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SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS WITH ONE ISOLATED SINGULARITY

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Abstract. If f is either given by $(1+u)^p$ for some $\frac{N+2}{N-2} < p < \frac{N+1}{N-3}$, $N \geq 3$ or if f is given by e^u when $N = 3$, we prove the existence of a positive weak solution of $\Delta u + \lambda f(u) = 0$ which is defined in the unit ball of $\mathbb{R}^N$, has 0 boundary data and has a nonremovable prescribed singularity at some point x_0 close to the origin.

1. Introduction. The structure of the set of solutions of the semilinear elliptic equation $\Delta u + \lambda e^u = 0$ in a domain of a 3-dimensional space, has been first studied by V.R. Emden in 1897 [12]. He was interested by physical applications, indeed, this simple equation is a model for the study of the density ρ of an isothermal gas sphere in gravitational equilibrium. The solution u and the density ρ are related by $\rho = e^{-u}$ and the positive constant λ is given by

$$\lambda = \frac{4\pi G\mu H}{kT}, \tag{1}$$

where G is Newton's constant, μ is the molecular weight of the gas, H is the mass of the proton, k is the Boltzmann constant and T is the absolute temperature. S. Chandrasekhar [9] used Emden's results to study the stellar structure and found radial solutions to the equation $\Delta u + \lambda e^u = 0$ which are singular at the origin. When the dimension of the space is greater than or equal to 3, very few results are known concerning either the existence or the regularity of weak solutions of

$$\Delta u + \lambda e^u = 0, \tag{2}$$

when the parameter $\lambda \geq 0$. In 1988, F. Mignot and J.P. Puel (cf. [19]) proved the existence and the uniqueness of a radial singular solution u of

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the problem $\Delta u + \lambda e^u = 0$ in $\mathbb{R}^N$ for a fixed positive constant λ. These results allowed them to prove the existence and the uniqueness of a radial singular solution (λ, u) of (2) in any ball of $\mathbb{R}^N$, with 0 boundary Dirichlet conditions, when $N \geq 3$. If the solutions are defined on a bounded set with 0 boundary data, T. Gallouet, F. Mignot and J.P. Puel [13] proved that this problem has no singular solution if the parameter λ is greater than some constant λ^*. Moreover, they proved some compactness result for the set of regular solutions. M.F. Bidaut-Véron and L. Véron [5] studied the asymptotic behaviour of a solution of (2) near an isolated singularity. In addition, they proved the existence of non radial singular solutions of (2) in $\mathbb{R}^3 \setminus \{0\}$. When the dimension of the space is greater than 10, F. Pacard [21] proved the existence of solutions to the problem (2) defined in a connex open set Ω of $\mathbb{R}^N$, with many isolated singularities.

Motivated by the above studies, this paper is devoted to the study of the existence of singular positive solutions of

$$\Delta u + \lambda f(u) = 0, \tag{3}$$

where $\lambda \geq 0$, $N \geq 3$ is the dimension of the space we work on and the nonlinear function $f(u)$ is defined by

$$f(u) = \begin{cases} (1+u)^p & \text{if} \quad N \geq 4 \quad \text{or} \\ e^u & \text{if} \quad N = 3. \end{cases}$$

The study of isolated singularities of solutions of $\Delta u + f(u) = 0$ has been the subject of a lot of work. Progresses have been done in two complementary directions: in one direction, some results are concerned with the existence of solutions with isolated singularities [23], [22], [16], [11] and, in another direction, many results are concerned with the asymptotic behaviour of the solutions near an isolated singularity [2], [6], [25], [7], [14].

The study of the asymptotic behaviour of solutions of $\Delta u + u^p = 0$ near an isolated singularity is much more difficult. In addition, the difficulty in this study seems to increase with the exponent of the nonlinearity. Such a study was achieved by P. Aviles when $p = N/(N-2)$ [2], by Caffarelli, B. Gidas and J. Spruck when $p \in (N/(N-2), (N+2)/(N-2))$ [7] and by B. Gidas and J. Spruck in the special case $p = (N+2)/(N-2)$ [14]. Later P. Aviles, N. Korevaar and R. Schoen (cf. [3]) recovered the result of B. Gidas and J. Spruck.

Concerning the existence of weak solutions which are singular at finitely many points, the case where $p = N/(N-2)$ was considered by F. Pacard in

[23]. Later, R. Mazzeo and F. Pacard, in their paper [16], proved in particular, the existence of positive weak solutions for the equation $\Delta u + u^p = 0$, they were able to prove the existence of solutions vanishing at the boundary and having prescribed isolated singularities when the exponent p is lying in the interval $(\frac{N}{N-2}, \frac{N+2}{N-2})$. Nearly at the same time, C.C. Chen and C.S. Lin proved the existence of positive weak solutions of $\Delta u + u^p = 0$ in their paper [11] when the exponent p is taken in the interval $(\frac{N}{N-2}, \frac{N+2\sqrt{N-1}}{N-4+2\sqrt{N-1}})$ and Ω is a smooth open set in $\mathbb{R}^N$ with $N \geq 3$. The main difference between the choice of the range in which p is allowed to vary, is due to the method used by the different authors. C.C. Chen and C.S. Lin have used a variational method, so their results depended on the stability of the singular solution, while in the paper of R. Mazzeo and F. Pacard, the proof relies on the study of the linearized operator near a singular solution, in weighted Hölder spaces.

In this paper, we are interested in the supercritical case; namely, when the nonlinearity involved in the equation is stronger than $u \longmapsto u^{\frac{N+2}{N-2}}$. The results here are much more difficult to obtain, or, to be more precise it does not seem possible to get results as general as the one obtained in the subcritical case.

The main purpose of this paper is to give proof of a result of H. Matano, which to our knowledge has already been proved by a different method but which has never been published. The problem is to prove the existence of solution for the problem $\Delta u + \lambda e^u = 0$ in the unit ball of a three dimensional space, with 0 boundary data and with an isolated nonremovable singularity at a given point close to the origin.

2. Statement of the results. The proof of the different results relies on the same analytical tool: the surjectivity of the linearized operator when defined between suitably chosen spaces.

Let us briefly recall what the radial singular solutions are. If $p > \frac{N+2}{N-2}$, there exists only one radial singular solution of (3) which is equal to 0 on the boundary of the unit ball. This solution is given by $u_0(x) = |x|^{-\frac{2}{p-1}} - 1$. The corresponding value of λ is $\lambda_0 = \frac{2}{p-1}(N - \frac{2p}{p-1})$. Notice that, since $p > \frac{N}{N-2}$, we have $\lambda_0 > 0$. In the special case $N = 3$, equation (2) has only one radial singular solution whose boundary data on the unit sphere is 0. This solution is given by $u_0(x) = -2\ \ln|x|$. In this case, we have $\lambda_0 = 2$.

Our first result concerns the existence of solutions of (3) with non constant boundary data, which are singular at the origin. More precisely we have :

Theorem 1. *Given p satisfying $\frac{N+2}{N-2} < p \le \frac{N+1}{N-3}$, there exists $\epsilon > 0$ such that, for all $\psi \in \mathcal{C}^{2,\alpha}(\partial B)$, if $||\psi||_{2,\alpha} \le \epsilon$, then the problem*

$$\begin{cases} \Delta u + \lambda(1+u)^p = 0 & \text{in} \quad B \setminus \{0\} \\ u = \psi & \text{on} \quad \partial B \end{cases} \tag{4}$$

has a solution (λ, u) where u has a nonremovable singularity at 0. A similar result holds in dimension 3, if the nonlinearity is replaced by λe^u.

A related result is proved in [25] using completely different techniques. Our second result is concerned with the existence of solutions which are singular at a given point close enough to the origin.

Theorem 2. *Given p satisfying $\frac{N+2}{N-2} < p \le \frac{N+1}{N-3}$, there exists $\epsilon \in (0,1)$ such that, for all $x_0 \in B_\epsilon$, there exists a solution (λ, u) of*

$$\begin{cases} \Delta u + \lambda(1+u)^p = 0 & \text{in} \quad B \setminus \{x_0\} \\ u = 0 & \text{on} \quad \partial B. \end{cases} \tag{5}$$

A similar result holds in dimension 3, if the nonlinearity is replaced by λe^u.

In the case where $N = 3$ and where the nonlinearity is λe^u, the result was announced by H. Matano. As far as we know, H. Matano's proof relies on infinite dimensional dynamical system technics. Our proof is somehow different and relies on purely elliptic techniques.

These two results contrast with the subcritical case. Indeed, when $p \in [N/(N-2), (N+2)/(N-2))$ the following theorem holds:

Theorem 3. [16] *Given p satisfying $\frac{N}{N-2} \le p < \frac{N+2}{N-2}$, for any $x_0 \in B_1$, there exists a one parameter family of solutions (λ, u) of*

$$\begin{cases} \Delta u + \lambda(1+u)^p = 0 & \text{in} \quad B \setminus \{x_0\} \\ u = 0 & \text{on} \quad \partial B \end{cases} \tag{6}$$

with a nonremovable singularity in x_0.

As a corollary of Theorem 1 and 2, we obtain:

Corollary 1. *There exists $\epsilon > 0$ such that, for all $\psi \in \mathcal{C}^{2,\alpha}(\partial B)$, if $||\psi||_{2,\alpha} \le \epsilon$, the set of solutions of (4) with one isolated singularity contains an N-dimensional manifold.*

If $p \in [N/(N-2), (N+2)/(N-2))$, then the result in [16] states that there exists $\epsilon > 0$ such that, for all $\psi \in \mathcal{C}^{2,\alpha}(\partial B)$, if $||\psi||_{2,\alpha} \le \epsilon$, the set of solutions of (4) with one isolated singularity contains a $(N+1)$-dimensional manifold.

One may wonder what happens when $p > \frac{N+1}{N-3}$. We have the result.

Theorem 4. *Given p satisfying $p > \frac{N+1}{N-3}$, there exists $\epsilon \in (0,1)$ such that, for all $\psi \in \mathcal{C}^{2,\alpha}(\partial B)$, if $||\psi||_{2,\alpha} \leq \epsilon$, there exist $x_\psi \in B$ and (λ, u) a solution of*

$$\begin{cases} \Delta u + \lambda(1+u)^p = 0 & in \quad B \setminus \{x_\psi\} \\ u = \psi & on \quad \partial B \end{cases} \tag{7}$$

with a nonremovable singularity at x_ψ.

This means that we may slightly perturb the boundary data provided we modify accordingly the singularity.

The proof of these results relies on the application of the implicit function theorem in a suitably defined space of functions. These spaces are weighted Hölder spaces which seem to provide the right framework for these kind of results. They were introduced in the context of nonlinear elliptic problems by L. Caffarelli, R. Hardt and L. Simon [8]. Later on these spaces were extensively used by R. Mazzeo and N. Smale [18], N. Smale [24], R. Mazzeo and F. Pacard [16].

One additional tool is needed to treat the limit cases $f(u) = e^u$ (in dimension 3) and $f(u) = u^{\frac{N+1}{N-3}}$ in higher dimension, this is the notion of deficiency space which was introduced by R. Mazzeo, D. Pollack and K. Uhlenbeck in [17].

Remark 1. In the remaining, we denote by $(r, \theta) \in \mathbb{R}_+^* \times S^{N-1}$, the spherical coordinates in $\mathbb{R}^N \setminus \{0\}$.

3. Preliminary computation. Let (λ_0, u_0) be the singular radial solution of (3) in the unit ball, whose boundary data is 0. We define the linear operator $\mathcal{L} : w \longmapsto \Delta w + \lambda_0 f'(u_0) w$. Clearly we have

$$\lambda_0 f'(u_0) = \frac{c_0}{r^2}, \tag{8}$$

where c_0 is a positive constant. Precisely,

$$c_0 = \begin{cases} \frac{2p}{p-1}(N - \frac{2p}{p-1}) & \text{if} \quad f(u) = (1+u)^p \\ 2 & \text{if} \quad f(u) = e^u. \end{cases}$$

Remark 2. The eigenvalues of the Laplacian on S^{N-1} are denoted by $\mu_0 = 0$, $\mu_1 = \mu_2 = \ldots = \mu_N = (N-1)$, $\mu_{N+1} = 2N, \ldots$ (cf. M. Berger,

P. Gauduchon and E. Mazet [4]) and we denote by $\varphi_j(\theta)$ the eigenfunction corresponding to μ_j which is normalized in such a way that

$$\int_{S^{N-1}} \varphi_j^2(\theta)d\theta = 1.$$

We define the indicial roots of $\mathcal{L}$ by

$$\gamma_j^\pm = \frac{2-N}{2} \pm \sqrt{\left(\tfrac{N-2}{2}\right)^2 + \mu_j - c_0}, \qquad j \geq 0. \tag{9}$$

Finally, for all $N \geq 3$, we define

$$p^* = \begin{cases} +\infty & \text{if} \quad N \leq 10 \\ \frac{N-2\sqrt{N-1}}{N-4-2\sqrt{N-1}} & \text{if} \quad N > 10. \end{cases}$$

We deduce the proposition from simple computations

Proposition 1. *The following inequalities hold:*

- *If* $\frac{N+2}{N-2} < p < p^*$, *then* $\Re(\gamma_0^\pm) = \frac{2-N}{2}$, $\Im(\gamma_0^\pm) \neq 0$.
- *If* $p > p^*$, *then* $\gamma_0^- < \frac{2-N}{2} < \gamma_0^+$.
- *If* $\frac{N+2}{N-2} < p < \frac{N+1}{N-3}$, *then* $\Re(\gamma_0^\pm) < -\frac{2}{p-1} < \gamma_1^+$.
- *Finally, if* $p > \frac{N+1}{N-3}$, *then* $\Re(\gamma_0^\pm) < \gamma_1^+ < -\frac{2}{p-1} < \gamma_{N+1}^+$.

Proof. Let us first notice that if we search the values of p for which $\gamma_0^\pm$ is real, we get that p must satisfy $p \geq p_1$ or $p \leq p_2$ where

$$p_1 = \frac{N-2\sqrt{N-1}}{N-4-2\sqrt{N-1}} \qquad \text{and} \qquad p_2 = \frac{N+2\sqrt{N-1}}{N-4+2\sqrt{N-1}}.$$

We remark that $p_1 < \frac{N}{N-2} < p_2 < \frac{N+2}{N-2}$ if $N < 10$ and $\frac{N}{N-2} < p_2 < \frac{N+2}{N-2} < \frac{N+1}{N-3} < p_1$ if $N > 10$. Then, it is immediate to conclude that if $\frac{N+2}{N-2} < p < p^*$, where p^* is defined in the statement of the result, then $\Re(\gamma_0^\pm) = \frac{2-N}{2}$ and $\Im(\gamma_0^\pm) \neq 0$ and if $p > p_1$, then we have $\gamma_0^- < \frac{2-N}{2} < \gamma_0^+$.

We also notice that for $p \geq \frac{N+2}{N-2}$, we have

$$\gamma_1^+ = \frac{2-N}{2} + \sqrt{\left(\tfrac{N}{2} - \tfrac{2p}{p-1}\right)^2} \quad \text{and} \quad \gamma_{N+1}^+ = \tfrac{2-N}{2} + \sqrt{\left(\tfrac{N}{2} - \tfrac{p+1}{p-1}\right)^2 + \tfrac{4p}{p-1}}.$$

It is easy to check that if $\frac{N+2}{N-2} < p < \frac{N+1}{N-3}$ then $\Re(\gamma_0^\pm) < -\frac{2}{p-1} < \gamma_1^+$ and if $p > \frac{N+1}{N-3}$ we have $\Re(\gamma_0^\pm) < \gamma_1^+ < -\frac{2}{p-1} < \gamma_{N+1}^+$. We also have if $p = \frac{N+1}{N-3}$ then $\gamma_1^+ = \frac{-2}{p-1}$ which ends the proof of the proposition. □

In the case $f(u) = e^u$ and $N = 3$, the previous proposition becomes

Proposition 2. *If $f(u) = e^u$ and $N = 3$, then*

(i) $\mathcal{R}e(\gamma_0^\pm) = \frac{2-N}{2}$, $\mathcal{I}m(\gamma_0^\pm) \neq 0$;

(ii) $\gamma_1^+ = 0$.

4. A right inverse for $\mathcal{L}$. For any $k \geq 0$, $\alpha \in (0,1)$ and $\nu \in \mathbb{R}$, we define some weighted Hölder spaces $\mathcal{C}_\nu^{k,\alpha}$ as follows

$$\mathcal{C}_\nu^{k,\alpha} = \left\{u \in \mathcal{C}_{loc}^{k,\alpha}(B \setminus \{0\}) \;/\; \| u \|_{\mathcal{C}_\nu^{k,\alpha}} = \sup_{r \leq \frac{1}{2}} \left(r^{-\nu} |u|_{k,\alpha,[r,2r]}\right) < +\infty\right\},$$

where, by definition

$$|u|_{k,\alpha,[r,2r]} = \sup_{r \leq |x| \leq 2r} \Big(\sum_{j=0}^{k} r^j |\nabla^j u|\Big) + r^{k+\alpha} \sup_{r \leq |x|,|y| \leq 2r; x \neq y} \frac{|\nabla^k u(y) - \nabla^k u(x)|}{|y-x|^\alpha}.$$

In addition, for all $j \geq 0$, we define

$$\mathcal{C}_{\nu,j}^{2,\alpha} = \{v \in \mathcal{C}_\nu^{2,\alpha} / v_{|\partial B} \in \operatorname{span}\{\varphi_0(\theta), \dots, \varphi_j(\theta)\}\}. \tag{10}$$

It follows from (8) that the linear operator $\mathcal{L}$ is well defined from $\mathcal{C}_\nu^{2,\alpha}$ into $\mathcal{C}_{\nu-2}^{0,\alpha}$. Here and in the sequel we assume that $\nu \geq \frac{2-N}{2}$. We describe in this section a rather general procedure to construct a right inverse for $\mathcal{L}$.

Proposition 3. *Assume that, for some $j \geq 1$ we have $\gamma_{j-1}^+ < \nu < \gamma_j^+$. Then, for all $g \in \mathcal{C}_{\nu-2}^{0,\alpha}$ there exists a unique $w \in \mathcal{C}_{\nu,j-1}^{2,\alpha}$ solution of $\mathcal{L}w = g$ in $B \setminus \{0\}$. In addition the mapping $g \in \mathcal{C}_{\nu-2}^{0,\alpha} \longrightarrow w \in \mathcal{C}_{\nu,j-1}^{2,\alpha}$ is bounded.*

Proof. Let us do the eigenfunction decomposition of both w and g. We set

$$w(x) = \sum_{i=0}^{+\infty} w_i(r)\varphi_i(\theta) \quad \text{and} \quad g(x) = \sum_{i=0}^{+\infty} g_i(r)\varphi_i(\theta).$$

Assuming that w already solves

$$\begin{cases} \mathcal{L}w = g \quad \text{in} \quad B \setminus \{0\} \\ w(1) \in \operatorname{Span}\{\varphi_0(\theta), \dots, \varphi_{j-1}(\theta)\} \end{cases}$$

and taking the projection over φ_i we obtain the ordinary differential equation satisfied by w_i

$$\begin{cases} \ddot{w}_i + \dfrac{N-1}{r}\dot{w}_i - \dfrac{\mu_i}{r^2} w_i + \lambda_0 f'(u_0) w_i = g_i \\ \omega_i(1) = 0 \quad \text{for} \quad i \geq j, \end{cases} \tag{11}$$

where g_i is bounded by a constant times $r^{\nu-2}$.

Step 1. We shall solve (11). Using the variation of the constant formula, it is possible to obtain a particular solution of the problem which is bounded by a constant times r^ν near 0.

As we have assumed that $\gamma_{j-1}^+ < \nu < \gamma_j^+$, we get the following explicit formula for all $i \geq j$ (notice that in this case, we always have $\gamma_i^+ \in \mathbb{R}$ since only $\gamma_0^\pm$ may fail to belong to $\mathbb{R}$)

$$w_i(r) = -r^{\gamma_i^+} \int_r^1 \kappa^{1-N-2\gamma_i^+} \int_0^\kappa s^{N-1+\gamma_i^+} g_i(s)\, ds\, d\kappa.$$

When $i < j$, we define

$$\omega_i(r) = \Re\big(r^{\gamma_i^+} \int_0^r \kappa^{1-N-2\gamma_i^+} \int_0^\kappa s^{N-1+\gamma_i^+} g_i(s)\, ds\, d\kappa\big),$$

It is straightforward to check that there exists some constant $c_i > 0$ such that for all $r \in (0,1]$, $|w_i(r)| \leq c_i ||g||_{\mathcal{C}^{0,\alpha}_{\nu-2}}$.

Step 2. We recall that we have assumed that $\nu \in (\gamma_{j-1}^+, \gamma_j^+)$. For $n \geq j$, we consider $\tilde{w}_n(x) = \sum_{i=j}^n w_i(r)\varphi_i(\theta)$ and $\tilde{g}_n(x) = \sum_{i=j}^n g_i(r)\varphi_i(\theta)$. We remark that $\tilde{w}_n$ is a solution of

$$\begin{cases} \mathcal{L}(\tilde{\omega}_n) = \tilde{g}_n & \text{in} \quad B \\ \tilde{\omega}_n = 0 & \text{on} \quad \partial B. \end{cases}$$

It follows from the previous step that

$$\sup_{x \in B} |x|^{-\nu} |\tilde{\omega}_n(x)| \leq C_n ||g||_{\mathcal{C}^{0,\alpha}_{\nu-2}}.$$

We claim that C_n is bounded independently of n. Indeed, if the claim were not true, there would exist a bounded sequence $(\tilde{g}_l)_{l\geq 0}$ in $\mathcal{C}^{0,\alpha}_{\nu-2}$ such that the solution v_l to the problem

$$\begin{cases} \mathcal{L}v_l = \tilde{g}_l & \text{in} \quad B \\ v_l = 0 & \text{on} \quad \partial B, \end{cases}$$

satisfies $\lim_{l\to+\infty}\sup_{x\in B}|v_l(x)||x|^{-\nu} = +\infty$ and also $v_l \in \mathrm{Span}\{r^{\gamma_i^+}\varphi_i(\theta) : i \geq j\}$. For every $l \in \mathbb{N}$, there exists a point $x_l \in B \setminus \{0\}$ such that

$$\frac{1}{2}\sup_{x\in B}(|x|^{-\nu}|v_l(x)|) \leq |x_l|^{-\nu}|v_l(x_l)| \leq \sup_{x\in B}(|x|^{-\nu}|v_l(x)|) \equiv A_l.$$

We define the function $\xi_l(x) = A_l^{-1}|x_l|^{-\nu}v_l(|x_l|x)$. We have $|x|^{-\nu}|\xi_l(x)| \leq 1$ for every $x \in B_{1/|x_l|}$. Moreover, ξ_l solves

$$\begin{cases} \mathcal{L}\xi_l = A_l^{-1}|x_l|^{-\nu+2}\tilde{g}_l(|x_l|x) & \text{in} \quad B_{1/|x_l|} \\ \xi_l = 0 & \text{on} \quad \partial B_{1/|x_l|}, \end{cases}$$

where $A_l^{-1}|x_l|^{-\nu+2}\tilde{g}_l(|x_l|x)$ tends to zero in $\mathcal{C}^{0,\alpha}_{\nu-2}$ when l tends to infinity. Up to a subsequence we may always assume that $(x_l)_{l\geq 0}$ converges in B (thanks to Schauder estimates we may assume that $\lim_{l\to+\infty}|x_l| < 1$). We distinguish two cases according to the value of $\lim_{l\to+\infty} x_l$:

- If $\lim_{l\to+\infty} x_l = 0$, then after further extracting some subsequence we get the existence of ξ, the limit of ξ_l, which satisfies

  $$\mathcal{L}\xi = 0 \quad \text{in} \quad \mathbb{R}^N.$$

  In addition, we also know that $\xi \in \mathrm{Span}\{r^{\gamma_i^+}\varphi_i(\theta) \mid i \geq j\}$ and that $\xi \neq 0$. This never happens since ξ would blow up at ∞ at least as fast as $|x|^{\gamma_j^+}$ and, by construction, $|\xi(x)| \leq |x|^{\nu}$.
- If $\lim_{l\to+\infty} x_l \in B \setminus \{0\}$, then after further extracting some subsequence we get the existence of $\xi(x)$, the limit almost everywhere of $\xi_l(x)$, which satisfies

  $$\begin{cases} \mathcal{L}\xi = 0 & \text{in} \quad B_{R_0} \\ \xi = 0 & \text{on} \quad \partial B_{R_0}, \end{cases}$$

  where $R_0 = \lim_{l\to+\infty} 1/|x_l|$. In addition, we also know that

  $$\xi \in \mathrm{Span}\{r^{\gamma_i^+}\varphi_i(\theta) : i \geq j\}$$

  and that $\xi \neq 0$. This never happens because if we make the expansion of ξ over the eigenspaces of the Laplacian, we see readily that all components of ξ have to be 0.

And the proof of the claim is complete.

Finally, since C_n is bounded independently of n, we have the estimates

$$\sup_{x\in B}|x|^{-\nu}|\tilde{\omega}_n(x)| \leq C \parallel g \parallel_{\mathcal{C}^{0,\alpha}_{\nu-2}(B)} .$$

This estimate allows us to pass to the limit and to get a solution $\tilde{w}$ of

$$\begin{cases} \mathcal{L}(\tilde{w}) = \tilde{g} & \text{in} \quad B \\ \tilde{w} = 0 & \text{on} \quad \partial B, \end{cases}$$

where $\tilde{g}(x) = \sum_{i=j}^{\infty} g_i(r)\varphi_i(\theta)$. Moreover, we have the estimate

$$\sup_{x\in B}|x|^{-\nu}|\tilde{w}(x)| \leq C \parallel g \parallel_{\mathcal{C}^{0,\alpha}_{\nu-2}(B)} . \tag{12}$$

Finally, we can define w, the solution of $\mathcal{L}w = g$ as follows

$$\begin{aligned} w(x) =& \sum_{i=0}^{j-1} \Re\big(r^{\gamma_i^+}\int_0^r \kappa^{1-N-2\gamma_i^+}\int_0^\kappa s^{N-1+\gamma_i^+} g_i(s)dsd\kappa\big)\varphi_i(\theta) \\ &+ \sum_{i=j}^{+\infty}\big(r^{\gamma_i^+}\int_r^1 \kappa^{1-N-2\gamma_i^+}\int_0^\kappa s^{N-1+\gamma_i^+} g_j(s)dsd\kappa\big)\varphi_i(\theta). \end{aligned} \tag{13}$$

We conclude from (12) and (13) that there exists some constant $C > 0$ independent of g such that

$$\sup_{x\in B}|x|^{-\nu}|w(x)| \leq C \parallel g \parallel_{\mathcal{C}^{0,\alpha}_{\nu-2}(B)} .$$

The estimates of the other derivatives are obtained using rescaled Schauder estimates (cf. D. Gilbarg and N.S. Trudinger [15]). □

Let us define

$$\begin{aligned} &\big(\mathcal{C}^{2,\alpha}_\nu \oplus r^{\gamma_1^+}\text{Span}\{\varphi_1(\theta), \dots, \varphi_N(\theta)\}\big)_0 \\ &\quad = \big\{w \in \mathcal{C}^{2,\alpha}_\nu \oplus r^{\gamma_1^+}\text{Span}\{\varphi_1(\theta), \dots, \varphi_N(\theta)\} | w_{|\partial B} \in \text{Span}\{1\}\big\}. \end{aligned} \tag{14}$$

As a Corollary of the previous Proposition, we have

Corollary 2. *Assume that* $\gamma_N^+ < \nu < \gamma_{N+1}^+$. *Then for all* $g \in \mathcal{C}^{0,\alpha}_{\nu-2}$ *there exists a unique solution of* $\mathcal{L}w = g$ *in* $B \setminus \{0\}$ *which belongs to the space* $(\mathcal{C}^{2,\alpha}_\nu \oplus r^{\gamma_1^+}\mathrm{Span}\{\varphi_1(\theta), \dots, \varphi_N(\theta)\})_0$. *In addition, the mapping* $g \in \mathcal{C}^{0,\alpha}_{\nu-2} \longrightarrow w \in (\mathcal{C}^{2,\alpha}_\nu \oplus r^{\gamma_1^+}\mathrm{Span}\{\varphi_1(\theta), \dots, \varphi_N(\theta)\})_0$ *is bounded.*

The same result would hold if the space

$$\left(\mathcal{C}^{2,\alpha}_\nu \oplus r^{\gamma_1^+}\mathrm{Span}\{\varphi_1(\theta), \dots, \varphi_N(\theta)\}\right)_0$$

is replaced by $(\mathcal{C}^{2,\alpha}_\nu \oplus r^{\gamma_1^-}\mathrm{Span}\{\varphi_1(\theta), \dots, \varphi_N(\theta)\})_0$.

5. The proof of Theorem 1.

5.1. The case where $f(u) = (1+u)^p$ and p is less than $\frac{N+1}{N-3}$. It follows from Proposition 1 that in this case it is always possible to choose

$$\Re(\gamma_0^+) \le -\frac{2}{p-1} < \nu < \gamma_1^+.$$

For all $\psi \in \mathcal{C}^{2,\alpha}(\partial B)$ we define $w_\psi(x) = \chi(r)\psi(\theta)$ where χ is some fixed regular function equal to 0 in $B_{1/2}$ and equal to 1 outside $B_{3/4}$.

We are going to find $v \in \mathcal{C}^{2,\alpha}_{\nu,0}$ solution of

$$\Delta(u_0 + v + w_\psi) + \lambda_0 f(u_0 + v + w_\psi) = 0 \quad \text{in} \quad B \setminus \{0\}. \tag{15}$$

To this end, we define, for all $(v, \psi) \in \mathcal{C}^{2,\alpha}_{\nu,0} \times \mathcal{C}^{2,\alpha}(\partial B)$

$$\mathcal{N}(v, \psi) \equiv \Delta(u_0 + v + w_\psi) + \lambda_0 f(u_0 + v + w_\psi).$$

It is easy to see that $\mathcal{N}$ is well defined from $\mathcal{C}^{2,\alpha}_{\nu,0} \times \mathcal{C}^{2,\alpha}(\partial B)$ into $\mathcal{C}^{0,\alpha}_{\nu-2}$. In addition, $\mathcal{N}(0,0) = 0$ and $D\mathcal{N}_{|(0,0)}(v, 0) = \mathcal{L}v$. For $p \in (\frac{N+2}{N-2}, \frac{N+1}{N-3})$ it follows easily from the implicit function theorem and Proposition 3 with $j = 0$ that all solutions of the equation (15) near $(0,0)$ are of the form (v_ψ, ψ) where $\psi \in \mathcal{C}^{2,\alpha} \longrightarrow v_\psi \in \mathcal{C}^{2,\alpha}_{\nu,0}$ is a regular mapping. Now that we have obtained a solution of (15) we explain how to find the solution of our original problem (4). To this aim, let us define

$$c_\psi = \frac{1}{|S^{N-1}|}\int_{S^{N-1}} v_\psi d\theta.$$

It is easy to see that $u_\psi = (1+c_\psi)^{-1}(u_0 + v_\psi + w_\psi - c_\psi)$ is a solution of

$$\begin{cases} \Delta u_\psi + \lambda_\psi f(u_\psi) = 0 & \text{in} \quad B \setminus \{0\} \\ u_\psi = (1+c_\psi)^{-1}\psi & \text{on} \quad \partial B, \end{cases}$$

where by definition $\lambda_\psi = \lambda_0(1+c_\psi)^{p-1}$. The mapping $\psi \longmapsto (1+c_\psi)^{-1}\psi$ is obviously a diffeomorphism from a neighborhood of 0 in $\mathcal{C}^{2,\alpha}(\partial B)$ into a neighborhood of 0 in $\mathcal{C}^{2,\alpha}(\partial B)$. This ends the proof of Theorem 1 in the case where $f(u) = (1+u)^p$ and p is below $\frac{N+1}{N-3}$.

5.2. The limit case $f(u) = (1+u)^{\frac{N+1}{N-3}}$ or $f(u) = e^u$ in dimension 3. In the case where $f(u) = (1+u)^{\frac{N+1}{N-3}}$ in dimension $N \geq 4$ or $f(u) = e^u$ in dimension 3, we have $\gamma_1^+ = \gamma_2^+ = ... = \gamma_N^+ = \frac{3-N}{2}$. The corresponding linearized operator is

$$\mathcal{L} = \Delta + \frac{N^2-1}{4}\frac{1}{r^2}.$$

In this case, we choose ν such that $\gamma_1^+ = \ldots = \gamma_N^+ < \nu < \gamma_{N+1}^+$ and we define the space

$$\mathbb{M} = \mathrm{Span}\{\varphi_1(\theta), \ldots, \varphi_N(\theta)\}. \tag{16}$$

Corollary 2 tells us that for all $g \in \mathcal{C}^{0,\alpha}_{\nu-2}$ the equation $\mathcal{L}w = g$ can always be solved in the space $\mathcal{C}^{2,\alpha}_{\nu,N} \oplus r^{\gamma_1^+}\mathbb{M}$. Unfortunately, the nonlinear operator $\mathcal{N}(v,\psi)$ is not well defined from $\mathcal{C}^{2,\alpha}_{\nu,N} \oplus r^{\gamma_1^+}\mathbb{M} \times \mathcal{C}^{2,\alpha}(\partial B)$ into $\mathcal{C}^{0,\alpha}_{\nu-2}$. To make the resolution of the nonlinear problem possible, we have to substitute to the finite dimensional space $\mathbb{M}$, a space more adapted to the nonlinearities.

The case where $f(u) = (1+u)^{\frac{N+1}{N-3}}$: We substitute a space more adapted to the nonlinearities for the space $\mathbb{M}$. For that, we look for singular solutions of the problem $-\Delta u = \lambda_0(1+u)^{\frac{N+1}{N-3}}$ such that $u(r,\theta) = r^{-\frac{N-3}{2}}w(\theta) - 1$. We have the following proposition from [6].

Proposition 4. *There exists $\mathcal{M} \subset \mathcal{C}^{2,\alpha}(S^{N-1})$, a N-dimensional manifold, such that for all $w \in \mathcal{M}$, the function $u(r,\theta) = r^{-\frac{N-3}{2}}w(\theta) - 1$ is a solution of $-\Delta u = \lambda_0(1+u)^{\frac{N+1}{N-3}}$ defined in $\mathbb{R}^N \setminus \{0\}$ which has a nonremovable singularity at 0. Moreover, the tangent space of $\mathcal{M}$ at the point $w = 1$ can be identified with $\mathbb{M}$.*

Proof. If we look for a singular solution of the problem $-\Delta u = \lambda_0(1+u)^{\frac{N+1}{N-3}}$ in $\mathbb{R}^N \setminus \{0\}$ of the form $u(r,\theta) = r^{-\frac{N-3}{2}}w(\theta) - 1$, we find that w must satisfy on S^{N-1} the equation

$$\Delta_{S^{N-1}} w - \frac{(N-1)(N-3)}{4}(w - w^{\frac{N+1}{N-3}}) = 0, \tag{17}$$

where $\Delta_{S^{N-1}}$ is the Laplace-Beltrami operator on (S^{N-1}, g_0). It is known that the set $\mathcal{M}$ of all positive solutions of the equation (17) has the structure of a N-dimensional non compact manifold associated to the Möbius group on S^{N-1} and that its elements are of the form $\psi_{\mu,a}(\theta) = \big(\frac{\sqrt{\mu^2-1}}{\mu - \cos d(a,\theta)}\big)^{\frac{N-3}{2}}$ for some $(\mu, a) \in (1, +\infty) \times S^{N-1}$ where d is the geodesic distance on S^{N-1} (cf. L.V. Ahlfors [1] and Obata [20]). We prove in what follows that $\mathcal{M}$ is of dimension N and that the tangent space at $w = 1$ can be identified with $\mathbb{M}$, since $\mathcal{M} = \mathcal{F}^{-1}(0)$, where $\mathcal{F}$ is the differentiable function given by $\mathcal{F}(w) = \Delta_{S^{N-1}} w - \frac{(N-1)(N-3)}{4}(w - w^{\frac{N+1}{N-3}})$. The differential of $\mathcal{F}$ at 1 is defined by $D\mathcal{F}_{|1}(v) = \Delta_{S^{N-1}} v + (N-1)v$. Consequently, we have

$$T_1\mathcal{M} = \ker D\mathcal{F}_{|1} = \mathrm{Span}\{\varphi_1, \varphi_2, ..., \varphi_N\} = \mathbb{M}.$$

This ends the proof. $\square$

Moreover, if we define the mapping $\Pi : \mathcal{M} \longrightarrow \mathbb{R}^N$ by

$$\Pi(w) \equiv \Big(\int_{S^{N-1}} w(\theta)\varphi_1(\theta) d\theta, \int_{S^{N-1}} w(\theta)\varphi_2(\theta) d\theta, \dots, \int_{S^{N-1}} w(\theta)\varphi_N(\theta) d\theta \Big),$$

it is easy to see that Π is a diffeomorphism from $\mathcal{U}$, an open neighbourhood of 1 in $\mathcal{M}$ onto $\mathcal{V}$ an open neighbourhood of 0 in $\mathbb{R}^N$.

The result of the Proposition 4 allows us to define a nonlinear mapping

$$\begin{aligned} \mathcal{N} : (v, a, \psi) \in \mathcal{C}^{2,\alpha}_{\nu,N} \times \mathbb{R}^N \times \mathcal{C}^{2,\alpha}(\partial B) \\ \rightarrow \Delta(u_a + v + w_\psi) + \lambda_0 f(u_a + v + w_\psi) \in \mathcal{C}^{0,\alpha}_{\nu-2}, \end{aligned}$$

where $u_a(r, \theta) = r^{-\frac{(N-3)}{2}} \Pi^{-1}(a) - 1$. This mapping is well defined and we have

$$\begin{aligned} D\mathcal{N}_{|(0,0,0)}(v, a, 0) &= \Delta\Big(v + r^{-\frac{(N-3)}{2}} \sum_{j=1}^{N} a_j \varphi_j\Big) \\ &\quad + \lambda_0 \frac{N+1}{N-3} r^{-2}\Big(v + r^{-\frac{(N-3)}{2}} \sum_{j=1}^{N} a_j \varphi_j\Big). \end{aligned}$$

Applying the implicit function theorem to $\mathcal{N}$, we get all solutions of

$$\mathcal{N}(v, a, \psi) = 0$$

in a neighborhood of $(0, 0, 0)$ in $\mathcal{C}^{2,\alpha}_{\nu,N} \times \mathbb{R}^N \times \mathcal{C}^{2,\alpha}(\partial B)$, which ends the proof of the results concerning the case $p = \frac{N+1}{N-3}$.

The case where $N = 3$ and $f(u) = e^u$: We use here arguments analogous to those used in the case $p = \frac{N+1}{N-3}$. In fact, we now look for all the singular solutions of the problem $-\Delta u = 2e^u$ of the form $u(x) = -2\ln(r) + 2w(\theta)$. This work is done in the paper of L. Véron, [25] and also in the paper of M.F. Bidaut-Véron and L. Véron [6], where they obtained the following result.

Proposition 5. *There exists a 3-dimensional manifold $\tilde{\mathcal{M}} \subset \mathcal{C}^{2,\alpha}(S^2)$, such that for all $w \in \tilde{\mathcal{M}}$, the function $u(x) = 2w(\frac{x}{|x|}) - 2\ln(r)$ is a solution of $\Delta u + 2e^u = 0$ defined on $\mathbb{R}^3 \setminus \{0\}$ which has a non removable singularity at 0. Moreover, the tangent space of $\tilde{\mathcal{M}}$ at 0 can be identified with $\mathbb{M}$.*

Proof. If we look for singular solution of the problem $-\Delta u = 2e^u$ defined on $\mathbb{R}^3 \setminus \{0\}$, which are of the form $u(r, \theta) = -2\ln(r) + 2w(\theta)$, we find that w must satisfy the equation

$$\Delta_{S^2} w + e^{2w} - 1 = 0. \tag{18}$$

This equation is equivariant under the conformal transformations of S^2 and the set $\tilde{\mathcal{M}}$ of solutions of $\Delta_{S^2} w + e^{2w} - 1 = 0$ is the set of functions $w = \frac{1}{2}\ln(\det|d\phi|)$ where ϕ is a conformal transformation of S^2 (cf. M.F. Bidaut-Véron and L. Véron [6], A. Chang and P.C. Yang [10]).

Using the fact that $\tilde{\mathcal{M}} = \{\omega \mid \Delta_{S^2}\omega + e^{2\omega} - 1 = 0\}$, we define $\tilde{\mathcal{F}}(w) = \Delta_{S^2}\omega + e^{2\omega} - 1$ and we get that the differential of $\tilde{\mathcal{F}}$ at zero is given by $D\tilde{\mathcal{F}}_{|0}(v) = \Delta_{S^2} v + 2v$. Therefore,

$$T_0\tilde{\mathcal{M}} = \ker D\tilde{\mathcal{F}}_{|0} = \mathrm{Span}\{\varphi_1, \varphi_2, \varphi_3\} = \mathbb{M}.$$

This completes the proof. □

Moreover, if we define the mapping $\Pi : \tilde{\mathcal{M}} \longrightarrow \mathbb{R}^3$ by

$$\Pi(w) \equiv \Big(\int_{S^2} w(\theta)\varphi_1(\theta)d\theta, \int_{S^2} w(\theta)\varphi_2(\theta)d\theta, \int_{S^2} w(\theta)\varphi_3(\theta)d\theta \Big),$$

it is easy to see that Π is a diffeomorphism from $\mathcal{U}$ an open neighbourhood of 0 in $\tilde{\mathcal{M}}$ onto $\mathcal{V}$ an open neighbourhood of 0 in $\mathbb{R}^3$.

The result of the Proposition 5 allows us to define a nonlinear mapping

$$\mathcal{N} : (v, a, \psi) \in \mathcal{C}^{2,\alpha}_{\nu,3} \times \mathbb{R}^3 \times \mathcal{C}^{2,\alpha}(\partial B) \longrightarrow \Delta(u_a + v + w_\psi) + 2e^{u_a+v+w_\psi} \in \mathcal{C}^{0,\alpha}_{\nu-2},$$

where $u_a(r, \theta) = -2\ln(r) + \Pi^{-1}(a)$. This mapping is well defined. In fact, for $(v, a, \psi) \in \mathcal{C}^{2,\alpha}_{\nu,3} \times \mathbb{R}^3 \times \mathcal{C}^{2,\alpha}(\partial B)$, we have $\Delta(u_a + v) + 2e^{u_a+v} = \Delta v + 2(e^{u_a+v} - e^{u_a}) \in \mathcal{C}^{0,\alpha}_{\nu-2}$. As before we have

$$D\mathcal{N}_{|(0,0,0)}(v, a, 0) = \Delta(v + \sum_{i=1}^{3} a_i\varphi_i) + 2e^{u_0}(v + \sum_{i=1}^{3} a_i\varphi_i).$$

It follows that we may apply the implicit function theorem to $\mathcal{N}$ to get all solutions of $\mathcal{N}(v, a, \psi) = 0$ in a neigborhood of $(0, 0, 0)$ in $\mathcal{C}^{2,\alpha}_{\nu,3} \times \mathbb{R}^3 \times \mathcal{C}^{2,\alpha}(\partial B)$. The end of the proof is similar to the end of the proof in the previous case considered, so we shall omit it.

6. Proof of Theorem 2. In order to prove Theorem 2 we have to solve the problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in} \quad B \setminus \{y\} \\ u = 0 & \text{on} \quad \partial B. \end{cases} \tag{19}$$

for y close to 0. Let $T(x, y)$ be a $\mathcal{C}^{2,\alpha}$ mapping from $B \times B_{1/4}$ into B such that for all $y \in B_{1/4}$ the mapping $T(\cdot, y)$ is a $\mathcal{C}^{2,\alpha}$ diffeomorphism from the unit ball into itself. Moreover we ask that

$$\begin{aligned} T(x, y) = x - y \quad & \text{for all} \quad x, y \in B_{1/4}, \\ T(x, y) = x \quad & \text{for all} \quad x \in B \setminus B_{3/4} \quad \text{and for all} \quad y \in B_{1/4} \end{aligned}$$

and finally

$$T(x, 0) = x.$$

Then solving (19) is equivalent to solve, for y close enough to 0

$$\begin{cases} \Delta(u \circ T(\cdot, y)) \circ T^{-1}(\cdot, y) + \lambda f(u) = 0 & \text{in} \quad B \setminus \{0\} \\ u = 0 & \text{on} \quad \partial B. \end{cases} \tag{20}$$

As before this existence result will follow from the implicit function theorem. More precisely, in the case where $f(u) = (1+u)^p$ with $p \in (\frac{N+2}{N-2}, \frac{N+1}{N-3})$, we choose $\nu \in (-\frac{2}{p-1}, \gamma_1^+)$ and we define the nonlinear mapping

$$\mathcal{N}(v,y) = \Delta((u_0 + v) \circ T(\cdot, y)) \circ T^{-1}(\cdot, y) + \lambda_0 f(u_0 + v).$$

It is easy to see that $\mathcal{N}$ is well defined from $\mathcal{C}^{2,\alpha}_{\nu,0} \times \mathbb{R}^N$ into $\mathcal{C}^{0,\alpha}_{\nu-2}$. In addition, $\mathcal{N}(0,0) = 0$ and

$$D\mathcal{N}_{|(0,0)}(v, 0) = \mathcal{L}v.$$

It follows easily from the implicit function theorem and Proposition 3 that all solutions of the equation $\mathcal{N}(v,y) = 0$ near $(0,0)$ are of the form (v_y, y) where

$$y \in \mathbb{R}^N \longrightarrow v_y \in \mathcal{C}^{2,\alpha}_{\nu,0}$$

is some regular mapping. Now that we have obtained a solution of

$$\Delta(u \circ T(\cdot, y)) \circ T^{-1}(\cdot, y) + \lambda_0 f(u) = 0 \quad \text{in} \quad B \setminus \{0\}, \tag{21}$$

we explain how to find the solution of our original problem (19). To this aim, let us define $u_y = (1+c_y)^{-1}(u_0 + v_y - c_y) \circ T(\cdot, y)$ where

$$c_y = \frac{1}{|S^{N-1}|} \int_{S^{N-1}} v_y(\theta) d\theta.$$

We have

$$\begin{cases} \Delta u_y + \lambda_y f(u_y) = 0 & \text{in} \quad B \setminus \{y\} \\ u_y = 0 & \text{on} \quad \partial B, \end{cases}$$

where by definition $\lambda_y = \lambda_0 (1 + c_y)^{p-1}$.

In the limit case ($f(u) = (1+u)^{\frac{N+1}{N-3}}$ in dimension $N \geq 4$ and $f(u) = e^u$ in dimension $N = 3$) the proof is nearly identical, we just give the main lines. This time we define the nonlinear mapping

$$\mathcal{N}(v, a, y) = \Delta((u_a + v) \circ T(\cdot, y)) \circ T^{-1}(\cdot, y) + \lambda_0 f(u_a + v),$$

where $u_a = u_0 + \Pi^{-1}(a)$. It is easy to see that $\mathcal{N}$ is well defined from $\mathcal{C}^{2,\alpha}_{\nu,0} \times \mathbb{R}^N \times \mathbb{R}^N$ into $\mathcal{C}^{0,\alpha}_{\nu-2}$. In addition, $\mathcal{N}(0,0,0) = 0$ and

$$D\mathcal{N}_{|(0,0,0)}(v, a, 0) = \mathcal{L}(v + r^{-\frac{N-3}{2}} \sum_{i=1}^{N} a_i \varphi_i).$$

The remainder of the proof is identical.

7. Proof of Theorem 4. Here we assume that $f(u) = (1+u)^p$ with $p > \frac{N+1}{N-3}$. We have

$$\gamma_1^- \leq \Re(\gamma_0^\pm) \leq \gamma_1^+ < -\frac{2}{p-1} < \gamma_{N+1}^+.$$

Moreover, we notice that $\gamma_1^- = -\frac{p+1}{p-1}$ if $\frac{N+1}{N-3} < p < \frac{N}{N-4}$ and $\gamma_1^+ = -\frac{p+1}{p-1}$ if $p \geq \frac{N}{N-4}$. We choose ν such that $-\frac{2}{p-1} < \nu < \gamma_{N+1}^+$ and, as before, we define the space $\mathbb{M}$ as follows:

$$\mathbb{M} = \{\varphi_1(\theta), \ldots, \varphi_N(\theta)\}.$$

Thanks to Corollary 2, for all $g \in \mathcal{C}^{0,\alpha}_{\nu-2}$, the problem

$$\mathcal{L}w = g \quad \text{in} \quad B \setminus \{0\}$$

has a solution in the space $(\mathcal{C}^{2,\alpha}_{\nu,N} \oplus r^{-\frac{p+1}{p-1}}\mathbb{M})_0$.

Given a function $\psi \in \mathcal{C}^{2,\alpha}(\partial B)$ we have to find a solution $(v,y) \in \mathcal{C}^{2,\alpha}_{\nu,0} \times \mathbb{R}^N$ of

$$\Delta((u_0 + w_\psi + v) \circ T(\cdot, y)) \circ T^{-1}(\cdot, y) + \lambda_0 f(u_0 + w_\psi + v) = 0 \quad \text{in} \quad B \setminus \{0\}.$$

We define the nonlinear mapping

$$\mathcal{N}(v,y,\psi) = [\Delta((u_0+w_\psi+v)\circ T(\cdot,y))+\lambda_0 f((u_0+w_\psi+v)\circ T(\cdot,y))]\circ T^{-1}(\cdot,y).$$

Obviously, $\mathcal{N}$ is well defined from $\mathcal{C}^{2,\alpha}_{\nu,0} \times \mathbb{R}^N \times \mathcal{C}^{2,\alpha}(\partial B)$ into the space $\mathcal{C}^{0,\alpha}_{\nu-2}$. We notice that $\mathcal{N}(0,0,0) = 0$. Furthermore

$$D\mathcal{N}_{|(0,0,0)}(v,0,0) = \mathcal{L}v$$

and, since $\Delta u_0 + \lambda_0 f(u_0) = 0$ in B,

$$\begin{aligned} D\mathcal{N}_{|(0,0,0)}(0,z,0) &= \Delta(Du_{0|x} \circ D_y T_{|(x,0)}(z) \\ &\quad + \lambda_0 p(1+u_0)^{p-1}(Du_{0|x} \circ D_y T_{|(x,0)}(z) \\ &= \mathcal{L}(Du_{0|x} \circ D_y T_{|(x,0)}(z)). \end{aligned}$$

Therefore

$$D\mathcal{N}(w,z,0)=\mathcal{L}(w-Du_{0|x}\circ D_yT_{|(x,0)}(z)).$$

Since $D_yT_{|(x,0)}(z)=0$ if $x\in B\backslash B_{3/4}$ and since $D_yT_{|(x,0)}(z)=-z$ if $x\in B_{1/4}$ we see that $(\mathcal{C}^{2,\alpha}_\nu\oplus r^{-\frac{p+1}{p-1}}\mathbb{M})_0=\mathcal{C}^{2,\alpha}_{\nu,0}\oplus \mathrm{Span}\{Du_{0|x}\circ D_yT_{|(x,0)}(z)/z\in\mathbb{R}^N\}$.

Therefore, we can use the implicit function theorem to prove that all solutions $\mathcal{N}(v,y,\psi)=0$ near $(0,0,0)$ are given by (v_ψ,y_ψ,ψ) where

$$\psi\in\mathcal{C}^{2,\alpha}\longrightarrow(v_\psi,y_\psi)\in\mathcal{C}^{2,\alpha}_{\nu,N}\times\mathbb{R}^N$$

is a regular mapping. We now define $u_\psi=(1+c_\psi)^{-1}(u_0+w_\psi+v_\psi-c_\psi)\circ T(.,y_\psi)$ where

$$c_\psi=\frac{1}{|S^{N-1}|}\int_{S^{N-1}}v_\psi(\theta)d\theta.$$

It is easy to see that u_ψ satisfies

$$\begin{cases}\Delta u_\psi+\lambda_\psi(1+u_\psi)^p=0 & \text{in}\quad B\setminus\{y_\psi\}\\ u_\psi=(1+c_\psi)^{-1}\psi & \text{on}\quad \partial B,\end{cases}$$

where $\lambda_\psi=\lambda_0(1+c_\psi)^{p-1}$. The end of the proof of Theorem 4 follows easily from the fact that the mapping $\psi\longmapsto(1+c_\psi)^{-1}\psi$ is a diffeomorphism from a neighborhood of 0 in $\mathcal{C}^{2,\alpha}(\partial B)$ into a neighborhood of 0 in $\mathcal{C}^{2,\alpha}(\partial B)$.

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