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# A NOTE ON RADIAL SYMMETRY OF POSITIVE SOLUTIONS FOR SEMILINEAR ELLIPTIC EQUATIONS IN $\mathbb{R}^n$

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**Abstract.** Symmetry properties of positive solutions of the equations

$$\Delta u + \phi(|x|)f(u) = 0$$

in $\mathbb{R}^n$ are considered. We employ the moving plane method based on the maximum principle on unbounded domains to obtain new results on symmetry.

**1. Introduction and statement of the results.** In this note we consider the symmetry properties of positive solutions for the equation of the form

$$\Delta u + \phi(|x|)f(u) = 0 \tag{1.1}$$

in $\mathbb{R}^n$, where $n \geq 3$, $\Delta$ is the $n$-dimensional Laplacian, and $|x|$ denotes the Euclidean length of $x \in \mathbb{R}^n$. In equation (1.1), we assume that $\phi \not\equiv 0$ is a locally Hölder continuous function on $[0, \infty)$ which satisfies

$$\phi(r) \geq 0 \text{ for } r \geq 0 \quad \text{and} \quad \phi(r) \text{ is nonincreasing in } r > 0,$$

and that $f \in C^1([0,\infty))$ with $f(u) > 0$ for $u > 0$.

The problem of existence of positive solutions of equation (1.1) has been studied extensively. It has been shown in [4, 5, 12] that if

$$\int_0^\infty r\phi(r)dr < \infty, \tag{1.2}$$

then, under some additional conditions on $f$, (1.1) has infinitely many bounded positive solutions in $\mathbb{R}^n$. Our main result is the following, which is a slight extension of [10, Theorem 5.16].

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**Theorem.** *Assume that* (1.2) *holds. Then all bounded positive solutions of* (1.1) *in* $\mathbb{R}^n$ *are radially symmetric about the origin.*

We give some corollaries of the theorem. First assume that (1.1) has a bounded positive solution $u$ in $\mathbb{R}^n$ satisfying

$$\liminf_{|x|\to\infty} u(x) > 0. \tag{1.3}$$

Then, by Lemma B.1 in Appendix B, we get (1.2). Thus we obtain the following:

**Corollary 1.** *Assume that* (1.1) *has a bounded positive solution* $u$ *in* $\mathbb{R}^n$ *satisfying* (1.3). *Then all bounded positive solutions are radially symmetric about the origin.*

Next, we consider the case where $f(0) > 0$. Assume that (1.1) has a bounded positive solution $u$ in $\mathbb{R}^n$. Then, by Lemma B.2 in Appendix B, we get (1.2). Thus we obtain the following:

**Corollary 2.** *Assume that* $f(0) > 0$. *Then all bounded positive solutions of* (1.1) *in* $\mathbb{R}^n$ *are radially symmetric about the origin.*

**Remark.** For the case $f(u) = e^{2u}$, precise existence and nonexistence criteria for positive solutions of (1.1) are obtained in [8, Theorems 1.4 and 1.5].

Symmetry properties of solutions of semilinear elliptic equations in $\mathbb{R}^n$ have been studied by several authors [1-3, 6-11, 16-18]. Their arguments are based on the moving plane method first developed by Serrin [16] in PDE theory, and later extended and generalized by Gidas, Ni, and Nirenberg [2, 3]. In this note, we present an approach based on the maximum principle on unbounded domains together with the method of moving plane. This approach helps us to improve the previous results and simplify the proofs.

In Section 2, we investigate the asymptotic behavior of positive solutions of (1.1). In Section 3, we prove the main Theorem by using the method of moving planes. We give the maximum principle on unbounded domains in Appendix A, and show the conditions which are equivalent to (1.2) in Appendix B.

**2. Asymptotic behavior of positive solutions.** We show the following proposition.

**Proposition.** *Assume that* (1.2) *holds. Let* $u$ *be a bounded positive solution of* (1.1) *in* $\mathbb{R}^n$. *Then* $\lim_{|x|\to\infty} u(x) = c$ *and* $u(x) > c$ *in* $\mathbb{R}^n$ *for some constant* $c \geq 0$.

In order to prove this, we first prove the following lemma.

**Lemma 1.** *Let* $g$ *be a continuous function in* $\mathbb{R}^n$, *and let* $w$ *be the Newtonian potential of* $g$, *i.e.*,

$$w(x) = c_n \int_{\mathbb{R}^n} \frac{g(y)}{|x-y|^{n-2}}\, dy,$$

*where* $c_n = [n(n-2)\omega_n]^{-1}$ *and* $\omega_n$ *is the volume of the unit ball in* $\mathbb{R}^n$. *Assume that there is a nonnegative nonincreasing function* $G$ *on* $[0,\infty)$ *satisfying*

$$g(x) \leq G(|x|), \quad x \in R^n; \quad \int_0^\infty rG(r)dr < \infty. \tag{2.1}$$

*Then* $w$ *is well defined and satisfies*

$$\lim_{|x|\to\infty} w(x) = 0. \tag{2.2}$$

**Proof.** By $(2.1)_2$ for any $\varepsilon > 0$ there exists $R > 0$ satisfying

$$c_n \int_R^\infty rG(r)\, dr < \tfrac{1}{3}\varepsilon \quad \text{and} \quad 3^{n-2} c_n \int_{3R}^\infty rG(r)\, dr < \tfrac{1}{3}\varepsilon. \tag{2.3}$$

From $(2.1)_1$, we have

$$|w(x)| \leq c_n \int_{\mathbb{R}^n} \frac{G(|y|)}{|x-y|^{n-2}}\, dy.$$

We decompose the integral as follows:

$$|w(x)| \leq c_n \left( \int_{\Omega_1} + \int_{\Omega_2} + \int_{\Omega_3} \right) \frac{G(|y|)}{|x-y|^{n-2}}\, dy \equiv I_1 + I_2 + I_3,$$

where $\Omega_1$, $\Omega_2$, and $\Omega_3$ are defined as

$$\begin{aligned}
\Omega_1 &= \{y \in \mathbb{R}^n : |y| \leq 3R\},\\
\Omega_2 &= \{y \in \mathbb{R}^n : |y| \geq 3R,\ |x-y| \geq \tfrac{1}{3}|y|\},\\
\Omega_3 &= \{y \in \mathbb{R}^n : |y| \geq 3R,\ |x-y| \leq \tfrac{1}{3}|y|\}.
\end{aligned}$$

We estimate $I_1$, $I_2$, and $I_3$ as follows. Since $\lim_{|x|\to\infty} I_1 = 0$, there exists $R_1 > 3R$ so that

$$I_1 < \tfrac{1}{3}\varepsilon \quad \text{for} \quad |x| > R_1. \tag{2.4}$$

From $(2.3)_2$ we obtain

$$I_2 \le 3^{n-2} c_n \int_{\Omega_2} \frac{G(|y|)}{|y|^{n-2}}\, dy \le 3^{n-2} c_n \int_{3R}^{\infty} rG(r)\, dr < \tfrac{1}{3}\varepsilon. \tag{2.5}$$

For $y \in \Omega_3$, since $|y| - |x| \le |y - x| \le \frac{1}{3}|y|$, we see that

$$\tfrac{2}{3}|y| \le |x|. \tag{2.6}$$

Then, for $y \in \Omega_3$ and $r \in [0, \frac{1}{3}|y|]$, we have

$$|x| - r \ge \tfrac{2}{3}|y| - \tfrac{1}{3}|y| = \tfrac{1}{3}|y| \ge r \quad \text{and} \quad |x| - \tfrac{1}{3}|y| \ge \tfrac{1}{3}|y| \ge R. \tag{2.7}$$

Since $G$ is nonincreasing and $|y| \ge |x| - |x - y|$, it follows that

$$I_3 \le c_n \int_{\Omega_3} \frac{G(|x| - |x - y|)}{|x - y|^{n-2}}\, dy = c_n \int_0^{\frac{1}{3}|y|} rG(|x| - r)\, dr.$$

From (2.7) and $(2.3)_1$ we obtain

$$\begin{aligned} I_3 &\le c_n \int_0^{\frac{1}{3}|y|} (|x| - r) G(|x| - r)\, dr = c_n \int_{|x| - \frac{1}{3}|y|}^{|x|} sG(s)\, ds \\ &\le c_n \int_R^{\infty} sG(s)\, ds < \tfrac{1}{3}\varepsilon. \end{aligned} \tag{2.8}$$

Then by (2.4), (2.5), and (2.8), we have $|w(x)| < \varepsilon$ for $|x| > R_1$. Since $\varepsilon > 0$ is arbitrary, we conclude that (2.2) holds.

**Proof of Proposition.** Let $v$ be the Newtonian potential of $\phi f(u)$, i.e.,

$$v(x) = c_n \int_{\mathbb{R}^n} \frac{\phi(|y|) f(u(y))}{|x - y|^{n-2}}\, dy.$$

Define $f_\infty = \max\{f(s) : 0 \le s \le \|u\|_{L^\infty(\mathbb{R}^n)}\}$. Then $\phi(|x|) f(u(x)) \le \phi(|x|) f_\infty$ in $\mathbb{R}^n$. Since $\phi$ is nonincreasing and (1.2) holds, we obtain

$$\lim_{|x|\to\infty} v(x) = 0 \tag{2.9}$$

by Lemma 1. It is easily seen that $v$ satisfies $\Delta v + \phi f(u) = 0$ in $\mathbb{R}^n$. We have $\Delta(u - v) = 0$ in $\mathbb{R}^n$ while $u - v$ is bounded in $\mathbb{R}^n$ by (2.9). Then by Liouville's theorem, we obtain

$$u(x) - v(x) \equiv c \quad \text{in} \;\; \mathbb{R}^n, \tag{2.10}$$

where $c$ is a constant. From (2.9) we conclude that $u(x) \to c$ as $|x| \to \infty$. Observe that $v$ satisfies $\Delta v = -\phi f(u) \leq 0$ and $v \geq 0$ in $\mathbb{R}^n$. By the maximum principle, we have $v > 0$ in $\mathbb{R}^n$. From (2.10) we conclude that $u(x) > c$ in $\mathbb{R}^n$.

**3. Proof of the theorem.** First, we introduce some notation. For $\lambda \in \mathbb{R}$, we define $T_\lambda$ and $\Sigma_\lambda$ as

$$T_\lambda = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_1 = \lambda\}$$

and

$$\Sigma_\lambda = \{x \in \mathbb{R}^n : x_1 < \lambda\}.$$

For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, let $x^\lambda$ be the reflection of $x$ with respect to the hyperplane $T_\lambda$, i.e., $x^\lambda = (2\lambda - x_1, x_2, \dots, x_n)$. It is easy to see that, if $\lambda > 0$,

$$|x^\lambda| - |x| > 0 \quad \text{for } x \in \Sigma_\lambda. \tag{3.1}$$

Let $u$ be a bounded positive solution of (1.1) in $\mathbb{R}^n$. By the proposition in Section 2, we have

$$\lim_{|x| \to \infty} u(x) = c \geq 0 \quad \text{and} \quad u(x) > c \quad \text{in } \mathbb{R}^n \tag{3.2}$$

for some constant $c$. We define

$$v_\lambda(x) = u(x) - u(x^\lambda) \quad \text{for } x \in \Sigma_\lambda.$$

**Lemma 2.** *Let $\lambda > 0$. Then $v_\lambda$ satisfies*

$$\Delta v_\lambda + c_\lambda(x) v_\lambda \leq 0 \quad \text{in } \Sigma_\lambda, \tag{3.3}$$

*where*

$$c_\lambda(x) = \phi(|x|) \int_0^1 f'\left(u(x^\lambda) + t(u(x) - u(x^\lambda))\right) dt. \tag{3.4}$$

We note that $c_\lambda(x)$ is well defined in $\mathbb{R}^n$.

**Proof.** Since $\phi$ in nonincreasing and (3.1) holds, it follows that

$$\begin{aligned} 0 &= \Delta u(x) + \phi(|x|) f(u(x)) - \Delta u(x^\lambda) - \phi(|x^\lambda|) f(u(x^\lambda)) \\ &\geq \Delta\left(u(x) - u(x^\lambda)\right) + \phi(|x|)\left(f(u(x)) - f(u(x^\lambda))\right) \\ &= \Delta v(x) + c_\lambda(x) v(x), \quad x \in \Sigma_\lambda, \end{aligned}$$

where $c_\lambda(x)$ is the function in (3.4).

**Lemma 3.** *Assume that* (1.2) *holds. Then there exists a positive function* $w(x)$ *on* $\{x \in \mathbb{R}^n : |x| \geq r_0\}$ *satisfying, for some* $r_0 > 0$ *and for any* $\lambda > 0$,

$$\Delta w + c_\lambda(x) w \leq 0 \quad \text{in } |x| > r_0 \quad \text{and} \quad \liminf_{|x|\to\infty} w(x) > 0. \tag{3.5}$$

**Proof.** Define

$$g_\infty = \max\{|f'(s)| : 0 \leq s \leq \|u\|_{L^\infty(\mathbb{R}^n)}\}.$$

Then from (3.4) we have

$$|c_\lambda(x)| \leq g_\infty \phi(|x|) \quad \text{in } \mathbb{R}^n \quad \text{for any } \lambda > 0. \tag{3.6}$$

Now consider the equation

$$\Delta w + g_\infty \phi(|x|) w = 0. \tag{3.7}$$

By applying Lemma B.1 in Appendix B to (3.7), we find that (3.7) has a positive solution $w$ on $\{|x| \geq r_0\}$ for some $r_0 > 0$, satisfying

$$\liminf_{|x|\to\infty} w(x) > 0.$$

By (3.6), $w$ satisfies (3.5).

Define $B_0 = \{x \in \mathbb{R}^n : |x| < r_0\}$, where $r_0$ is the constant appearing in Lemma 3.

**Lemma 4.** *Let* $\lambda > 0$*. Assume that* $v_\lambda(x) > 0$ *on* $\partial B_0 \cap \Sigma_\lambda$*. Then* $v_\lambda(x) > 0$ *in* $\Sigma_\lambda \setminus \overline{B_0}$*.*

**Proof.** By Lemma 2 we obtain

$$\Delta v_\lambda + c_\lambda(x) v_\lambda \leq 0 \quad \text{in } \Sigma_\lambda \setminus \overline{B_0}, \qquad v_\lambda > 0 \quad \text{on } \partial B_0 \cap \Sigma_\lambda.$$

By Lemma 3, there is a positive function $w$ satisfying

$$\Delta w + c_\lambda(x) w \leq 0 \quad \text{in } \Sigma_\lambda \setminus \overline{B_0}.$$

From (3.2) and (3.5) we see that

$$\frac{v_\lambda(x)}{w(x)} \leq \frac{u(x) - c}{w(x)} \to 0 \quad \text{as } |x| \to \infty.$$

By applying Lemma A in Appendix A with $\Omega = \Sigma_\lambda \setminus \overline{B_0}$, we get $v_\lambda > 0$ in $\Sigma_\lambda \setminus \overline{B_0}$.

Define $\Lambda = \{\lambda \in (0, \infty) : v_\lambda(x) > 0 \text{ in } \Sigma_\lambda\}$.

**Lemma 5.** *If $\lambda \notin \Lambda$, then there exists $x_0 \in \Sigma_\lambda \cap \overline{B_0}$ such that $v_\lambda(x_0) \le 0$.*

**Proof.** Assume to the contrary that $v_\lambda(x) > 0$ on $\Sigma_\lambda \cap \overline{B_0}$. Then by Lemma 4 we have $v_\lambda(x) > 0$ in $\Sigma_\lambda \setminus \overline{B_0}$. Therefore, $v_\lambda(x) > 0$ in $\Sigma_\lambda$, which contradicts the assumption $\lambda \notin \Lambda$.

**Lemma 6.** *Let $\lambda \in \Lambda$. Then $\partial u/\partial x_1 < 0$ on $T_\lambda$.*

**Proof.** By Lemma 1, we have (3.3) and $v_\lambda > 0$ in $\Sigma_\lambda$. Since $v_\lambda = 0$ on $T_\lambda$, we obtain $\partial v_\lambda/\partial x_1 < 0$ on $T_\lambda$ by the Hopf boundary Lemma ([2, Lemma H]). Therefore,

$$\frac{\partial u}{\partial x_1} = \frac{1}{2}\frac{\partial v_\lambda}{\partial x_1} < 0 \quad \text{on } T_\lambda.$$

**Proof of the theorem.** Since (3.2) holds, there exists $r_1 > r_0$ such that

$$\max\{u(x) : |x| \ge r_1\} < \min\{u(x) : |x| \le r_0\}, \tag{3.8}$$

where $r_0$ is the constant appearing in Lemma 3. We now divide the proof into several steps.

**Step 1.** $[r_1, \infty) \subset \Lambda$.

Let $\lambda \ge r_1$. We note that $\overline{B_0} \subset \Sigma_\lambda$. From (3.8), we have $v > 0$ in $\overline{B_0}$. Then by Lemma 4 we have $v_\lambda > 0$ in $\Sigma_\lambda \setminus \overline{B_0}$. Therefore, $v > 0$ in $\Sigma_\lambda$, i.e., $\lambda \in \Lambda$. This implies that $[r_1, \infty) \subset \Lambda$.

**Step 2.** Let $\lambda_0 \in \Lambda$. Then there exists $\varepsilon > 0$ such that $(\lambda_0 - \varepsilon, \lambda_0] \subset \Lambda$.

Assume to the contrary that there exists an increasing sequence $\{\lambda_i\}$, $i = 1, 2, \ldots$, such that $\lambda_i \notin \Lambda$ and $\lambda_i \to \lambda_0$ as $i \to \infty$. By Lemma 5 there exists a sequence $\{x_i\}$, $i = 1, 2, \ldots$, such that $x_i \in \Sigma_{\lambda_i} \cap \overline{B_0}$ and $v_{\lambda_i}(x_i) \le 0$. Then there is a subsequence, which we again call $\{x_i\}$ which converges to some point $x_0 \in \overline{\Sigma_{\lambda_0}} \cap \overline{B_0}$. We have $v_{\lambda_0}(x_0) \le 0$. Since $v_{\lambda_0} > 0$ in $\Sigma_{\lambda_0}$, we must have $x_0 \in T_{\lambda_0}$.

By the mean value theorem, there exists a point $y_i$ satisfying $(\partial u/\partial x_1)(y_i) \ge 0$ on the straight segment joining $x_i$ to $x_i^{\lambda_i}$, for each $i = 1, 2, \ldots$. Since $y_i \to x_0$ as $i \to \infty$, we have $(\partial u/\partial x_1)(x_0) \ge 0$. On the other hand, since $x_0 \in T_{\lambda_0}$ we have $(\partial u/\partial x_1)u(x_0) < 0$ by Lemma 6. This is a contradiction, and Step 2 is established.

**Step 3.**

$$u(x) \ge u(x^0) \quad \text{in } \Sigma_0. \tag{3.9}$$

Let $\lambda_1 = \inf\{\lambda > 0 : (\lambda, \infty) \subset \Lambda\}$. We show that $\lambda_1 = 0$. Assume to the contrary that $\lambda_1 > 0$. From the continuity of $u$, we have $v_{\lambda_1}(x) =$

$u(x)-u(x^{\lambda_1}) \geq 0$ in $\Sigma_{\lambda_1}$. By Lemma 2, we obtain (3.3) with $\lambda = \lambda_1$. Hence, by the maximum principle ([2]), we have either

$$v_{\lambda_1} \equiv 0 \;\text{ in } \Sigma_{\lambda_1}, \text{ i.e., } \; u(x) \equiv u(x^{\lambda_1}) \;\text{ in } \Sigma_{\lambda_1}, \quad \text{or} \tag{3.10}$$

$$v_{\lambda_1} > 0 \;\text{ in } \Sigma_{\lambda_1}, \text{ i.e., } \; u(x) > u(x^{\lambda_1}) \;\text{ in } \Sigma_{\lambda_1}. \tag{3.11}$$

If (3.10) occurs, by (1.1) we have $\phi(|x|)f(u(x)) \equiv \phi(|x^{\lambda_1}|)f(u(x))$ for $x \in \Sigma_{\lambda_1}$. Because $f(u(x)) > 0$, we have $\phi(|x|) \equiv \phi(|x^{\lambda_1}|)$ in $\Sigma_{\lambda_1}$. Since $\phi$ is nonincreasing, we see that $\phi(r) \equiv \phi(0)$ for $r \geq 0$. By (1.2), $\phi(r) \equiv 0$ for $r \geq 0$. This contradicts the assumption $\phi \not\equiv 0$. Therefore (3.10) cannot happen.

On the other hand, if (3.11) occurs. Then, $\lambda_1 \in \Lambda$. From Step 2, there exists $\varepsilon > 0$ such that $(\lambda_1 - \varepsilon, \lambda_1] \subset \Lambda$. This contradicts the definition of $\lambda_1$.

Therefore, we conclude that $\lambda_1 = 0$. Thus, $u(x) > u(x^{\lambda})$ in $\Sigma_\lambda$ for $\lambda > 0$. By the continuity of $u$, we obtain (3.9).

We can repeat the previous Steps 1-3 for the negative $x_1$-direction to conclude that $u(x) \leq u(x^0)$ for $x \in \Sigma_0$. Hence, from (3.9), $u$ must be symmetric about the plane $x_1 = 0$. Since the equation in (1.1) is invariant under rotation, we may take any direction as the $x_1$-direction and conclude that $u$ is symmetric in every direction. Therefore, $u$ must be radially symmetric about the origin.

**Appendix A.** Let $\Omega$ be an unbounded domain in $\mathbb{R}^n$, and let $Lu \equiv \Delta u + c(x)u$, where $c \in \mathrm{L}^\infty(\Omega)$.

**Lemma A.** *Suppose that $u$ satisfies $Lu \leq 0$ in $\Omega$ and $u \geq 0$ on $\partial\Omega$. Suppose, furthermore, that there exists a function $w$ such that $w > 0$ on $\Omega \cup \partial\Omega$ and $Lw \leq 0$ in $\Omega$. If*

$$\frac{u(x)}{w(x)} \to 0 \quad \textit{as } |x| \to \infty, \; x \in \Omega, \tag{A.1}$$

*then $u > 0$ in $\Omega$.*

**Remark.** If $\Omega$ is bounded, we do not require the condition (A.1). See [15, Chap. 2, Theorem 10].

**Proof.** First we show that $u \geq 0$ in $\Omega$. Assume to the contrary that $u(x_0) < 0$ for some $x_0 \in \Omega$. Choose $\delta > 0$ so that

$$u(x_0) + \delta w(x_0) = 0. \tag{A.2}$$

From (A.1), there exists $R > |x_0|$ satisfying $u(x) + \delta w(x) \geq 0$ on $\{|x| = R\} \cap \Omega$. Define $B_R = \{x \in \mathbb{R}^n : |x| < R\}$. Then $u+\delta w$ satisfies $L(u+\delta w) \leq 0$ on $\Omega \cap B_R$ and $u + \delta w \geq 0$ on $\partial(\Omega \cap B_R)$. By [15, Chap. 2, Theorem 10], $(u + \delta w)/w$ cannot attain a nonpositive minimum at an interior point of $\Omega \cap B_R$ unless it is a constant. This contradicts (A.2). Therefore, $u \geq 0$ in $\Omega$. By the maximum principle ([2]), we conclude that $u > 0$ in $\Omega$.

**Appendix B.** Conditions which are equivalent to (1.2).

**Lemma B.1.** *Equation* (1.1) *has a bounded positive solution* $u$ *on* $\{x \in \mathbb{R}^n : |x| \geq r_0\}$ *for some* $r_0 > 0$ *satisfying*

$$\liminf_{|x|\to\infty} u(x) > 0 \tag{B.1}$$

*if and only if* (1.2) *holds.*

**Proof.** Assume that $u$ is a bounded solution of (1.1) on $\{|x| \geq r_0\}$ satisfying (B.1). Let $\overline{u}$ be the spherical mean of $u$, i.e.,

$$\overline{u}(r) = \frac{1}{n\omega_n r^{n-1}} \int_{|x|=r} u(x)dS \quad \text{for} \quad r \geq r_0,$$

where $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$. Then, $\overline{u}$ satisfies

$$(r^{n-1}\overline{u}')' + r^{n-1}\phi(r)h(r) = 0, \quad r > r_0, \tag{B.2}$$

where

$$h(r) = \frac{1}{n\omega_n r^{n-1}} \int_{|x|=r} f(u(x))dS \quad \text{for} \quad r \geq r_0.$$

(See, e.g., [13, 14].) Since $u$ is bounded, by integrating (B.2) we obtain

$$\int_{r_0}^{\infty} r^{1-n} \int_{r_0}^{r} s^{n-1}\phi(s)h(s)dsdr = \frac{1}{n-2} \int_{r_0}^{\infty} s\phi(s)h(s)ds < \infty. \tag{B.3}$$

From (B.1), there exists a constant $u_0 > 0$ satisfying $u(x) \geq u_0$ for $|x| \geq r_0$. Define $u_\infty$ and $f_0$ as $u_\infty = \max\{u(x) : |x| \geq r_0\}$ and $f_0 = \min\{f(s) : 0 < u_0 \leq s \leq u_\infty\}$. We see that $f_0 > 0$ and $h(r) \geq f_0$ for $r \geq r_0$. By (B.3) we have (1.2).

Conversely, assume that (1.2) holds. Let $c > 0$. Define $f_c = \max\{f(s) : c \le s \le 2c\}$. Choose $r_0 > 0$ so large that

$$\int_{r_0}^{\infty} s\phi(s)ds < \frac{(n-2)c}{f_c}.$$

Let $C([r_0,\infty))$ denote the Fréchet space of continuous functions on $[r_0,\infty)$ with the topology of uniform convergence on any compact subinterval of $[r_0,\infty)$. Consider the set $U = \{u \in C([r_0,\infty)) : c \le u(r) \le 2c,\ r \ge r_0\}$, which is a closed convex subset of $C([r_0,\infty))$. We define the operator $F$ on $U$ by

$$Fu(r) = c + \int_r^{\infty} s^{1-n} \int_{r_0}^{s} t^{n-1}\phi(t)f(u(t))dtds, \quad r \ge r_0.$$

If $u \in U$, then $Fu(r) \ge c$ and

$$Fu(r) \le c + \frac{f_c}{n-2}\int_{r_0}^{\infty} s\phi(s)ds \le 2c, \quad r \ge r_0.$$

Thus the operator $F$ maps $U$ into itself. It is easy to see that $F$ is continuous on $U$ and $FU$ is relatively compact in the topology of $C([r_0,\infty))$. By the Schauder-Tychonoff fixed point theorem, $F$ has an element $u \in U$ such that $u = Fu$, i.e., $u(r) = Fu(r)$ for $r \ge r_0$. Then $u = u(|x|)$ is a positive solution of (1.1) on $\{|x| \ge r_0\}$ and satisfies $\lim_{|x|\to\infty} u(x) = c$. This completes the proof of Lemma B.1.

**Lemma B.2.** *Assume that $f(0) > 0$. Then,* (1.1) *has a bounded positive solution $u$ on $\{x \in \mathbb{R}^n : |x| \ge r_0\}$ for some $r_0 > 0$ if and only if* (1.2) *holds.*

**Proof.** Assume that $u$ is a bounded positive solution of (1.1) on $\{|x| \ge r_0\}$. Let $\overline{u}$ be the spherical mean of $u$. Then by the argument in the proof of Lemma B.1 we have (B.3). Define $u_\infty$ and $f_0$ as $u_\infty = \max\{u(x) : |x| \ge r_0\}$ and $f_0 = \min\{f(s) : 0 \le s \le u_\infty\}$. We see that $f_0 > 0$ since $f(s) > 0$ for $s \ge 0$, and that $h(r) \ge f_0$ for $r \ge r_0$. By (B.3) we have (1.2).

Conversely, assume that (1.2) holds. Then, by the argument in the proof of Lemma B.1, we obtain a bounded positive solution of (1.1) on $\{|x| \ge r_0\}$.

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