$, one and only one of the following holds.*

(i) *u is a constant function and there exists $\gamma \in \{0, k_1, \ldots, k_n, r_1, \ldots, r_{n-1}\}$ such that $|u|^2 = \gamma$.*

(ii) *There exist $j \in \{1, \ldots, n\}$, $x_0 \in \mathbb{R}$ and $(\varepsilon_1, \varepsilon_2) \in \{-1, 1\} \times \{-1, 1\}$ such that $u(x) = \varepsilon_1 v_j(\varepsilon_2(x - x_0))$, where v_j is the unique solution of*

$$\begin{cases} v_j^{'} = \frac{1}{\sqrt{2}} \prod_{i=1}^{j-1}(v_j^2 - k_i) \prod_{s=j}^{n}(k_s - v_j^2) > 0 \\ v_j(0) = \sqrt{r_{j-1}} \quad \text{if} \quad j > 1, \qquad v_1(0) = 0 \quad \text{if} \quad j = 1. \end{cases} \tag{1.3}$$

The next Proposition gives some properties of the v_j's, solutions of (1.3).

Proposition 1.7. *The solution v_j satisfies $k_{j-1} < v_j^2 < k_j$ if $j > 1$ and $v_1^2 < k_1$, moreover, there exist positive constants $\alpha_j^+, \alpha_j^-, C_j^+, C_j^-$ such that:*

$$v_j(x) = \sqrt{k_j} - C_j^+ e^{-\alpha_j^+ x} + o(e^{-\alpha_j^+ x}) \qquad \text{as} \quad x \to +\infty$$
$$v_j(x) = L(j) + C_j^- e^{\alpha_j^- x} + o(e^{\alpha_j^- x}) \qquad \text{as} \quad x \to -\infty,$$

where

$$L(j) := \begin{cases} \sqrt{k_{j-1}} & \text{if } j > 1 \\ -\sqrt{k_1} & \text{if} \quad j = 1. \end{cases}$$

Finally v_1 is an odd function while no symmetry property holds for $j > 1$ since, in general, $\alpha_j^+ \neq \alpha_j^-$.

Remark 1.8. The constants $\alpha_j^+, \alpha_j^-, C_j^+, C_j^-$ depend only on $k_1, \cdots, k_n$ and can be explicitly calculated (see the proof of Proposition 1.7).

Theorems 1.1 and 1.2 state that any finite-energy solution u of (0.1) satisfies $|u|^2 \leq k_n$ on $\mathbb{R}^K$; the following question arises in a natural way "does any solution of (0.1) satisfy the bound $|u|^2 \leq k_n$ on $\mathbb{R}^K$? " The next Proposition answers this question.

Proposition 1.9. *Any solution u of* (0.1) *is smooth and satisfies $|u|^2 \leq k_n$ on $\mathbb{R}^K$.*

The proofs are given in Sections 3 ,4, 5 and 6.

2. Preliminary results.

In this section we introduce some auxiliary result which will be used in the sequel.

Lemma 2.1. *For any solution* $u = (u_1, \cdots, u_M)$ *of* (0.1) *there exists* $\delta > 0$ *such that* $\forall i = 1, \cdots, M$:

$$\Delta(|u_i| - \sqrt{k_n})^+ \geq \delta(|u_i| - \sqrt{k_n})^+ \quad \text{in} \quad \mathcal{D}^{'}(\mathbb{R}^K). \tag{2.1}$$

Proof. The proof is in two steps.

Step 1. For any $i \in \{1, \cdots, M\}$ we set $\psi_i := u_i - \sqrt{k_n}$ so that $\psi_i \in L^1_{loc}$ and $\Delta\psi_i \in L^1_{loc}$. By Kato's inequality (see [10]) we have:

$$\begin{aligned}
\Delta\psi_i^+ &\geq \text{sign}^+(\psi_i)\Delta\psi_i = \text{sign}^+(u_i - \sqrt{k_n})u_i P_n^{'}(|u|^2) \\
&= \text{sign}^+(u_i - \sqrt{k_n})u_i \prod_{j=1}^{n}(|u|^2 - k_j)Q_n(|u|^2) \\
&\geq \text{sign}^+(u_i - \sqrt{k_n})u_i \prod_{j=1}^{n}(|u_i|^2 - k_j)Q_n(|u_i|^2)
\end{aligned}$$

the latter follows from the obvious fact that $Q_n(t)$ as well as $\prod_{j=1}^{n}(t - k_j)$ are increasing for $t \geq k_n$. Hence

$$\Delta\psi_i^+ \geq \psi_i^+ u_i(|u_i| + \sqrt{k_n}) \prod_{j=1}^{n-1}(|u_i|^2 - k_j)Q_n(|u_i|^2) = S(|u_i|)\psi_i^+, \tag{2.2}$$

where

$$S(t) := \begin{cases} t(t + \sqrt{k_n}) \prod_{j=1}^{n-1}(t^2 - k_j)Q_n(t^2) & \text{if} \quad t > \sqrt{k_n} \\ 2k_n \prod_{j=1}^{n-1}(k_n - k_j)Q_n(k_n) & \text{if} \quad t \leq \sqrt{k_n}. \end{cases}$$

Taking $0 < \delta \leq S(\sqrt{k_n})$ we have $\Delta\psi_i^+ \geq \delta\psi_i^+$ in $\mathcal{D}^{'}(\mathbb{R}^K)$.

Step 2. Note also that the function $-u$ is a solution of (0.1), then Step 1 applied to $-u$ yields $\Delta(-u_i - \sqrt{k_n})^+ \geq \delta(-u_i - \sqrt{k_n})^+$in $\mathcal{D}^{'}(\mathbb{R}^K)$. To prove (2.1) we remark that $(|u_i| - \sqrt{k_n})^+ = (u_i - \sqrt{k_n})^+ + (-u_i - \sqrt{k_n})^+$. □

Now we prove a maximum principle-type result for system (0.1) which is fundamental in the proof of our main results.

Proposition 2.2. (i) *Suppose* u *satisfies* (0.1) *and there are* $x_0 \in \mathbb{R}^K$, $R_0 > 0$, $C > 0$ *and* $h \in N$ *such that* $\forall R > R_0$:

$$\int_{B_R(x_0)} (\prod_{j=1}^{n}(|u|^2 - k_j)^+)^2 \leq CR^h \tag{2.3}$$

then $|u|^2 \le k_n$. *In particular* u *is smooth.*

(ii) *Suppose* u *satisfies* (0.1) *and there are* $x_0 \in \mathbb{R}^K$, $R_0 > 0, C > 0, p \in [1, +\infty]$ *and* $h \in \mathbf{N}$ *such that* $\forall\ R > R_0$*:*

$$\int_{B_R(x_0)} |\nabla u|^p \le CR^h \quad \text{if} \quad p < +\infty, \ \left\|\nabla u\right\|_{L^\infty_{B_R(x_0)}} \le CR^h \quad \text{if} \quad p = \infty \quad (2.4)$$

then $|u|^2 \le k_n$. *In particular* u *is smooth.*

Proof. (i) The proof is in two steps.

Step 1. For any $i \in \{1, \cdots, M\}$ we set $\phi_{i,n} := (|u_i| - \sqrt{k_n})^+$, then there is a positive constant $\tilde{C} = \tilde{C}(k_1, \cdots, k_n)$ such that $\phi_{i,n} \le \tilde{C} \prod_{j=1}^n (|u|^2 - k_j)^+$ in $\mathbb{R}^K$. From (2.3) and Cauchy-Schwartz inequality we have:

$$\forall\, R > R_0, \quad \int_{B_R(x_0)} \phi_{i,n} \le C_0 R^{\frac{h+K}{2}}, \tag{2.5}$$

where C_0 is a positive constant which depends only $k_1, \cdots, k_n, K$ and C.

Now by mathematical induction we can prove that $\forall m \in N$ there exists $C_m > 0$:

$$\forall\, R > R_0, \quad \int_{B_R(x_0)} \phi_{i,n} \le C_m R^{\frac{h+K}{2} - 2m}. \tag{2.6}$$

To prove (2.6) we use (2.1) and the test functions $\xi_R(x) := \xi_0(\frac{x-x_0}{R})$ where ξ_0 is a smooth fixed function satisfying $0 \le \xi \le 1$, $\xi(x) = 1$ for $|x| \le 1$ and $\xi(x) = 0$ for $|x| \ge 2$. More precisely we have, $\forall\, R > R_0$,

$$\int_{B_R(x_0)} \phi_{i,n} \le \int_{\mathbb{R}^K} \phi_{i,n} \xi_R \le \delta \int_{\mathbb{R}^K} \phi_{i,,n} \Delta \xi_R \le \frac{C}{R^2} \int_{B_{2R}(x_0)} \phi_{i,n},$$

where C is a constant depending only on $k_1, \cdots, k_n$ and $\|\Delta \xi_0\|_{L^\infty}$ but not on R. The induction hypothesis applied to the latter inequality leads immediately to (2.6).

Now fixing $m > \frac{h+K}{4}$ and letting $R \to +\infty$ in (2.6) we obtain:

$$\phi_{i,n} \equiv 0, \quad \forall i = 1, \cdots, M \quad \text{that is} \quad |u_i| \le \sqrt{k_n}, \quad \forall i = 1, \cdots, M. \tag{2.7}$$

Step 2. From (2.7), making use of the Gagliardo-Nirenberg inequality and regularity results for weak and "very-weak " solutions of elliptic problems (see [1], [3], [7]), we deduce the smoothness of u and then we can apply

Kato's inequality to the function $\psi := |u|^2 - k_n$. By the same method used in Lemma 2.1 we obtain

$$\Delta(|u|^2 - k_n)^+ \geq \sigma(|u|^2 - k_n)^+ \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^K), \tag{2.8}$$

where $\sigma > 0$. Now, as already done in Step 1, we prove the validity of (2.6) for the function ψ^+ and we conclude that $\psi^+ \equiv 0$, that is $|u|^2 \leq k_n$.

(ii) From (2.4), making use of distributional inequality (2.1) and test functions ξ_R, already used in the first part of this proof, we have:

$$\forall\, R > R_0 > 1, \quad \int_{B_R(x_0)} \phi_{i,n} \leq C_0 R^{h+K-1}, \tag{2.9}$$

where C_0 is a positive constant which depends only on $k_1, \cdots, k_n, N, p$ and C. Note that (2.9) is (2.5) (where $\frac{h+K}{2}$ is replaced by $h+K-1$). Using the same inductive method as in Step 1 we can conclude that $|u|^2 \leq k_n$.

Lemma 2.3. *Let u be a smooth solution of* (0.1) *and* (0.3). *Given $x_0 \in \mathbb{R}^K$, $R_0 > 0$ and $m \in \mathbf{N}^+$ there are $C_1 > 0, C_2 > 0$ such that:*

$$\forall\, R > R_0, \quad \int_{B_R(x_0)} |u|^2 (P_n'(|u|^2))^{2m} \leq C_1 + C_2 R^{\alpha_m}, \tag{2.10}$$

where $\alpha_m := 2^{-m}(K - 2^{m+1} + 2)$ and C_1, C_2 do not depend on R.

Proof. Since u is smooth we have:

$$\tfrac{1}{2}\Delta|u|^2 = |u|^2 P_n'(|u|^2) + |\nabla u|^2. \tag{2.11}$$

Multiplying (2.11) by $P_n'(|u|^2)\xi_R$, where ξ_R is the test function used in the proof of Proposition 2.2, and integrating by parts over $\mathbb{R}^K$ yields, $\forall\, R > R_0$,

$$\begin{aligned}
&\int_{B_R(x_0)} |u|^2 (P_n'(|u|^2))^2 \leq \int_{\mathbb{R}^K} |u|^2 (P_n'(|u|^2))^2 \xi_R \\
&= -\frac{1}{2}\int_{\mathbb{R}^K} (\nabla|u|^2 \cdot \nabla P_n'(|u|^2))\xi_R - \frac{1}{2}\int_{\mathbb{R}^K} (\nabla|u|^2 \cdot \nabla\xi_R) P_n'(|u|^2) \\
&\quad - \int_{\mathbb{R}^K} |\nabla u|^2 P_n'(|u|^2)\xi_R \\
&\leq \frac{1}{2}\int_{B_{2R}(x_0)} |\nabla|u|^2|^2 P_n''(|u|^2)\xi_R + A\Big(\int_{B_{2R}(x_0)} |\nabla|u|^2|^2\Big)^{\frac{1}{2}} R^{\frac{K}{2}-1} + B,
\end{aligned}$$

where A, B are constants independent of R. This proves (2.10) for $m = 1$. To prove (2.10) for $m > 1$ we use induction. It is enough to multiply the identity (2.11) by $(P_n'(|u|^2))^{2m+1}\xi_R$, integrate by parts and use the induction hypothesis with together (0.3).

3. Quantization effects.

Proof of Theorem 1.1. Assertion (i) follows immediately from (0.2) and Proposition 2.2, in particular u is smooth. Going back to (0.1)and using the fact that u is smooth and bounded together with standard elliptic estimates (see [7]) we obtain $\nabla u \in L^\infty(\mathbb{R}^2)$. To prove (ii) we claim that for $\epsilon > 0$ small enough and all $i = 1, \cdots, n-1$, the sets

$$B^{0,\epsilon} := \{x \in \mathbb{R}^2 : |u(x)|^2 \leq k_1 - \epsilon\},$$
$$B^{i,\epsilon} := \{x \in \mathbb{R}^2 : k_i + \epsilon \leq |u(x)|^2 \leq k_{i+1} - \epsilon\}$$

are bounded. Suppose not, then there exist $l \in \{0, \cdots, n-1\}$ and a sequence $\{x_n\} \in B^{l,\epsilon}$ such that $|x_n| \to \infty$. Let $M := \|\nabla |u|^2\|_{L^\infty}$, then:

$$|u(y)|^2 - M|y-x| \leq |u(x)|^2 \leq |u(y)|^2 + M|y-x| \quad \forall\ (x,y) \in \mathbb{R}^K \times \mathbb{R}^K$$

thus, we have for x in $B_{\frac{\epsilon}{2M}}(x_n)$,

$$|u(x)|^2 \leq k_1 - \frac{\epsilon}{2} \quad \text{if} \quad l = 0, \quad k_l + \frac{\epsilon}{2} \leq |u(x)|^2 \leq k_{l+1} - \frac{\epsilon}{2} \quad \text{if} \quad l > 1.$$

Hence,

$$\int_{B_{\frac{\epsilon}{2M}}(x_n)} P_n(|u|^2) \geq \left(\frac{\epsilon}{2}\right)^{2n+2} \frac{\pi}{M^2}$$

which contradicts (0.2) since $|x_n| \to \infty$.

Suppose $\bigcup_{l=0}^{n-1} B^{i,\epsilon} \subset B_{\lambda_\epsilon}(0)$. Since $\mathbb{R}^2 \setminus B_{\lambda_\epsilon}(0)$ is a connected set we deduce that there is an integer $j \in \{1, \cdots, n\}$ and $\overline{\epsilon} > 0$ such that

$$\forall\ 0 < \epsilon < \overline{\epsilon}, \quad \exists\ R_\epsilon > 0 : k_j - \epsilon \leq |u(x)|^2 \leq k_j + \epsilon\ \forall\, x \in \mathbb{R}^2 : |x| > R_\epsilon \quad (3.1)$$

then the desired conclusion follows.

To prove (iii) we follow [5]. From (ii) we deduce that there exists $R_0 > 0$ such that $\frac{4}{5}k_j \leq |u(x)|^2 \leq \frac{3}{2}k_j$ when $|x| > R_0$; then the topological degree of u, $d := deg(u, S_R)$ is well-defined and independent of R for $R \geq R_0$. Without loss of generality we may assume that $d \geq 0$ (the general case follows by complex conjugation). Hence we may write $u(x) = |u(x)|e^{i(\theta d + \psi(x))} = \rho(x)e^{i\varphi(x)}$ where $\rho(x) = |u(x)|$ and $\varphi(x) = \theta d + \psi(x)$. Note that φ is well-defined as a single-valued function only locally on $\{|x| \geq R_0\}$ while ψ and $\nabla\varphi$ are smooth and well-defined globally on $\{|x| \geq R_0\}$. A basic estimate is the following:

Proposition 3.1. *We have*

$$\int_{\mathbb{R}^2\setminus B_{R_0}} |\nabla\rho|^2 + |\nabla\psi|^2 < \infty. \tag{3.2}$$

Proof. Writing (0.1) in terms of ρ and φ and then separating the real and imaginary parts we get:

$$\begin{cases} \rho\Delta\varphi + 2\nabla\rho\cdot\nabla\varphi = div(\rho^2\nabla\varphi) = 0 & \text{for } |x| > R_0 \quad (3.3) \\ \Delta\rho - \rho|\nabla\varphi|^2 = \rho P_n^{'}(\rho^2) & \text{for } |x| > R_0. \quad (3.4) \end{cases}$$

Now following [5] (see Prop. 1 therein) we get, for every $R > R_0$,

$$\int_{S_R} \rho^2 \frac{\partial\psi}{\partial\nu} = 0, \tag{3.5}$$

where $S_R := \{x \in \mathbb{R}^2 : |x| = R\}$. Set $A_R := B_R \setminus B_{R_0}$, then from (3.3) and (3.5) we get

$$\begin{aligned} \frac{4}{5}k_j \int_{A_R} |\nabla\psi|^2 &\le \int_{A_R} \rho^2|\nabla\psi|^2 \qquad (3.6) \\ &\le \frac{3}{2}k_j \int_{S_R} |\frac{\partial\psi}{\partial\nu}||\psi - \psi_R| + \int_{A_R} |k_j - \rho^2|\frac{d}{r}|\nabla\psi| + C \end{aligned}$$

and then (see [5])

$$\int_{A_R} |\nabla\psi|^2 \le \frac{R}{\beta}\int_{S_R} |\nabla\psi|^2 + C_1\big(\int_{A_R} |\nabla\psi|^2\big)^{\frac{1}{2}} + C_2 \text{ with } \beta = \frac{16}{15} > 1. \tag{3.7}$$

Moreover, $\forall\ R > R_0$, we have:

$$\int_{A_R} |\nabla\psi|^2 \le \frac{2}{k_1}\int_{B_R} |\nabla u|^2 \le CR \quad \text{with} \quad C \quad \text{independent of } R. \tag{3.8}$$

From (3.7), (3.8) and Lemma 1 in [5] we get

$$\int_{\mathbb{R}^2\setminus B_{R_0}} |\nabla\psi|^2 < \infty. \tag{3.9}$$

It is immediate to prove that

$$\int_{\mathbb{R}^2\setminus B_{R_0}} |\sqrt{k_j}-\rho||\nabla\varphi|^2 \ <\infty$$

then the method used in [5] yields:

$$\int_{\mathbb{R}^2\setminus B_{R_0}} |\nabla\rho|^2<\infty. \tag{3.10}$$

With (3.9) and (3.10) we conclude the proof of Proposition 3.1.

Now we are in position to prove assertion (iii). The Pohozaev method applied to (0.1) asserts that, for every $r>0$,

$$\frac{1}{2}\int_{S_r}\big|\frac{\partial u}{\partial\nu}\big|^2+\frac{1}{r}\int_{B_r}P_n(|u|^2)=\frac{1}{2}\int_{S_r}\big|\frac{\partial u}{\partial\tau}\big|^2+\frac{1}{2}\int_{S_r}P_n(|u|^2). \tag{3.11}$$

Set

$$E=\int_{\mathbb{R}^2}P_n(|u|^2)\quad\text{and}\quad E(r)=\int_{B_r}P_n(|u|^2)$$

then

$$E(R)\to E \text{ and } \frac{1}{\log R}\int_0^R\frac{E(r)}{r}dr\to E \text{ as } R\to\infty. \tag{3.12}$$

Integrating (3.11) for $r\in(0,R)$ and dividing by $\log R$ we have

$$\frac{1}{2\log R}\int_{B_R}\Big|\frac{\partial u}{\partial\nu}\Big|^2+\frac{1}{\log R}\int_0^R\frac{E(r)}{r}dr=\frac{1}{2\log R}\int_{B_R}\Big|\frac{\partial u}{\partial\tau}\Big|^2+\frac{E(R)}{2\log R}. \tag{3.13}$$

Note that, for $r>R_0$, (as in [5]):

$$\Big|\frac{\partial u}{\partial\nu}\Big|^2\le|\nabla\rho|^2+k_n|\nabla\psi|^2 \tag{3.14}$$

$$\Big|\frac{\partial u}{\partial\tau}\Big|^2=\Big|\frac{\partial\rho}{\partial\tau}\Big|^2+\rho^2\Big|\frac{\partial\varphi}{\partial\tau}\Big|^2=\Big|\frac{\partial\rho}{\partial\tau}\Big|^2+\rho^2\Big(\frac{d}{r}+\frac{\partial\psi}{\partial\tau}\Big)^2 \tag{3.15}$$

hence

$$\bigg|\Big|\frac{\partial u}{\partial\tau}\Big|^2-\frac{k_jd^2}{r^2}\bigg|\le|\nabla\rho|^2+\frac{|k_j-\rho^2|d^2}{r^2}+\frac{2k_nd}{r}|\nabla\psi|+k_n|\nabla\psi|^2. \tag{3.16}$$

From (3.14) and Proposition 3.1 we deduce that

$$\int_{B_R} \left|\frac{\partial u}{\partial \nu}\right|^2 \le C \quad \text{as} \quad R \to \infty, \tag{3.17}$$

and similarly

$$\int_{B_R \setminus B_{R_0}} \left| \left|\frac{\partial u}{\partial \tau}\right|^2 - \frac{k_j d^2}{r^2} \right| \le C(\log R)^{\frac{1}{2}} \quad \text{as} \quad R \to \infty. \tag{3.18}$$

Hence

$$\frac{1}{2\log R} \int_{B_R} \left|\frac{\partial u}{\partial \tau}\right|^2 \to \pi k_j d^2 \quad \text{as} \quad R \to \infty. \tag{3.19}$$

Combining (3.12), (3.13) and (3.17), (3.19) we obtain $E = \pi k_j d^2$.

This completes the proof of (iii).

4. Liouville-type results and Pohozaev-type identities.

Proof of Theorem 1.2. The proof is in three steps.

Step 1.

Lemma 4.1. *For every solution u of* (0.1) *and* (0.3) *with $K > 1$ one and only one of the following holds:*

(i) $u \in L^2(\mathbb{R}^K, \mathbb{R}^M)$ *and* $|u| \to 0$ *as* $|x| \to \infty$.

(ii) *There exists* $\gamma \in \{k_1, \cdots, k_n, r_1, \cdots, r_{n-1}\}$ *such that*

$$(|u|^2 - \gamma) \in L^2(\mathbb{R}^K) \quad \textit{and} \quad |u|^2 \to \gamma \quad \textit{as} \quad |x| \to \infty.$$

Proof. The proof is in two steps.

Step 1. From Proposition 2.2 we know that u is smooth and bounded, then $\nabla u \in L^\infty$.

Consider inequality (2.10) and fix $m \in \mathbf{N}$ such that $\alpha_m \le 0$ (for example $m \ge \log_2(K+2) - 1$). Then

$$\int_{\mathbb{R}^K} |u|^2 \prod_{j=1}^{n} (|u|^2 - k_n)^{2m} \prod_{s=1}^{n-1} (|u|^2 - r_s)^{2m} < \infty. \tag{4.1}$$

Note that, $\forall \epsilon > 0$ small enough and $\forall i = 1, \cdots, n$, $\forall j = 1, \cdots, n-1$, the sets

$$B^{i,\epsilon} := \{x \in \mathbb{R}^K : r_{i-1} + \epsilon < |u|^2 < k_i - \epsilon\},$$
$$X^{j,\epsilon} := \{x \in \mathbb{R}^K : k_j + \epsilon < |u|^2 < r_j - \epsilon\}$$

are bounded (here $r_0 = 0$). This follows easily from (4.1) and the fact that $\nabla u \in L^\infty$.

Suppose $(\cup_{i=1}^n B^{i,\epsilon}) \bigcup (\cup_{j=1}^{n-1} X^{j,\epsilon}) \subset B_{\lambda_\epsilon}(0)$. Since $\mathbb{R}^K \backslash B_{\lambda_\epsilon}(0)$ is connected we deduce that there are $\gamma \in \{0, k_1, \cdots, k_n, r_1, \cdots, r_{n-1}\}$ and $\overline{\epsilon} > 0$ such that

$$\forall\, 0 < \epsilon < \overline{\epsilon},\ \exists\, R_\epsilon > 0 : \gamma - \epsilon \leq |u(x)|^2 \leq \gamma + \epsilon,\ \forall\, x \in \mathbb{R}^K : |x| > R_\epsilon. \quad (4.2)$$

If $\gamma = 0$ this proves (i) while when $\gamma \neq 0$ we have

$$(|u|^2 - \gamma) \in L^{2m}(\mathbb{R}^K) \quad \text{and} \quad |u|^2 \to \gamma \quad \text{as} \quad |x| \to \infty. \quad (4.3)$$

Step 2. Here we prove (ii). By (4.3) and induction we prove that $\forall\, m \in \mathbf{N}^+$ there are $R_0 > 0, C_1 > 0, C_2 > 0$ such that $\forall\ R > R_0$:

$$\int_{B_R(0)} (|u|^2 - \gamma)^2 \leq C_1 + C_2 R^{\alpha_m}, \quad (4.4)$$

where α_m was defined in Lemma 2.3. From Step 1, there are $R_0 > 0$, $\beta > 0$ such that $(|u|^2 - \gamma)^2 \leq \beta |u|^2 (P_n^{'}(|u|^2))^2$ when $|x| > R_0$. Hence, $\forall\ R > R_0$,

$$\begin{aligned} \int_{B_R(0)} (|u|^2 - \gamma)^2 &= \int_{B_{R_0}(0)} (|u|^2 - \gamma)^2 + \int_{B_R(0) \backslash B_{R_0}(0)} (|u|^2 - \gamma)^2 \\ &\leq A + \beta \int_{B_R(0) \backslash B_{R_0}(0)} |u|^2 (P_n^{'}(|u|^2))^2 \\ &\leq A + \beta \int_{\mathbb{R}^K} |u|^2 (P_n^{'}(|u|^2))^2 \xi_R, \end{aligned} \quad (4.5)$$

where ξ_R is the cut-off function defined in the proof of Prop. 2.2 (with $x_0 = 0$) and A is a constant independent of R. Now, from (2.11), we get

$$|u|^2 (P_n^{'}(|u|^2))^2 \xi_R = \frac{1}{2} (\Delta |u|^2) P_n^{'}(|u|^2) \xi_R - |\nabla u|^2 P_n^{'}(|u|^2) \xi_R. \quad (4.6)$$

Combining (4.5) and (4.6) we obtain:

$$\int_{B_R(0)}(|u|^2-\gamma)^2 \le A+\frac{\beta}{2}\int_{\mathbb{R}^K}(\Delta|u|^2)P_n^{'}(|u|^2)\xi_R-\beta\int_{\mathbb{R}^K}|\nabla u|^2P_n^{'}(|u|^2)\xi_R$$
$$\le A-\frac{\beta}{2}\int_{\mathbb{R}^K}(\nabla|u|^2\cdot\nabla P_n^{'}(|u|^2))\xi_R$$
$$-\frac{\beta}{2}\int_{\mathbb{R}^K}(\nabla|u|^2\cdot\nabla\xi_R)P_n^{'}(|u|^2)-\beta\int_{\mathbb{R}^K}|\nabla u|^2P_n^{'}(|u|^2)\xi_R$$
$$\le A+B\int_{B_{2R}(0)}|\nabla|u|^2|^2+C(\int_{B_{2R}(0)}|\nabla|u|^2|^2)^{\frac{1}{2}}R^{\frac{K}{2}-1},$$

where A, B, C are constants independent of R. This proves (4.4) for $m=1$. Inequality (4.4) for $m>1$ is now proved by induction, making use of (4.5) and (4.6).

Step 2. From Lemma 4.1, we deduce that there exists $\gamma\in\{0,k_1,\cdots,k_n, r_1,\cdots,r_{n-1}\}$ such that $P_{n,\gamma}(|u|^2)\in L^1(\mathbb{R}^K)$. Hence, we can prove a Pohozaev type identity for the system (0.1). Taking the inner product in $\mathbb{R}^K$, of both sides of (0.1) with $(x\cdot\nabla u)\xi_R$, where ξ_R is the cut-off function defined in the proof of Prop. 2.2, integrating by parts over $\mathbb{R}^K$ we obtain

$$\begin{aligned}\int_{\mathbb{R}^K}\Delta u\cdot(x\cdot\nabla u)\xi_R &= \big(\frac{k}{2}-1\big)\int_{\mathbb{R}^K}|\nabla u|^2\xi_R\\ &+\frac{1}{2}\int_{\mathbb{R}^K}(x\cdot\nabla\xi_R)|\nabla u|^2-\int_{\mathbb{R}^K}(\nabla u\cdot x)\cdot(\nabla u\cdot\nabla\xi_R)\end{aligned}\tag{4.7}$$

$$\begin{aligned}\int_{\mathbb{R}^K}uP_n^{'}(|u|^2)(x\cdot\nabla u)\xi_R &= \frac{1}{2}\int_{\mathbb{R}^K}(x\cdot\nabla P_{n,\gamma}(|u|^2))\xi_R\\ &=-\frac{K}{2}\int_{\mathbb{R}^K}P_{n,\gamma}(|u|^2)\xi_R-\frac{1}{2}\int_{\mathbb{R}^K}P_{n,\gamma}(|u|^2)(x\cdot\nabla\xi_R).\end{aligned}\tag{4.8}$$

Note that

$$\begin{aligned}&\big|\frac{1}{2}\int_{\mathbb{R}^K}(x\cdot\nabla\xi_R)|\nabla u|^2-\int_{\mathbb{R}^K}(\nabla u\cdot x)\cdot(\nabla u\cdot\nabla\xi_R)\big|\\ &\le C\int_{\mathbb{R}^K}|x||\nabla u|^2|\nabla\xi_R|\le C\int_{B_{2R}\setminus B_R}|\nabla u|^2\to 0 \text{ as } R\to\infty\end{aligned}\tag{4.9}$$

since $\nabla u\in L^2(\mathbb{R}^K)$. Moreover,

$$\big|\int_{\mathbb{R}^K}P_{n,\gamma}(|u|^2)(x\cdot\nabla\xi_R)\big|\le C\int_{B_{2R}\setminus B_R}|P_{n,\gamma}(|u|^2)|\to 0 \text{ as } R\to\infty \tag{4.10}$$

since $P_{n,\gamma}(|u|^2) \in L^1(\mathbb{R}^K)$. Combining (0.1), (4.7), (4.8), (4.9) and (4.10) we deduce the following Pohozaev identity

$$\left(\frac{K}{2}-1\right)\int_{\mathbb{R}^K}|\nabla u|^2 = -\frac{K}{2}\int_{\mathbb{R}^K}P_{n,\gamma}(|u|^2). \tag{4.11}$$

Step 3. Suppose that $\gamma \in \{k_1, \cdots, k_n\}$, then $P_{n,\gamma} = P_n \geq 0$. Hence $|u|^2 = \gamma$ and $\nabla u = 0$ follow immediately from (4.11) when $K > 2$ (the left-hand side is non-negative while the right-hand side is non-positive). For $K = 2$, $|u|^2 = \gamma$ still holds and we conclude applying Liouville's Theorem to equation (0.1). Suppose $\gamma \in \{0, r_1, \cdots, r_{n-1}\}$ then assertions (ii) and (iii) follows from (0.3), Prop.2.2, Lemma 4.1 and (4.11). This completes the proof of Theorem 1.2.

Proof of Corollary 1.3. (i) Let u be a solution of the Ginzburg-Landau system $-\Delta u = u(1-|u|^2)$ in $\mathbb{R}^K$ satisfying (0.3) with K>1. Taking the inner product of $-\Delta u = u(1-|u|^2)$ and $u\xi_R$, integrating by parts over $\mathbb{R}^K$ we get:

$$-\int_{\mathbb{R}^K}|\nabla u|^2 = \int_{\mathbb{R}^K}|u|^2(|u|^2-1). \tag{4.12}$$

Combining (4.12) and Pohozaev identity (4.11) we get $|u| = 0$.

(ii) From (0.4) we deduce that there exists $j \in \{1, \cdots, n\}$ such that $|u|^2 \to k_j$ as $|x| \to \infty$, then the conclusion follows invoking Theorem 1.2.

5. The one-dimensional case.

Proof of Theorem 1.4. From (2.10) with $m = 1$ we have

$$\int_{\mathbb{R}}|u|^2\prod_{j=1}^{n}(|u|^2-k_n)^2\prod_{s=1}^{n-1}(|u|^2-r_s)^2 < \infty, \tag{5.1}$$

then there are $\gamma^\pm \in \{0, k_1, \cdots, k_n, r_1, \cdots, r_{n-1}\}$ such that $|u|^2 \to \gamma^\pm$ as $x \to \pm\infty$. Integrating (0.1) we obtain

$$|u^{'}|^2 = P_{n,\gamma^\pm}(|u|^2) + c^\pm,$$

where $c^\pm$ are constants. Since $|u^{'}|^2 \in L^1(\mathbb{R})$, we have

$$|u^{'}|^2 = P_{n,\gamma^\pm}(|u|^2). \tag{5.2}$$

Moreover, we remark that

$$P_{n,0}(t) \leq 0 \quad \forall\, t \in [0, k_n], \tag{5.3}$$
$$P_{n,r_s}(t) < 0 \quad \forall\, t \in (k_s, k_{s+1}) \setminus \{r_s\}, \quad \forall\, s = 1, \cdots, n-1. \tag{5.4}$$

If γ^+ or $\gamma^- \in \{0, r_1, \cdots, r_{n-1}\}$, from (5.3), (5.4), we obtain that u is a constant function and $|u|^2 = \gamma^+ = \gamma^-$ (i.e. u satisfies (i) of Theorem 1.4). Now, suppose $\gamma^\pm \in \{k_1, \cdots, k_n\}$. If u is not a constant function it is immediate to check that u satisfies assertion (ii).

Proof of Theorem 1.6. In view of Theorem 1.4 it is enough to prove the existence of functions v_j satisfying the Cauchy problem (1.3) and check that any non-constant solution u of (0.1) and (0.3) is of the form $u(x) = \varepsilon_1 v_j(\varepsilon_2(x - x_0))$ where $x_0 \in \mathbb{R}$ and $(\varepsilon_1, \varepsilon_2) \in \{-1, 1\} \times \{-1, 1\}$. For any $j \in \{1, \cdots, n\}$ existence and uniqueness of v_j is an immediate consequence of ordinary differential equations theory. We remark that equation (0.1) is invariant with respect x-axis and y-axis reflections, then we may assume $u^{'} > 0$ and $\lim_{x\to+\infty} u(x) \in \{\sqrt{k_1}, \cdots, \sqrt{k_n}\}$. It follows that $\lim_{x\to-\infty} u(x) = -\sqrt{k_1}$, if j=1 and $\lim_{x\to-\infty} u(x) = \sqrt{k_{j-1}}$, if j> 1. Hence, there is $x_0 \in \mathbb{R}$ such that $u(x_0) = 0$, if j=1 and $u(x_0) = \sqrt{r_{j-1}}$, if j>1. Now we use the translation invariance of (0.1) and the uniqueness of the solution in a Cauchy problem to conclude that $u(x) = v_j(x - x_0)$.

Proof of Proposition 1.7. Existence and uniqueness for the Cauchy problem (1.3) follow from ordinary differential equations theory, moreover solution v_j is defined over all $\mathbb{R}$. It is immediate to check that: ${v_j}^{'}(x) > 0,\ \forall j = 1, \cdots, n\ ; k_{j-1} < v_j^2 < k_j$ if $j > 1$ and $v_1^2 < k_1, \quad \forall x \in \mathbb{R}$. Hence

$$x(v_j) = \int_{\sqrt{r_{j-1}}}^{v_j} \frac{\sqrt{2}}{\prod_{i=1}^{j-1}(t^2 - k_i)\prod_{s=j}^{n}(k_s - t^2)}\, dt, \tag{5.5}$$

where $r_0 = 0$. From (5.5) it follows that $x \to +\infty$ iff $v_j \to \sqrt{k_j}$ and $x \to -\infty$ iff $v_j \to L(j)$. It remains to prove the rate of convergence of v_j to ∞. To do so we write the function $\frac{\sqrt{2}}{\prod_{i=1}^{j-1}(t^2-k_i)\prod_{s=j}^{n}(k_s-t^2)}$ in a different way. It is easy to prove that there are real constants B_i and C_s, depending only on $k_1, \cdots, k_n$, such that:

$$\frac{\sqrt{2}}{\prod_{i=1}^{j-1}(t^2 - k_i)\prod_{s=j}^{n}(k_s - t^2)} = \sum_{i=1}^{j-1} \frac{B_i}{(t^2 - k_i)} + \sum_{s=j}^{n} \frac{C_s}{(k_s - t^2)} \tag{5.6}$$

then a direct integration yields

$$
\begin{aligned}
x(v_j) &= \sum_{i=1}^{j-1} \frac{B_i}{2\sqrt{k_i}} \big[\log \big(\frac{v_j - \sqrt{k_i}}{v_j + \sqrt{k_i}} \big) - \log \delta_i \big] \\
&\quad + \sum_{s=j}^{n} \frac{C_s}{2\sqrt{k_s}} \big[\log \big(\frac{\sqrt{k_s} + v_j}{\sqrt{k_s} - v_j} \big) - \log \sigma_s \big] \qquad (5.7) \\
&= \sum_{i=1}^{j-1} \log \big[\big(\frac{1}{\delta_i} \frac{v_j - \sqrt{k_i}}{v_j + \sqrt{k_i}} \big)^{\beta_i} \big] + \sum_{s=j}^{n} \log \big[\big(\frac{1}{\sigma_s} \frac{\sqrt{k_s} + v_j}{\sqrt{k_s} - v_j} \big)^{\gamma_s} \big],
\end{aligned}
$$

where $\beta_i := \frac{A_i}{2\sqrt{k_i}}$, $\delta_i := \frac{\sqrt{r_{j-1}} - \sqrt{k_i}}{\sqrt{r_{j-1}} + \sqrt{k_i}} > 0$, $\gamma_s := \frac{B_s}{2\sqrt{k_s}}$, $\sigma_s := \frac{\sqrt{k_s} + \sqrt{r_{j-1}}}{\sqrt{k_s} - \sqrt{r_{j-1}}} > 0$. Hence,

$$
x(v_j) = \log \big[\theta \prod_{i=1}^{j-1} \big(\frac{v_j - \sqrt{k_i}}{v_j + \sqrt{k_i}} \big)^{\beta_i} \prod_{s=j}^{n} \big(\frac{\sqrt{k_s} + v_j}{\sqrt{k_s} - v_j} \big)^{\gamma_s} \big], \qquad (5.8)
$$

where $\theta := \prod_{i=1}^{j-1} \delta_i{}^{-\beta_i} \prod_{s=j}^{n} \sigma_s{}^{-\gamma_s} > 0$.

Let us consider the case when x goes to $+\infty$ (the case when x goes to $-\infty$ is treated in the same way). From (5.8) we have

$$
e^{-x} \theta \prod_{i=1}^{j-1} \big(\frac{v_j(x) - \sqrt{k_i}}{v_j(x) + \sqrt{k_i}} \big)^{\beta_i} \prod_{s=j+1}^{n} \big(\frac{\sqrt{k_s} + v_j(x)}{\sqrt{k_s} - v_j(x)} \big)^{\gamma_s} = \big(\frac{\sqrt{k_j} - v_j(x)}{\sqrt{k_j} + v_j(x)} \big)^{\gamma_j} \qquad (5.9)
$$

since $v_j \to \sqrt{k_j}$ as $x \to +\infty$ we have that $\gamma_j > 0$ and

$$
\begin{aligned}
0 &\le \sqrt{k}_j - v_j(x) \qquad (5.10) \\
&= \theta^{\frac{1}{\gamma_j}} e^{-\frac{x}{\gamma_j}} \prod_{i=1}^{j-1} \big(\frac{v_j(x) - \sqrt{k_i}}{v_j(x) + \sqrt{k_i}} \big)^{\frac{\beta_i}{\gamma_j}} \prod_{s=j+1}^{n} \big(\frac{\sqrt{k_s} + v_j(x)}{\sqrt{k_s} - v_j(x)} \big)^{\frac{\gamma_s}{\gamma_j}} (\sqrt{k_j} + v_j),
\end{aligned}
$$

hence, the desired conclusion follows taking $\alpha_j^+ = \frac{1}{\gamma_j}$ and

$$
\begin{aligned}
C_j^+ &= 2\sqrt{k_j} \prod_{i=1}^{j-1} \big[\frac{(\sqrt{r_{j-1}} + \sqrt{k_i})(\sqrt{k_j} - \sqrt{k_i})}{(\sqrt{r_{j-1}} - \sqrt{k_i})(\sqrt{k_j} + \sqrt{k_i})} \big]^{\frac{\beta_i}{\gamma_j}} \\
&\quad \times \prod_{s=j+1}^{n} \big[\frac{(\sqrt{k_s} - \sqrt{r_{j-1}})(\sqrt{k_s} + \sqrt{k_j})}{(\sqrt{k_s} + \sqrt{r_{j-1}})(\sqrt{k_s} - \sqrt{k_j})} \big]^{\frac{\gamma_s}{\gamma_j}}.
\end{aligned}
$$

To conclude we note that v_1 is an odd function, since both v_1 and $w = -v_1(-x)$ are solutions of the Cauchy problem (1.3) with $j = 1$, while no symmetry property holds for $j > 1$ since, in general, $\alpha_j^+ \neq \alpha_j^-$ as it can be easily seen from our proof (or by a direct calculation).

6. Boundedness of solutions. The proof of the Proposition 1.9 is based on the following result proved by H. Brezis in [4].

Lemma 6.1. *Let $p > 1$ and assume $v \in L^p_{loc}(\mathbb{R}^K)$ satisfies*

$$-\Delta v + |v|^{p-1}v \leq 0 \quad in \quad \mathcal{D}'(\mathbb{R}^K). \tag{6.1}$$

Then $v \leq 0$ a.e. on $\mathbb{R}^K$.

Proof of Proposition 1.9. The proof is in two steps.

Step 1. We claim that there exists $\delta > 0$ such that $\forall\ i = 1, \cdots, M$

$$-\Delta(|u_i| - \sqrt{k_n})^+ + \delta\big[(|u_i| - \sqrt{k_n})^+\big]^2 \leq 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^K). \tag{6.2}$$

For any $i \in \{1, \cdots, M\}$ we set $\psi_i := u_i - \sqrt{k_n}$ so that (2.2) holds true. Then we have

$$\Delta\psi_i^+ \geq \psi_i^+ u_i(|u_i| + \sqrt{k_n}) \prod_{j=1}^{n-1}(|u_i|^2 - k_j)Q_n(|u_i|^2)$$

$$= \psi_i^+(u_i - \sqrt{k_n})(|u_i| + \sqrt{k_n}) \prod_{j=1}^{n-1}(|u_i|^2 - k_j)Q_n(|u_i|^2)$$

$$+ \sqrt{k_n}\psi_i^+(|u_i| + \sqrt{k_n}) \prod_{j=1}^{n-1}(|u_i|^2 - k_j)Q_n(|u_i|^2) \geq \tilde{S}(|u_i|)\big(\psi_i^+\big)^2,$$

where

$$\tilde{S}(t) := \begin{cases} (t + \sqrt{k_n})\prod_{j=1}^{n-1}(t^2 - k_j)Q_n(t^2) & \text{if} \quad t > \sqrt{k_n} \\ 2\sqrt{k_n}\prod_{j=1}^{n-1}(k_n - k_j)Q_n(k_n) & \text{if} \quad t \leq \sqrt{k_n}. \end{cases}$$

Taking $0 < \delta \leq \tilde{S}(\sqrt{k_n})$ we have $-\Delta\psi_i^+ + \delta\big(\psi_i^+\big)^2 \leq 0$ in $\mathcal{D}'(\mathbb{R}^K)$.

Now, with the same method used in the proof of Lemma 2.1 (Step 2), we obtain (6.2).

Applying Lemma 6.1 to $v = \delta(|u_i| - \sqrt{k_n})^+$, with $p = 2$, we get:

$$|u_i| \leq \sqrt{k_n} \quad \forall\ i = 1, \cdots, M. \tag{6.3}$$

Step 2. From (6.3), as already done in the proof of Prop. 2.2 (part (i), Step 2), we deduce the smoothness of u and then we can apply Kato's inequality to the function $\psi := |u|^2 - k_n$. By the same method used in Step 1 we obtain

$$-\Delta\psi^+ + \sigma(\psi^+)^2 \leq 0 \quad \text{in} \quad \mathcal{D}^{'}(\mathbb{R}^K),$$

where $\sigma > 0$. The desired conclusion follows applying Lemma 6.1 to $\sigma\psi^+$ with $p = 2$.

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REFERENCES

[1] S. Agmon, "Lectures on Elliptic Boundary Values Problems," Van Nostrand, 1965.
[2] F. Bethuel, H. Brezis, and F. Hélein, "Ginzburg-Landau Vortices," Birkhauser, 1994.
[3] H. Brezis, "Analyse Fonctionelle, Théorie et Applications," Editions, Masson, Paris 1983.
[4] H. Brezis, *Semilinear equations in* $\mathbb{R}^N$ *without conditions at infinity,* Appl. Math. Optim., 12(3) (1984), 271–282.
[5] H. Brezis, F. Merle, and T. Rivière, *Quantization effects for* $-\Delta u = (1 - |u|^2)u$ *in* $\mathbb{R}^2$, Arch. Rat. Mech. Anal., 126 (1994), 35–58.
[6] P.H. Damgaard, and U.M. Heller, *Observation of the Meissner effect in the lattice higgs models,* Phys. Rev. Lett., 60 (1988), 1246–1249.
[7] D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order,* Berlin, Heidelberg, New York, Springer, 1982.
[8] V.L. Ginzburg and L.D. Landau, *On the theory of superconductivity,* Zh. Èksper. Teoret. Fiz., 20 (1950), 1064–1082; (English translation in Men of Physics: L.D. Landau, I (D. ter Haar. ed.) pp. 138–167, Pergamon, New York and Oxford, 1965).
[9] V.L. Ginzburg and L. Pitaevskii, *On the theory of superfluidity,* Soviet Phys. JETP, 1958, pp. 858–861.
[10] T. Kato, *Schrödinger operators with singular potentials,* Israel J. Math., 13 (1972), 135–148.