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# NONNEGATIVE WEAK SOLUTIONS OF A POROUS MEDIUM EQUATION WITH STRONG ABSORPTION*

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**Abstract.** This paper studies the nonnegative weak solutions of a porous medium equation with strong absorption. We prove an apriori $L^\infty$ estimate through Moser iteration and obtain a compactness theorem and an integral-type Harnack inequality. Using these fundamental results we prove the existence of initial traces of weak solutions and obtain the existence of a fundamental solution and the nonexistence of a very singular solution, as byproducts. As an another application of our apriori estimates we prove the finiteness of the propagation speed without using comparison principle.

**1. Introduction and statement of results.** In this paper we study some qualitative properties of nonnegative solutions of a nonlinear degenerate parabolic equation

$$u_t = \Delta[u^m] - b\,u^p \tag{1.1}$$

in $\mathbb{R}^n \times (0,\infty)$. Here $m > 1$, $b \in [0,1]$ and $p \in (0,1)$ are given constants. This is a mathematical model describing a nonlinear process of gas diffusion with absorption in a homogeneous isotropic porous medium, where $u$ is a density or concentration, $[mu^{m-1}]$ represents the diffusivity, and $[bu^{p-1}]$ is the absorption coefficient. Notice that as $u$ goes to 0, the diffusivity also goes to 0, while the absorption coefficient goes to infinity. So we are in presence of slow diffusion and strong absorption. This equation can be used to model the heat propagation. For more discussions of related models, see [7,8,9,17].

Since (1.1) has a degeneracy where $u = 0$, the concept of classical solution is too restrictive, so we consider weak solutions.

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**Definition.** A nonnegative measurable function $u : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$ is a weak solution of (1.1) if

1. For each $[t_1, t_2] \subset (0,\infty)$ and $\Omega \subset\subset \mathbb{R}^n$
$$\sup_{t\in(t_1,t_2)} \|u\|^2_{2,\Omega}(t) + \|\nabla u^m\|^2_{2,\Omega\times(t_1,t_2)} < \infty. \tag{1.2}$$
2. For each $\varepsilon > 0$ $\sup_{t>\varepsilon} \|u\|_{1,\mathbb{R}^n}(t) < \infty$.
3. It satisfies (1.1) in a distribution sense, that is, for each bounded interval $[t_1, t_2] \subset (0,\infty)$, $u$ satisfies that
$$\begin{aligned}&\int u(x,t_2)\zeta(x,t_2)\,dx - \int u(x,t_1)\zeta(x,t_1)\,dx\\ &= \int_{t_1}^{t_2}\int \left\{u\zeta_t - mu^{m-1}[\nabla u\cdot\nabla\zeta] - b\,u^p\zeta\right\}\,dx\,dt\end{aligned} \tag{1.3}$$
for any smooth function $\zeta$ on $\mathbb{R}^n\times[t_1,t_2]$ such that for each $t \in [t_1,t_2]$, $\zeta(\cdot,t)$ is compactly supported in $\mathbb{R}^n$.

In the above, $\|\cdot\|_1$ and $\|\cdot\|_2$ denotes the usual $L^1$-norm and $L^2$-norm, respectively, on a corresponding domain. If the integral is considered in the full region, $\mathbb{R}^n$ for the space, $(0,\infty)$ for the time, we will often omit them, for notational simplicity.

There have been so many studies for the equations of the type (1.1). The basic results and historical remarks can be found in [9]. In particular, for the problem (1.1), the following fundamental results are well known:

1. Existence and uniqueness: The Cauchy problem for (1.1) with bounded and nonnegative initial data has a unique solution, c.f. see [13, Theorem 2.1].
2. Comparison theorem: Let $u$ and $v$ be two solutions of (1.1) with corresponding initial data $u_0, v_0 \in L^\infty(\mathbb{R}^n)$. If $u_0 \ge v_0$, then $u \ge v$ on $\mathbb{R}^n \times (0,\infty)$. c.f. see [13, Theorem 2.2]
3. Finite extinction: Let $u$ be a solution of (1.1) with initial data $u_0 \in L^\infty(\mathbb{R}^n)$. Then there exists a finite time $T_E$ such that $u(x,t) \equiv 0$ for $t \ge T_E$.
4. Finite propagation: Let $u$ be a solution of (1.1) which is continuous in $\mathbb{R}^n\times[0,\infty)$ such that the initial data $u_0$ is compactly supported. Then, for each time $t > 0$, the function $u(\cdot,t)$ is also compactly supported.

The finite propagation property is due to the degeneracy of (1.1). It can be easily verified by comparing with a solution of porous medium equation without absorption, which has the same property, c.f. see [2, 9]. The finite extinction is the most striking property in the case of strong absorption. This also can be seen by comparing with a suitable flat solution, c.f. see [19, Theorem 1.2].

In this paper, we prove various apriori estimates for the weak solutions of (1.1), by which a compactness theorem and a Harnack principle are proved. As applications, we prove the existence of initial trace and the finite propagation property. As we have indicated, the latter is already known by a comparison argument. Here, we illustrate how the same result can be obtained without using comparison argument.

The initial trace problem is a kind of inverse problem and concerns the asymptotic behavior as time goes to zero. The question is whether for a nonnegative weak solution $u$ there exists a limit of $u(\cdot,t)$, in a suitable sense, as time decreases to zero. To be specific, we adopt the following

**Definition.** A Radon measure $\mu$ on $\mathbb{R}^n$ is the initial trace of a weak solution $u$ if

$$\lim_{t\to 0}\int u(x,t)\zeta(x)\,dx = \int \zeta(x)\,d\mu(x) \tag{1.4}$$

for any $\zeta \in C_0^\infty(\mathbb{R}^n)$. In particular, a weak solution whose initial trace is the Dirac mass is called a fundamental solution or a source-type solution. If a weak solution $u$ satisfies $\lim_{t\to 0}\|u\|_{1,\{|x|<R\}}(t)=\infty$, for all $R>0$, then it is called a very singular solution.

It is to be remarked that the nonexistence of a very singular solution and the existence of a fundamental solution are already well known. The nonexistence can be proved by comparison with solutions for $p=1$ in $\{u \geq 1\}$. On the other hand, the existence of a fundamental solution of (1.1) is proved by Zhao [20] in a more general situation, where he showed the compactness of the family $\{u_k(x,t)\}$ of solutions whose initial data sequence $\{u_k(x,0)\}$ converges to the Dirac measure.

Further, in this paper, we prove a general compactness theorem and that there exists a initial trace for any weak solution. So, the nonexistence of a very singular solution and the existence of a fundamental solution are obtained as corollaries.

There are so many works regarding the same questions for the similar related problems. See [3, 4, 7] for the classical heat equation ($m=1$). For the porous medium equation ($m > 1$) and fast diffusion equation ($m < 1$) without absorption or with weak absorption ($p > 1$), see [5, 10, 11, 14, 18].

This paper is organized as follows. In Section 2 we prove by Moser iteration that the maximum of a weak solution is controlled by its spatial $\mathrm{L}^1$ norm. This is the basic estimate throughout this paper.

In section 3 we introduce $\mathcal{S}_{\mathrm{M}}$ as the class of all solutions of (1.1) satisfying $\sup_{t\in(0,\infty)} \|u\|_{1,\mathbb{R}^n}(t) \le \mathrm{M}$ and prove the following compactness theorem.

**Theorem 1.1.** *Let $\{u_k\}$ be a sequence in $\mathcal{S}_{\mathrm{M}}$, where each $u_k$ is a solution of* (1.1) *with corresponding $b_k \in [0,1]$ and with corresponding initial trace $\mu_k$. Suppose that $\mu_k$ converges to $\mu$ weakly and that $b_k$ converges to $b \in [0,1]$. Then there exists a subsequence of $\{u_k\}$ which converges uniformly on each compact subset of $\mathbb{R}^n \times (0,\infty)$ to a solution $u$ of $u_t = \Delta[u^m] - b\,u^p$ whose initial trace is $\mu$.*

The existence of a fundamental solution follows as a corollary. Furthermore, Theorem 1.1 is our main device to prove the following integral-type Harnack principle.

**Theorem 1.2.** *Suppose that $u$ is a solution of* (1.1). *Then there exist constants $C_{\mathrm{H}}$ and $\mathrm{T}_0$ depending only on $m$ and $n$ such that for each $0 < t_1 < t_2 < \infty$ and $R \ge \big[(t_2 - t_1)/\mathrm{T}_0\big]^{(m-p)/[2(1-p)]}$,*

$$\fint_{B_R(x_0)} u(x,t_1)\,dx \le C_{\mathrm{H}}\Big\{\Big[\frac{R^2}{t_2 - t_1}\Big]^{\frac{1}{m-1}} + \Big[\frac{t_2 - t_1}{R^2}\Big]^{\frac{n}{2}}[u(x_0,t_2)]^{\frac{\varkappa}{2}}\Big\},$$

*where the left hand side denotes the integral average of $u(x,t_1)$ in $B_R(x_0)$ and $\varkappa = (m-1)n + 2$.*

Notice that the finite extinction property prevents us from obtaining a classical Harnack inequality such as $u(x_0,t_0) \le C \inf_{|x-x_0|\le R} u(x, t_0 + h)$, where $C > 0$. The compactness argument adopted here is developed by Dahlberg-Kenig [5] to study a generalized porous medium equation. This is particularly useful when we do not know the explicit form of fundamental solutions.

In Section 4 the existence of initial traces of weak solutions is proved using our Harnack principle.

**Theorem 1.3.** *Suppose $u$ is a solution of* (1.1) *in $\mathbb{R}^n \times (0,\infty)$. Then there is a unique Radon measure $\mu$ on $\mathbb{R}^n$ which is the initial trace of $u$. Furthermore,*

$$\sup_{R \ge C_{13}} R^{-\frac{\varkappa}{m-1}} \int_{B_R(0)} d\mu < C(m,n,u(0,1)), \quad C_{13} = \mathrm{T}_0^{\frac{p-m}{2(1-p)}}, \tag{1.5}$$

*where* $\mathrm{T}_0$ *is the generic constant described in Theorem* 1.2.

The nonexistence of a very singular solution of (1.1) follows as a corollary. Notice that (1.5) gives a necessary condition for the existence of a solution to the Cauchy problem of (1.1) involving measures as initial conditions.

In Section 5 we describe the propagation of positive set of nonnegative weak solutions which is continuous on $\mathbb{R}^n \times [0, \infty)$. The following theorem states that the propagation speed of positivity set $\{u > 0\}$ is finite.

**Theorem 1.4.** *Suppose that* $u$ *is a solution of* (1.1) *which is continuous on* $\mathbb{R}^n \times [0, \infty)$ *and* $\operatorname{supp} u_0 \subset\subset B_R(0)$*. Then there exists a constant* $C$ *depending only on* $m$ *and* $n$ *such that* $u(x,t) = 0$ *for all* $(x,t)$ *satisfying*

$$t \leq \min\Big\{ (|x| - R)^{\frac{2(1-p)}{m-p}} \, , \, \frac{C}{\|u_0\|_1^{m-1}} (|x| - R)^{\varkappa} \Big\}.$$

*Consequently we have, for each* $t$*,* $\operatorname{supp} u(\cdot, t)$ *is contained in the ball in* $\mathbb{R}^n$ *centered at the origin with radius*

$$\max\Big\{ R + \big(C \|u_0\|_1^{m-1} t\big)^{\frac{1}{\varkappa}} \, , \, R + t^{\frac{m-p}{2(1-p)}} \Big\}.$$

Notice that the propagation speed estimate in Theorem 1.4 makes sense locally in time, since the solution vanishes after the finite extinction time $\mathrm{T}_\mathrm{E}$. As we have remarked the finite propagation property itself can be verified by a comparison argument. By contrast, our proof depends only on the various apriori estimates and does not use comparison argument, so it might be applied to another problem without comparison principle.

Now we close this section by introducing some notations that will be used in the subsequent sections. We write $z = (x,t) \in \mathbb{R}^n \times [0, \infty)$, where $x$ and $t$ denote the space and time variable, respectively. We introduce the cylindrical domains

$$Q_{R,h}(x_0, t_0) = B_R(x_0) \times (t_0 - h^2, t_0)$$
$$S_{R,h}(x_0, t_0) = B_R(x_0) \times (t_0 - h^2, t_0 + h^2)$$

where $B_R(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < R\}$. For notational simplicity, we also use the notations $Q_R$ and $S_R$, which will denote $Q_{R,R}$ and $S_{R,R}$, respectively. In some cases it is convenient to use a cylindrical domain of another type

$$Q_R^h(x_0, t_0) = B_R(x_0) \times (t_0, t_0 + h).$$

If there is no confusion, we drop out the reference point $(x_0, t_0)$ in various expressions. We use $C$ to denote the generic constants depending only on $m$, $n$ and $p$, although they may have changed in the same proof.

**2. Apriori estimates.** In this section we prove various apriori estimates which are useful to pointwise estimates for nonnegative weak solutions of (1.1). Various Moser type iterations are main tools here.

We begin with observing the scaling structure of (1.1). Let $u$ be a solution of (1.1). Then it is easily verified by direct computation that the map

$$v(x,t) = [R^{-1}h]^{2/(m-1)}u(x_0 + Rx, t_0 + h^2 t)$$

becomes a solution of

$$v_t = \Delta[v^m] - b[R^{p-1}h^{m-p}]^{2/(m-1)}v^p \quad \text{in } \mathbb{R}^n \times (-t_0, \infty). \tag{2.1}$$

Notice that if $R \geq h^{(m-p)/(1-p)}$, then the coefficient of the absorption term for this new equation (2.1) is less than or equal to 1 and hence is of the type of (1.1). In the following we make use of this property to simplify the calculations.

**Lemma 2.1.** *Suppose $u$ is a solution of* (1.1) *in a domain containing $Q_1(0,0)$. Then there exists a constant $C$ depending only on $m$ and $n$ such that*

$$\|u\|_{\infty, Q_{\frac{1}{2}}} \leq C\Big[\fint_{Q_1} u^{m+1}\,dz + 1\Big]^{1/2}.$$

**Proof.** Let $1/2 < r < \rho \leq 1$ and let $\eta$ be a standard cut-off function such that

$$0 \leq \eta \leq 1;\ \eta \equiv 1 \text{ on } Q_r;\ \eta = 0 \text{ on } \partial_{\mathrm{p}} Q_\rho;\ \max\{|\nabla\eta|^2, |\eta_t|\} \leq (\rho - r)^{-2}c_0,$$

where $c_0$ is a fixed positive constant independent of $r$ and $\rho$. Here $\partial_{\mathrm{p}} Q_\rho$ denotes the usual parabolic boundary of $Q_\rho$, that is, $\partial_{\mathrm{p}} Q_\rho = \partial Q_\rho - \{(x,t) \in \overline{Q_\rho} : t = 0\}$.

Let $t \in (-\rho^2, 0)$ and $\alpha \geq 0$. We take $\big[\chi_{[-\rho^2,t]}u^{\alpha+1}\eta^2\big]$, where $\chi$ denotes the usual characteristic function, as a test function to (1.1) to get

$$\begin{aligned}\frac{1}{\alpha+2}\int u^{\alpha+2}(x,t)\eta^2(x,t)\,dx + b\int u^{p+\alpha+1}\eta^2\,dz& \\ + m(\alpha+1)\int u^{m-1+\alpha}|\nabla u|^2\eta^2\,dz& \\ = -2m\int u^{m+\alpha}[\nabla u \cdot \nabla\eta]\eta\,dz + \frac{2}{\alpha+2}\int u^{\alpha+2}\eta_t\eta\,dz.&\end{aligned}$$

Notice that each term of LHS is nonnegative since $u$ is nonnegative. Further, applying Young's inequality to the first term on RHS we obtain

$$\frac{1}{\alpha+2}\sup_{t\in[-\rho^2,0]}\|u^{\alpha+2}\eta^2\|_{1,\mathbb{R}^n}(t)+\frac{1}{2}m(\alpha+1)\int u^{m+\alpha-1}|\nabla u|^2\eta^2\,dz$$
$$\leq\frac{2}{\alpha+2}\int u^{\alpha+2}\eta|\eta_t|\,dz+\frac{4m}{\alpha+1}\int u^{m+\alpha+1}|\nabla\eta|^2\,dz. \tag{2.2}$$

Once we have (2.2) the remaining process is rather standard. Decomposing

$$u^{m+\alpha+1+\frac{2}{n}(\alpha+2)}\eta^{2(1+\frac{2}{n})}=\left[u^{\alpha+2}\eta^2\right]^{\frac{2}{n}}\left[u^{m+\alpha+1}\eta^2\right]$$

it follows from Hölder inequality, Sobolev inequality and (2.2) that

$$\int u^{m+\alpha+1+\frac{2}{n}(\alpha+2)}\eta^{2(1+\frac{2}{n})}\,dz=\int\left[u^{\alpha+2}\eta^2\right]^{\frac{2}{n}}\left[u^{m+\alpha+1}\eta^2\right]dz$$
$$\leq\int\Big[\int u^{\alpha+2}\eta^2\,dx\Big]^{\frac{2}{n}}\Big[\int\big(u^{\frac{m+\alpha+1}{2}}\eta\big)^{\frac{2n}{n-2}}\,dx\Big]^{\frac{n-2}{n}}\,dt$$
$$\leq C(n)\Big[\sup_{t\in[-\rho^2,0]}\|u^{\alpha+2}\eta^2\|_1(t)\Big]^{\frac{2}{n}}\int|\nabla\big(u^{\frac{m+\alpha+1}{2}}\eta\big)|^2\,dz$$
$$\leq(c_1\alpha+c_2)^{c_3}\Big[\frac{1}{(\rho-r)^2}\int_{Q_\rho}\big(u^{m+\alpha+1}+1\big)\,dz\Big]^{1+2/n} \tag{2.3}$$

for some constants $c_1$, $c_2$ and $c_3$ depending only on $m$, and $n$. Let us define sequences $\{\rho_{\mathrm{J}}\}_{\mathrm{J}=0}^{\infty}$, $\{\alpha_{\mathrm{J}}\}_{\mathrm{J}=0}^{\infty}$ and $\{\Phi_{\mathrm{J}}\}_{\mathrm{J}=0}^{\infty}$ by

$$\rho_{\mathrm{J}}=\frac{1}{2}(1+2^{-\mathrm{J}});\ \ \alpha_0=0,\ \ \alpha_{\mathrm{J}+1}=\alpha_{\mathrm{J}}+\frac{2}{n}(\alpha_{\mathrm{J}}+2);\ \ \Phi_{\mathrm{J}}=\fint_{Q_{\rho_{\mathrm{J}}}}u^{m+\alpha_{\mathrm{J}}+1}\,dz.$$

Observe that $m+\alpha_{\mathrm{J}}+1+\frac{2}{n}(\alpha_{\mathrm{J}}+2)=m+\alpha_{\mathrm{J}+1}+1$. Then we deduce from (2.3) that

$$\Phi_{\mathrm{J}+1}\leq c^{\mathrm{J}+1}\Phi_{\mathrm{J}}^{\gamma}+c^{\mathrm{J}+1},\quad\gamma=1+2/n, \tag{2.4}$$

for some large constant $c$. Iterating (2.4) we get

$$\Phi_{\mathrm{J}}\leq c^{\mathrm{J}}\Phi_{\mathrm{J}-1}^{\gamma}+c^{\mathrm{J}}\leq\ 2^{\gamma^{\mathrm{J}-1}+\gamma^{\mathrm{J}-2}+\cdots+\gamma}c^{\mathrm{J}+(\mathrm{J}-1)\gamma+\cdots+\gamma^{\mathrm{J}-1}}\Phi_0^{\gamma^{\mathrm{J}}}$$
$$+\mathrm{J}2^{\gamma^{\mathrm{J}-1}+\gamma^{\mathrm{J}-2}+\cdots+\gamma}c^{\mathrm{J}+(\mathrm{J}-1)\gamma+\cdots+\gamma^{\mathrm{J}-1}}. \tag{2.5}$$

Note that $\lim_{\mathrm{J}\to\infty}(m+\alpha_{\mathrm{J}}+1)/\gamma^{\mathrm{J}}=2$. Sending $\mathrm{J}\to\infty$ in (2.5) and after suitable adjustment we get the result. □

Now we improve our $\mathrm{L}^\infty$ estimate. In fact, we can estimate the $\mathrm{L}^\infty$ norm of a weak solution by its spatial $\mathrm{L}^1$ norm. This is achieved by the previous lemma and further analysis using the cylinder $S_R$ instead of $Q_R$.

**Lemma 2.2.** *Suppose $u$ is a solution of (1.1) in a domain containing $S_2(0,0)$. Then*

$$\|u\|_{\infty,Q_{\frac{1}{2}}} \le C\Big[\sup_{|t|\le 4}\|u\|_{1,B_2}(t)+1\Big]^{\delta},$$

*for some constant $C$ and $\delta$ depending only on $m$ and $n$.*

**Proof.** Let $1<\rho<r\le 2$ and $\eta$ a standard cut-off function such that $0\le \eta\le 1$, $\eta=0$ on $\partial S_r$, $\eta(z)=1$ for all $z\in S_\rho$, $\max\{|\nabla\eta|^2,|\eta_t|\}\le (r-\rho)^{-2}c_0$, where $c_0$ is a fixed positive constant.

Let us denote $u_1=u+1$. For any number $\alpha$ such that

$$\alpha\ne -1,\quad \alpha\ne -2,\quad m+\alpha>-1, \tag{2.6}$$

we take $\big[u_1^{\alpha+1}\eta^2\big]$ as a test function to (1.1) and get

$$\begin{aligned} &m(\alpha+1)\int u^{m-1}u_1^{\alpha}|\nabla u|^2\eta^2\,dz+b\int u^p u_1^{\alpha+1}\eta^2\,dz\\ =&\frac{2}{\alpha+2}\int u_1^{\alpha+2}\eta_t\eta\,dz-2m\int u^{m-1}u_1^{\alpha+1}[\nabla u\cdot\nabla\eta]\eta\,dz.\end{aligned}$$

Then by Young's inequality we have

$$\begin{aligned} &\frac{1}{2}m|\alpha+1|\int u^{m-1}u_1^{\alpha}|\nabla u|^2\eta^2\,dz \\ \le&\frac{2}{|\alpha+2|}\int u_1^{\alpha+2}\eta|\eta_t|\,dz+\frac{8m}{|\alpha+1|}\int u^{m-1}u_1^{\alpha+2}|\nabla\eta|^2\,dz+b\int u^p u_1^{\alpha+1}\eta^2\,dz.\end{aligned} \tag{2.7}$$

Define $\mathcal{G}=\sup_{|t|\le 4}\|u\|_{1,B_2}(t)$. As in Lemma 2.1 writing $u^{m+1+\frac{2}{n}}u_1^{\alpha}\eta^{2[1+\frac{2}{n}]}$ as

$$u^{m+1+\frac{2}{n}}u_1^{\alpha}\eta^{2[1+\frac{2}{n}]}=\big[u\eta^2\big]^{\frac{2}{n}}\big[u^{m+1}u_1^{\alpha}\eta^2\big]$$

we obtain from Hölder inequality, Sobolev inequality and the estimate (2.7) that

$$\int u^{m+1+\frac{2}{n}}u_1^{\alpha}\eta^{2[1+\frac{2}{n}]}\,dz\le C(\alpha)\frac{\mathcal{G}^{\frac{2}{n}}}{(r-\rho)^2}\int\big[u^{m-1}u_1^{\alpha+2}+u_1^{\alpha+2}+u^p u_1^{\alpha+1}\big]dz. \tag{2.8}$$

Since we are assuming $p<1<m$, we deduce from (2.8) that for each $1<\rho<r\le 2$,

$$\int_{S_\rho}u_1^{m+\alpha+1+\frac{2}{n}}\,dz\le C(\alpha)\frac{(\mathcal{G}+1)^{\frac{2}{n}}}{(r-\rho)^2}\int_{S_r}u_1^{m+\alpha+1}\,dz. \tag{2.9}$$

Now we define finite sequences $\{\alpha_{\mathrm{J}}\}_{\mathrm{J}=0}^{s}$ and $\{r_{\mathrm{J}}\}_{\mathrm{J}=0}^{s}$ by $r_{\mathrm{J}} = 1 + 2^{-\mathrm{J}}$ and $\alpha_{\mathrm{J}} = 2\mathrm{J}/n - (m + \varepsilon_0)$, where $\varepsilon_0$ is a sufficiently small number such that each $\alpha_{\mathrm{J}}$ satisfies (2.6) and $s$ is the first integer satisfying $\alpha_s \geq 0$. It follows from (2.9) that

$$\int_{S_{r_{\mathrm{J}+1}}} u_1^{m+\alpha_{\mathrm{J}+1}+1}\, dz \leq C(\mathcal{G}+1)^{2/n} \int_{S_{r_{\mathrm{J}}}} u_1^{m+\alpha_{\mathrm{J}}+1}\, dz.$$

From this recurrence relation we get

$$\int_{S_{r_s}} u_1^{m+\alpha_s+1}\, dz \leq C(\mathcal{G}+1)^{2s/n} \int_{S_{r_0}} u_1^{m+\alpha_0+1}\, dz,$$

and hence

$$\int_{S_1} u_1^{m+1}\, dz \leq C\big[\sup_{|t|\leq 4} \|u\|_{1,B_2}(t) + 1\big]^{2s/n+1},$$

since $r_s \geq 1$, $\alpha_s \geq 0$, $r_0 = 2$ and $m + \alpha_0 + 1 = 1 - \varepsilon_0 \leq 1$. Consequently we have

$$⨍_{Q_1} (u+1)^{m+1}\, dz \leq C\big[\sup_{|t|\leq 4} \|u\|_{1,B_2}(t) + 1\big]^{\delta}, \tag{2.10}$$

for some positive constants $C$ and $\delta$ depending only on $m$ and $n$. Combining Lemma 2.1 and (2.10) we get the lemma. $\square$

The main result in this section is the following improvement of Lemma 2.2.

**Theorem 2.3.** *Suppose $u$ is a solution of* (1.1) *in a domain containing $S_2(0,0)$. Then there are positive constants $C$, $\delta_1$ and $\delta_2$ depending only on $m$ and $n$ such that*

$$\|u\|_{\infty,Q_{\frac{1}{4}}} \leq C\big\{\mathcal{G}^{\delta_1} + \mathcal{G}^{\delta_2}\big\},$$

*where $\mathcal{G} = \sup_{|t|\leq 4} \|u\|_{1,B_2}(t)$.*

**Proof.** Let $\varepsilon_0$ be a fixed positive constant. Notice that if $\mathcal{G} \geq \varepsilon_0$, then $1 \leq \varepsilon_0^{-\delta}\mathcal{G}^{\delta}$. So, by Lemma 2.2, we get $\|u\|_{\infty,Q_{1/4}} \leq C(\varepsilon_0)\mathcal{G}^{\delta_1}$, where we set $\delta_1 = \delta$, the generic constant appeared in Lemma 2.2.

Now we assume that $\mathcal{G} < \varepsilon_0$. Then, by Lemma 2.2, we obtain $\|u\|_{\infty,Q_{1/2}} \leq C(\varepsilon_0)$, and hence from (2.2) we can deduce

$$\begin{aligned} &\frac{1}{\alpha+2} \sup_{t\in[-r^2,0]} \|u^{\alpha+2}\eta^2\|_{1,B_r}(t) + \frac{1}{2} m(\alpha+1) \int_{Q_r} u^{m+\alpha-1} |\nabla u|^2 \eta^2\, dz \\ &\leq C(\varepsilon_0)\Big[\frac{2}{\alpha+2} + \frac{4m}{\alpha+1}\Big] \frac{1}{(r-\rho)^2} \int_{Q_r} u^{\alpha}\, dz, \end{aligned} \tag{2.11}$$

for each $1/4 < \rho < r \le 1/2$ and for each $\alpha \ge 0$, where $\eta$ is a standard cut-off function as in Lemma 2.1. By Hölder inequality, Sobolev inequality and (2.11) we get

$$\int_{Q_\rho} u^{(m-1)+(\alpha+2)(1+\frac{2}{n})}\, dz \le C_0(\varepsilon_0)\Big[\frac{1}{(r-\rho)^2}\int_{Q_r} u^\alpha\, dz\Big]^{1+2/n},$$

for each $1/4 < \rho < r \le 1/2$. Iterating this we get $\|u\|_{\infty,Q_{1/4}} \le C(\varepsilon_0)\,\mathcal{G}^{\delta_2}$, for some positive constant $\delta_2$ depending only on $m$ and $n$. The proof is completed. □

By the scaling argument we obtain the following

**Corollary 2.4.** *Suppose $u$ is a solution of* (1.1) *and $S_{2R,2h}(x_0,t_0) \subset \mathbb{R}^n \times (0,\infty)$ with $R \ge h^{(m-p)/(1-p)}$. Then*

$$\|u\|_{\infty,Q_{\frac14 R,\frac14 h}} \le C\big[Rh^{-1}\big]^{\frac{2}{m-1}}\big\{\mathcal{G}^{\delta_1}_{2R,2h} + \mathcal{G}^{\delta_2}_{2R,2h}\big\},$$

*where*

$$\mathcal{G}_{R,h} = \big[R^{-\varkappa}h^2\big]^{\frac{1}{m-1}} \sup_{|t-t_0|\le h^2} \|u\|_{1,B_R(x_0)}(t),$$

*for some constants $C$, $\delta_1$ and $\delta_2$ depending only on $m$ and $n$.*

**Proof.** Let $\gamma = [R^{-1}h]^{2/(m-1)}$ and $v(x,t) = \gamma u(x_0 + Rx, t_0 + h^2 t)$. Then $v$ is a solution of

$$v_t = \Delta[v^m] - b\left[R^{p-1}h^{m-p}\right]^{2/(m-1)} v^p.$$

Note that if $R \ge h^{(m-p)/(1-p)}$, then $b\,[R^{p-1}h^{m-p}]^{2/(m-1)} \in [0,1]$. Also,

$$\|v\|_{\infty,Q_{\frac14}(0,0)} = \gamma\,\|u\|_{\infty,Q_{\frac14 R,\frac14 h}(x_0,t_0)}$$

and

$$\sup_{|t|\le 4}\|v\|_{1,B_2(0)}(t) = \gamma R^{-n} \sup_{|t-t_0|\le 4h^2}\|u\|_{1,B_{2R}(x_0)}(t).$$

Applying Theorem 2.3 to $v$ and scaling back we obtain the result. □

It is easily verified from the definition of weak solutions that the spatial $\mathrm{L}^1$ norm is decreasing with respect to time.

**Proposition 2.5.** *Suppose $u$ is a solution of* (1.1). *Then for each $t_2 > t_1 > 0$ we have*

$$\|u\|_1(t_2) \le \|u\|_1(t_1).$$

**Proof.** We will show that

$$\|u\|_{1,B_R(0)}(t_2) \le \|u\|_1(t_1) \quad \text{for all} \quad R > 0 \tag{2.12}$$

from which the lemma follows. Given $R > 0$, we construct a sequence $\{\zeta_k\}$ of cut-off functions on $\mathbb{R}^n$ satisfying $0 \le \zeta_k \le 1$, $\zeta_k \equiv 1$ on $B_R(0)$, $\zeta_k \equiv 0$ on $\mathbb{R}^n - B_{(1+k^{-1})R}(0)$ and $|\nabla \zeta_k| \le c_0 R^{-1} k$, for some fixed constant $c_0$ not depending on $k$. Taking a test function $[\chi_{[t_1,t_2]}\zeta_k]$ to (1.1) we get

$$\|u\zeta_k\|_1(t_2) - \|u\zeta_k\|_1(t_1) \le \|\nabla u^m\|_{2,\Omega_k \times [t_1,t_2]} \|\nabla \zeta_k\|_{2,\Omega_k\times[t_1,t_2]}, \tag{2.13}$$

where $\Omega_k = B_{(1+k^{-1})R}(0) - B_R(0)$. Note that $\|\nabla\zeta_k\|_{2,\Omega_k\times[t_1,t_2]} \le C$ for some constant $C$ independent of $k$. Then from the definition of weak solutions the RHS of (2.13) goes to zero as $k$ goes to infinity, since $|\Omega_k \times [t_1, t_2]| \to 0$. We also note that for each $k$

$$\|u\|_{1,B_R(0)}(t_2) - \|u\|_1(t_1) \le \|u\zeta_k\|_1(t_2) - \|u\zeta_k\|_1(t_1). \tag{2.14}$$

From (2.13) and (2.14) we get (2.12) and we complete the proof. □

**3. Compactness and Harnack principle.** In this section we prove a compactness theorem (Theorem 1.1) and a Harnack inequality (Theorem 1.2). The existence of a fundamental solution will be obtained as a corollary.

We start with a definition. For fixed $m > 1$ and $p \in (0, 1)$ we denote by $\mathcal{S}_{\mathrm{M}}$ the class of all nonnegative weak solutions of $u_t = \Delta[u^m] - b\,u^p$ for some $b \in [0, 1]$ satisfying

$$\sup_{t\in(0,\infty)} \|u\|_1(t) \le \mathrm{M}.$$

For the class $\mathcal{S}_{\mathrm{M}}$ we get the following uniform estimate from our apriori estimates.

**Lemma 3.1.** *Suppose $u \in \mathcal{S}_{\mathrm{M}}$. Then, for each $t \in (0, 1)$, we have*

$$\|u\|_{\infty,\mathbb{R}^n}(t) \le C(\mathrm{M})\, t^{-n/\varkappa}$$

*for some positive constant* $C(\mathrm{M})$ *depending only on* $m$, $n$ *and* $\mathrm{M}$.

**Proof.** Let $(x,t)\in\mathbb{R}^n\times(0,1)$. We set $R=(\sqrt{t}/2)^{2/\varkappa}$ and $h=\sqrt{t}/2$. Then we see that $S_{2R,2h}=S_{2R,2h}(x,t)\subset\mathbb{R}^n\times(0,\infty)$ and $R\ge h^{(m-p)/(1-p)}$. Furthermore

$$\mathcal{G}_{2R,2h}=2^{-n}\sup_{|s-t|\le t}\|u\|_{1,B_{2R}}(s)\le \mathrm{M}.$$

Thus, by Corollary 2.4, we have

$$u(x,t)\le\|u\|_{\infty,Q_{\frac14 R,\frac14 h}}\le C\big[(\sqrt{t}/2)^{2/\varkappa-1}\big]^{2/(m-1)}\big\{\mathrm{M}^{\delta_1}+\mathrm{M}^{\delta_2}\big\},$$

from which the lemma follows. □

**Lemma 3.2.** *Suppose* $u\in\mathcal{S}_{\mathrm{M}}$. *Then for each* $t\in(0,1)$ *and for any* $\Omega\subset\subset\mathbb{R}^n$

$$\|\nabla u^m\|_{1,\Omega\times(0,t)}\le C(\mathrm{M})\,t^{1/\varkappa} \tag{3.1}$$

*for some positive constant* $C(\mathrm{M})$ *depending only on* $m$, $n$ *and* $\mathrm{M}$.

**Proof.** Let us set $\varepsilon_1=5/4-3m/4$ and $\varepsilon_2=1/4$. Then, since $|\nabla u^m|=mu^{m-1}|\nabla u|$, by Young's inequality, we get

$$\text{LHS of (3.1)}=m\int_0^t\int_\Omega\big[s^{\varepsilon_2}u^{-\varepsilon_1}|\nabla u|\big]\,\big[s^{-\varepsilon_2}u^{m-1+\varepsilon_1}\big]\,dx\,ds \tag{3.2}$$
$$\le\Big\{\frac{m}{2}\int_0^t\int_\Omega s^{2\varepsilon_2}u^{-2\varepsilon_1}|\nabla u|^2\,dx\,ds\Big\}+\Big\{\frac{m}{2}\int_0^t\int_\Omega s^{-2\varepsilon_2}u^{2(m-1+\varepsilon_1)}\,dx\,ds\Big\}.$$

We write the RHS of (3.2) as $\{\mathrm{I}\}+\{\mathrm{II}\}$ and estimate each term separately. First we estimate $\{\mathrm{II}\}$. By Lemma 3.1 and the fact

$$\sup_{s\in(0,t)}\|u\|_{1,\Omega}(s)\le\mathrm{M},\quad\text{for all }\Omega\subset\mathbb{R}^n, \tag{3.3}$$

we have

$$\{\mathrm{II}\}\le\frac{m}{2}\mathrm{M}\int_0^t s^{-2\varepsilon_2}\|u\|_\infty^{2(m-1+\varepsilon_1)-1}(s)\,ds$$
$$\le C(\mathrm{M})\int_0^t s^{-2\varepsilon_2-\frac{n}{\varkappa}(2m-3+2\varepsilon_1)}\,ds\le C(\mathrm{M})\,t^{1/\varkappa}. \tag{3.4}$$

To estimate $\{\mathrm{I}\}$ we let $\psi$ be a standard cut-off function in $\mathbb{R}^n$ such that $\psi = 1$ on $\Omega$ and take $\left[t^{2\varepsilon_2} u^{2-(m+2\varepsilon_1)}\psi^2\right]$ as a test function. Then, by Young's inequality, (3.3) and Lemma 3.1, we get

$$\begin{aligned}
\{\mathrm{I}\} &\le C\int_0^t\int \left[s^{2\varepsilon_2-1}u^{3-(m+2\varepsilon_1)}\psi^2 + s^{2\varepsilon_2}u^{2-2\varepsilon_1}|\nabla\psi|^2\right]dx\,ds \\
&\le C\mathrm{M}\int_0^t \left[s^{2\varepsilon_2-1}\|u\|_\infty^{2-(m+2\varepsilon_1)}(s) + s^{2\varepsilon_2}\|u\|_\infty^{1-2\varepsilon_1}(s)\right]ds \qquad (3.5)\\
&\le C(\mathrm{M})\left[t^{2\varepsilon_2-\frac{n}{\varkappa}[2-(m+2\varepsilon_1)]} + t^{1+2\varepsilon_2-\frac{n}{\varkappa}(1-2\varepsilon_1)}\right] \\
&\le C(\mathrm{M})\left[t^{1/\varkappa} + t^{3/\varkappa}\right].
\end{aligned}$$

Combining (3.2), (3.4) and (3.5) we get the lemma. $\square$

**Lemma 3.3.** *Suppose $u \in \mathcal{S}_{\mathrm{M}}$. Then for each $t \in (0,1)$ and for any $\Omega \subset\subset \mathbb{R}^n$,*

$$\|u^p\|_{1,\Omega\times(0,t)} \le C(\mathrm{M},|\Omega|)\,t^{\delta},$$

*where $C(\mathrm{M},|\Omega|)$ is a positive constant depending only on $m$, $n$, $p$, $\mathrm{M}$ and $|\Omega|$, and*

$$\delta = \delta(m,n,p) = \begin{cases} 1-\frac{n}{\varkappa}p & \text{if} \quad p < \frac{\varkappa}{n} \\ \frac{2}{3} & \text{if} \quad p = \frac{\varkappa}{n} \\ \frac{np-\varkappa}{2np-\varkappa} & \text{if} \quad p > \frac{\varkappa}{n}. \end{cases}$$

**Proof.** This is the consequence of Lemma 3.1 and Hölder inequality. The first case is rather simple. We have from Lemma 3.1 that

$$\begin{aligned}
\|u^p\|_{1,\Omega\times(0,t)} &\le |\Omega|\int_0^t \|u\|_\infty^p(s)\,ds \le C(\mathrm{M},|\Omega|)\int_0^t s^{-\frac{n}{\varkappa}p}\,ds \\
&\le C(\mathrm{M},|\Omega|)\,t^{1-np/\varkappa}.
\end{aligned}$$

We see that if $\varkappa \ge n$, then the remaining two cases are vacuous, since we are assuming $p < 1$. So we assume $\varkappa < n$ and proceed the proof. Now we consider the second case. For notational convenience, we set $\varepsilon = 1/2$. Then, by Hölder inequality, (3.3) and Lemma 3.1, we have

$$\begin{aligned}
\|u^{\varkappa/n}\|_{1,\Omega\times(0,t)} &\le |\Omega|^{1-\frac{\varkappa}{n(1+\varepsilon)}}\int_0^t\left[\int_\Omega u^{1+\varepsilon}\,dx\right]^{\frac{\varkappa}{n(1+\varepsilon)}}ds \\
&\le |\Omega|^{1-\frac{\varkappa}{n(1+\varepsilon)}}\int_0^t\left[\mathrm{M}\|u\|_\infty^{\varepsilon}(s)\right]^{\frac{\varkappa}{n(1+\varepsilon)}}ds \\
&\le C(\mathrm{M},|\Omega|)\int_0^t s^{-\frac{\varepsilon}{1+\varepsilon}}\,ds \le C(\mathrm{M},|\Omega|)\,t^{2/3}.
\end{aligned}$$

Finally let $p > \varkappa/n$. We set $\varepsilon = [1/p + n/(np-\varkappa)]/2$. It is verified through direct calculation that $p\varepsilon > 1$. Since we are assuming $p < 1$, we deduce $\varepsilon > 1$. It follows from Hölder inequality, (3.3) and Lemma 3.1 that

$$\begin{aligned}\|u^p\|_{1,\Omega\times(0,t)} &\le |\Omega|^{1-\frac{1}{\varepsilon}} \int_0^t \Big[\int_\Omega u^{p\varepsilon}\,dx\Big]^{\frac{1}{\varepsilon}} ds \le |\Omega|^{1-\frac{1}{\varepsilon}} \int_0^t \big[\mathrm{M}\|u\|_\infty^{p\varepsilon-1}(s)\big]^{\frac{1}{\varepsilon}} ds \\ &\le C(\mathrm{M},|\Omega|) \int_0^t s^{-\frac{n}{\varkappa}(p-\frac{1}{\varepsilon})}\,ds \le C(\mathrm{M},|\Omega|)\, t^{(np-\varkappa)/(2np-\varkappa)}.\end{aligned}$$

The proof is completed. □

Finally we need the local equicontinuity for the class $\mathcal{S}_\mathrm{M}$. This is a consequence of Lemma 3.1 and the regularity theorem of DiBenedetto-Friedman [6].

**Lemma 3.4.** *Suppose $u \in \mathcal{S}_\mathrm{M}$. Then, for each $\Omega \times (\varepsilon_1, \varepsilon_2) \subset\subset \mathbb{R}^n \times (0,\infty)$*

$$\|u\|_{C^\alpha(\Omega\times[\varepsilon_1,\varepsilon_2])} \le C(\mathrm{M}, \varepsilon_1, \varepsilon_2),$$

*for some positive constant $C(\mathrm{M}, \varepsilon_1, \varepsilon_2)$ depending only on $m, n, \varepsilon_1, \varepsilon_2, \mathrm{M}$.*

**Proof.** Let $\Omega \subset\subset B_{R-1}(x_0) \subset\subset B_R(x_0)$ and set $h = \sqrt{\varepsilon_2 - \varepsilon_1/4}$. Note that $\Omega \times (\frac{1}{2}\varepsilon_1, \varepsilon_2) \subset Q_{R,h}(x_0, \varepsilon_2)$. Do the same calculation within this cylinder as in Lemma 2.1 to get

$$\|\nabla u^m\|^2_{2,\Omega\times(\frac{1}{2}\varepsilon_1,\varepsilon_2)} \le C\big\{\|u^{m+1}\|_{1,\mathbb{R}^n\times(\frac{1}{4}\varepsilon_1,\varepsilon_2)} + \|u^{2m}\|_{1,\mathbb{R}^n\times(\frac{1}{4}\varepsilon_1,\varepsilon_2)}\big\},$$

for some positive constant $C$ depending only on $m$ and $n$, see (2.2). On the other hand, for each $t \in (\frac{1}{2}\varepsilon_1, \varepsilon_2)$, we have

$$\|u\|^2_{2,\Omega}(t) \le \|u^2\|_{1,\mathbb{R}^n}(t).$$

Thus we conclude from (3.3) and Lemma 3.1 that

$$\sup_{t\in(\frac{1}{2}\varepsilon_1,\varepsilon_2)} \|u\|^2_{2,\Omega}(t) + \|\nabla u^m\|^2_{2,\Omega\times(\frac{1}{2}\varepsilon_1,\varepsilon_2)} \le C(\mathrm{M}, \varepsilon_1, \varepsilon_2),$$

for some constant $C(\mathrm{M}, \varepsilon_1, \varepsilon_2)$ depending only on $m$, $n$, $\varepsilon_1$, $\varepsilon_2$ and M. Then by Theorem 1.2 in [6] we get the estimate. □

Now we are ready to prove our compactness theorem.

**Proof of Theorem 1.1.** Since the sequence $\{u_k\}$ is contained in $\mathcal{S}_{\mathrm{M}}$, it follows from Lemma 3.1 that the sequence is locally uniformly bounded. Furthermore by Lemma 3.4 we know that these are locally equicontinuous. Hence it is enough to show that whenever $u$ is locally the uniform limit of $\{u_k\}$, $u$ is a solution of $u_t = \Delta[u^m] - b\,u^p$ with initial trace $\mu$. It is easily verified that $u$ is a solution. Now, to prove that $\mu$ is the initial trace of $u$, let $0 < t_1 < t_2 < 1$ and $\zeta \in C_0^\infty(\mathbb{R}^n)$. Taking $\left[\chi_{[t_1,t_2]}\zeta\right]$ as a test function to (1.1) we see from (1.3) and Lemma 3.2 and Lemma 3.3 that

$$\begin{aligned} &|\int u_k(x,t_2)\zeta(x)\,dx - \int u_k(x,t_1)\zeta(x)\,dx| \\ &\le C\int_{t_1}^{t_2}\int \left\{[u_k]^{m-1}|\nabla u_k||\nabla\zeta| + [u_k]^p\zeta\right\}dx\,ds \\ &\le C(\mathrm{M}, \|\nabla\zeta\|_\infty, \|\zeta\|_\infty, |\mathrm{supp}\,\zeta|)\,t_2^{\tilde\delta}, \end{aligned} \tag{3.6}$$

for some positive constant $\tilde\delta$ depending only on $m$, $n$ and $p$. In fact,

$$\tilde\delta = \tilde\delta(m,n,p) = \begin{cases} \min\{\frac{1}{\varkappa}, 1 - \frac{n}{\varkappa}p\} & \text{if} \quad p < \frac{\varkappa}{n} \\ \min\{\frac{1}{\varkappa}, \frac{2}{3}\} = \frac{1}{\varkappa} & \text{if} \quad p = \frac{\varkappa}{n} \\ \min\{\frac{1}{\varkappa}, \frac{np-\varkappa}{2np-\varkappa}\} & \text{if} \quad p > \frac{\varkappa}{n}. \end{cases}$$

Sending $t_1$ to zero and then sending $k$ to infinity on (3.6) we obtain

$$|\int u(x,t_2)\zeta(x)\,dx - \int \zeta(x)\,d\mu| \le C(\mathrm{M}, \|\nabla\zeta\|_\infty, \|\zeta\|_\infty, |\mathrm{supp}\,\zeta|)t_2^{\tilde\delta}.$$

Sending $t_2$ to zero we obtain (1.4) and thus the proof is completed. □

The existence of a fundamental solution follows as a direct consequence of the compactness theorem.

**Corollary 3.5.** *There exists a fundamental solution of* (1.1).

**Proof.** Recall that for any continuous and compactly supported function $u_0$ in $\mathbb{R}^n$, the Cauchy problem of (1.1) has a unique solution $u$. Furthermore by Proposition 2.5 we have

$$\|u\|_1(t) \le \|u_0\|_1 \quad \text{for all} \quad t \in (0,\infty),$$

so that $u \in \mathcal{S}_{\|u_0\|_1}$. Now let $\Psi \in C_0^\infty(\mathbb{R}^n)$ be such that $\|\Psi\|_1 = 1$ and construct a sequence $\{\Psi_k\}$ in $C_0^\infty(\mathbb{R}^n)$ satisfying $\|\Psi_k\|_1 = 1$ by

$$\Psi_k(x) = k^n \Psi(kx).$$

We also construct a sequence $\{u_k\}$ in $\mathcal{S}_1$ by solving the Cauchy problem

$$[u_k]_t = \Delta\big[u_k\big]^m - b\big[u_k\big]^p; \quad u_k(x,0) = \Psi_k(x).$$

Noticing that $\Psi_k$ converges weakly to Dirac mass we know from Theorem 1.1 that there exists a fundamental solution. □

Turning to the proof of Harnack inequality we prepare two lemmas. It is well known that the fundamental solution of $u_t = \Delta[u^m]$ is unique, c.f. see [15].

**Lemma 3.6.** *Let $Q$ be the unique fundamental solution of $u_t = \Delta[u^m]$. Then, there is a $\mathrm{T}_0 < 1$ that can be taken to depend only on $m$ and $n$ such that $Q(0, \mathrm{T}_0) > 1/2$.*

**Proof.** It follows from a result of Vazquez[18] that $Q$ is radial and decreasing in $|x|$, for each $t > 0$. Now let $\zeta$ be a fixed element in $C_0^\infty(\mathbb{R}^n)$ which is nonnegative and

$$\zeta(0) = \max \zeta = 1, \quad \int \zeta \, dx = 1.$$

By (3.6) and since the initial trace of $Q$ is Dirac mass, we have

$$\Big| \int Q(x,t)\zeta(x)\, dx - \zeta(0) \Big| \le C t^\delta \tag{3.7}$$

for some positive constants $C$ and $\delta$ depending on $m$ and $n$. Let us choose $\mathrm{T}_0$ to be a number so that $C\mathrm{T}_0^\delta < 1/2$, where $C$ is the generic constant in (3.7). Then we have from (3.7) that

$$Q(0, \mathrm{T}_0) = \int Q(0, \mathrm{T}_0)\zeta(x)\, dx \ge \int Q(x, \mathrm{T}_0)\zeta(x)\, dx \ge 1/2.$$

The proof is completed. □

In fact, we know the explicit form of the fundamental solution of $u_t = \Delta[u^m]$ :

$$Q(x,t) = t^{-n/\varkappa}\big[\big(1 - \frac{m-1}{2m\varkappa} t^{-2/\varkappa}|x|^2\big)_+\big]^{\frac{1}{m-1}},$$

which is called the Barenblatt-Prattle solution [1,16]. So one can find the $\mathrm{T}_0$ explicitly as a function of $m$ and $n$. For example, $\mathrm{T}_0 = 1/2$ is a possible choice.

The following lemma is essential in the proof of Harnack principle.

**Lemma 3.7.** *Suppose that $u$ is a solution of* (1.1) *and* $u_0 \in \mathrm{L}^1(\mathbb{R}^n) \cap \mathrm{L}^\infty(\mathbb{R}^n)$. *Define a map*

$$H(s) = \begin{cases} 1 & \text{if} \quad s \leq 1 \\ s^{\varkappa/2} & \text{if} \quad s \geq 1. \end{cases}$$

*Then there is a constant $C$ depending only on $m$ and $n$ such that*

$$\|u_0\|_{1,B_1(0)} \leq CH(u(0,\mathrm{T}_0)), \tag{3.8}$$

*where* $\mathrm{T}_0$ *is the generic constant in Lemma 3.6.*

**Proof.** First, we prove under the additional assumption that

$$\operatorname{supp} u(x,0) \subset \{x \in \mathbb{R}^n : |x| < 1\}. \tag{3.9}$$

If (3.8) does not hold, then for each $k = 1, 2, 3, \cdots$, there exist a $b_k \in [0,1]$ and a nonnegative weak solution $u_k$ of

$$[u_k]_t = \Delta[u_k]^m - b_k[u_k]^p$$

satisfying (3.9) and

$$I_k = \|u_k\|_{1,\mathbb{R}^n}(0) \geq kH(u_k(0,\mathrm{T}_0)).$$

From the hypothesis we know that each $I_k$ is finite. Define $\alpha_k$ by

$$[\alpha_k]^{\varkappa/(m-1)} = I_k.$$

Then

$$v_k(x,t) = [\alpha_k]^{-2/(m-1)} u_k(\alpha_k x, t)$$

is a solution of

$$[v_k]_t = \Delta[v_k]^m - b_k\,[\alpha_k]^{2(p-1)/(m-1)}[v_k]^p,$$

and

$$\operatorname{supp} v_k(x,0) \subset \{x \in \mathbb{R}^n : |x| < \alpha_k^{-1}\}, \quad \|v_k\|_{1,\mathbb{R}^n}(0) = 1.$$

We note that $I_k$ goes to infinity as $k$ goes to infinity, since $H \geq 1$. Thus $\alpha_k$ goes to infinity and $v_k(x,0)$ converges weakly to Dirac mass, and then, we see that $b_k[\alpha_k]^{2(p-1)/(m-1)}$ goes to zero as $k$ goes to infinity since we are assuming $p < 1$. Finally we see from Proposition 2.5 and the observation $\|v_k\|_1(0) = 1$ that each $v_k$ is an element of $\mathcal{S}_1$. By Theorem 1.1, $v_k$ converges uniformly on each compact subset of $\mathbb{R}^n \times (0, \infty)$ to a weak solution $Q$ of $v_t = \Delta[v^m]$ with initial trace Dirac mass. So we have

$$\lim_{k\to\infty} \alpha_k^{-\frac{2}{m-1}} u_k(0, \mathrm{T}_0) = \lim_{k\to\infty} v_k(0, \mathrm{T}_0) = Q(0, \mathrm{T}_0) > 1/2 \tag{3.10}$$

by Lemma 3.6. Since $\lim_{k\to\infty} \alpha_k = \infty$, it follows that $\lim_{k\to\infty} u_k(0, \mathrm{T}_0) = \infty$, and hence, for sufficiently large $k$, we have

$$[u_k(0, \mathrm{T}_0)]^{\frac{\varkappa}{2}} = H(u_k(0, \mathrm{T}_0)) \le k^{-1} I_k = k^{-1} \alpha_k^{\frac{2}{m-1}\left[\frac{\varkappa}{2}\right]}$$

by our assumption, from which it follows that

$$Q(0, \mathrm{T}_0) = \lim_{k\to\infty} \alpha_k^{-\frac{2}{m-1}} u_k(0, \mathrm{T}_0) \le \lim_{k\to\infty} k^{-1} = 0.$$

This contradicts to (3.10) and we obtain the desired estimate.

To remove the assumption (3.9), let $\psi \in C_0^\infty(B_1)$ with $0 \le \psi \le 1$. Let $w$ solves the equation

$$w_t = \Delta[w^m] - b\, w^p; \quad w(x, 0) = \psi(x) u_0(x).$$

By comparison theorem we know $w \le u$ on $\mathbb{R}^n \times (0, \infty)$. In particular, we have $w(0, \mathrm{T}_0) \le u(0, \mathrm{T}_0)$. By applying the previous arguments to $w$ we get

$$\|u_0 \psi\|_{1, B_1(0)} = \|w\|_{1, B_1(0)}(0) \le CH(w(0, \mathrm{T}_0)) \le CH(u(0, \mathrm{T}_0))$$

since $H$ is increasing. Since $\psi$ is an arbitrary smooth function, the lemma follows. $\square$

Now we can prove the Harnack principle.

**Proof of Theorem 1.2.** Let us set $\mathrm{T}_0$ as the constant in Lemma 3.6. We note that

$$v(x, t) = \left[\frac{t_2 - t_1}{\mathrm{T}_0 R^2}\right]^{\frac{1}{m-1}} u(x_0 + Rx, t_1 + \frac{t_2 - t_1}{\mathrm{T}_0} t)$$

is a solution of

$$v_t = \Delta[v^m] - b\left[\frac{t_2 - t_1}{\mathrm{T}_0}\right]^{\frac{m-p}{m-1}} R^{\frac{2(p-1)}{m-1}} v^p$$

and $v(x,0)$ is an element of $\mathrm{L}^1(\mathbb{R}^n)\cap\mathrm{L}^\infty(\mathbb{R}^n)$ from the definition of weak solutions and Lemma 3.1. Notice that the coefficient of $v^p$ is less than or equal to 1 since we are assuming $R\geq\big[(t_2-t_1)/\mathrm{T}_0\big]^{(m-p)/[2(1-p)]}$. So, it follows from Lemma 3.7 that

$$\begin{aligned}
&R^{-n}\Big[\frac{t_2-t_1}{\mathrm{T}_0R^2}\Big]^{\frac{1}{m-1}}\|u\|_{1,B_R(x_0)}(t_1)=\|v\|_{1,B_1(0)}(0)\\
&\leq CH(v(0,\mathrm{T}_0))\leq C\big\{1+[v(0,\mathrm{T}_0)]^{\frac{\varkappa}{2}}\big\}\\
&=C\Big\{1+\Big[\frac{t_2-t_1}{\mathrm{T}_0R^2}\Big]^{\frac{(m-1)n+2}{2(m-1)}}\big[u(x_0,t_2)\big]^{\frac{\varkappa}{2}}\Big\},
\end{aligned}$$

where $C$ is a constant depending only on $m$ and $n$, and thus

$$⨍_{B_R(x_0)}u(x,t_1)\,dx\leq C\Big\{\Big[\frac{R^2}{t_2-t_1}\Big]^{\frac{1}{m-1}}+\Big[\frac{t_2-t_1}{R^2}\Big]^{\frac{n}{2}}[u(x_0,t_2)]^{\frac{\varkappa}{2}}\Big\}.$$

The proof is completed. □

**4. Existence of initial trace.** This section is devoted to the existence of initial traces for nonnegative weak solutions of (1.1). The Harnack principle is the main idea.

We need similar apriori estimates as Lemma 3.1 and Lemma 3.2 without the assumption $\sup_{t\in(0,\infty)}\|u\|_1(t)<\infty$. We can obtain the desired estimates by modifying the arguments in section 3.

Let us begin with introducing a quantity measuring the growth of solutions. Suppose $u$ is a nonnegative weak solution of (1.1). We define

$$\mathrm{M}_{R,h}(x_0)=\sup_{t\in(0,\frac12 h)}\big[R^{-\frac{\varkappa}{m-1}}\|u\|_{1,B_R(x_0)}(t)\big],\quad\text{for}\quad R>\Big[\frac{h}{\mathrm{T}_0}\Big]^{\frac{m-p}{2(1-p)}},$$

where $\mathrm{T}_0$ is the generic constant described in Theorem 1.2.

**Lemma 4.1.** *Suppose $u$ is a solution of (1.1). Then, for each $t\in(0,\frac14 h)$,*

$$\|u\|_{\infty,B_R(x_0)}(t)\ \leq\ C(\mathrm{M}_{8R,h}(x_0),R)\,t^{-\frac{1-\delta}{m-1}}\tag{4.1}$$

$$\|\nabla u^m\|_{1,B_R(x_0)\times(0,t)}\leq C(\mathrm{M}_{8R,h}(x_0),R)\,t^{\frac12\delta}\tag{4.2}$$

*for some positive number $\delta$ depending only on $m$ and $n$. Here, $C(\mathrm{M}_{8R,h}(x_0), R)$ is a positive constant depending only on $m$, $n$, $\mathrm{M}_{8R,h}(x_0)$ and $R$.*

**Proof.** We notice that if $t \in (0, \frac{1}{4}h)$, then $S_{2(4R),2(\frac{1}{4}\sqrt{t})}(x_0, t) \subset \mathbb{R}^n \times (0, \infty)$. Noting

$$\mathcal{G}_{2(4R),2(\frac{1}{4}\sqrt{t})} = \frac{(\frac{1}{2}\sqrt{t})^{\frac{2}{m-1}}}{(8R)^{\frac{\varkappa}{m-1}}} \sup_{|s-t| \le \frac{1}{4}t} \|u\|_{1,B_{8R}(x_0)}(s) \le C\mathrm{M}_{8R,h}(x_0) t^{\frac{1}{m-1}}$$

we obtain (4.1) by Corollary 2.4.

To prove (4.2) using Young's inequality we calculate as in Lemma 3.2

$$\text{LHS of (4.2)} \le \Big\{\frac{m}{2}\int_0^t \int_{B_R} s^{2\varepsilon_2} u^{-2\varepsilon_1} |\nabla u|^2 \, dx \, ds\Big\} + \\ + \Big\{\frac{m}{2}\int_0^t \int_{B_R} s^{-2\varepsilon_2} u^{2(m-1+\varepsilon_1)} \, dx \, ds\Big\} := \{\mathrm{I}\} + \{\mathrm{II}\}, \tag{4.3}$$

where $\varepsilon_1 = 5/4 - 3m/4$ and $\varepsilon_2 = 1/4$. First we estimate $\{\mathrm{II}\}$. Using the fact

$$\sup_{s\in(0,t)} \|u\|_{1,B_R(x_0)}(s) \le \sup_{s\in(0,t)} \|u\|_{1,B_{8R}(x_0)}(s) \le \mathrm{M}_{8R,h}(x_0) R^{\varkappa/(m-1)} \tag{4.4}$$

and (4.1) we have

$$\{\mathrm{II}\} \le C(\mathrm{M}_{8R,h}(x_0), R) \int_0^t s^{-2\varepsilon_2} \|u\|_\infty^{2(m-1+\varepsilon_1)-1}(s) \, ds \\ \le C(\mathrm{M}_{8R,h}(x_0), R) \int_0^t s^{-\frac{1-\delta}{m-1}(2m-3+2\varepsilon_1) - 2\varepsilon_2} ds \le C(\mathrm{M}_{8R,h}(x_0), R) t^{\frac{1}{2}\delta}. \tag{4.5}$$

To estimate $\{\mathrm{I}\}$ we let $\psi$ be a standard cut-off function in $\mathbb{R}^n$ such that $\psi = 1$ on $B_R(x_0)$ and take $\big[t^{2\varepsilon_2} u^{2-(m+2\varepsilon_1)} \psi^2\big]$ as a test function. Then by Young's inequality, (4.4) and (4.1) we get

$$\{\mathrm{I}\} \le C \int_0^t \int_{B_R} \big[s^{2\varepsilon_2 - 1} u^{3-(m+2\varepsilon_1)} \psi^2 + s^{2\varepsilon_2} u^{2-2\varepsilon_1} |\nabla \psi|^2\big] dx \, ds \\ \le C(\mathrm{M}_{8R,h}(x_0), R) \int_0^t \big[s^{2\varepsilon_2-1} \|u\|_\infty^{2-(m+2\varepsilon_1)}(s) + s^{2\varepsilon_2} \|u\|_\infty^{1-2\varepsilon_1}(s)\big] \, ds \\ \le C(\mathrm{M}_{8R,h}(x_0), R) \big[t^{2\varepsilon_2 - \frac{1-\delta}{m-1}[2-(m+2\varepsilon_1)]} + t^{1+2\varepsilon_2 - \frac{1-\delta}{m-1}(1-2\varepsilon_1)}\big] \\ \le C(\mathrm{M}_{8R,h}(x_0), R) \big[t^{\frac{1}{2}\delta} + t^{\frac{3}{2}\delta}\big]. \tag{4.6}$$

Combining (4.3), (4.5) and (4.6) we get (4.2). The proof is completed. □

**Proof of Theorem 1.3.** First, we claim that for each $B_{R_0}(x_0) \subset \mathbb{R}^n$, there exists a constant $C$ such that

$$|\int u(x,t)\psi(x)\,dx| \leq C, \quad \forall \psi \in C_0^\infty(B_{R_0}(x_0)) \text{ s.t. } \|\psi\|_\infty \leq 1, \quad \forall t \in (0, \tfrac{1}{4}).$$

Without loss of generality we may assume that $R_0 \geq \mathrm{T}_0^{(p-m)/[2(1-p)]}$, where $\mathrm{T}_0$ is the constant in Theorem 1.2.

Recall that $u$ is continuous in $\mathbb{R}^n \times (0,\infty)$. In particular, $u(x_0,1)$ is finite. Hence from the Harnack estimate we have that

$$R_0^{-\frac{\varkappa}{m-1}}\|u\|_{1,B_{R_0}(x_0)}(t) \leq C_{\mathrm{H}}\big\{\big[\frac{1}{1-t}\big]^{\frac{1}{m-1}} + \frac{(1-t)^{\frac{n}{2}}}{R_0^{\frac{\varkappa}{m-1}}}[u(x_0,1)]^{\frac{\varkappa}{2}}\big\}, \qquad (4.7)$$

for all $t \in (0,1)$. It follows from (4.7) that for each $t \in (0, \frac{1}{4})$ and for any $\psi \in C_0^\infty(B_{R_0}(x_0))$ such that $\|\psi\|_\infty \leq 1$,

$$|\int u(x,t)\psi(x)\,dx| \leq \|\psi\|_\infty \|u\|_{1,B_{R_0}(x_0)}(t) \leq C(u(x_0,1), R_0).$$

Then, by the weak compactness of Radon measures, we know that there exists a Radon measure $\mu$ and a sequence $\{t_j\} \downarrow 0$ such that

$$\lim_{j\to\infty} \int u(x,t_j)\zeta(x)\,dx = \int \zeta(x)\,d\mu, \quad \text{for all } \zeta \in C_0^\infty(\mathbb{R}^n),$$

that is, (1.4) holds along the sequence $\{t_j\}$. Now we have to show that if there exists another Radon measure $\nu$ and a sequence $\{\tilde{t}_j\} \downarrow 0$ which has the same property as $\mu$ and $\{t_j\}$, then $\nu = \mu$. It suffices to show that the equality holds for each ball. Let $x_0 \in \mathbb{R}^n$ and $R > 0$ be arbitrarily fixed. Set $h = \mathrm{T}_0 R^{2(1-p)/(m-p)}$. It follows from Harnack principle that for each $t \in (0, \frac{1}{2}h)$,

$$\frac{1}{(8R)^{n+\frac{2}{m-1}}} \int\limits_{B_{8R}(x_0)} u(x,t)\,dx \leq C_{\mathrm{H}}\big\{\big[\frac{1}{h-t}\big]^{\frac{1}{m-1}} + \frac{(h-t)^{\frac{n}{2}}}{(8R)^{n+\frac{2}{m-1}}}[u(x_0,h)]^{\frac{\varkappa}{2}}\big\}.$$

Thus we have

$$\mathrm{M}_{8R,h}(x_0) < C(u(x_0,h), R).$$

Then, by (4.2), for each $t \in (0, \frac{1}{4}h)$, we get

$$\int_0^t \int_{B_R(x_0)} u^{m-1}|\nabla u|\,dx\,ds \leq C(u(x_0,h),R)t^{\frac{1}{2}\delta}. \tag{4.8}$$

Now let $\zeta$ be a nonnegative cut-off function in $B_R(x_0)$ such that $0 \leq \zeta \leq 1$. Taking $[\chi_{[s,t]}\zeta]$ as a test function to (1.1) we get

$$\begin{aligned}\int u(x,t)\zeta(x)\,dx - \int u(x,s)\zeta(x)\,dx &\leq C\|\nabla\zeta\|_\infty \int_s^t \int_{B_R(x_0)} u^{m-1}|\nabla u|\,dx\,ds \\ &\leq C(u(x_0,h),R,\|\nabla\zeta\|_\infty)t^{\frac{1}{2}\delta}\end{aligned}$$

by using the estimate (4.8). Taking $s = t_j$ and sending $j$ to infinity we get

$$\int u(x,t)\zeta(x)\,dx \;\leq\; \int \zeta(x)\,d\mu + C(u(x_0,h),R,\|\nabla\zeta\|_\infty)t^{\frac{1}{2}\delta}.$$

Taking $t = \tilde{t}_j$ and sending $j$ to infinity, and then taking a supremum on $\zeta$ we get $\nu(B_R(x_0)) \leq \mu(B_R(x_0))$. The converse inequality also holds from the symmetry of argument, and hence $\nu(B_R(x_0)) = \mu(B_R(x_0))$. Because $R$ and $x_0$ are arbitrary we have $\nu = \mu$. Finally (1.5) comes directly from (4.7). The proof is completed. □

**Corollary 4.2.** *There is no very singular solution of* (1.1).

**Proof.** Let $u$ be a weak solution of (1.1). Then, by Theorem 1.3, there exists a unique Radon measure $\mu$ satisfying (1.4) holds, which implies that for any $R > 0$,

$$\lim_{t\to 0} \|u\|_{1,B_R(0)}(t) = \mu(B_R(0)) < \infty,$$

where we have used the fact that a Radon measure of any compact subset is finite. This completes the proof.

**5. Propagation of positivity set.** In this section we assume that $u$ is a nonnegative weak solution of the Cauchy problem (1.1) with compactly supported initial data $u_0$ and that $u$ is continuous on $\mathbb{R}^n \times [0,\infty)$.

In the following we prove Theorem 1.4 which implies that the propagation speed is finite, that is, if $\operatorname{supp} u_0 \subset\subset \mathbb{R}^n$, then, for each $t$, we also have $\operatorname{supp} u(\cdot,t) \subset\subset \mathbb{R}^n$. To show this we use only apriori estimates. Recall the notation

$$Q_R^h(x_0,t_0) = B_R(x_0) \times (t_0, t_0+h).$$

The following two lemmas can be proved by modifying the estimates in Section 2.

**Lemma 5.1.** *Suppose that $u(x,t_0) = 0$ for all $x \in B_R(x_0)$. Then there exists a constant $C_{51}$ depending only on $m$ and $n$ such that*

$$\|u\|_{\infty, Q^h_{\frac{1}{2}R}} \le C_{51}\Big[\frac{h}{R^2}\Big]^{1/2}\Big[\fint_{Q_R^h} u^{m+1}\,dz\Big]^{1/2}.$$

**Proof.** For calculational simplicity we use scaling argument. Define

$$v(x,t) = \big[R^{-2}h\big]^{1/(m-1)} u(x_0+Rx, t_0+ht).$$

Then $v$ is a solution of

$$v_t = \Delta[v^m] - b\big[h^{\frac{m-p}{m-1}} R^{\frac{2(p-1)}{m-1}}\big] v^p.$$

Let $1/2 < r < \rho \le 1$ and $\zeta \in C_0^\infty(B_\rho)$ a standard cut-off function such that $\zeta \equiv 1$ on $B_r$ and $|\nabla\zeta| \le (\rho-r)^{-1}c_0$, for some fixed constant $c_0$ independent of $r$ and $\rho$. Let $t \in (0,1)$ and $\alpha \ge 0$. We take $\big[\chi_{[0,t]} v^{\alpha+1}\zeta^2\big]$ as a test function to (1.1) to get

$$\begin{aligned}&\frac{1}{\alpha+2}\int v^{\alpha+2}(x,t)\zeta^2(x)\,dx + m(\alpha+1)\int v^{m-1+\alpha}|\nabla v|^2\zeta^2\,dz\\ &+ b\big[h^{\frac{m-p}{m-1}} R^{\frac{2(p-1)}{m-1}}\big]\int v^{p+\alpha+1}\zeta^2\,dz + 2m\int v^{m+\alpha}\big[\nabla v\cdot\nabla\zeta\big]\zeta\,dz = 0,\end{aligned}$$

since $v(x,0)=0$ in $B_1(0)$. It follows from Young's inequality that

$$\begin{aligned}&\frac{1}{\alpha+2}\sup_{t\in[0,1]}\|v^{\alpha+2}\zeta^2\|_{1,\mathbb{R}^n}(t) + \frac{1}{2}m(\alpha+1)\int v^{m+\alpha-1}|\nabla v|^2\zeta^2\,dz\\ &\quad\le \frac{4m}{\alpha+1}\int v^{m+\alpha+1}|\nabla\zeta|^2\,dz. \qquad (5.1)\end{aligned}$$

As in Lemma 2.1 writing $v^{m+\alpha+1+\frac{2}{n}(\alpha+2)}\zeta^{2(1+\frac{2}{n})} = \big[v^{\alpha+2}\zeta^2\big]^{\frac{2}{n}}\big[v^{m+\alpha+1}\zeta^2\big]$ and using Hölder inequality, Sobolev inequality and (5.1) we obtain

$$\Big[\int_{Q_r^1} v^{m+\alpha+1+\frac{2}{n}(\alpha+2)}\,dz\Big] \le (c_1\alpha + c_2)^{c_3}\Big[\frac{1}{(\rho-r)^2}\int_{Q_\rho^1} v^{m+\alpha+1}\,dz\Big]^{1+2/n} \tag{5.2}$$

for some constants $c_1$, $c_2$ and $c_3$ depending only on $m$, and $n$. Let us define two sequences $\{\alpha_{\rm J}\}_{\rm J=0}^\infty$ and $\{\rho_{\rm J}\}_{\rm J=0}^\infty$ by $\alpha_0 = 0$, $\alpha_{\rm J+1} = \alpha_{\rm J} + \frac{2}{n}(\alpha_{\rm J}+2)$; $\rho_{\rm J} = \frac{1}{2}(1+2^{-\rm J})$. Observing $m + \alpha_{\rm J} + 1 + \frac{2}{n}(\alpha_{\rm J}+2) = m + \alpha_{\rm J+1} + 1$ we deduce from (5.2) that

$$\Big[\fint_{Q^1_{\rho_{\rm J+1}}} v^{m+\alpha_{\rm J+1}+1}\,dz\Big] \le c^{\rm J+1}\Big[\fint_{Q^1_{\rho_{\rm J}}} v^{m+\alpha_{\rm J}+1}\,dz\Big]^{1+2/n}$$

and iterating this we obtain

$$\|v\|_{\infty,Q^1_{\frac{1}{2}}} \le C\Big[\fint_{Q_1^1} v^{m+1}\,dz\Big]^{1/2},$$

for some constant $C$ depending only on $m$ and $n$. Scaling back we complete the proof. □

Comparing with Lemma 2.1 there is no constant 1 on the RHS of the estimate in Lemma 5.1. This is possible from the further assumption that $u$ vanishes on the bottom of the cylinder.

**Lemma 5.2.** *Suppose $u(x,t_0) = 0$ in $B_R(x_0)$. Then, for any $\varepsilon > 0$, there exists a constant $C_{52}$ depending only on $m$, $n$, and $\varepsilon$ such that*

$$\fint_{Q^h_{\frac{1}{2}R}} u^{m+1}\,dz \le C_{52}(\varepsilon)\Big[\sup_{t_0\le t\le t_0+h}\fint_{B_R} u(x,t)\,dx\Big]^{m+1} + \varepsilon\Big[\frac{R^2}{h}\Big]^{\frac{m+1}{m-1}}$$

*whenever $R \ge h^{(m-p)/[2(1-p)]}$.*

**Proof.** Here, we use the scaling argument again. Let $v$ and $\zeta$ be as in the previous lemma. Let us set $v_\delta = v + \delta$, for $\delta > 0$. First, we claim that, for each $\alpha \in \mathbb{R}^- - \{-1,-2\}$,

$$\int_{Q_r^1} v^{m-1}v_\delta^\alpha|\nabla v|^2\,dz \le C(\alpha)\Big[\frac{1}{(\rho-r)^2}\int_{Q_\rho^1} v^{m-1}v_\delta^{\alpha+2}\,dz + \mathcal{G} + \delta^{\alpha+1} + \delta^{\alpha+2}\Big], \tag{5.3}$$

where $\mathcal{G} = \sup_{t\in[0,1]} \|v\|_{1,B_1}(t)$. Take $\big[\chi_{[0,1]} v_\delta^{\alpha+1}\zeta^2\big]$, $\alpha < 0$, as a test function to (1.1) to get

$$\begin{aligned} m(\alpha+1)\int v^{m-1}v_\delta^{\alpha}|\nabla v|^2\zeta^2\,dz + \frac{1}{\alpha+2}\|v_\delta^{\alpha+2}\zeta^2\|_1(1) \\ + b\big[h^{\frac{m-p}{m-1}}R^{\frac{2(p-1)}{m-1}}\big]\int v^p v_\delta^{\alpha+1}\zeta^2\,dz \\ = -\,2m\int v^{m-1}v_\delta^{\alpha+1}\big[\nabla v\cdot\nabla\zeta\big]\zeta\,dz + \frac{\delta^{\alpha+2}}{\alpha+2}\|\zeta^2\|_1, \end{aligned} \tag{5.4}$$

since $v(x,0) = 0$ in $B_1(0)$. Now we divide all the possible cases as $\alpha > -1$; $\alpha < -2$; $\alpha \in (-2,-1)$, and prove (5.3) for each case separately. For the first case it is easily verified since all the terms on LHS of (5.4) are positive. In fact, from Young's inequality we have

$$\int v^{m-1}v_\delta^{\alpha}|\nabla v|^2\zeta^2\,dz \le C(\alpha)\big[\int v^{m-1}v_\delta^{\alpha+2}|\nabla\zeta|^2\,dz + \delta^{\alpha+2}\big],$$

which gives the assertion. Next, consider the case $\alpha < -2$. By Young's inequality we have

$$\int v^{m-1}v_\delta^{\alpha}|\nabla v|^2\zeta^2 dz \le C(\alpha)\big[\int v^{m-1}v_\delta^{\alpha+2}|\nabla\zeta|^2\,dz + \delta^{\alpha+2} + \int v^p v_\delta^{\alpha+1}\zeta^2\,dz\big], \tag{5.5}$$

where we have used the observation that the coefficient of the last term on LHS of (5.4) is less than or equal to 1 under our hypothesis. Furthermore, since $0 \le \zeta \le 1$, $(\rho-r)^2 \le 1$, and $v^p v_\delta^{\alpha+1} \le [v^{m-1}v_\delta^{\alpha+2} + \delta^{\alpha+1}]$, (5.5) implies (5.3). Now let $\alpha \in (-2,-1)$. By Young's inequality we obtain

$$\begin{aligned} &\int v^{m-1}v_\delta^{\alpha}|\nabla v|^2\zeta^2 dz \\ &\le C(\alpha)\big[\int v^{m-1}v_\delta^{\alpha+2}|\nabla\zeta|^2 dz + \|v_\delta^{\alpha+2}\zeta^2\|_1(1) + \int v^p v_\delta^{\alpha+1}\zeta^2 dz\big]. \end{aligned} \tag{5.6}$$

Notice that we have used the fact that the first term on LHS and the last term on RHS are of different sign in this case. As in the second case we can replace the last term by the terms on the RHS of (5.3). On the other hand, since we are assuming $0 < \alpha + 2 < 1$, we have

$$\|v_\delta^{\alpha+2}\zeta^2\|_1(1) \le 2\|v^{\alpha+2}\zeta^2\|_1(1) + 2\delta^{\alpha+2} \le 2\mathcal{G} + 2\delta^{\alpha+2},$$

where we have used the Hölder inequality which is justified by the condition $\alpha + 2 < 1$. Therefore (5.6) implies (5.3) and this completes the proof of our claim.

Writing $v^{m+1+\frac{2}{n}} v_\delta^\alpha \zeta^{2(1+\frac{2}{n})} = \big[v\zeta^2\big]^{\frac{2}{n}} \big[v^{m+1} v_\delta^\alpha \zeta^2\big]$ it follows from Hölder inequality, Sobolev inequality, the estimate (5.3) that

$$\int_{Q_r^1} v^{m+1+\frac{2}{n}} v_\delta^\alpha \, dz \le C\mathcal{G}^{\frac{2}{n}} \Big[\frac{1}{(\rho - r)^2} \int_{Q_\rho^1} v^{m-1} v_\delta^{\alpha+2} \, dz + \mathcal{G} + \delta^{\alpha+1} + \delta^{\alpha+2}\Big],$$

where we also used the fact $v^{m-1} v_\delta^\alpha |\nabla v|^2 \ge v^{m+1} v_\delta^{\alpha-2} |\nabla v|^2$. Thus we get

$$\begin{aligned} &\int_{Q_r^1} v_\delta^{m+1+\frac{2}{n}+\alpha} \, dz \\ &\le C\mathcal{G}^{\frac{2}{n}} \Big[\frac{1}{(\rho - r)^2} \int_{Q_\rho^1} v_\delta^{m+1+\alpha} \, dz + \mathcal{G} + \delta^{\alpha+1} + \delta^{\alpha+2}\Big] + \delta^{m+1+\frac{2}{n}+\alpha}, \end{aligned} \tag{5.7}$$

for some constant $C$ depending only on $m$, $n$, and $\alpha$, where we have used the fact $\alpha < 0$. By iterations of (5.7) starting from $\alpha_0 = -m$ and applying Young's inequality several times we conclude

$$\fint_{Q_{\frac{1}{2}}^1} v^{m+1} \, dz \le C(\varepsilon) \Big[\sup_{t \in [0,1]} \fint_{B_1} v(x,t) \, dx\Big]^{m+1} + \varepsilon.$$

with an appropriate choice of $\delta$ for each given $\varepsilon > 0$. Scaling back we get the lemma. □

From Lemma 5.1 we deduce the following recurrence relation

$$\big[\|u\|_{\infty, Q_{\frac{1}{2}R}^h}\big] \le C_{51} \big[R^{-2} h\big]^{1/2} \big[\|u\|_{\infty, Q_R^h}\big]^{(m+1)/2}. \tag{5.8}$$

By applying Lemma 5.6 of [12] to (5.8) and using Lemma 5.1 again we obtain

**Lemma 5.3.** *Suppose that $u(x, t_0) = 0$ for all $x \in B_R(x_0)$. Then there exists a positive constant $C_{53}$ depending only on $m$ and $n$ such that if*

$$\fint_{Q_R^h} u^{m+1} \, dz \le C_{53} \Big[\frac{R^2}{h}\Big]^{(m+1)/(m-1)},$$

*then* $u \equiv 0$ *in* $Q^h_{\frac{1}{4}R}$.

Now we can prove Theorem 1.4.

**Proof of Theorem 1.4.** Let $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$. Notice that $u_0(x) = 0$ in $B_{|x_0|-R}(x_0)$. From Lemma 5.3 we know that if

$$\fint_{Q^{t_0}_{\frac{1}{2}(|x_0|-R)}} u^{m+1}\,dz \le C_{53}\Big[\frac{(|x_0|-R)^2}{4t_0}\Big]^{\frac{m+1}{m-1}},$$

then $u \equiv 0$ in $Q^{t_0}_{\frac{1}{8}(|x_0|-R)}(x_0, 0)$. Hence if $u(x_0, t_0) \neq 0$, then

$$\fint_{Q^{t_0}_{\frac{1}{2}(|x_0|-R)}} u^{m+1}\,dz > C_{53}\Big[\frac{(|x_0|-R)^2}{4t_0}\Big]^{\frac{m+1}{m-1}}.$$

On the other hand we know from Lemma 5.2 that for any $\varepsilon > 0$,

$$\fint_{Q^{t_0}_{\frac{1}{2}(|x_0|-R)}} u^{m+1}\,dz \le C_{52}(\varepsilon)\Big[\sup_{t\in[0,t_0]}\fint_{B_{|x_0|-R}} u(x,t)\,dx\Big]^{m+1} + \varepsilon\Big[\frac{(|x_0|-R)^2}{t_0}\Big]^{\frac{m+1}{m-1}}$$

whenever $|x_0| - R \ge t_0^{(m-p)/[2(1-p)]}$. Hence if we choose $\varepsilon = \frac{1}{2}4^{(m+1)/(1-m)}C_{53}$, then

$$\frac{1}{2}C_{53}\Big[\frac{(|x_0|-R)^2}{4t_0}\Big]^{\frac{m+1}{m-1}} \le C\Big[\frac{1}{(|x_0|-R)^n}\sup_{t\in[0,t_0]}\int_{B_{|x_0|-R}} u(x,t)\,dx\Big]^{m+1}.$$

Since $\|u\|_1(t) \le \|u_0\|_1$, for all $t$, we conclude that

$$t_0 > \frac{C}{\|u_0\|_1^{m-1}}(|x_0|-R)^{\varkappa}.$$

The proof is completed. $\square$

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## REFERENCES

[1] G.I. Barenblatt, *On some unsteady motions of a fluid and a gas in a porous medium*, Prikl. Mat. Mekh. **16** (1952), 67–78.

[2] M. Bertsch, R.Kersner and L.A. Peletier, *Positivity versus localization in degenerate diffusion equations*, Nonlinear Anal. TMA **9** (1985), 987–1008.
[3] H. Brezis and A. Friedman, *Nonlinear parabolic equations involving measures as initial conditions*, J. Math. Pures Appl. **62** (1983), 73–97.
[4] H. Brezis, L.A. Peletier and D. Terman, *A very singular solution of the heat equation with absorption*, Arch. Rational Mech. Anal. **75** (1986), 185–209.
[5] B.E.J. Dahlberg and C.E. Kenig, *Non-negative solutions of generalized porous medium equations*, Revista Math. Iber. **2** (1986), 267–305.
[6] E. DiBenedetto and A. Friedman, *Hölder estimates for nonlinear degenerate parabolic systems*, J. Reine Angew. Math. **357** (1985), 1-22.
[7] V.A. Galaktionov, S.P. Kurdyumov, and A.A. Samarskii, *On asymptotic "eigenfunctions" of the Cauchy problem for a nonlinear parabolic equation*, Math. USSR Sbornik **54** (1986), 421–455.
[8] V.A. Galaktionov and J.L. Vazquez, *Extinction for a quasilinear heat equation with absorption I. Technique of intersection comparison*, Comm. Partial Diff. Eq. **19** (1994), 1075–1106.
[9] A.S. Kalashnikov, *Some problems of the qualitative theory of non-linear degenerate second-order parabolic equations*, Russian Math. Surveys **42** (1987), 169–222.
[10] S. Kamin and L.A. Peletier, *Source type solutions of degenerate diffusion equations with absorption*, Israel J. Math. **50** (1985), 219–230.
[11] S. Kamin, L.A. Peletier and J.L. Vazquez, *Classification of singular solutions of a nonlinear heat equation*, Duke Math. J. **58** (1989), 601–615.
[12] O.A.Ladyzenskaja, V.A. Solonnikov & N.N. Ural'ceva, Linear and quasi-linear equations of parabolic type, Transl. Math. Monographs 23, Amer. Math. Soc., Providence, RI, 1968.
[13] L.A. Peletier and J. Zhao, *Large time behaviour of solutions of the porous medium equation with absorption : the fast diffusion case*, Nonlinear Anal. TMA **17** (1991), 991–1009.
[14] L.A. Peletier and D. Terman, *A very singular solution of the porous media equation with absorption*, J. Diff. Eq. **65** (1986), 396–410.
[15] M. Pierre, *Uniqueness of the solutions of* $u_t - \Delta\phi(u) = 0$ *with initial datum a measure*, Nonlinear Anal. TMA **6** (1982), 175–187.
[16] R.E. Prattle, *Diffusion from an instantaneous point source with concentration-dependent coefficient*, Quart. J. Mech. Appl. Math. **12** (1959), 407–409.
[17] A.A. Samarskii, V.A. Galaktionov, S.P. Kurdyumov, and A.P. Mikhailov, Blow-up in Quasilinear Parabolic Equations, Walter de Gruyter, Berlin/New York, 1995.
[18] J.L. Vazquez, *Symétrization pour* $u_t = \Delta\phi(u)$ *et applications*, C.R. Acad. Paris **295** (1982), 71–74.
[19] H. Yuan, *Generalized Harnack inequality for bounded weak solutions of the porous medium equation with absorption*, Preprint.
[20] J. Zhao, *Source-type solutions of degenerate quasilinear parabolic equations*, J. Diff. Eq. **92** (1991), 179–198.