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INFLUENCE OF DISSIPATIVE FORCES ON THE STABILITY BEHAVIOR OF THE STEADY MOTIONS OF LAGRANGIAN MECHANICAL SYSTEMS WITH CYCLIC COORDINATES*

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Abstract. We are concerned with stability problems for a holonomic system $\mathcal{S}$ with n degrees of freedom having $n-m$ cyclic coordinates, $m<n$. Let x be the set of the acyclic coordinates and let v be the set of the generalized velocities corresponding to the cyclic coordinates. For $\mathcal{S}$ conservative or subject to a dissipation restricted to the acyclic coordinates, we revisit classical stability results concerning the steady motions of the system and give some new contribution. When $\mathcal{S}$ is strictly dissipative with respect to all the coordinates, the integrals of momenta disappear and so do the steady motions. In this case, under suitable conditions there exist motions for which x is constant and, consequently, $v \to 0$ as $t \to +\infty$ (pseudosteady motions). We analyze the stability properties with respect to $(x, \dot{x})$ of these motions. Such properties define a stable or unstable behavior with respect to $(x, \dot{x})$ of corresponding steady motions of the conservative system under the influence of strictly dissipative perturbing forces.

1. Introduction.

1.1. The present paper is concerned with an analysis of the stability behavior of the steady motions of conservative holonomic systems with ignorable coordinates under dissipative perturbations of the governing system.

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Consider preliminarily the autonomous differential system in $\mathbf{R}^n$, $n \geq 1$,

$$\dot{z} = f(z), \tag{1.1}$$

where f is continuous in an open set $\mathcal{D}$ of $\mathbf{R}^n$ and satisfies conditions ensuring uniqueness of solutions. Let y be an m-dimensional component of z, $m \leq n$. Denote by $z(t, t_0, z_0)$, $t_0 \geq 0$, the noncontinuable solution satisfying $z(t_0, t_0, z_0) = z_0$. We assume, as known [6], the stability concepts with respect to y concerning a given solution $\tilde{z}(t)$ existing for all $t \geq 0$. These concepts will be referred to as y-stability, y-uniform stability, y-instability, etc. For example, we say that $\tilde{z}(t)$ is: (i) y-stable if for any $\varepsilon > 0$ and $t_0 \geq 0$ there exists $\delta = \delta(t_0, \varepsilon) \in (0, \varepsilon)$ such that $\|z_0 - \tilde{z}(t_0)\| < \delta$ implies $\|y(t, t_0, z_0) - y(t, t_0, \tilde{z}(t_0))\| < \varepsilon$ for $t \geq t_0$; (ii) y-uniformly stable if it is stable and δ is independent of t_0. The property which defines the effective y-stable behavior is the y-uniform stability. Hence, it is natural to refer to the solution $\tilde{z}$ as effectively y-unstable if it is not y-uniformly stable. The constant

$$\begin{aligned} R_y^{(e)} = \sup\{h \geq 0 : \exists(\tau_i), \tau_i \geq 0, \exists(z_i) \subset \mathcal{D}, \|z_i - \tilde{z}(\tau_i)\| \to 0, \exists(t_i), \\ t_i \geq \tau_i, \text{ such that } \|y(t_i, \tau_i, z_i) - \tilde{y}(t_i)\| \geq h \text{ for each i} \in \mathbf{N}\} \end{aligned}$$

will be called radius of effective y-instability. Clearly, $\tilde{z}$ is effectively y-unstable if and only if $R_y^{(e)} > 0$. In this case, the value of $R_y^{(e)}$ induces a natural ordering for this property. We also consider the function defined in $[0, +\infty)$

$$\begin{aligned} R_y(t_0) = \sup\{h \geq 0 : \exists(z_i) \subset \mathcal{D}, \|z_i - \tilde{z}(t_0)\| \to 0, \exists(t_i), t_i \geq t_0, \\ \text{such that } \|y(t_i, t_0, z_i) - \tilde{y}(t_i)\| \geq h \text{ for each i} \in \mathbf{N}\}, \end{aligned}$$

which will be called radius of y-instability. Clearly, $\tilde{z}$ is y-unstable if and only if $R_y(t_0) > 0$ for some t_0. Since system (1.1) is autonomous, R_y reduces to a constant when $\tilde{z}$ is a static solution.

Consider now a family of differential systems in $\mathbf{R}^n$

$$\dot{z} = g(z, \eta), \tag{Σ_η}$$

depending on a parameter η belonging to a subset $\mathcal{F}$ of a (finite or infinite dimensional) vector space containing the origin. Assume that $g(\cdot, \eta) \in$

$C(\mathcal{D}, \mathbf{R}^n)$ and satisfies conditions ensuring uniqueness of solutions for any $\eta \in \mathcal{F}$. System (Σ_0) will be called the unperturbed system. Let y be again an m-dimensional component of z, $m \leq n$. Let (s) be a static solution of (1.1), $z(t) \equiv \bar{z}$, and let (s_η) be the solution $z_\eta(t, 0, \bar{z})$ of (Σ_η). Letting $\Delta = \mathcal{F} - \{0\}$, we need some concepts concerning the influence of the perturbations $\eta \in \Delta$ on the y-stability behavior of (s) in the case that the constant character of y is preserved when (1.1) is replaced by any (Σ_η).

Definition 1.1. Assume that $y_\eta(t, 0, \bar{z}) \equiv \bar{y}$ for any $\eta \in \Delta$. Then (s) is said to be

(i) effectively y-stable under Δ if (s_η) is y-uniformly stable for any $\eta \in \Delta$;

(ii) effectively y-asymptotically stable under Δ if (s_η) is y-uniformly asymptotically stable for any $\eta \in \Delta$;

(iii) effectively y-unstable under Δ if (s_η) is effectively y-unstable for any $\eta \in \Delta$ and the radius $R_y^{(e)}(\eta)$ is greater than a positive number $\gamma > 0$ independent of η.

Needless to say, (iii) is not in general the inversion of (i).

1.2. Let $\mathcal{S}$ be a material system subject to bilateral, ideal, and time independent holonomic constraints, and let $q = (q_1, \dots, q_n)$ be a system of Lagrangian coordinates for $\mathcal{S}$. Let the generalized force acting on $\mathcal{S}$ be composed of a conservative positional force and of a time independent dissipative force D. Suppose that the coordinates $q_{m+1}, \dots, q_n$, $m < n$, are cyclic; that is, the potential energy Π, D, and the coefficients of the kinetic energy T are independent of $(q_{m+1}, \dots, q_n)$. Let $x = (q_1, \dots, q_m)$, $\xi = (q_{m+1}, \dots, q_n)$, $u = (\dot{q}_1, \dots, \dot{q}_m)$, $v = (\dot{q}_{m+1}, \dots, \dot{q}_n)$. Thus, we have $\Pi = \Pi(x)$, $D = D(x, u, v)$, $T(x, u, v) = (1/2)\langle (u, v), A(x)(u, v)\rangle$ where A is a symmetric and positive definite $n \times n$ matrix. Finally we set $\Gamma = (D_1, D_2, \dots, D_m)$, $\Lambda = (D_{m+1}, D_{m+2}, \dots, D_n)$, and $D = (\Gamma, \Lambda)$.

The determination of (x, u, v) along the motion may be obtained by the $n + m$ dimensional system

$$\begin{aligned} &\dot{x} = u \\ &\frac{d}{dt}\frac{\partial T}{\partial u} - \frac{\partial T}{\partial x} = -\frac{\partial \Pi}{\partial x} + \Gamma \\ &\frac{d}{dt}\frac{\partial T}{\partial v} = \Lambda, \end{aligned} \tag{S_D}$$

which does not depend on the cyclic coordinates. We suppose that Π and the elements of A are C^1 functions defined in an open set Ω of $\mathbf{R}^m$ containing the origin. Moreover, we suppose that $D \in C(\Omega \times \mathbf{R}^n, \mathbf{R}^n)$ and that A, Π, and D satisfy conditions ensuring uniqueness of solutions. When system (S_D) is integrated, the cyclic coordinates will be obtained by solving the system $\dot{q}_\alpha = v_\alpha$ $(\alpha = m+1, m+2, \dots, n)$ where v_α are known functions of t and the initial values of x, u, v.

The property $\langle D(x,u,v), (u,v) \rangle \leq 0$ for all (x,u,v) and the continuity of $D(x,u,v)$ at $u = 0$, $v = 0$ imply $D(x,0,0) \equiv 0$. We will include the conservative case, $D = 0$, as well as the case that D is strictly dissipative; that is, $\langle D(x,u,v), (u,v) \rangle < 0$ for $(u,v) \neq (0,0)$. We denote by (S_0) the system (S_D) relative to $D = 0$ and regard (S_D) as a perturbed system of (S_0).

Consider any static solution of (S_0). This solution may be represented by the constant values $x(t) \equiv 0$, $u(t) \equiv 0$, $v(t) \equiv \bar{v}$, and will be denoted by $(s_{\bar{v}})$. In Section 2 the modified version in [7] of the Routh theorem is reviewed and a further generalization is given. In Section 3.1 the case of a dissipative perturbing force $D = (\Gamma, 0)$ is considered. Under this perturbation the solution $(s_{\bar{v}})$ persists. The above generalization is still applicable and the stability is preserved. This improves a result in [8] on the same matter. Necessary and sufficient conditions for stability are also given under appropriate conditions on Γ.

The most interesting case occurs when the perturbing force D is strictly dissipative (Sections 3.3, 3.4). In this case, the integrals of momenta are lost and (S_D) does not have static solutions for which $\bar{v} \neq 0$. Assume that for such a $\bar{v}$ system (S_D) admits the solution $(s_{\bar{v}D})$ for which $x(t) \equiv 0$, $u(t) \equiv 0$, $v(0) = \bar{v}$. Along $(s_{\bar{v}D})$ one has $v(t) \to 0$ as $t \to +\infty$. Then $x = 0$ must be a critical point of Π. When $x = 0$ is an isolated critical point of Π we find that $(s_{\bar{v}D})$ is uniformly asymptotically stable if Π has a relative minimum at $x = 0$ and is effectively (x,u)-unstable if Π does not have such a minimum.

Let now Δ be the set of all strictly dissipative forces D with $\Gamma(0,0,v) \equiv 0$. Consider the following condition:

(P) for any $\bar{v}$ in $\mathbf{R}^{n-m}$ system (S_0) has the solution $(s_{\bar{v}})$ and every system (S_D), $D \in \Delta$, has the solution $(s_{\bar{v}D})$.

This condition clearly requires that $x = 0$ is a critical point of Π and will be easily characterized in terms of T. Under (P) we find that if $x = 0$ is an isolated critical point of Π, then for any $\bar{v}$ in $\mathbf{R}^{n-m}$ the solution $(s_{\bar{v}})$ of system (S_0) is (x,u)-effectively asymptotically stable or effectively (x,u)-unstable under Δ, depending on whether Π has or does not have a relative minimum at $x = 0$. It is remarkable that in each of these two cases the stability properties of $(s_{\bar{v}D})$ are independent of $\bar{v}$ and D while for the solution $(s_{\bar{v}})$ of (S_0) one may have stability or instability according to suitable values of $\bar{v}$. Thus, $(s_{\bar{v}})$ may be effectively (x,u)-stable under Δ even if it is an (x,u)-unstable solution of (S_0).

Section 4 concerns the influence of strictly dissipative forces on the stability behavior of the steady motions of a heavy rigid body whose inertia ellipsoid at the center of mass G is an ellipsoid of revolution and a point $O \neq G$ of the axis of revolution is fixed. In this case, condition (P) is satisfied for each of the critical points of the potential energy. The case of a strictly dissipative Lagrangian top in particular is considered.

2. The conservative case. Consider system (S_0):

$$\begin{aligned} &\dot{x} = u \\ &\frac{d}{dt}\frac{\partial T}{\partial u} - \frac{\partial T}{\partial x} = -\frac{\partial \Pi}{\partial x} \\ &\frac{d}{dt}\frac{\partial T}{\partial v} = 0. \end{aligned} \tag{S_0}$$

Equation $(S_0)_3$ may be written as

$$A_v v + Mu = c, \tag{2.1}$$

where A_v is the $(n-m)\times(n-m)$ symmetric matrix constituted by the last $n-m$ rows and columns of A, M is the $(n-m)\times m$ matrix constituted by the last $n-m$ rows and the first m columns of A, and c is an arbitrary $n-m$ dimensional constant vector. Since the matrix A_v is invertible, (2.1) may be solved with respect to v,

$$v = A_v^{-1}(c - Mu), \tag{2.1$'$}$$

and T may be expressed in terms of x, u, c. One obtains

$$T = \mathcal{T} + \mathcal{T}_0, \tag{2.2}$$

where $\mathcal{T}_0 = (1/2)\langle A_v^{-1}c, c\rangle$ and $\mathcal{T}$ is a positive definite quadratic form in u with coefficients depending on x only. In particular, $T(x,u,c)$ does not contain linear terms in u. It is known that the determination of (x,u) relative to the motions for which c has a given value may be obtained by the so called Routh equations

$$\begin{aligned} &\dot{x} = u \\ &\frac{d}{dt}\frac{\partial \mathcal{T}}{\partial u} - \frac{\partial \mathcal{T}}{\partial x} = -\frac{\partial W}{\partial x} + Gu, \end{aligned} \tag{R_c}$$

where $W = W(x,c)$ is the effective potential $W(x,c) = \Pi(x) + \mathcal{T}_0(x,c)$, and $G = G(x,c)$ has a gyrostatic character; that is, G is an antisymmetric $m \times m$ matrix for each c. We denote by $\mathcal{H}$ the total energy $H = T + \Pi$ expressed in terms of x,u,c; that is, we set

$$\mathcal{H}(x,u,c) = \mathcal{T}(x,u) + W(x,c).$$

Let us consider any static solution of (S_0). By a change of the coordinates $q_1, \dots, q_m$ this solution may be always written as:

$$x(t) \equiv 0, \quad u(t) \equiv 0, \quad v(t) \equiv \bar{v}, \tag{$s_{\bar{v}}$}$$

where $\bar{v}$ is a constant vector. Clearly, the property that $(s_{\bar{v}})$ is a solution of (S_0) is equivalent to the property that the identities

$$x(t) \equiv 0, \quad u(t) \equiv 0, \quad c = \bar{c}, \tag{$2.3_{\bar{c}}$}$$

with $\bar{c} = A_v(0)\bar{v}$, define a solution of the system

$$\begin{aligned} &\dot{x} = u \\ &\frac{d}{dt}\frac{\partial \mathcal{T}}{\partial u} - \frac{\partial \mathcal{T}}{\partial x} = -\frac{\partial W}{\partial x} + Gu \\ &\dot{c} = 0. \end{aligned} \tag{2.4}$$

This last property is equivalent to the assumption that

$$\frac{\partial W}{\partial x}(0,\bar{c}) = 0.$$

We may assume $W(0,\bar{c}) = 0$. Relative to the static solution $(s_{\bar{v}})$ of (S_0), there corresponds an $(n-m)$-dimensional family of motions of the material system $\mathcal{S}$ for which x, u, v assume the constant values $(0,0,\bar{v})$ and $\xi = \bar{v}t + \xi_0$, with ξ_0 an arbitrary $n-m$ constant vector. These motions are the so called steady or merostatic motions. Clearly, the stability of $(s_{\bar{v}})$, or equivalently the stability of the solution $(2.3)_{\bar{c}}$ of (2.4) is nothing but the partial stability of any of these motions with respect to x, u, v.

If $W(x,\bar{c})$ has a strict relative minimum at $x = 0$, then the solution $x(t) \equiv 0$, $u(t) \equiv 0$ of $(R_{\bar{c}})$ is stable. Indeed, the function $\mathcal{H}_{\bar{c}}(x,u) \equiv \mathcal{H}(x,u,\bar{c})$ is positive definite and along the solutions of $(R_{\bar{c}})$, one has $\dot{\mathcal{H}}_{\bar{c}}(x,u) \equiv 0$. Hence, $\mathcal{H}_{\bar{c}}$ satisfies for this solution the conditions in the Liapunov theorem on nonasymptotic stability. The result is equivalent to the fact that the solution $(s_{\bar{v}})$ of (S_0) is stable with respect to the initial perturbations x_0,u_0,v_0 for which $c = \bar{c}$ (Routh theorem). The restriction that $c = \bar{c}$ on the stability behavior is not essential; that is, if $W(\cdot,\bar{c})$ has a strict relative minimum at $x = 0$, then $(s_{\bar{v}})$ is stable. This is the modified version of the Routh theorem to which we have alluded to in Section 1. We give now a generalization of this result which is similar to a well known generalization of the Lagrange-Dirichlet theorem [5]. First we prove a theorem concerning a class of autonomous differential equations.

Theorem 2.1. *Let $\mathcal{D}$ be an open set of $\mathbf{R}^r$, $r \geq 1$, containing the origin. Consider the differential system*

$$\dot{y} = g(y,a), \dot{a} = 0, \tag{2.5}$$

where g is defined and continuous in $\mathcal{D} \times \mathbf{R}^p$, $p \geq 1$, satisfies conditions ensuring the uniqueness of solutions, and $g(0,\bar{a}) = 0$ for some $\bar{a} \in \mathbf{R}^p$. Suppose there exists for (2.5) a function $V \in C^1(\mathcal{D} \times \mathbf{R}^p, \mathbf{R})$, $V(0,\bar{a}) = 0$, and a fundamental family F of closed neighborhoods of $y = 0$ contained in $\mathcal{D}$, such that: (i) $\dot{V}(y,a) \leq 0$ for all $(y,a) \in \mathcal{D} \times \mathbf{R}^p$; (ii) for every $\mathcal{A} \in F$ and $y \in \partial\mathcal{A}$, one has $V(y,\bar{a}) > 0$. Then the solution $y(t) \equiv 0$, $a = \bar{a}$ of (2.5) is stable.

Proof. Let $\mathcal{A}$ be any bounded set in F and let $m = \min\{V(y,\bar{a}), y \in \partial\mathcal{A}\}$. Since $m > 0$, $V(0,\bar{a}) = 0$, and V is continuous, there exist $\sigma_1 = \sigma_1(\mathcal{A})$, $0 < \sigma_1 < \operatorname{dist}(0,\partial\mathcal{A})$, and $\sigma_2 = \sigma_2(\mathcal{A}) > 0$, such that

$$|V(y,a)| < \frac{m}{2} \ \text{ if } \|y\| < \sigma_1 \ \text{ and } |a - \bar{a}| < \sigma_2,$$

$$V(y,a) \geq \frac{m}{2} \ \text{ if } y \in \partial\mathcal{A} \ \text{ and } |a - \bar{a}| < \sigma_2.$$

Let y_0,a be such that $\|y_0\| < \sigma_1$ and $|a - \bar{a}| < \sigma_2$. Then the solution $y(t,y_0,a)$ of $(2.5)_1$, $y(0,y_0,a) = y_0$, satisfies $y(t,y_0,a) \in \operatorname{int}(\mathcal{A})$ for all $t \geq 0$. Indeed, the existence of $t_1 > 0$ such that

$$y(t, y_0, a) \in \mathrm{int}(\mathcal{A}) \ \text{ for each } t \in [0, t_1) \ \text{ and } \ y(t_1, y_0, a) \in \partial\mathcal{A}$$

would imply

$$\frac{m}{2} > |V(y_0, a)| \geq V(y_0, a) \geq V(y(t_1, y_0, a), a) \geq \frac{m}{2},$$

a contradiction. This completes the proof, since F is a fundamental family.

We are now able to prove the following theorem concerning the stability of the solution $(s_{\bar{v}})$ of system (S_0).

Theorem 2.2. *Assume that there exists in the x-space a fundamental family Φ of closed neighborhoods of $x = 0$ contained in Ω, such that for every $\phi \in \Phi$ and $x \in \partial\phi$, one has $W(x, \bar{c}) > 0$. Then $(s_{\bar{v}})$ is stable.*

Proof. It is the same to prove that the solution $(2.3)_{\bar{c}}$ of (2.4) is stable. For any ϕ in Φ let $\chi = \mathrm{dist}(0, \partial\phi)$ and consider in the u-space the ball $B[\chi] = \{u : \|u\| \leq \chi\}$. The family $F = \{\phi \times B[\chi] : \phi \text{ in } \Phi\}$ of sets in the (x, u)-space forms a fundamental family of closed neighborhoods of the origin. We have $\mathcal{H}(0, 0, \bar{c}) = 0$ and, for each $\mathcal{A} \in F$ and $(x, u) \in \partial\mathcal{A}$, we have clearly $\mathcal{H}(x, u, \bar{c}) > 0$. Since $\dot{\mathcal{H}}(x, u, c) \equiv 0$, we see that $\mathcal{H}$ satisfies all the conditions of the function V in Theorem 2.1 for the solution $(2.3)_{\bar{c}}$ of system (2.4). The proof is complete.

If $W(\cdot, \bar{c})$ does not have a strict relative minimum at $x = 0$, the problem of stability is very hard, even harder than the similar problem concerning the equilibrium. Indeed, the stability may occur due to the gyrostatic part Gu which clearly does not contribute to the integral of energy (gyrostatic stabilization). In contrast, if $x = 0$ is an isolated critical point of $W(\cdot, \bar{c})$, under a suitable dissipation restricted to the acyclic coordinates, the solution $(s_{\bar{v}})$ is preserved and the problem of stability is completely solved through the inspection of $W(\cdot, \bar{c})$ near the origin (see Section 3.1).

3. The dissipative case

3.1. We begin by considering the case $D = (\Gamma, 0)$. System (S_D) will be now denoted by (S_Γ):

$$\begin{aligned} &\dot{x} = u \\ &\frac{d}{dt}\frac{\partial T}{\partial u} - \frac{\partial T}{\partial x} = -\frac{\partial \Pi}{\partial x} + \Gamma \\ &\frac{d}{dt}\frac{\partial T}{\partial v} = 0. \end{aligned} \tag{S_Γ}$$

Since D is dissipative and $\Lambda = 0$, then $\Gamma(x, 0, v) \equiv 0$. Hence, systems (S_0) and (S_Γ) have the same static solutions. The Routh system becomes

$$\begin{aligned} \dot{x} &= u \\ \frac{d}{dt}\frac{\partial \mathcal{T}}{\partial u} - \frac{\partial \mathcal{T}}{\partial x} &= -\frac{\partial W}{\partial x} + Gu + \mathcal{G}, \end{aligned} \qquad (R_{c\Gamma})$$

where $\mathcal{G}(x, u, c) = \Gamma(x, u, A_v^{-1}(c - Mu))$. Since now along the solutions we have $\dot{\mathcal{H}}(x, u, c) \le 0$, Theorem 2.2 holds for the static solutions of (S_Γ), again as a consequence of Theorem 2.1. The force $(\Gamma, 0)$ is said to be strictly dissipative in the acyclic coordinates if $\langle \Gamma(x, u, v), u \rangle < 0$ for $u \ne 0$. The following theorem holds.

Theorem 3.1. *Let us assume that $x = 0$ is an isolated critical point of $W(\cdot, \bar{c})$ and Γ is strictly dissipative in the acyclic coordinates. Then $(s_{\bar{v}})$, considered as a solution of (S_Γ), is stable if and only if $W(\cdot, \bar{c})$ has a relative minimum at $x = 0$. Moreover, if this minimum occurs, in addition to its unconditional stability, $(s_{\bar{v}})$ is (x, u)-asymptotically stable with respect to the perturbations for which $c = \bar{c}$.*

Proof. If $W(\cdot, \bar{c})$ has a relative minimum at $x = 0$, the stability is a consequence of the above extension of Theorem 2.2. The remaining part of Theorem 3.1 is obtained by considering the solutions of system $(R_{\bar{c}\Gamma})$. We set again $\mathcal{H}_{\bar{c}}(x, u) \equiv \mathcal{H}(x, u, \bar{c})$. Let $a > 0$ be such that $x = 0$ is the only critical point of $W(\cdot, \bar{c})$ in the ball $B'[a] = \{x : \|x\| \le a\}$. Consider in $\mathbf{R}^{2m}$ the ball $B[a] = \{(x, u) : \|(x, u)\| \le a\}$. Then in the set $\mathcal{M} = \{(x, u) \in B[a] : (x, u) \ne (0, 0), \dot{\mathcal{H}}_{\bar{c}}(x, u) = 0\}$, there are no orbits of $(R_{\bar{c}\Gamma})$. Indeed, since $\mathcal{G}(x, u, \bar{c}) = 0$ only if $u = 0$, one has $\mathcal{M} = \{(x, u) \in B[a] : (x, u) \ne (0, 0), u = 0\}$. If $\mathcal{M}$ contains an orbit of $(R_{\bar{c}\Gamma})$, then along this orbit we would have $u(t) \equiv 0$ and then in $B'[a]$ there would exist a critical point different from $x = 0$, a contradiction. By using the theorem of Barbasin-Krasovskii [2] on asymptotic stability and that of Krasovskii [3] on instability, and taking into account the properties of $\dot{\mathcal{H}}_{\bar{c}}$, we immediately obtain that $(s_{\bar{v}})$ is asymptotically stable or unstable according to whether $W(\cdot, \bar{c})$ has or does not have a relative minimum at $x = 0$. The proof is complete.

3.2. Let us consider now the case of a strictly dissipative force $D = (\Gamma, \Lambda)$. Letting $z = (x, u, v)$, for fixed D we denote by $z(t, t_0, z_0)$ the noncontinuable solution of (S_D) through (t_0, z_0) and put $z(t, z_0) = z(t, 0, z_0)$.

System (S_D) admits the null solution

$$x(t) \equiv 0, \quad u(t) \equiv 0, \quad v(t) \equiv 0 \tag{σ}$$

if and only if $x = 0$ is an isolated critical point of Π. Relative to (σ) there corresponds an $(n-m)$-dimensional family of equilibrium positions of the material system $\mathcal{S}$ for which $x = 0$ and ξ is any fixed element in $\mathbf{R}^{n-m}$. Thus, each of these positions is not isolated. Hence, the well known theorems on asymptotic stability and instability of equilibria for systems subject to strictly dissipative forces are not applicable. Nevertheless, when $x = 0$ is an isolated critical point of Π these properties occur for the static solution (σ) of system (S_D) and thus they will imply the asymptotic stability or the instability of each of the above equilibrium positions with respect to (x, u, v). Moreover, when the instability of (σ) occurs, then (σ) will be unstable with respect to (x, u). Precisely the following theorem holds.

Theorem 3.2. *Suppose that $x = 0$ is an isolated critical point of Π. One has: (a) if Π has a relative minimum at $x = 0$, then (σ) is (uniformly) asymptotically stable; (b) if Π does not have a relative minimum at $x = 0$, then (σ) is (x, u)-unstable and the radius of (x, u)-instability is greater than a constant $\gamma > 0$ independent of D.*

Proof. Let $a > 0$ be such that in the ball $B'[a] = \{x : \|x\| \leq a\}$, $x = 0$ is the only critical point of Π. Consider in $\mathbf{R}^{m+n}$ the ball $B[a] = \{z : \|z\| \leq a\}$. Then in the set $\mathcal{M} = \{z \in B[a] : z \neq 0, \dot{H}(z) = 0\}$, there are no orbits of system (S_D). Indeed, since D is strictly dissipative one has $\mathcal{M} = \{z \in B[a] : z \neq 0, (u, v) = (0, 0)\}$. If $\mathcal{M}$ contains an orbit of (S_D), then along this orbit we would have $u(t) \equiv 0$ and then in $B'[a]$ there would exist a critical point different from $x = 0$, a contradiction. By using again the above theorems of Barbasin and Krasovskii, and taking into account the properties of the total energy $H(z)$, we immediately obtain that (σ) is asymptotically stable or unstable depending on whether Π has or does not have a relative minimum at $x = 0$.

We now prove that in the last case (σ) is (x, u)-unstable and the radius of (x, u)-instability is greater than a constant $\gamma > 0$ independent of D. Since (σ) is unstable there exists $\alpha \in (0, a)$, a sequence $\{\zeta_i\}$ in the z-space, $\zeta_i \to 0$ and a sequence $\{t_i\}, t_i > 0$, such that $\|z(t, \zeta_i)\| < \alpha$ for $t \in [0, t_i)$ and $\|z(t_i, \zeta_i)\| = \alpha$. Let $z_0 = (x_0, u_0, v_0)$ be anyone of the ζ_i's and denote by $\hat{t}$ the corresponding t_i. We recall that T is a quadratic form, positive definite in (u, v). Hence, $T(x, u, v) \geq b(\|(u, v)\|)$ for all $z \in B[\alpha]$. Here b is a strictly increasing continuous mapping from $\mathbf{R}^+$ into $\mathbf{R}^+$ with $b(0) = 0$. We may assume $\Pi(0) = 0$. Then there exists $\gamma \in (0, \alpha/2)$ for which

$$z \in B(\gamma) \Longrightarrow |\Pi(x)| < \frac{1}{2} b(\frac{\alpha}{2}) \;\text{ and } |H(z)| < \frac{1}{2} b(\frac{\alpha}{2}).$$

Consider any $\delta \in (0, \gamma)$ and choose the above z_0 in $B(\delta)$. For our proof it will be obviously sufficient to show that $\|x(\hat{t}, z_0)\| \geq \gamma$. Assume not; that is, $\|x(\hat{t}, z_0)\| < \gamma$. Then we have $\|(u(\hat{t}, z_0), v(\hat{t}, z_0))\| \geq \alpha - \gamma > \alpha/2$. Thus, it follows that

$$b(\frac{\alpha}{2}) < T(z(\hat{t}, z_0)) \leq H(z_0) - \Pi(x(\hat{t}, z_0)) < \frac{1}{2} b(\frac{\alpha}{2}) + \frac{1}{2} b(\frac{\alpha}{2}) = b(\frac{\alpha}{2}),$$

a contradiction. This completes the proof of (b).

3.3. Suppose that for some $\bar{v} \in \mathbf{R}^{n-m}$ system (S_D) admits the solution for which

$$x(t) \equiv 0, \quad u(t) \equiv 0, \quad v(0) = \bar{v}. \tag{p}$$

This solution will be denoted by $(s_{\bar{v}D})$ and called pseudostatic. Each motion of the material system $\mathcal{S}$ corresponding to $(s_{\bar{v}D})$ will be called a pseudosteady or pseudomerostatic motion. We now prove the following proposition.

Proposition 3.1. *Along $(s_{\bar{v}D})$ one has $v(t) \to 0$ as $t \to +\infty$. Therefore if (S_D) admits a pseudostatic solution, (S_D) admits also the null solution and $x = 0$ will be a critical point of Π.*

Proof. Equation $(S_D)_3$ for $x(t) \equiv 0$, $u(t) \equiv 0$ reduces to

$$A_v(0)\dot{v} = \Lambda(0, 0, v). \tag{3.1}$$

If $\bar{v} = 0$, then from (3.1) it follows that $v(t) \equiv 0$ and the assertion holds. Suppose that $\bar{v} \neq 0$ and consider the function $V \in C^1(\mathbf{R}^n, \mathbf{R})$ defined by

$$V(v) = \frac{1}{2}\langle A_v(0)v, v\rangle.$$

V is then a positive definite quadratic form and one has

$$\dot{V}(v) = \langle A_v(0)\dot{v}, v\rangle = \langle \Lambda(0, 0, v), v\rangle \leq 0.$$

Then $v(t)$ belongs for all $t \geq 0$ to the set $\{v : V(v) \leq V(\bar{v})\}$ which is a compact neighborhood of $v = 0$. Since $\langle \Lambda(0, 0, v), v\rangle = 0$ only for $v = 0$, the first part of our assertion is obtained by an immediate application of the LaSalle invariance principle [4]. As a consequence, the subset $\{z = 0\}$ of $\mathbf{R}^{n+m}$ is invariant for (S_D) and the proof is complete.

Theorem 3.3. *Suppose that $x = 0$ is an isolated critical point of Π. We have: (a) if Π has a relative minimum at $x = 0$, then $(s_{\bar{v}D})$ is uniformly asymptotically stable; (b) if Π does not have a relative minimum at $x = 0$, then $(s_{\bar{v}D})$ is effectively (x, u)-unstable with a radius greater than the constant $\gamma > 0$ (independent of D) of Theorem 3.2.*

Proof. Letting $z = (x, u, v)$ we denote by $\tilde{z}(t)$ the solution $(s_{\bar{v}D})$ of (S_D) and again we use the notation $z(t, z_0)$ for $z(t, 0, z_0)$.

Suppose that Π has a relative minimum at $x = 0$. From Theorem 3.2 it follows that the solution $z(t) \equiv 0$ of (S_D) is (uniformly) asymptotically stable. Hence, we have (i) for any $\varepsilon > 0$ there exists $\delta(\varepsilon) \in (0, \varepsilon)$ with the property that $\|z_0\| < \delta(\varepsilon)$ implies $\|z(t, z_0)\| < \varepsilon$ for every $t \geq 0$; (ii) there exists $\sigma > 0$ with the property that for any $\nu > 0$ there exists a number $h = h(\nu) > 0$ for which if $\|z_0\| < \sigma$, then $\|(z(t, z_0)\| < \nu$ for all $t \geq h$. Let $\varepsilon > 0$ and $\delta(\varepsilon) \in (0, \varepsilon)$ as in (i). By virtue of Proposition 3.1 there exists $\tau \geq 0$ such that $\|\tilde{z}(t)\| < \delta(\varepsilon/2)/2$ for any $t \geq \tau$. Let $t_0 \geq \tau$ and choose z_0 such that $\|z_0 - \tilde{z}(t_0)\| < \delta(\varepsilon/2)/2$. Thus, it follows $\|z_0\| < \delta(\varepsilon/2)$. We have $z(t, t_0, z_0) = z(t - t_0, z_0)$ and, because of (i), $\|z(t, t_0, z_0)\| < \varepsilon/2$ for all $t \geq t_0$. Consequently, $\|z(t, t_0, z_0) - \tilde{z}(t)\| < \varepsilon$ for all $t \geq t_0$.

Consider now any $t_0 \in [0, \tau]$. Since t_0 varies in a compact set, we may find a number $\delta'(\varepsilon) \in (0, \delta(\varepsilon))$ (independent of t_0) for which if $\|z_0 - \tilde{z}(t_0)\| < \delta'(\varepsilon)$, then $\|z(\tau, t_0, z_0) - \tilde{z}(\tau)\| < \delta(\varepsilon/2)/2$. Thus, $\tilde{z}$ is uniformly stable. To prove attractivity let $\sigma > 0$ be as in (ii) and let $\tau \geq 0$ be such that $\|\tilde{z}(t)\| < \sigma/2$ for any $t \geq \tau$. Let $t_0 \geq \tau$ and choose z_0 so that $\|z_0 - \tilde{z}(t_0)\| < \sigma/2$. Then $\|z_0\| < \sigma$ and, because of (ii), for any $\nu > 0$ we have $\|z(t, t_0, z_0)\| < \nu/2$ for any $t \geq t_0 + h(\nu/2)$. Moreover, there exists $h' \geq h(\nu/2)$ such that $\|\tilde{z}(t)\| < \nu/2$ for any $t \geq t_0 + h'$ and consequently, $\|z(t, t_0, z_0) - \tilde{z}(t)\| < \nu$ for all $t \geq t_0 + h'$. Again, if $t_0 \in [0, \tau]$, we may find σ' independent of t_0, such that $\|z_0 - \tilde{z}(t_0)\| < \sigma'$ implies $\|z(\tau, t_0, z_0) - \tilde{z}(\tau)\| < \sigma/2$. The proof of (a) is now complete.

Suppose that Π does not have a relative minimum at $x = 0$ and let $y = (x, u)$. From Theorem 3.2 it follows that the solution $z(t) \equiv 0$ of (S_D) is y-unstable. Thus, for the number $\gamma > 0$ of Theorem 3.2, we have for each $\delta \in (0, \gamma)$ the existence of a z_0, $\|z_0\| < \delta$, and of a $\tilde{t} > 0$ such that $\|y(\tilde{t}, z_0)\| \geq \gamma$. Let $\tau \geq 0$ be such that $\|\tilde{z}(\tau)\| < \delta$. One has $\|z_0 - \tilde{z}(\tau)\| < 2\delta$. Since $y(\tau + \tilde{t}, \tau, z_0) = y(\tilde{t}, z_0)$, then $\|y(\tau + \tilde{t}, \tau, z_0)\| \geq \gamma$. The proof of (b) is clearly complete.

3.4. Let now Δ be the set of all strictly dissipative forces D with

$\Gamma(0,0,v) \equiv 0$ and consider the condition (P) of Section 1.2: for any $\bar{v}$ in $\mathbf{R}^{n-m}$ system (S_0) has the solution $(s_{\bar{v}})$ and every system (S_D), $D \in \Delta$, has the solution $(s_{\bar{v}D})$. The following proposition gives a characterization of the systems (S_0) for which this condition is satisfied. The quantities involved include Π and the matrices A_v, M of Section 2. The proof is trivial and will be omitted.

Proposition 3.2. *System (S_0) has the property (P) if and only if $x = 0$ is a critical point of Π and $M(0) = 0$, $A'_v(0) = 0$, with*

$$A'_v = (\partial A_v/\partial x_1, \dots, \partial A_v/\partial x_m).$$

Thus, Theorem 3.3 and Definition 1.1 imply the following result.

Corollary 3.1. *Suppose that $x = 0$ is an isolated critical point of Π and condition (P) is satisfied. For any $\bar{v}$ in $\mathbf{R}^{n-m}$ we have: (a) if Π has a relative minimum at $x = 0$, then the solution $(s_{\bar{v}})$ of (S_0) is effectively (x,u)-asymptotically stable under Δ; (b) if Π does not have a relative minimum at $x = 0$, then $(s_{\bar{v}})$ is effectively (x,u)-unstable under Δ, and the radius of effective (x,u)-instability of each $(s_{\bar{v}D})$ is greater than the constant $\gamma > 0$ of Theorem 3.2.*

We emphasize that in each of the two cases in this corollary, Π having or not having a relative minimum at $x = 0$, the stability properties of the solutions $(s_{\bar{v}D})$ are the same for all $\bar{v} \in \mathbf{R}^{n-m}$ and $D \in \Delta$ while for the solution $(s_{\bar{v}})$ of (S_0) one may have stability or instability according to suitable values of $\bar{v}$. Consider for example the following Lagrangian functions of two systems with two degrees of freedom for which one of the coordinates, say ξ, is cyclic

$$\mathcal{L}_1 = \tfrac{1}{2}[u^2 + \exp(-x^2)v^2 + x^2]$$
$$\mathcal{L}_2 = \tfrac{1}{2}[u^2 + \exp(x^2)v^2 - x^2],$$

where $v = \dot{\xi}$. In both cases condition (P) is satisfied. In the first case the solution $(s_{\bar{v}D})$ is always effectively (x,u)-unstable, while $(s_{\bar{v}})$ is (x,u)-unstable if $\bar{v} < 1$ and (uniformly) stable if $\bar{v} > 1$. In the second case, $(s_{\bar{v}D})$ is always uniformly asymptotically stable, while $(s_{\bar{v}})$ is stable if $\bar{v} < 1$ and (x,u)-unstable if $\bar{v} > 1$.

Finally, we remark that under (P) the set $\mathcal{N} = \{(x,u,v) : x = 0, u = 0, v \in \mathbf{R}^{n-m}\}$ is a set of stationary points for (S_0) and is invariant for every system (S_D), $D \in \Delta$. The stability properties in (a) and (b) of Corollary 3.1 may be considered as properties of $\mathcal{N}$. We will refer to them in the next section by saying that $\mathcal{N}$ is effectively asymptotically stable under Δ in case (a) and effectively unstable under Δ in case (b). Clearly, these concepts are not in the spirit of the usual ones regarding the stability properties of a set.

4. An application to the mechanics of inertially symmetric rigid bodies. Let $\mathcal{S}$ be a rigid body whose inertia ellipsoid at the center of mass G is an ellipsoid of revolution. Denote by a the axis of revolution. Suppose that a point $O \neq G$ of a is fixed without friction. Refer $\mathcal{S}$ to two sets of rectangular right handed Cartesian axes: (i) a set of axes $O\xi\eta\zeta$ fixed in the space with η vertically upwards; (ii) a set of axes $OXYZ$ fixed in $\mathcal{S}$ with Z coinciding with the axis a oriented from O towards G. Let θ, ψ, φ be the Euler angles of $OXYZ$ with respect to $O\xi\eta\zeta$. Precisely, $\theta \in (0, \pi)$ is the angle between the axes ζ and Z (rotation angle), $\psi \in [0, 2\pi)$ is the angle between the nodal axis X_1 and ξ (precession angle) and $\varphi \in [0, 2\pi)$ that between X_1 and X (proper rotation angle). The coordinates ψ and φ can be considered as angles mod 2π. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be the moments of inertia with respect to the axes X, Y, Z, respectively. Since also the inertia ellipsoid of $\mathcal{S}$ at O is an ellipsoid of revolution around a, one has $\mathcal{A} = \mathcal{B}$. The kinetic energy T and the potential energy Π corresponding to the gravitational force mg are independent of φ and given by:

$$T = \frac{1}{2}\mathcal{A}(\dot{\theta}^2 + \dot{\psi}^2 \cos^2\theta) + \frac{1}{2}\mathcal{C}(\dot{\varphi} + \dot{\psi}\cos\theta)^2, \tag{4.1}$$

$$\Pi = mgl \sin\theta \cos\psi, \tag{4.2}$$

where l is the length of OG. Let D be a φ-independent dissipative force. Then $\mathcal{S}$ when subject to the forces mg and D has φ as the only cyclic coordinate. Letting $x = (\theta, \psi)$, $u = \dot{x}$, $v = \dot{\varphi}$, we may write $T = T(x, u, v)$, $D = D(x, u, v)$, and $D = (\Gamma, \Lambda)$. Now Γ is a two dimensional vector function and Λ is a scalar function. Let $x_1 = (\pi/2, \pi)$, $x_2 = (\pi/2, 0)$. Clearly, for any configuration of $\mathcal{S}$ for which Z is vertical downwards (resp. upwards) one has $x = x_1$ (resp. $x = x_2$). We denote again by (S_D) the governing differential system for x, u, v and by (S_0) this system for $D = 0$. Let Δ be the set of all strictly dissipative forces $D = (\Gamma, \Lambda)$ such that $\Gamma(x, 0, v) \equiv 0$ for $x = x_1$ and $x = x_2$. In the case that $\mathcal{S}$ is constituted by a finite number of points and a is an axis of material symmetry, examples of such forces are the forces of friction due to viscous homogeneous media. We have immediately from

(4.1), (4.2) that x_1 and x_2 are (isolated) critical points of Π and that the conditions in Proposition 3.2 are satisfied for both points. Therefore, for any $\bar{v}$ in $\mathbf{R}$ system (S_0) has the static solutions

$$(s^1_{\bar{v}}):\ x(t) \equiv x_1, \quad u(t) \equiv 0, \quad v(t) \equiv \bar{v},$$
$$(s^2_{\bar{v}}):\ x(t) \equiv x_2, \quad u(t) \equiv 0, \quad v(t) \equiv \bar{v},$$

and for any $\bar{v}$ in $\mathbf{R}$ and D in Δ, system (S_D) has the pseudostatic solutions $(s^1_{\bar{v}D})$ and $(s^2_{\bar{v}D})$ defined by

$$(s^1_{\bar{v}D}):\ x(t) \equiv x_1, \quad u(t) \equiv 0, \quad v(0) = \bar{v},$$
$$(s^2_{\bar{v}D}):\ x(t) \equiv x_2, \quad u(t) \equiv 0, \quad v(0) = \bar{v},$$

respectively. Letting $\mathcal{N}_i = \{(x,u,v) : x = x_i,\, u = 0,\, v \in \mathbf{R}\}$, $i = 1,2$, we obtain from (4.2) and Theorem 3.3:

Theorem 4.1. *For any $\bar{v}$ in $\mathbf{R}$ and D in Δ, $(s^1_{\bar{v}D})$ is uniformly asymptotically stable while $(s^2_{\bar{v}D})$ is effectively (x,u)-unstable with a radius greater than a constant independent of $\bar{v}$, D. Thus, $\mathcal{N}_1$ is effectively (x,u)-asymptotically stable under Δ and $\mathcal{N}_2$ is effectively (x,u)-unstable under Δ.*

In contrast, $(s^1_{\bar{v}})$ is a stable solution of (S_0) for every $\bar{v}$ in $\mathbf{R}$, while there exists a number $\lambda > 0$ such that $(s^2_{\bar{v}})$ is a stable or an unstable solution of (S_0) according to $|\bar{v}| \geq \lambda$ or $|\bar{v}| < \lambda$, respectively. These properties are well known (see for instance [1], p. 148–155). They are recognized by using as coordinates the Euler angles, say $\theta^\star, \psi^\star, \varphi^\star$, relative to $OXYZ$ and a set of fixed axes $O\xi^\star\eta^\star\zeta^\star$ with $\zeta^\star$ vertically upwards. Since these coordinates are singular at $\theta^\star = 0$ we have not used them in analyzing the dissipative case. However, we remark that by our choice of coordinates, the stability of $(s^2_{\bar{v}})$ for $|\bar{v}| \geq \lambda$ cannot be recognized by an application of the Routh theorem. Indeed, in our treatment $\mathcal{T}_0$ is independent of x and hence the effective potential energy $W(x,c) = \Pi(x) + \mathcal{T}_0(c)$ has a relative maximum at $x = x_2$ for any fixed c. In terms of θ, ψ, φ, the stability appears to be due to the gyrostatic part in the Routh system.

We conclude by observing that the fact that for any $\bar{v}$ the top in the air falls down may be represented by the above effective instability of $\mathcal{N}_2$ under Δ.

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