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DISCONTINUOUS IMPLICIT ELLIPTIC BOUNDARY VALUE PROBLEMS*

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Abstract. In this paper we provide an existence result for an implicitly given elliptic differential equation of the form $f(x, u, Lu) = 0$ under Dirichlet boundary conditions. The peculiarity of this implicit equation is that the function f may be discontinuous in all its arguments. The main tool used to treat this problem is a fixed point result in partially ordered sets based on a generalized iteration method combined with an appropriately modified method of upper and lower solutions.

1. Introduction. In this paper we consider the following implicit elliptic boundary value problem (BVP)

$$f(x, u, Lu) = 0 \quad \text{in} \;\; \Omega, \qquad u = 0 \quad \text{on} \;\; \partial\Omega, \tag{1.1}$$

where L is a semilinear elliptic operator of the form

$$Lu := -\Delta u + a(x, u), \tag{1.2}$$

and $\Omega \subset \mathbb{R}^N$ is a bounded domain having a sufficiently smooth boundary $\partial\Omega$. The Laplacian in (1.2) can also be replaced by any linear second order strongly elliptic operator with sufficiently smooth coefficients. In addition to

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the implicitness of the BVP (1.1) the governing nonlinearity $f\colon \Omega\times\mathbb{R}^2\to\mathbb{R}$ may be discontinuous in all its arguments.

There are two main approaches to deal with implicit equations. The first approach is to solve the equation

$$f(x,u,v)=0 \tag{1.3}$$

for v and reduce the implicit elliptic equation in (1.1) to a semilinear elliptic BVP of the form

$$Lu=h(x,u)\ \text{ in }\ \Omega,\quad u=0\ \text{ on }\ \partial\Omega.$$

However, this approach requires a global continuous solution $v=h(x,u)$ of (1.3) to be defined for all $(x,u)\in\Omega\times\mathbb{R}$ which, even for a continuous function f, can rarely be obtained.

The second approach which is much more general reduces the implicit equation to a differential inclusion of the form

$$Lu\in H(x,u), \tag{1.4}$$

where the multifunction H is required to be a lower semicontinuous multiselection of the multifunction $\mathcal{H}$ defined by

$$\mathcal{H}(x,u)=\{v\in\mathbb{R}\mid f(x,u,v)=0\}. \tag{1.5}$$

Based on a result by Ricceri in [12] on lower semicontinuous inclusions this approach has been used only recently e.g. by Marano in [11]. In [11] the operator L was assumed to be linear and the nonlinearity f has been assumed to be continuous in the argument standing for Lu which is more restrictive than the assumptions of this paper.

Due to the discontinuous behavior of f of equation (1.1) also this second approach cannot, in general, be applied to the implicit BVP to be considered in this paper. In fact, we provide an example which shows that the multifunction $\mathcal{H}$ has neither a lower nor an upper semicontinuous multiselection, but which, nevertheless, can be treated by our method.

The aim of the present paper is to prove the existence of solutions of the BVP (1.1) by assuming an existence of appropriately defined upper and lower solutions. In contrast to the rather implicit hypotheses usually imposed in papers dealing with implicit differential equations and their related multivalued differential inclusions, cf. e.g. [5, 6, 9, 10, 11], the assumptions made in this paper are explicit and can readily be verified.

2. Notations, assumptions and preliminary results. We denote by $W^{k,2}(\Omega)$ the usual Sobolev space of square integrable real-valued functions having square integrable generalized derivatives up to order $k \in \mathbb{N}$. In particular, $W^{0,2}(\Omega) \hat{=} L^2(\Omega)$. By $W_o^{1,2}(\Omega)$ we denote the subspace of $W^{1,2}(\Omega)$ whose elements have generalized homogeneous boundary values.

Definition 2.1. A function $u \in W^{2,2}(\Omega) \cap W_o^{1,2}(\Omega)$ is called a solution of the BVP (1.1) if

$$f(x, u(x), Lu(x)) = 0 \quad \text{for almost all (a.a)} \;\; x \in \Omega.$$

Definition 2.2. A function $\bar{u} \in W^{2,2}(\Omega)$ is called an upper solution of the BVP (1.1) if $f(\cdot, \bar{u}, L\bar{u}) \in L^2(\Omega)$ and

(i) $\bar{u} \geq 0$ on $\partial\Omega$;
(ii) $f(x, \bar{u}(x), L\bar{u}(x)) \geq 0$ for a.a $x \in \Omega$.

Similarly, a lower solution of the BVP (1.1) is defined by reversing the inequalities in (i) and in (ii).

We shall impose the following hypotheses:

(H1) The BVP (1.1) has a lower solution $\underline{u}$ and an upper solution $\bar{u}$ such that $L\underline{u} \leq L\bar{u}$.

(H2) There is a function $\mu \colon \Omega \times \mathbb{R}^2 \to (0, \infty)$ such that the function $\mu \cdot f$ is sup-measurable, and that the function $(x, u, v) \to v - (\mu \cdot f)(x, u, v)$ is monotone nondecreasing in u and in v for a.a. $x \in \Omega$.

(H3) The nonlinearity $a \colon \Omega \times \mathbb{R} \to \mathbb{R}$ of L is a Carathéodory function which is monotone nondecreasing in its second argument and satisfies $a(\cdot, \underline{u})$, $a(\cdot, \bar{u}) \in L^2(\Omega)$.

We are going to show that conditions (H1)–(H3) are sufficient for the solvability of (1.1). In condition (H2) no continuity assumption is imposed on the function f in any of its variables.

The following result is needed in the definition of a partial ordering in $W^{2,2}(\Omega)$ which is suitable for our purposes.

Lemma 2.1. *Let u, $v \in W^{2,2}(\Omega)$ satisfy $Lu \leq Lv$ in Ω and $u \leq v$ on $\partial\Omega$ and assume that* (H3) *holds with $\underline{u}$, $\bar{u}$ replaced by u, v. Then $u \leq v$ in Ω.*

Proof. Applying the inequality $Lu \le Lv$ and using integration by parts we get

$$\int_\Omega \nabla(u-v)\nabla\varphi\,dx + \int_\Omega (a(\cdot,u)-a(\cdot,v))\varphi\,dx \le 0$$

for all $\varphi \in W_o^{1,2}(\Omega)$ with $\varphi \ge 0$. The assumption $u \le v$ on $\partial\Omega$ implies that

$$\varphi = (u-v)^+ = \max\{u-v,0\} \in W_o^{1,2}(\Omega).$$

Taking this special test function φ we get

$$\begin{aligned} 0 \le \int_\Omega |\nabla(u-v)^+|^2\,dx &\le \int_\Omega \nabla(u-v)\nabla(u-v)^+\,dx \\ &+ \int_\Omega (a(\cdot,u)-a(\cdot,v))(u-v)^+\,dx \le 0, \end{aligned}$$

which implies that

$$\nabla(u-v)^+ = 0.$$

Consequently, since $(u-v)^+ \in W_o^{1,2}(\Omega)$, it follows that $(u-v)^+ = 0$ in Ω, i.e., $u \le v$ in Ω.

Remark 2.1. From hypotheses (H1) and (H3) it readily follows by Lemma 2.1 that $\underline{u}(x) \le \bar{u}(x)$ for a.a. $x \in \Omega$.

Let us define the following truncated function $\hat{a}$ which is related with $a\colon \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$\hat{a}(x,u) = \begin{cases} a(x,\bar{u}(x)), & \text{if } \ u > \bar{u}(x), \\ a(x,u), & \text{if } \ \underline{u}(x) \le u \le \bar{u}(x), \\ a(x,\underline{u}(x)), & \text{if } \ u < \underline{u}(x), \end{cases} \tag{2.1}$$

and denote

$$\hat{L}u = -\Delta u + \hat{a}(x,u). \tag{2.2}$$

Obviously, $Lu = \hat{L}u$ whenever $\underline{u}(x) \le u \le \bar{u}(x)$ for a.a. $x \in \Omega$.

Remark 2.2. The function $\hat{a}\colon \Omega \times \mathbb{R} \to \mathbb{R}$, defined in (2.1), satisfies hypothesis (H3), and moreover, for any $L^2(\Omega)$-function u we have $\hat{a}(\cdot,u) \in L^2(\Omega)$.

We shall introduce in $W^{2,2}(\Omega)$ an order relation '$\preceq$' defined by

$$u \preceq w \ \text{ iff } \hat{L}u \le \hat{L}w \text{ in } \Omega \text{ and } u \le w \text{ on } \ \partial\Omega. \tag{2.3}$$

This relation is obviously reflexive and transitive, and by means of Lemma 2.1 also antisymmetric. Thus (2.3) defines a partial order relation $\preceq$ in $W^{2,2}(\Omega)$. By hypothesis (H1) the given upper and lower solutions satisfy $\underline{u} \preceq \bar{u}$, and thus the order interval defined by

$$[\underline{u}, \bar{u}] = \{u \in W^{2,2}(\Omega) \mid \underline{u} \preceq u \preceq \bar{u}\}$$

is nonempty.

Lemma 2.2. *Let hypotheses* (H1)–(H3) *be satisfied. Given* $v \in [\underline{u}, \bar{u}]$ *we define*

$$Fv(x) := \hat{L}v(x) - (\mu \cdot f)(x, v(x), \hat{L}v(x)), \; x \in \Omega. \tag{2.4}$$

Then the semilinear elliptic BVP

$$\hat{L}z = Fv \quad in \quad \Omega, \qquad z = 0 \quad on \quad \partial\Omega \tag{2.5}$$

has a unique solution $z = \hat{L}^{-1} \circ Fv$ *with* $z \in [\underline{u}, \bar{u}]$. *Furthermore,* $u \in [\underline{u}, \bar{u}]$ *is a solution of the implicit BVP* (1.1) *if and only if* $u = \hat{L}^{-1} \circ Fu$.

Proof. Let $v \in [\underline{u}, \bar{u}]$, i.e., $\hat{L}\underline{u} \le \hat{L}v \le \hat{L}\bar{u}$ in Ω and $\underline{u} \le v \le \bar{u}$ on $\partial\Omega$, which implies by Lemma 2.1 also $\underline{u} \le v \le \bar{u}$ in Ω. Since $\hat{L}\underline{u} = L\underline{u}$ we get by definition of the lower solution and using (H2)

$$\hat{L}\underline{u} \le \hat{L}\underline{u} - (\mu \cdot f)(\cdot, \underline{u}, \hat{L}\underline{u}) = F\underline{u} \le Fv.$$

Analogously we have $\hat{L}\bar{u} \ge F\bar{u} \ge Fv$. This proves that $Fv \in L^2(\Omega)$ and that $\underline{u}$ and $\bar{u}$ are lower and upper solutions, respectively, for the BVP (2.5). By applying the upper and lower solution method (cf. e.g. [1, 2, 3, 4]) combined with a regularity result given in [7, Theorem 8.12] the existence of a weak solution $z \in W^{2,2}(\Omega) \cap W_o^{1,2}(\Omega)$ satisfying $\underline{u}(x) \le z(x) \le \bar{u}(x)$ can be ensured. Due to the monotonicity of the nonlinear function $\hat{a}$ in $\hat{L}$ this solution must be unique.

Finally, let $u \in [\underline{u}, \bar{u}]$ satisfy

$$u = \hat{L}^{-1} \circ Fu.$$

This means that

$$\hat{L}u = Fu \;\; \text{in} \;\; \Omega, \quad u = 0 \;\; \text{on} \;\; \partial\Omega.$$

From $u \in [\underline{u}, \bar{u}]$, which means

$$\hat{L}\underline{u} \le \hat{L}u \le \hat{L}\bar{u} \quad \text{in} \quad \Omega$$

and

$$\underline{u} \le u \le \bar{u} \quad \text{on} \quad \partial\Omega,$$

we deduce just in the same way as in Lemma 2.1 the inequality $\underline{u} \le u \le \bar{u}$ in Ω. Hence $\hat{L}u = Lu$, and from $\hat{L}u = Fu$ it follows by (2.4) that $f(x, u, Lu) = 0$ in Ω, i.e., u is a solution of the BVP (1.1). Conversely, any solution u of the BVP (1.1) satisfying $u \in [\underline{u}, \bar{u}]$ is a solution of $u = \hat{L}^{-1} \circ Fu$.

Lemma 2.3. *Under the hypotheses* (H1)–(H3) *the operator* $G = \hat{L}^{-1} \circ F\colon [\underline{u}, \bar{u}] \to [\underline{u}, \bar{u}]$ *is nondecreasing.*

Proof. Let $u_1, u_2 \in [\underline{u}, \bar{u}]$ satisfy $u_1 \preceq u_2$. By definition (2.3) of $\preceq$ and by Lemma 2.1 it follows that $\underline{u} \le u_1 \le u_2 \le \bar{u}$ in Ω. Denote

$$z_i = Gu_i = \hat{L}^{-1} \circ Fu_i,$$

i.e.,

$$\hat{L}z_i = Fu_i \;\; \text{in} \;\; \Omega, \quad z_i = 0 \;\; \text{on} \;\; \partial\Omega.$$

By the monotonicity of F according to hypothesis (H2) we get

$$\hat{L}\underline{u} \le F\underline{u} \le Fu_1 \le Fu_2 \le F\bar{u} \le \hat{L}\bar{u},$$

and thus

$$\hat{L}\underline{u} \le \hat{L}z_1 \le \hat{L}z_2 \le \hat{L}\bar{u} \;\; \text{in} \;\; \Omega, \qquad z_i = 0 \;\; \text{on} \;\; \partial\Omega.$$

This implies that

$$z_1 \preceq z_2$$

and that

$$z_1, z_2 \in [\underline{u}, \bar{u}].$$

Now we define a metric d in the space $W^{2,2}(\Omega)$ by

$$d(u, w) = \|u - w\|_{L^2(\partial\Omega)} + \|\hat{L}u - \hat{L}w\|_{L^2(\Omega)}. \tag{2.6}$$

Obviously, the space $W^{2,2}(\Omega)$, equipped with this metric and the partial ordering '$\preceq$' becomes an ordered metric space, denoted by

$$\mathcal{M} = (W^{2,2}(\Omega), d, \preceq\,).$$

Lemma 2.4. *Let hypotheses* (H1)–(H3) *be satisfied, and let* $(u_n)_{n=o}^{\infty}$ *be a monotone sequence in* $[\underline{u}, \bar{u}]$. *Then there is* $w \in W^{2,2}(\Omega)$ *such that* $(Gu_n)_{n=o}^{\infty}$ *converges to* w *in* $\mathcal{M}$.

Proof. Let $(u_n)_{n=o}^{\infty}$ be a monotone nondecreasing sequence in the order interval $[\underline{u}, \bar{u}]$ of $\mathcal{M}$. Then by Lemma 2.3 $(Gu_n)_{n=o}^{\infty}$ is monotone nondecreasing and belongs to $[\underline{u}, \bar{u}]$. This and Lemma 2.1 imply that the sequence $(Gu_n)_{n=o}^{\infty}$ satisfies

$$\underline{u}(x) \le Gu_n(x) \le Gu_{n+1}(x) \le \bar{u}(x)$$

for all $n \in \mathbb{N}$ and for a.a. $x \in \Omega$. Thus the almost everywhere pointwise limit w, given by

$$w(x) = \lim_{n \to \infty} Gu_n(x) \quad \text{for a.a. } \ x \in \Omega \tag{2.7}$$

exists. We are going to show that $w \in W^{2,2}(\Omega)$, and that $d(w, Gu_n) \to 0$ as $n \to \infty$. Let

$$w_n = Gu_n,$$

i.e., for all $n \in \mathbb{N}$ we have

$$\hat{L}w_n = Fu_n \ \text{ in } \ \Omega, \quad w_n = 0 \ \text{ on } \ \partial\Omega. \tag{2.8}$$

Since $u_n \in [\underline{u}, \bar{u}]$ and $(u_n)_{n=o}^{\infty}$ is a nondecreasing sequence with respect to the partial ordering $\preceq$, it follows that

$$\underline{u}(x) \le u_n(x) \le u_{n+1}(x) \le \bar{u}(x)$$

for all $n \in \mathbb{N}$ and for a.a. $x \in \Omega$. This implies that $Fu_n \in L^2(\Omega)$ and

$$F\underline{u}(x) \le Fu_n(x) \le Fu_{n+1}(x) \le F\bar{u}(x)$$

for all $n \in \mathbb{N}$ and for a.a. $x \in \Omega$. By the dominated convergence theorem it follows that

$$Fu_n \to z \in L^2(\Omega) \quad \text{as } \ n \to \infty. \tag{2.9}$$

From (2.8) we obtain

$$-\Delta(w_n - w_m) + \hat{a}(\cdot, w_n) - \hat{a}(\cdot, w_m) = Fu_n - Fu_m \quad \text{in } \ \Omega,$$

$$w_n - w_m = 0 \quad \text{on } \ \partial\Omega,$$

which yields

$$\int_\Omega |\nabla(w_n - w_m)|^2 dx + \int_\Omega (\hat{a}(\cdot, w_n) - \hat{a}(\cdot, w_m))(w_n - w_m)dx \\ \le \|Fu_n - Fu_m\|_{L^2(\Omega)} \|w_n - w_m\|_{L^2(\Omega)},$$

and by the monotonicity of the function $s \mapsto \hat{a}(x, s)$ we get

$$\|w_n - w_m\|_{W_o^{1,2}(\Omega)} \le c\,\|Fu_n - Fu_m\|_{L^2(\Omega)}, \tag{2.10}$$

where c is some positive constant. By means of (2.9) from (2.10) it follows that $w_n \to \tilde{w}$ in $W_o^{1,2}(\Omega)$. Since $w_n = Gu_n \to w$ pointwise by (2.7), then

$$w_n \to w \;\text{ in } W_o^{1,2}(\Omega) \text{ as } \; n \to \infty. \tag{2.11}$$

Furthermore, by the regularity result of [7, Theorem 8.12] we obtain with some positive constant c the following estimate

$$\|w_n - w_m\|_{W^{2,2}(\Omega)} \le c \big(\|w_n - w_m\|_{L^2(\Omega)} + \|Fu_n - Fu_m\|_{L^2(\Omega)} \\ + \|\hat{a}(\cdot, w_n) - \hat{a}(\cdot, w_m)\|_{L^2(\Omega)}\big), \tag{2.12}$$

which by (2.9), (2.11) and because of the continuity of the Nemytskij operator generated by $\hat{a}$ implies that

$$w_n \to w \;\text{ in } W^{2,2}(\Omega) \text{ as } \; n \to \infty. \tag{2.13}$$

Since $\hat{a}(\cdot, w_n) \to \hat{a}(\cdot, w)$ in $L^2(\Omega)$, it follows from (2.13) that $\hat{L}w_n \to \hat{L}w$ in $L^2(\Omega)$, which shows that

$$w_n = Gu_n \to w \in W^{2,2}(\Omega) \quad \text{in} \;\; \mathcal{M}.$$

The proof for a monotone nonincreasing sequence can be done analogously.

3. Main result. We call a solution $\hat{u}$ of the BVP (1.1) *minimal* with respect to the order interval $[\underline{u}, \bar{u}]$ if $\underline{u} \preceq \hat{u} \preceq u$ for any other solution $u \in [\underline{u}, \bar{u}]$, and *maximal* in $[\underline{u}, \bar{u}]$ if $u \preceq \hat{u} \preceq \bar{u}$ for any other solution $u \in [\underline{u}, \bar{u}]$. If the BVP (1.1) has both the maximal and the minimal solution in $[\underline{u}, \bar{u}]$, then they are called *extremal solutions* of (1.1) in $[\underline{u}, \bar{u}]$.

The main result of this paper is given by the following theorem.

Theorem 3.1. *Let the hypotheses* (H1)–(H3) *be satisfied. Then the discontinuous implicit BVP* (1.1) *has the extremal solutions in* $[\underline{u}, \bar{u}]$ *with respect to the underlying partial ordering '$\preceq$', defined by* (2.3).

As we already know from Lemma 2.2, every solution of (1.1) in $[\underline{u}, \bar{u}]$ is a fixed point of the operator $G = \hat{L}^{-1} \circ F$ and vice versa. Therefore the proof will be based on a fixed point result in partially ordered sets which has been proved in [8, Theorem 1.2.2]. In its concrete form which meets the situation given here it reads as follows.

Lemma 3.1. *Let* $[\underline{u}, \bar{u}]$ *be a nonempty order interval in the partially ordered metric space* $\mathcal{M} = (W^{2,2}(\Omega), d, \preceq)$*, and let* $G\colon [\underline{u}, \bar{u}] \to [\underline{u}, \bar{u}]$ *be a monotone nondecreasing mapping. If* $(Gu_n)_{n=o}^{\infty}$ *converges in* $\mathcal{M}$ *whenever* $(u_n)_{n=o}^{\infty}$ *is a monotone sequence in* $[\underline{u}, \bar{u}]$*, then* G *has the least fixed point* u_* *and the greatest fixed point* u^* *in* $[\underline{u}, \bar{u}]$*. Furthermore, these extremal fixed points can be characterized by*

$$u_* = \min\{w \in [\underline{u}, \bar{u}] \mid Gw \preceq w\},$$
$$u^* = \max\{w \in [\underline{u}, \bar{u}] \mid w \preceq Gw\}.$$

Proof of Theorem 3.1. The results of Lemmata 2.3 and 2.4 ensure that the hypotheses of the fixed point theorem given by Lemma 3.1 are satisfied when the operator G is given by $G = \hat{L}^{-1} \circ F$, which is defined in Lemma 2.2. Thus by Lemma 3.1 the operator G has the least fixed point u_* and the greatest fixed point u^* in $[\underline{u}, \bar{u}]$. According to Lemma 2.2 this means that u_* and u^* are the minimal and the maximal solutions of the BVP (1.1) in $[\underline{u}, \bar{u}]$.

Example 3.1. Consider the following simple but nontrivial implicit discontinuous BVP

$$\begin{aligned} -u''(x) &= -x + \frac{1}{2}[-u''(x)] \quad \text{a.e. in} \quad \Omega = (-1, 1), \\ u(-1) &= u(1) = 0, \end{aligned} \tag{3.1}$$

where $[v]$ means the greatest integer less or equal to v. In our notation we have

$$Lu = -u'',$$

and

$$f(x, u, v) = v + x - \frac{1}{2}[v].$$

Obviously, $f(x,u,v) = 0$ cannot be solved uniquely for v. However, by an elementary calculation one can construct explicitly the multifunction $\mathcal{H}$ defined in the introduction of this paper and given by

$$\mathcal{H}(x,u) = \{v \in \mathbb{R} \mid f(x,u,v) = 0\}.$$

One can show that $\mathcal{H}$ does not possess either a lower or an upper semicontinuous multiselection. Thus none of the two approaches described in the introduction can be applied to (3.1). Nevertheless, this problem can be treated very easily by the theory developed in this paper. To this end lower and upper solutions have to be constructed and the hypotheses (H1)–(H3) to be verified. Hypothesis (H1) requires upper and lower solutions $\bar{u}$ and $\underline{u}$, respectively, which satisfy

$$L\underline{u} \le L\bar{u},$$

i.e.,

$$-\underline{u}''(x) + x - \frac{1}{2}[-\underline{u}''(x)] \le 0 \text{ a.e. in } (-1,1), \quad \underline{u}(-1) \le 0,\ \underline{u}(1) \le 0,$$
$$-\bar{u}''(x) + x - \frac{1}{2}[-\bar{u}''(x)] \ge 0 \text{ a.e. in } (-1,1), \quad \bar{u}(-1) \ge 0,\ \bar{u}(1) \ge 0,$$

and

$$-\underline{u}''(x) \le -\bar{u}''(x) \quad \text{a.e. in} \quad (0,1).$$

For this let $z \in W^{2,2}(-1,1)$ be defined as the solution of the following BVP:

$$-z''(x) = 2|x| + 2 \text{ a.e. in } (-1,1), \quad z(-1) = z(1) = 0. \tag{3.2}$$

The unique (and nonnegative) solution z satisfying (3.2) can be calculated immediately, and it can be verified that $\bar{u} = z$ and $\underline{u} = -z$ are upper and lower solutions of (3.1) which obviously satisfy also $L\underline{u} \le L\bar{u}$.

Hypothesis (H2) can be fulfilled by taking $\mu(x,u,v) \equiv 1$, since then we have

$$v - (\mu \cdot f)(x,u,v) = -x + \frac{1}{2}[v],$$

which is nondecreasing with respect to u and v. The measurability conditions are trivially satisfied.

Since in our example $a(x,u) \equiv 0$, also (H3) is trivially satisfied. By Theorem 3.1 there exist solutions of the BVP (3.1) in the order interval $[-z,z]$, where z is given by (3.2).

In a similar way one can treat the following implicit elliptic equation defined on the unit ball in $\mathbb{R}^N$

$$-\Delta u = -|x| + \frac{1}{2}[-\Delta u] \quad \text{in } B = \{x \in \mathbb{R}^N \mid |x| < 1\}, \quad u = 0 \text{ on } \partial B,$$

where $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^N$ and where as above $[v]$ means the greatest integer less or equal to v. Following the approach of the previous example upper and lower solutions can be obtained by means of the unique solution of the BVP

$$-\Delta z = |x| + 2 \quad \text{in } B\,, \quad u = 0 \text{ on } \partial B,$$

taking $\bar{u} = z$ and $\underline{u} = -z$.

Remark 3.1. (i) In view of hypothesis (H2) it should be noted that the describing discontinuous nonlinearity f of the BVP (1.1) need not to be monotone in any of its arguments.

(ii) The assumptions imposed on f and a in hypotheses (H2) and (H3), respectively, can be relaxed further. If we suppose the function $(x, u, v) \to v - (\mu \cdot f)(x, u, v)$ to be nondecreasing only for those u and v satisfying $\underline{u}(x) \le u \le \bar{u}(x)$ and $L\underline{u}(x) \le v \le L\bar{u}(x)$, respectively, as well as the function $u \to a(x, u)$ to be nondecreasing only for u satisfying $\underline{u}(x) \le u \le \bar{u}(x)$, then our main result still remains valid.

(iii) As a special case problem (1.1) includes explicit semilinear BVP of the form

$$Lu = g(x, u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where g may be discontinuous in both its arguments.

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