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CLASSIFICATIONS OF NONNEGATIVE SOLUTIONS TO SOME ELLIPTIC PROBLEMS

Yuan Lou
Department of Mathematics, Ohio State University, Columbus, OH 43210

Meijun Zhu
Department of Mathematics and Statistics, McMaster University
Hamilton, Ontario, Canada L8S 4K1

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Abstract. The main purpose of this paper is to show that all nonnegative solutions to $\Delta u = 0$ (or $\Delta u = u^p$) in the n-dimensional upper half space $H = \{(x', t)|x' \in \mathbb{R}^{n-1}, t > 0\}$ with boundary condition $\partial u/\partial t = u^q$ on ∂H must be linear functions of t (or $u \equiv 0$) when $n \geq 2$ and $q > 1$ (or $n \geq 2$ and $p, q > 1$).

1. Introduction. Let $H = \{(x', t) \mid x' = (x_1, x_2, \dots, x_{n-1}) \in \mathbb{R}^{n-1},\ t > 0\}$ be the upper half space in $\mathbb{R}^n$ with $n \geq 2$. We are interested in the following problem.

$$\begin{cases} \Delta u = 0, \quad u \geq 0 \quad \text{in} \quad H, \\ \dfrac{\partial u}{\partial t} = c u^q \quad \text{on} \quad \partial H. \end{cases} \tag{1}$$

Recently there have been many works concerning the symmetry property about the solutions in a half space; see, for example, [2], [4], [6], [7], [8], [9] and the references therein. In [6], Hu studied (1) when $c < 0, 1 < q < n/(n-2)$ and $n \geq 3$. Using the moving-plane method (see, e.g., [5]), he proved that $u \equiv 0$ is the only solution. Later, in [7], all solutions of (1) are classified when $q = n/(n-2)$ and $n \geq 3$ by using the method of moving spheres regardless of the sign of c. In this note, by using some technical lemmas developed in [7] and the moving-plane method we shall classify all solutions when $c > 0$ and $q > 1$. To this end, after rescaling and by the Strong Maximum Principle, it suffices to consider the following problem.

$$\begin{cases} \Delta u = 0, \quad u > 0 \quad \text{in} \quad H, \\ \dfrac{\partial u}{\partial t} = u^q \quad \text{on} \quad \partial H. \end{cases} \tag{2}$$

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Our first result can be stated as follows.

Theorem 1.1. *If* $u(x) \in C^2(H) \cap C^1(\bar{H})$ *solves* (2) *and* $q > 1$, *then* $u = at + b$ *for some positive constants* a, b *satisfying* $a = b^q$.

There are two ingredients in our proof. First, by virtue of the boundary condition we can prove Theorem 1.1 in the supercritical case, i.e., $q > n/(n-2)$ and $n \geq 3$, by the moving-plane method. In the proof some technical lemmas will play an important role as in [7]. Secondly, when q is subcritical or $q > 1$ and $n = 2$, we observe that if (2) is put into a high-dimensional space setting (its meaning will be made precise in the proof), any superlinear power becomes a supercritical power. This reduces all the cases to the supercritical one.

It turns out that these two ideas combined is also useful in treating some other problems with a so-called "good sign" coefficient in front of the nonlinearity. Among them, the following problem serves as a good example.

$$\begin{cases} \Delta u = u^p, \quad u \geq 0 \quad \text{in} \quad H, \\ \dfrac{\partial u}{\partial t} = u^q \quad \text{on} \quad \partial H. \end{cases} \tag{3}$$

For solutions of (3) we have the following result.

Theorem 1.2. *If* $u(x) \in C^2(H) \cap C^1(\bar{H})$ *solves* (3) *for* $p, q > 1$, *then* $u \equiv 0$.

Our paper is organized as follows: Theorem 1.1 will be established in Section 2. In this section, we first prove Theorem 1.1 in the supercritical case (Theorem 2.1 below), and then pursue the second ingredient to complete the proof of Theorem 1.1 (Theorem 2.7 below). The proof of Theorem 1.2 will be presented in Section 3. In the last section we shall discuss some related things.

2. Proof of Theorem 1.1.

2.1. Case $q > n/(n-2)$, $n \geq 3$. We assume $n \geq 3$ and $q > n/(n-2)$ throughout this subsection.

Theorem 2.1. *If* $u(x) \in C^2(H) \cap C^1(\bar{H})$ *solves* (2) *and* $q > n/(n-2)$, *then* $u = at + b$ *with* $a = b^q$ *and* $b > 0$.

Since there is no assumption on the decay rate of $u(x)$ at infinity, as usual we perform the Kelvin transformation on u, i.e., set

$$v(x) = \frac{1}{|x|^{n-2}} u\Big(\frac{x}{|x|^2}\Big).$$

Then $v(x)$ satisfies

$$\begin{cases} \Delta v = 0, \quad v(x) > 0 \quad \text{in} \quad H, \\ \dfrac{\partial v}{\partial t} = |x|^{\alpha} v^{q} \quad \text{on} \quad \partial H \setminus \{0\}, \end{cases} \tag{4}$$

where $\alpha = (n-2)q - n > 0$. Our purpose is to obtain some symmetry properties of $v(x)$, and we achieve this goal by using moving planes which are parallel to t-axis. Our first lemma, which is a modification of Lemma 2.1 in [7], will be used to handle the possible singular point.

Lemma 2.2. *Let $v \in C^2(H) \cap C^1(\bar{H}) \setminus \{0\}$ satisfy (4). Then for all $0 < \epsilon < \min\{1, \min_{\partial B_1^+ \cap \partial B_1} v\}$, we have $v(x) \geq \frac{\epsilon}{2}$ for all $x \in \overline{B_1^+} \setminus \{0\}$.*

Proof. For $0 < r < 1$, we introduce an auxiliary function

$$\varphi_0(x) = \frac{\epsilon}{2} - \frac{r^{n-2}\epsilon}{|x|^{n-2}} + \frac{\epsilon t}{2}, \qquad x \in B_1^+ \setminus B_r^+.$$

Set $P_0 = v - \varphi_0$. Clearly, P_0 satisfies

$$\begin{cases} \Delta P_0 = 0 \quad \text{in} \quad B_1^+ \setminus B_r^+, \\ \dfrac{\partial P_0}{\partial t} = |x|^{\alpha} v^{q} - \dfrac{\epsilon}{2} \quad \text{on} \quad \partial(\overline{B_1^+} \setminus \overline{B_r^+}) \cap \partial H. \end{cases} \tag{5}$$

We will show that

$$P_0 \geq 0 \quad \text{in} \quad \overline{B_1^+} \setminus \overline{B_r^+}. \tag{6}$$

On $\partial B_r^+ \cap \partial B_r$,

$$P_0 = v - (\frac{\epsilon}{2} - \epsilon + \frac{\epsilon t}{2}) > v > 0;$$

on $\partial B_1^+ \cap \partial B_1$,

$$P_0 = v - (\frac{\epsilon}{2} - \frac{r^{n-2}\epsilon}{|x|^{n-2}} + \frac{\epsilon t}{2}) > v - \epsilon > 0.$$

Suppose that (6) fails; it follows from the Strong Maximum Principle that there exists some $\bar{x} = (\bar{x}', 0)$ with $r < |\bar{x}'| < 1$ such that

$$P_0(\bar{x}) = \min_{\overline{B_1^+} \setminus \overline{B_r^+} P_0} < 0.$$

Therefore, $\frac{\partial P_0}{\partial t}(\bar{x}) \geq 0$. By using the boundary condition of P_0, we have $v(\bar{x})^q \geq \frac{\epsilon}{2}$. Hence

$$P_0(\bar{x}) = v(\bar{x}) - \left(\frac{\epsilon}{2} - \frac{r^{n-2}\epsilon}{|\bar{x}'|^{n-2}}\right) > v(\bar{x}) - \frac{\epsilon}{2} > 0,$$

which contradicts $P_0(\bar{x}) < 0$. This establishes (6). For $x \in B_1^+ \setminus \{0\}$, it follows from (6) that for all $0 < r < |x|$ we have $P_0(x) \geq 0$. Let $r \to 0$; this finishes the proof of Lemma 2.2.

Corollary 2.3 (scaled version)**.** *Let* $v \in C^2(H) \cap C^1(\bar{H}) \setminus \{0\}$ *solve* (4). *Then for all* $0 < \epsilon < \min\{R^{(2-n)/2}, \min_{\partial B_R^+ \cap \partial B_R} v\}$*, we have* $v(x) > \frac{\epsilon}{2}$ *for all* $x \in \overline{B_R^+} \setminus \{0\}$.

Proof. It follows easily from applying Lemma 2.2 to $\bar{v}(x) = R^{\frac{n-2}{2}} v(Rx)$.

For $\lambda < 0$ we define

$$\Sigma_\lambda = \{x \mid t > 0,\ x_1 > \lambda\},\ T_\lambda = \{x \mid t \geq 0,\ x_1 = \lambda\},$$
$$\tilde{\Sigma}_\lambda = \overline{\Sigma}_\lambda \setminus \{0\}, \quad x^\lambda \text{ is the reflection point of } x \text{ about } T_\lambda,$$
$$v_\lambda(x) = v(x^\lambda),\ w_\lambda = v(x) - v_\lambda(x).$$

Then $w_\lambda(x)$ satisfies

$$\begin{cases} \Delta w_\lambda = 0 \quad \text{in} \ \ \Sigma_\lambda \\ \dfrac{\partial w_\lambda}{\partial t} \leq c_1(x) w_\lambda \ \ \text{on} \ \ \partial H \cap \tilde{\Sigma}_\lambda, \end{cases} \tag{7}$$

where $c_1(x) = q|x|^\alpha \cdot \xi_1^{q-1}(x)$, ξ_1 is a positive function between v_λ and v. Now we are ready to apply the moving-plane method.

Proposition 2.4. *If* $|\lambda|$ *is large enough, then* $w_\lambda(x) \geq 0$ *for all* $x \in \tilde{\Sigma}_\lambda$.

Proof. We argue by contradiction. Suppose that w_λ is negative somewhere in $\tilde{\Sigma}_\lambda$. Since $v(x) \to 0$ as $|x| \to +\infty$, and for any fixed λ, $|x^\lambda| \to +\infty$ as $|x| \to +\infty$, we know that $w_\lambda(x) \to 0$ as $|x| \to +\infty$. It follows from Lemma 2.2 that if $|\lambda|$ is large enough, $w_\lambda(x) > 0$ for x near the origin. Thus there exists some point $\bar{x}$ such that $w_\lambda(\bar{x}) = \min_{x \in \tilde{\Sigma}_\lambda} w_\lambda(x) < 0$. By the Strong Maximum Principle, $\bar{x} \in \partial H \cap \tilde{\Sigma}_\lambda$, but this contradicts the boundary condition of (7).

Now we can define

$$\lambda_0 = \sup\{\lambda < 0 : w_\mu(x) \geq 0 \ \text{ in } \tilde{\Sigma}_\mu \ \text{ for all } -\infty < \mu < \lambda\}. \tag{8}$$

Proposition 2.5. $\lambda_0 = 0$.

Proof. We again argue by contradiction. Suppose that $\lambda_0 < 0$; then we claim that

$$w_{\lambda_0} \equiv 0. \tag{9}$$

This will contradict (4) since $\alpha > 0$. Hence from now on it suffices to prove (9) under the assumption $\lambda_0 < 0$. Suppose that (9) fails; then by the definition of λ_0 and the Strong Maximum Principle, $w_{\lambda_0} > 0$ in $\tilde{\Sigma} \setminus T_{\lambda_0}$. The following result is needed to treat the singular point, and we relegate its proof to the end.

Lemma 2.6. *For $r_0 < \min\{\frac{1}{2}|\lambda_0|, 1\}$, there exists some positive constant γ depending only on λ_0 and r_0 such that $w_{\lambda_0}(x) > \gamma$ in $B^+_{r_0}(0)$.*

We continue the proof of Proposition 2.5. By the definition of λ_0, there is a sequence $\lambda_k \to \lambda_0$ with $\lambda_k > \lambda_0$ such that $\inf_{\tilde{\Sigma}_{\lambda_k}} w_{\lambda_k} < 0$. It is not difficult to see from Lemma 2.6 and the continuity of $v(x)$ away from the origin that for k large enough, we have $w_{\lambda_k}(x) \geq \gamma/2$, $\forall x \in \overline{B^+_{r_0}} \setminus \{0\}$. We also know that $w_{\lambda_k}(x) \to 0$ as $|x| \to +\infty$, and it follows that there exists $x_k = (x'_k, t_k) \in \tilde{\Sigma}_{\lambda_k} \setminus \overline{B^+_{r_0}}$ such that $w_{\lambda_k}(x_k) = \min_{\tilde{\Sigma}_{\lambda_k}} w_{\lambda_k} < 0$. It is clear that $r_0 < |x_k|$ and due to the boundary condition, $t_k > 0$. Since w_{λ_k} is a harmonic function in $\tilde{\Sigma}_{\lambda_k}$, we thus reach a contradiction. Hence (9) is verified and this finishes the proof of Proposition 2.5.

Proof of Lemma 2.6. We know that w_{λ_0} satisfies

$$\begin{cases} \Delta w_{\lambda_0} = 0, \quad w_{\lambda_0} > 0, \text{ in } B^+_{r_0}(0) \setminus \{0\}, \\ \dfrac{\partial w_{\lambda_0}}{\partial t} = |x|^\alpha v^q - |x^{\lambda_0}|^\alpha v^q_{\lambda_0} \text{ on } \overline{B^+_{r_0}}(0) \cap \partial H \setminus \{0\}. \end{cases} \tag{10}$$

Let $\min_{\partial B^+_{r_0}(0)} w_{\lambda_0} \geq \epsilon$ for some $0 < \epsilon < 1$. Due to the continuity of v in $\mathbb{R}^n_+ \setminus \{0\}$, there exists some positive constant C_1 such that

$$v_{\lambda_0}(x) < C_1 < +\infty \text{ for } x \in \overline{B^+_{r_0}}(0) \setminus \{0\}. \tag{11}$$

Let

$$\varphi_1(x) = \frac{\epsilon\mu}{2} - \frac{r^{n-2}\epsilon}{|x|^{n-2}} + \frac{\epsilon t(1-\mu)}{2} \text{ in } B^+_{r_0}(0) \setminus B^+_r(0),$$

where $\mu < 1$ will be chosen later. Set $P_1(x) = w_{\lambda_0}(x) - \varphi_1(x)$. $P_1(x)$ satisfies

$$\begin{cases} \Delta P_1(x) = 0 \quad \text{in} \quad B^+_{r_0}(0) \setminus B^+_r(0) \\ \dfrac{\partial P_1(x)}{\partial t} = |x|^\alpha v^q - |x^{\lambda_0}|^\alpha v^q_{\lambda_0} - \dfrac{\epsilon(1-\mu)}{2} \quad \text{on} \quad \overline{B^+_{r_0}}(0) \setminus \overline{B^+_r}(0) \cap \partial H. \end{cases} \tag{12}$$

On $\partial B^+_{r_0}(0) \cap \partial B_{r_0}(0) : P_1(x) \geq \epsilon - (\epsilon - \frac{r^{n-2}\epsilon}{|x|^{n-2}}) > 0$; on $\partial B^+_r(0) \cap \partial B_r(0)$: $P_1(x) > w_{\lambda_0}(x) \geq 0$. If there exists some minimum point $\bar{x}$ of $P_1(x)$ such that $P_1(\bar{x}) < 0$, we must have $\bar{x} \in \overline{B^+_{r_0}}(0) \setminus \overline{B^+_r}(0) \cap \partial H$ and $\frac{\partial P_1(x)}{\partial t}(\bar{x}) \geq 0$. From $P_1(\bar{x}) < 0$ we have $v(\bar{x}) - v_{\lambda_0}(\bar{x}) - \varphi_1(\bar{x}) < 0$, and then

$$v(\bar{x}) < C_2 < +\infty \tag{13}$$

for some constant C_2 depending only on C_1. Again by $P_1(\bar{x}) < 0$,

$$w_{\lambda_0}(\bar{x}) < \frac{\epsilon\mu}{2} - \frac{r^{n-2}\epsilon}{|\bar{x}|^{n-2}} < \frac{\epsilon\mu}{2}. \tag{14}$$

By (11), (13) and the Mean Value Theorem we have $|\bar{x}|^\alpha v^q(\bar{x}) - |\bar{x}^\lambda|^\alpha v^q_{\lambda_0}(\bar{x}) \leq C_3 w_{\lambda_0}(\bar{x})$ for some positive constant C_3 depending only on C_1, C_2 and λ_0. Thus from $\frac{\partial P_1}{\partial t}(\bar{x}) \geq 0$ we have

$$w_{\lambda_0}(\bar{x}) \geq \frac{\epsilon}{2C_3} \cdot (1-\mu). \tag{15}$$

Combining (14)and (15) we have

$$\frac{\epsilon\mu}{2} > \frac{\epsilon}{2C_3} \cdot (1-\mu);$$

i.e., $\mu > \frac{1}{1+C_3}$. If μ is chosen in such a way that $\mu < \frac{1}{1+C_3}$ from the beginning, we can then deduce that $P_1(x) \geq 0$ in $\overline{B^+_{r_0}}(0) \setminus \overline{B^+_r}(0)$. Let $r \to 0$; we thus establish Lemma 2.6 by choosing $\gamma = \frac{\epsilon\mu}{2}$ for some $\mu < 1/(1+C_3)$.

Proof of Theorem 2.1. By Proposition 2.5 and also by moving planes from the positive direction of x_1, we see that $v(x', t)$ is symmetric with respect to x_1. Clearly the above argument also applies to any direction perpendicular to t-axis, therefore we conclude that $v(x', t)$ is symmetric with respect to x'. It follows that $u(x', t)$ is also symmetric with respect to x' due to the inverse Kelvin transformation. Since we can choose the origin arbitrarily on the hyperplane $t = 0$, it is easy to see that $u(x', t)$ is independent of x'. This reduces (2) to solving an ordinary differential equation, and the proof of Theorem 2.1 is now complete.

2.2. Case $1 < q \leq n/(n-2), n \geq 3$ or $q > 1, n = 2$.

Theorem 2.7. *If* $u(x) \in C^2(H) \cap C^1(\bar{H})$ *solves* (2), $1 < q \le n/(n-2)$ *and* $n \ge 3$, *or* $q > 1$ *and* $n = 2$, *then* $u = at + b$ *for some positive constants* a, b *with* $a = b^q$.

Proof. It suffices to show that $u(x)$ is independent of x'. For fixed $q > 1$ and $n \ge 2$, we choose a positive integer m so large that $q > \frac{n+m}{n+m-2}$. Set

$$w(x_1, x_2, \dots, x_{n-1}, x_{n+1}, \dots, x_{n+m}, t) = u(x_1, \dots, x_{n-1}, t),$$

where $x_i \in \mathbb{R}$ for $i = 1, \dots, n-1, n+1, \dots, n+m$, $t \ge 0$. Then w satisfies

$$\begin{cases} \Delta w = 0, \quad w \ge 0, \quad \text{in} \quad \mathbb{R}^{n+m}_+, \\ \dfrac{\partial w}{\partial t} = w^q \quad \text{on} \quad \partial\mathbb{R}^{n+m}_+. \end{cases}$$

By the choice of m we see that q is a supercritical power in this new setting. Then Theorem 2.1 applies and we know that w is independent of x_i for $i = 1, \dots, n-1, n+1, \dots, n+m$. Thus u is independent of x_i for $i = 1, \dots, n-1$.

3. Proof of Theorem 1.2. In this section we focus on establishing the following result.

Theorem 3.1. *Suppose that* $n \ge 3$. *If* $u(x) \in C^2(H) \cap C^1(\bar{H})$ *solves* (3) *for* $q > n/(n-2)$ *and* $p > (n+2)/(n-2)$, *then* $u \equiv 0$.

Remark 3.2. Theorem 1.2 follows from Theorem 3.1 and some argument similar to the proof of Theorem 2.7.

Remark 3.3. When $p = (n+2)/(n-2)$ and $q = n/(n-2)$, Theorem 1.2 has already been established in [4].

As usual, we set

$$v(x) = \frac{1}{|x|^{n-2}} u\Big(\frac{x}{|x|^2}\Big);$$

then $v(x)$ satisfies

$$\begin{cases} \Delta v = |x|^\tau v^p, \quad v(x) > 0 \quad \text{in} \quad H, \\ \dfrac{\partial v}{\partial t} = |x|^\beta v^q \quad \text{on} \quad \partial H \setminus \{0\}, \end{cases} \tag{16}$$

where $\tau = p(n-2) - (n+2) > 0$, $\beta = q(n-2) - n > 0$. We need the following lemma to take care of the possible singular point.

Lemma 3.4. *Let $v \in C^2(H) \cap C^1(\bar{H}) \setminus \{0\}$ satisfy* (16). *Then for all $0 < \epsilon < \min\{1, \min_{\partial B^+_{1/2} \cap \partial B_{1/2}} v\}$, we have $v(x) > \frac{\epsilon}{2}$ for all $x \in \overline{B^+_{1/2}} \setminus \{0\}$.*

Proof. Set

$$\varphi_2(x) = \frac{\epsilon}{2} - \frac{r^{n-2}\epsilon}{|x|^{n-2}} + \frac{\epsilon t}{2} + \frac{\epsilon t^2}{2}, \quad x \in B^+_{1/2} \setminus B^+_r$$

and $P_2(x) = v(x) - \varphi_2(x)$. Then we have

$$\begin{cases} \Delta P_2 = |x|^\tau v^p - \epsilon \quad \text{in } \ B^+_{1/2} \setminus B^+_r, \\ \dfrac{\partial P_2}{\partial t} = |x|^\beta v^q - \dfrac{\epsilon}{2} \quad \text{on } \ \partial(\overline{B^+_{1/2}} \setminus \overline{B^+_r}) \cap \partial \mathbb{R}^n_+. \end{cases} \tag{17}$$

We want to show that

$$P_2 \geq 0 \quad \text{in } \ \overline{B^+_{1/2}} \setminus \overline{B^+_r}. \tag{18}$$

Let

$$S = \{x \ : \ |x|^\tau v^p - \epsilon > 0\} \cap (B_{1/2}(0) \setminus B_r(0)),$$
$$S^c = \{x \ : \ |x|^\tau v^p - \epsilon \leq 0\} \cap (B_{1/2}(0) \setminus B_r(0)).$$

In S, since $1 \geq |x| > 0$, we have $v^p(x) > \epsilon/|x|^\tau > \epsilon$. It follows that $v(x) > \epsilon \geq \varphi_2$ for $x \in S$; i.e., $P_2 \geq 0$ in S; in S^c, $\Delta P_2 \leq 0$. Therefore, arguing similarly as in the proof of Lemma 2.2, we can show that $P_2 \geq 0$ in S^c. This thus establishes (18). Let $r \to 0$; we obtain Lemma 3.4.

For $\lambda < 0$, we recall that Σ_λ, T_λ, x^λ, v_λ and w_λ are defined as in Section 2. It is easy to check that $w_\lambda(x)$ satisfies

$$\begin{cases} \Delta w_\lambda \leq c_2(x) w_\lambda \quad \text{in } \ \Sigma_\lambda, \\ \dfrac{\partial w_\lambda}{\partial t} \leq c_3(x) w_\lambda \ \text{ on } \ \partial H \cap \tilde{\Sigma}_\lambda, \end{cases} \tag{19}$$

where $c_2(x) = p|x|^\tau \cdot \xi_2^{p-1}(x)$, $c_3(x) = q|x|^\beta \cdot \xi_3^{q-1}(x)$, ξ_2 and ξ_3 are two functions between v_λ and v. In the following we again apply the moving-plane method.

Proposition 3.5. *If $|\lambda|$ is large enough, then $w_\lambda(x) \geq 0$ for all $x \in \tilde{\Sigma}_\lambda$.*

Proof. Suppose that $w_\lambda(x)$ is negative somewhere in $\tilde{\Sigma}_\lambda$. Note that

$$w_\lambda(x) \to 0 \ \text{ as } \ x \to \infty,$$

thus by Lemma 3.4 we know for some $|\lambda|$ large enough, there exists $\bar{x}_\lambda$ such that $w_\lambda(\bar{x}_\lambda) = \min_{x\in\tilde{\Sigma}_\lambda} w_\lambda(x) < 0$. Since $c_2(x)$ is nonnegative, by the Strong Maximum Principle we know that $\bar{x}_\lambda \in \partial H \cap \tilde{\Sigma}_\lambda$, but this contradicts the boundary condition in (19). This finishes the proof.

Then we can define

$$\lambda_0 = \sup\{\lambda < 0 : w_\mu(x) \geq 0 \text{ in } \tilde{\Sigma}_\mu \ \text{ for all } -\infty < \mu < \lambda\}.$$

Proposition 3.6. $\lambda_0 = 0$.

Proof. The proof is similar to that of Proposition 2.5. It suffices to show that if $\lambda_0 < 0$, then

$$w_{\lambda_0} \equiv 0. \tag{20}$$

Suppose that (20) fails, then by the Strong Maximum Principle we have $w_{\lambda_0} > 0$ in $\tilde{\Sigma} \setminus T_{\lambda_0}$. The following lemma is needed to treat the possible singular point, and we postpone its proof till the end.

Lemma 3.7. *For $r_0 < \min\{\frac{1}{2}|\lambda_0|, 1/2\}$, there exists some positive constant γ depending only on λ_0 and r_0 such that $w_{\lambda_0}(x) > \gamma$ in $B^+_{r_0}(0)$.*

We now continue the proof of Proposition 3.6. By the definition of λ_0, there exists a sequence $\lambda_k \to \lambda_0$ with $\lambda_k > \lambda_0$ such that $\inf_{\tilde{\Sigma}_{\lambda_k}} w_{\lambda_k} < 0$. From Lemma 3.7 and the continuity of $v(x)$ away from the origin it follows that for k large enough, there exists $x_k = (x'_k, t_k) \in \tilde{\Sigma}_{\lambda_k} \setminus \overline{B^+_{r_0}}$ such that

$$w_{\lambda_k}(x_k) = \min_{\tilde{\Sigma}_{\lambda_k}} w_{\lambda_k} < 0. \tag{21}$$

Clearly $|x_k| > r_0$ and $t_k > 0$ as from Lemma 3.7 and the boundary condition. But (21) contradicts the Strong Maximum principle since $c_2(x) \geq 0$ in (19). This finishes the proof of Proposition 3.6.

Proof of Lemma 3.7. Set

$$\varphi_3(x) = \frac{\epsilon\mu}{2} - \frac{r^{n-2}\epsilon}{|x|^{n-2}} + \frac{\epsilon t(1-\mu)}{2} + \frac{\epsilon t^2(1-\mu)}{2} \quad \text{in} \quad B^+_{r_0}(0) \setminus B^+_r(0),$$

where ϵ is some small positive constant such that $\min_{\partial B^+_{r_0}(0)} w_{\lambda_0} \geq \epsilon$. $\mu < 1$ will be chosen later in the proof. Let $P_3(x) = w_{\lambda_0}(x) - \varphi_3(x)$; then $P_3(x)$ satisfies the following equation.

$$\begin{cases} \Delta P_3(x) = |x|^\tau v^p - |x^{\lambda_0}|^\tau v^p_{\lambda_0} - \epsilon(1-\mu) \ \text{ in } \ B^+_{r_0}(0) \setminus B^+_r(0) \\ \dfrac{\partial P_3(x)}{\partial t} = |x|^\tau v^q - |x^{\lambda_0}|^\tau v^q_{\lambda_0} - \dfrac{\epsilon(1-\mu)}{2} \ \text{ on } \ \overline{B^+_{r_0}}(0) \setminus \overline{B^+_r}(0) \cap \partial H. \end{cases} \tag{22}$$

On $\partial B^+_{r_0}(0) \cap \partial B_{r_0}(0)$ and $\partial B^+_r(0) \cap \partial B_r(0)$, we know $P_3(x) \geq 0$. Let $\bar{x}$ be one of the minimum points of $P_3(x)$. If $P_3(\bar{x}) < 0$, arguing similarly as in the proof of Lemma 2.6 we know that there exists some $\mu_1 = \mu_1(\lambda_0) > 0$ such that $\bar{x}$ can not be on ∂H for all $\mu \leq \mu_1$. Hence for $\mu \leq \mu_1$, if $P_3(\bar{x}) < 0$, then $\Delta P_3(\bar{x}) \geq 0$. Using the same method as in the proof of Lemma 2.6, we can derive a contradiction if $\mu \leq \mu_2$ for some $\mu_2 = \mu_2(\lambda_0) > 0$. Set $\mu = min\{\mu_1, \mu_2\}$; then we know that $P_3(x) \geq 0$ in $\overline{B^+_{r_0}}(0) \setminus \overline{B^+_r}(0)$. Let $r \to 0$; we thus obtain Lemma 3.7 with $\gamma = \epsilon\mu/2$.

Proof of Theorem 3.1. Arguing similarly as in the proof of Theorem 2.1, we can show that $u(x)$ depends only on t, and thus (3) becomes

$$\begin{cases} u'' = u^p, \quad u > 0 \ \text{ for } \ t > 0 \\ u'(0) = u^q(0). \end{cases} \tag{23}$$

Since $p > 1$, it is easy to check that (23) has no global solution (see [4] for details).

4. Miscellaneous results.

4.1. Sign-changing solutions. Brezis ([1]) proved that $u = 0$ is the only solution to the following entire space problem.

$$\begin{cases} -\Delta u + |u|^{p-1}u = 0 \quad \text{ in } \ \mathbb{R}^n, \\ u \in L^p_{loc}(\mathbb{R}^n), \quad p > 1. \end{cases}$$

Thus one may wonder whether for $q > 1$, all solutions to the following equation

$$\begin{cases} \Delta u = 0 \ \text{ in } \ H, \\ \dfrac{\partial u}{\partial t} = |u|^{q-1}u \ \text{ on } \ \partial H \end{cases} \tag{24}$$

depend only on t. It turns out that this is not always the case as the following example shows.

Example 4.1. $u(x) = x_1^3 t - x_1 t^3 + x_1$ solves (24) with $q = 3$.

In view of this example, it seems that the set of sign-changing solutions to (24) is more complicated than that of nonnegative solutions.

4.2. Some other results. Consider the following equation:

$$\begin{cases} \Delta u - u^p = 0, \quad u \geq 0 \ \text{ in } \ \mathbb{R}^n, \\ u \in C^2(\mathbb{R}^n), \quad 1 < p < +\infty. \end{cases} \tag{25}$$

The result of Brezis in [1] implies the following theorem.

Theorem 4.2. *If $u(x)$ solves* (25), *then* $u \equiv 0$.

We would like to present a different proof of Theorem 4.2 here. This does not mean that our method is simpler or more flexible (actually, it seems that the moving-plane method does not apply to sign-changing solutions). The point which we want to make is that the sign of the coefficient for the nonlinear term in (25) is a "good" sign, which allows us to apply the moving-plane method for the supercritical case.

The proof of Theorem 4.2 is similar to that of Theorem 1.2. It suffices to prove it for the case when $p > (n+2)/(n-2)$, $n \geq 3$, since all the other cases can be treated similarly as in the proof of Theorem 2.7. As before we let

$$v(x) = \frac{1}{|x|^{n-2}} u\Big(\frac{x}{|x|^2}\Big);$$

then $v(x)$ satisfies

$$\Delta v - |x|^{\delta} v^p = 0, \quad v(x) > 0 \quad \text{in } \ \mathbb{R}^n \setminus \{0\} \tag{26}$$

where $\delta = p(n-2) - (n+2) > 0$. We set out to show that $v(x)$ is symmetric with respect to the origin. The following result will be needed to take care of the singularity.

Lemma 4.3. *Assume that $v(x)$ satisfies* (26). *If for some $0 < \epsilon < 1$, $v(x) \geq \epsilon$ on $\partial B_1(0)$, then $v(x) \geq \epsilon/2$ in $B_1(0) \setminus \{0\}$.*

Proof. For small r, Set

$$\varphi_4(x) = \frac{\epsilon}{2} - \frac{r^{n-2}\epsilon}{|x|^{n-2}} + \frac{t^2\epsilon}{2}, \quad x \in B_1(0) \setminus B_r(0).$$

Then $\Delta\phi = \epsilon$, and $\Delta(u - \phi) = |x|^{\delta} u^p - \epsilon$. Let

$$S = \{x \ : \ |x|^{\delta} u^p - \epsilon > 0\} \cap (B_1(0) \setminus B_r(0)),$$
$$S^c = \{x \ : \ |x|^{\delta} u^p - \epsilon \leq 0\} \cap (B_1(0) \setminus B_r(0)).$$

Since $|x| \leq 1$ in S, we have $u^p > \epsilon/|x|^{\delta} > \epsilon$. It follows that $u(x) > \epsilon \geq \varphi_4$ for $x \in S$; in S^c, $\Delta(u - \varphi_4) \leq 0$. It is easy to check the following facts: On $\{x : |x|^{\delta} u^p - \epsilon = 0\}$ and ∂B_1, $u \geq \varphi_4$; on ∂B_r, $u > \varphi_4$. By the Maximum Principle, we know that $u \geq \varphi_4$ in S^c. Thus we always have $u \geq \varphi_4$ in $B_1 \setminus B_r$. Let $r \to 0$; we obtain $u \geq \frac{\epsilon}{2}$. This proves Lemma 4.3.

Once we establish Lemma 4.3, we can apply the moving-plane method to prove Theorem 4.2. We refer readers to [3], [5] for details about such a process.

In our proofs of Theorems 1.1 and 1.2, the assumption $p, q > 1$ plays an essential role in the adding dimension argument. In the following we present a simple example to show that Theorem 4.2 breaks down for $p = 1$. The same example also serves as a counterexample to Theorem 1.2 for $p = q = 1$.

Example 4.4. $u(x) = e^t$ solves (25) with $p = 1$.

Moreover, there exist positive solutions to (25) which depend on more than one variable.

Example 4.5. $u(x) = e^{(x_1+x_2)/\sqrt{2}}$ solves (25) with $p = 1$.

Thus it seems interesting to ask whether Theorem 1.1 still holds for $q = 1$.

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