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AN INTRINSIC APPROACH TO LJUSTERNIK–SCHNIRELMAN THEORY FOR LIGHT RAYS ON LORENTZIAN MANIFOLDS*

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Abstract. In this paper it is proven the existence of light–like geodesics joining an event p and a *time–like vertical curve* γ of a Lorentzian manifold $\mathcal{M}$ endowed with a Universal Time Function T, under a certain compactness condition. Moreover, it is developed a Ljusternik–Schnirelman theory for light rays, using which it is shown that, if the topology of $\mathcal{M}$ satisfies a *non–triviality* condition, then there are multiple light rays joining p with γ. The results are obtained under intrinsic assumptions on the manifold $\mathcal{M}$, that do not involve the coefficients of the Lorentzian metric.

1. Introduction and statement of the results. The aim of this paper is to study the existence and multiplicity of light-like geodesics on a time-oriented Lorentzian manifolds $\mathcal{M}$, using an intrinsic approach, i.e., a method that does not require assumptions on the coefficients of the metric of $\mathcal{M}$. We will assume that $\mathcal{M}$ is endowed with a time function T; this property is stable under *small* perturbations of the metric of $\mathcal{M}$.

The advantage of a coordinate–free approach consists primarily in the fact that some significant results obtained so far by many authors, some of whom have been cited in the paper, can be extended to more general situations; secondarily, the effort in trying to state the results in purely geometrical or topological terms has led the authors to a better understanding of the geometrical meaning of some previously used analytical conditions.

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We will investigate the existence of light–like geodesics that join an event p of $\mathcal{M}$ with a time–like curve γ. They represent in General Relativity the trajectories of the light rays received by an observer from a given fixed source of light. The multiplicity result is related with the topology of the space of light-like curves joining the source of light and the observer.

The first results in this direction were proved in [16], where the author constructed a Morse theory on the space of the null, piecewise smooth paths joining p with γ, assuming $\mathcal{M}$ a globally hyperbolic manifold and a nonintrinsic growth condition on the coefficients of the metric g of $\mathcal{M}$.

In [2] the authors proved existence and multiplicity results for light rays on stationary space-time manifolds with boundary and obtained Morse relations relating the set of light rays to the topology of the space-time. The results of [2] were improved in [11] using an intrinsic compactness conditions similar to the one of Definition 1.3, given in terms of the Arrival Time functional, introduced by Perlick in [9, 10]. In [5], the authors extend the results of [16] under intrinsic assumptions, by reducing to work on a global orthogonal splitting (see Section 3 for the definition).

In this paper, we prove the same results of [5] without using a global orthogonal splitting coordinate system. Indeed, our arguments are intrinsic, in the sense that the proofs either do not use coordinate systems, or use only a specific local orthogonal splitting coordinate system (whose existence is guaranteed by the presence of a time function). In particular, the variational Fermat principle used in the proofs (and announced in [1]) is formulated intrinsically.

The motivation of this paper consists in the fact that our techniques (different from the ones used in [5]) allow to treat also cases where a global orthogonal splitting does not necessarily exists. This occurs, for example, in the case of a manifold with boundary.

Before stating our results we recall some basic notions on Lorentzian geometry. We give here a very brief account of the theory, referring the reader to [7] for a complete exposition of the subject.

A Lorentzian manifold is a couple $(\mathcal{M}, g)$, where $\mathcal{M}$ is a smooth manifold and g is a smooth, second order metric tensor on $\mathcal{M}$, i.e., for any $z \in \mathcal{M}$, $g(z)[\cdot,\cdot] = \langle \cdot\,,\cdot\rangle_z$ is a nondegenerate bilinear form on $T_z\mathcal{M}$ having index equal to 1. The points of $\mathcal{M}$ are called *events*. If $f\colon \mathcal{M} \longmapsto \mathbb{R}$ is a smooth function, we will denote by ∇f its (Lorentzian) gradient, that is the vector field on $\mathcal{M}$ satisfying:

$$f'(z)[\zeta] = \langle \nabla f(z)\,, \zeta\rangle_z, \quad \forall\, z \in \mathcal{M}, \forall\, \zeta \in T_z\mathcal{M}.$$

A geodesic on $\mathcal{M}$ is a smooth curve $\gamma : [a,b] \longmapsto \mathcal{M}$ satisfying

$$\nabla_s \dot\gamma(s) = 0, \qquad \forall\, s \in [a,b]$$

where $\nabla_s \dot\gamma(s)$ is the covariant derivative of $\dot\gamma(s)$ along γ induced by the Levi–Civita connection of g. Equivalently, γ is a geodesic if it is a critical point for the *action* functional, given by

$$f(\gamma) = \frac{1}{2}\int_a^b \langle \dot\gamma\,,\dot\gamma\rangle \,\mathrm{d}s.$$

It is well known that if γ is a geodesic on $\mathcal{M}$, there exists a constant $E_\gamma \in \mathbb{R}$ such that

$$E_\gamma = \langle \dot\gamma(s)\,,\dot\gamma(s)\rangle \qquad \forall s \in (a,b).$$

A geodesic γ is called space-like, light-like (null) or time-like if E_γ is greater, equal or less than zero respectively.

In General Relativity, light-like geodesics represent the trajectories of light rays under the action of the gravitational field.

A smooth curve $\gamma : [a,b] \longrightarrow \mathcal{M}$ is called time–like (resp. light–like, space–like) if for any $s \in (a,b)$, the tangent vector $\dot\gamma(s)$ is time–like (resp. light–like, space–like), i.e., if $g(\gamma(s))[\dot\gamma(s),\dot\gamma(s)]$ is negative (resp. null, positive). The curve γ in $\mathcal{M}$ is said to be *causal* if the tangent vector $\dot\gamma(s)$ is either time–like or light–like for every s. A continuous curve γ is said to be causal if every pair of points in $\operatorname{supp}(\gamma)$ can be joined by a smooth causal curve.

For $p, q \in \mathcal{M}$, we say that p *is in the past of* q and we write $p < q$ if p can be connected to q by a future pointing causal curve in $\mathcal{M}$. As customary, we denote by $I^+(p) = \{q \in \mathcal{M} \,\big|\, p < q\}$ and by $I^-(p) = \{q \in \mathcal{M} \,\big|\, q < p\}$.

We recall that a Lorentzian manifold $\mathcal{M}$ is said to be *time–oriented* if there exists a smooth, time–like vector field Y on $\mathcal{M}$. If $\mathcal{M}$ is time–oriented, a smooth curve $z : [0,1] \longrightarrow \mathcal{M}$ (with non vanishing gradient) is called *future pointing* (*past pointing*) if $\langle \dot z(s), Y(z(s))\rangle$ is negative (positive) for all $s \in [0,1]$.

If γ is a time–like curve and p is an event in $\mathcal{M}$, we denote by $\mathcal{L}^+_{p,\gamma}$ ($\mathcal{L}^-_{p,\gamma}$) the set of all C^2, future pointing (past pointing), light–like curves $z : [0,1] \longrightarrow \mathcal{M}$ joining p and γ. More precisely, we set

$$\mathcal{L}^+_{p,\gamma} = \big\{ z \in C^2([0,1],\mathcal{M}) \,\big|\, \langle \dot z(s), \dot z(s)\rangle = 0\,, \langle \dot z(s), Y(z(s))\rangle \le 0 \\ \text{for all } s \in (0,1);\; z(0) = p,\; z(1) \in \operatorname{supp}(\gamma)\, \big\};$$

the space $\mathcal{L}^-_{p,\gamma}$ being defined in the obvious way.

Observe that the space $\mathcal{L}^+_{p,\gamma}$ may be empty. For instance, if $\mathcal{M} = \mathbb{R} \times \mathbb{R}$ is equipped with the Lorentzian metric:

$$\mathrm{d}s^2 = (1+t^2)^2\,\mathrm{d}x^2 - \mathrm{d}t^2,$$

then, there exists no light-like curve joining $p = (0,0)$ with the time like curve $\gamma(s) = (\frac{\pi}{2}, s)$. We will assume throughout the paper that p and γ are chosen so that $\mathcal{L}^+_{p,\gamma}$ ($\mathcal{L}^-_{p,\gamma}$) is not empty.

Notice that light rays pointing in the past of an event are meaningful in General Relativity. Roughly speaking, if γ is interpreted as a continuous light or radio source, like a star or a quasar, a past pointing light ray going from p to γ represents the image of the source of light that is observed at p. Under some circumstances in which the topology of the space–time becomes complicated, like the presence of a deflecting medium, there may be multiplicity of light rays between p and γ. The multiplicity of past pointing light rays joining p and γ gives rise to the phenomenon of *gravitational lensing effect* (see [15] for a reference), for which an observer on the earth sees simultaneously more than one image of the same source. The phenomenon has indeed been observed by astronomers in the last years; the parity of the number (if finite) of images received is one thing astronomers seem to make a big fuzz about. For those who are familiar with the Internet features, a nice image of gravitational lensing effect in Galaxy Cluster Abell 2218 can be found in [17].

In [1], the authors have introduced the following functionals on $\mathcal{L}^+_{p,\gamma}$:

$$F(z) = -\int_0^1 \langle \dot{z}(s), Y(z(s))\rangle\,\mathrm{d}s \tag{1.0.1}$$

and

$$Q(z) = \int_0^1 \langle \dot{z}(s), Y(z(s))\rangle^2\,\mathrm{d}s. \tag{1.0.2}$$

We will consider a Lorentzian manifold $\mathcal{M}$, endowed with a time function T, i.e., a smooth function on $\mathcal{M}$, whose gradient ∇T is a time–like vector field along $\mathcal{M}$. The vector field $Y = -\nabla T$ gives a time orientation of $\mathcal{M}$, and the functional F can be written as

$$F(z) = \int_0^1 \langle \dot{z}(s)\,, \nabla T(z(s))\rangle \mathrm{d}s = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\,(T(z(s))\,\mathrm{d}s = T(z(1)) - T(z(0)). \tag{1.0.3}$$

Remark 1.1. Observe that $F(z)$ does not depend on the parametrization chosen for z, whereas $Q(z)$ does.

A smooth curve z in $\mathcal{M}$ is called *pregeodesic* if there exists a reparametrization of z which is a geodesic in $\mathcal{M}$, or, equivalently, if $\nabla_s \dot{z}$ is parallel to $\dot{z}$. As it has been proven in [1], the critical points of the functionals F and Q in $\mathcal{L}^+_{p,\gamma}$ are pre-geodesics. We recall here the results obtained in [1]:

Theorem 1.1. *Let $\mathcal{M}$ be a Lorentzian manifold with a time function T, F be the functional given by* (1.0.3) *and z be in $\mathcal{L}^+_{p,\gamma}$. Assume that $\dot{z}(s) \neq 0$ for every s. Then, z is a critical point of F if and only if z is a pre–geodesic. An analogous result holds for past pointing light–like pre-geodesics in $\mathcal{L}^-_{p,\gamma}$.*

Theorem 1.2. *Let $\mathcal{M}$ be a Lorentzian manifold with a time function T, and let Q be the functional given by* (1.0.2)*, where $Y = -\nabla T$. A curve $z \in \mathcal{L}^+_{p,\gamma}$ with $\dot{z}(s) \neq 0$ is a critical point of Q if and only if z is a pre–geodesic such that $\langle \dot{z}(s)\,, Y(z(s))\rangle$ is constant.*
An analogous result holds for past pointing pre-geodesics.

There is a remarkable correspondence between the functionals F and Q with the Length and the Energy functionals on a Riemannian manifold, for which results similar to the ones of Theorems 1.1 and 1.2 hold. For instance, if $\mathcal{M}_0$ is a Riemannian manifold and $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ is a static manifold, then F and Q give precisely the length and the energy functional on $\mathcal{M}_0$.

The compactness condition on the manifold $\mathcal{L}^+_{p,\gamma}$ that we need is the following:

Definition 1.3. Let c be a real number. The manifold $\mathcal{L}^+_{p,\gamma}$ is called c–precompact if for every sequence $\{z_n\}$ in $\mathcal{L}^+_{p,\gamma}$ such that $F(z_n) \leq c$, there exists a subsequence $\{z_{n_k}\}$ which is uniformly convergent in $\mathcal{M}$, up to a reparametrization.

A similar condition is defined in [5, 11] using the Arrival Time of Perlick (see [9]). An alternative compactness condition on $\mathcal{L}^+_{p,\gamma}$ equivalent to the c-precompactedness is discussed in Section 4.

For $c \in \mathbb{R}$, we will denote by $Q^c \cap \mathcal{L}^+_{p,\gamma}$ the c-sublevel of Q in $\mathcal{L}^+_{p,\gamma}$, i.e., the subset of $\mathcal{L}^+_{p,\gamma}$ consisting of curves z for which $Q(z) \leq c$.

Definition 1.4. Let $\mathcal{M}$ be a Lorentzian manifold time–oriented by the vector field Y. A smooth curve $\gamma\colon (a,b) \longmapsto \mathcal{M}$ is called a *time–like vertical curve* if γ is a maximal solution of the equation:

$$\dot{\gamma} = Y(\gamma).$$

It is easy to see that a time–like vertical curve γ is a closed embedding of a real interval in $\mathcal{M}$.

We have the following result regarding the existence of light rays joining an event with a vertical time–like curve:

Theorem 1.5. *Let $\mathcal{M}$ be a Lorentzian manifold with a time function, $p \in \mathcal{M}$ and γ a time–like vertical curve in $\mathcal{M}$, with $p \notin supp(\gamma)$. Suppose that $\mathcal{L}^+_{p,\gamma}$ is non-empty, and c–precompact, for every $c \in \mathbb{R}$. Then, there exists a light–like, future pointing geodesic in $\mathcal{M}$ joining p with γ. An analogous result holds for past pointing geodesics.*

The multiplicity of light rays is given in terms of Ljusternik–Schnirelman category of the space $\mathcal{L}^+_{p,\gamma}$ ($\mathcal{L}^-_{p,\gamma}$), which is a homological invariant. We recall here its definition:

Definition 1.6. Let X be a topological space, and $Y \subseteq X$ a subspace. The Ljusternik–Schnirelman category $\mathrm{cat}_X\big(Y\big)$ of Y in X is the minimal number (possibly infinite) of closed, contractible subsets of X covering Y. As usual, we set $\mathrm{cat}\,(X) = \mathrm{cat}_X\big(X\big)$.

The second main result of this paper is given by the following:

Theorem 1.7. *Under the hypothesis of Theorem 1.5, there are at least $\mathrm{cat}\left(\mathcal{L}^+_{p,\gamma}\right)$ light–like, future pointing geodesics joining p and γ. Moreover, if $\mathrm{cat}\left(\mathcal{L}^+_{p,\gamma}\right) = +\infty$, then there exists a sequence $\{z_n\}$ of light–like geodesics in $\mathcal{L}^+_{p,\gamma}$ such that*

$$\lim_{n\to\infty} F(z_n) = +\infty.$$

An analogous result holds for past pointing geodesics.

The paper is organized as follows: Section 2 contains a few algebraic results on the problem of orthogonalization in a Lorentzian vector space. These results are used in Section 3, where it is shown (on a Lorentzian manifold endowed with a time function) how to construct a local space–like decomposition, where the time function is used as a local coordinate.

Section 4 is devoted to the introduction of the functional spaces $\hat{\mathcal{L}}_{p,\gamma}$ and $\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$, $\varepsilon > 0$, consisting respectively of light–like and time–like absolutely continuous curves, with fixed energy. The main difficulty of the paper arises from the fact that $\hat{\mathcal{L}}_{p,\gamma}$, the *natural* space for the search of light–like geodesics, is not a regular manifold, and we will have to use an approximation method using the regular manifolds $\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$. Also, a suitable characterization of the c- precompactness property is discussed, which allows to extend it to the approximating manifolds.

In Section 5, we will discuss a few technical properties of a time function, and we will prove that, under the assumption of the c- precompactness , a time function can be rescaled in order to satisfy a crucial property of negativity of its Hessian; of course, rescaling the time function does not change our setup.

In Section 6, we will be concerned with the above mentioned approximation scheme, and we will show, among other things, the existence of continuous and injective maps between the sublevels of Q in $\hat{\mathcal{L}}_{p,\gamma}$ and $\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$.

The main analytical part of the work is found in Sections 7 and 8, where the results of existence and multiplicity for light rays are proven. The main tool is provided by the fact that the Palais–Smale condition of Q on $\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$ holds uniformly with respect to ε, so that the classical Ljusternik–Schnirelman theory, including deformation theorems and the min-max argument, is extended to $\hat{\mathcal{L}}_{p,\gamma}$.

2. Orthogonalization in a Lorentzian vector space. In this section we collect a few results on the purely algebraic problem of orthogonalization in a Lorentzian vector space. Some of the results presented here are contained in similar forms in [7], but our emphasis will be in the orthogonalization procedure.

Let V be an $(n+1)$–dimensional real vector space, with $n > 0$, endowed with a bilinear, symmetric, non-degenerate form of index 1 on $V \times V$, denoted by $\langle \cdot \,, \cdot \rangle$. Such spaces are called *Lorentzian vector spaces*.

A vector $v \in V$ is called time–like, light–like or space–like respectively if $\langle v \,, v \rangle$ is negative, null or positive. More in general, a subspace $W \subset V$ is called time–like, light–like or space–like if all its non–zero vectors are. It is easily seen that every non trivial time–like or light–like subspace of a Lorentzian space is one dimensional.

The first result is the Lorentzian counterpart of the Schwartz's inequality for Hilbert spaces.

Lemma 2.1. *If v_1 is a time–like vector in V, then for every $v_2 \in V$ one has:*

$$\langle v_1 \,, v_2 \rangle^2 \geq \langle v_1 \,, v_1 \rangle \langle v_2 \,, v_2 \rangle.$$

The equality holds if and only if v_2 is a multiple of v_1.

Proof. Clearly, if v_2 is a multiple of v_1 one has $\langle v_1 \,, v_2 \rangle^2 = \langle v_1 \,, v_1 \rangle \langle v_1 \,, v_2 \rangle$, so we can assume that v_1 and v_2 are linearly independent. In this case, the

2-dimensional space V' generated by v_1 and v_2 is itself a Lorentzian space, with the same inner product. This follows from the fact that V' contains both time–like vectors (v_1) and space–like vectors, since it has dimension 2. The restriction of $\langle \cdot\,, \cdot \rangle$ to V' is represented, in the basis $\{v_1, v_2\}$, by a 2×2 matrix $G = (g_{i,j})$, with $g_{i,j} = \langle v_i\,, v_j \rangle$, $i = 1, 2$. G has exactly one positive an one negative eigenvalue, so its determinant is negative:

$$0 > g_{1,1}g_{2,2} - g_{1,2}^2 = \langle v_1\,, v_1 \rangle \langle v_2\,, v_2 \rangle - \langle v_1\,, v_2 \rangle^2,$$

which proves the thesis. $\square$

We would like to point out the fact that the first part of the Lemma could be easily deduced from Proposition 30, ch. 5 of [7], whereas our result is stronger in the second part, where no assumption was made on the causality of the vector v_2.

In analogy with Hilbert spaces, we define *orthogonal* a basis $\{v_1, v_2, \ldots, v_{n+1}\}$ of V if $\langle v_i\,, v_j \rangle = 0$ for every pair of distinct indices i and j. The main difference with the case of a positive definite inner product is that orthogonality does not imply linear independence. For instance, a light–like vector is self–orthogonal. However, orthogonality with a time–like vector guarantees the linear independence, as the following immediate corollary to Lemma 2.1 establishes:

Corollary 2.2. *If $w \neq 0$ is a vector orthogonal to a time–like vector v, then w is space–like, and in particular, linear independent from v.*

Proof. The linear independence is obvious, since any non–zero multiple of v satisfies $\langle \lambda v\,, v \rangle = \lambda \langle v\,, v \rangle \neq 0$. Then, the condition of orthogonality and Lemma 2.1 say that $0 = \langle v\,, w \rangle^2 > \langle v\,, v \rangle \langle w\,, w \rangle$, so that $\langle w\,, w \rangle > 0$. $\square$

The next result gives a causal characterization of the orthogonal basis:

Lemma 2.3. *If $\{v_1, v_2, \ldots, v_{n+1}\}$ is an orthogonal basis of V, then there exists a unique i such that v_i is time–like, and v_j is space–like for every $j \neq i$. In particular, an orthogonal basis does not contain light–like vectors. Conversely, if $\{v_1, v_2, \ldots, v_{n+1}\}$ is an orthogonal family that does not contain any light–like vector, then $\{v_1, v_2, \ldots, v_{n+1}\}$ is an orthogonal basis.*

Proof. Suppose $\{v_1, v_2, \ldots, v_{n+1}\}$ an orthogonal basis. If it were $\langle v_i\,, v_i \rangle \geq 0$ for every i, then for every $v \in V$, $v = \sum \lambda_i v_i$ it would follow:

$$\langle v\,, v \rangle = \sum_{i,j} \lambda_i \lambda_j \langle v_i\,, v_j \rangle = \sum_i \lambda_i^2 \langle v_i\,, v_i \rangle \geq 0,$$

which would contradict the existence of time–like vectors. Let now i be such that v_i is time like, and consider $j \neq i$. From Lemma 2.1 it follows:

$$0 = \langle v_i \,, v_j \rangle^2 > \langle v_i \,, v_i \rangle \langle v_j \,, v_j \rangle,$$

which implies that v_j is space–like. Conversely, if $\{v_1, v_2, \ldots, v_{n+1}\}$ is an orthogonal family such that $\langle v_i \,, v_i \rangle \neq 0$ for every i, then if $\sum \lambda_i v_i = 0$, taking the inner product with v_j, one gets $\lambda_j \langle v_j \,, v_j \rangle = 0$, which implies $\lambda_j = 0$, so that the v_i's are linearly independent. □

An easy consequence of the Lemma 2.3 is that two time–like vectors are never orthogonal. It is important to observe that the causality conditions are valid only for orthogonal basis; it is indeed possible to prove that there exist algebraic basis with any number of time–like, light–like and space–like vectors.

Observe also that, due to the absence of light–like vectors in every orthogonal basis, it makes sense to define *orthonormal* an orthogonal basis for which $|\langle v_i \,, v_j \rangle| = \delta_{i,j}$. Clearly, every orthogonal basis can be normalized by dividing each v_i by $\pm\sqrt{|\langle v_i \,, v_i \rangle|} \neq 0$.

The next result guarantees the existence of many orthogonal basis for a Lorentzian space, and gives an algorithm to produce them. It has to be interpreted as the analogous of the Gram–Schmidt orthogonalization procedure for Hilbert spaces.

Proposition 2.4. *Let $\{v_1, v_2, \ldots, v_{n+1}\}$ be any algebraic basis of V, with v_1 a time–like vector. Then, there exists $\{e_1, e_2, \ldots, e_{n+1}\}$ an orthogonal (orthonormal) basis of V, such that each of the e_k's is in the subspace of V generated by the first k elements $\{v_1, \ldots, v_k\}$. If $\{e'_1, e'_2, \ldots, e'_{n+1}\}$ is another such basis, then there exist scalars λ_k such that $e'_k = \lambda_k e_k$, $k = 1, 2, \ldots, n+1$; in particular if the two basis are orthonormal, then $|\lambda_k| = 1$ for every k.*

Proof. We will show the orthogonalization procedure, the normalization being obvious. Regarding e_1, for the properties requested in the thesis, it is compulsory to choose a multiple of v_1. We set $e_1 = v_1$. For the choice of e_2, we set $e_2 = v_2 + \alpha e_1$, where α is a scalar to be determined. Imposing the orthogonality with e_1, one obtains:

$$0 = \langle e_1 \,, e_2 \rangle = \langle e_1 \,, v_2 \rangle + \alpha \langle e_1 \,, e_1 \rangle. \tag{2.4.1}$$

Since $\langle e_1 \,, e_1 \rangle < 0$, this equation is solved by $\alpha = -\langle e_1 \,, v_2 \rangle \langle e_1 \,, e_1 \rangle^{-1}$. Observe that the uniqueness of the solution for the equation (2.4.1) implies that any other vector in the subspace generated by v_1 and v_2, which is orthogonal to e_1, has to be a multiple of e_2. It is clear that e_1 and e_2 are linearly independent, and from Lemma 2.1, we also get $0 = \langle e_1 , e_2\rangle^2 > \langle e_1 , e_1\rangle\langle e_2 , e_2\rangle$, which says that e_2 is a space–like vector. We proceed by induction, and we assume to have determined j pairwise orthogonal vectors $e_1, e_2, \ldots, e_j$ such that each e_k lies in the subspace generated by $v_1, \ldots, v_k$, with e_1 time–like and v_k space–like for $k > 1$. We set

$$e_{j+1} = v_{j+1} + \sum_{k=1}^{j} \alpha_k e_k,$$

where the α_k's are scalars to be determined by imposing the j linear conditions: $\langle e_{j+1} , e_k\rangle = \langle v_{j+1} , e_k\rangle + \alpha_k\langle e_k , e_k\rangle = 0$, $k = 1, 2, \ldots, j$. These equations are uniquely solved, since $\langle e_k , e_k\rangle \neq 0$ for every k. Once again, notice that e_{j+1} is not a multiple of e_1, therefore Lemma 2.1 gives:

$$0 = \langle e_1 , e_2\rangle^2 > \langle e_1 , e_1\rangle\langle e_{j+1} , e_{j+1}\rangle,$$

which says that e_{j+1} is space–like. Thus, we build an orthogonal family $\{e_1, e_2, \ldots, e_{n+1}\}$ that does not contain any light–like vector, so that, by Lemma 2.3, it is an orthogonal basis. This concludes the proof. $\square$

We conclude the section with a result that gives the conditions for a given family of orthogonal vectors to be completed to an orthogonal basis.

Proposition 2.5. *Let $B_1 = \{v_1, v_2, \ldots, v_k\}$ be an orthogonal family in V, with $k \leq n + 1$. Then, there exists a family $B_2 = \{w_1, w_2, \ldots, w_{n-k+1}\}$ in V such that $B_1 \cup B_2$ is an orthogonal basis for V if and only if all the v_i's are space–like, except, at the most, for one, that is time–like.*

Proof. We have already seen the necessity of the condition, since orthogonal basis do not contain light–like vectors nor more than one time–like vector. To prove the sufficiency, one sees that the v_i's are linearly independent and they generate a k–dimensional subspace V' of V. Since $\langle \cdot , \cdot\rangle$ is nondegenerate, then V can be written as direct sum $V = V' \oplus (V')^\perp$. Then, either B_1 contains a time–like vector, or $(V')^\perp$ does. Thus, if B_1 does not contain a time–like vector, we can add to it a time–like vector from $(V')^\perp$ and we can assume that B_1 is an orthogonal family that contains exactly one time–like vector, say v_1. Then we can complete B_1 to any basis and use the orthogonalization procedure described in Proposition 2.4. It is clear that the orthogonalization does not affect the elements of B_1, and this proves the Proposition. $\square$

3. The metric structure of a time oriented Lorentzian manifold. In this section we will assume that $\mathcal{M}$ is a Lorentzian manifold of dimension $n+1$, and that T is a time function on $\mathcal{M}$. The main result is that T gives rise to a local *orthogonal splitting* of $\mathcal{M}$, i.e., around every point p of $\mathcal{M}$ it can be chosen a coordinate system $(x,t)=(x_1,\ldots,x_n,t)$, with the coordinate t coinciding with the time function T, such that the metric of $\mathcal{M}$ is written in terms of these coordinate as:

$$\mathrm{d}s^2=\sum_{i,j=1}^{n}\alpha_{i,j}(x,t)\,\mathrm{d}x_i\otimes\mathrm{d}x_j-\beta(x,t)\,\mathrm{d}t^2,$$

where $\alpha(x,t)=(\alpha_{i,j}(x,t))$ is a positive definite matrix. In the second part of the section, we will discuss the invariance of some properties with respect to rescaling of the time function T. Finally, the last result is the introduction of a suitable Riemannian structure on $\mathcal{M}$.

We recall that a Lorentzian manifold $(\mathcal{M}',\langle\cdot\,,\cdot\rangle')$ is called an orthogonal splitting if there exists a Riemannian manifold $(\mathcal{M}_0,\langle\cdot\,,\cdot\rangle^o)$ and a positive smooth scalar field β on $\mathcal{M}'$ such that

(1) $\mathcal{M}'=\mathcal{M}_0\times\mathbb{R}$;
(2) $\langle(\xi_1,\tau_1)\,,(\xi_2,\tau_2)\rangle_{(x,t)}=\langle\alpha(x,t)(\xi_1)\,,\xi_2\rangle^o_x-\beta(x,t)\tau_1\tau_2$,

for every $x\in\mathcal{M}_0$, $t\in\mathbb{R}$. and $(\xi_i,\tau_i)\in T_{(x,t)}\mathcal{M}\simeq T_x\mathcal{M}_0\times T_t\mathbb{R}$, $i=1,2$, where $\alpha(x,t)$ is a positive automorphism of $T_x\mathcal{M}_0$, depending smoothly on x and t.

We give the basic definitions needed for the study of the local metric properties of $\mathcal{M}$.

Definition 3.1. A smooth submanifold $\mathcal{N}$ of $\mathcal{M}$ is called *space–like* if $T_q\mathcal{N}$ is a space–like subspace of $T_q\mathcal{M}$ for every $q\in\mathcal{N}$. Evidently, a space–like submanifold of $\mathcal{M}$ is a Riemannian manifold.

We recall that if U is an open subset of $\mathcal{M}$, a *k-dimensional distribution* on U is a function $\Delta\colon z\longmapsto\Delta_z$ on U, where Δ_z is a k-dimensional subspace of $T_z\mathcal{M}$, such that Δ_z is locally generated by k linearly independent smooth vector fields. A distribution Δ is said space–like if, for every $z\in\mathcal{M}$, Δ_z is a space–like subspace of $T_z\mathcal{M}$. A k-dimensional submanifold $\mathcal{N}$ of $\mathcal{M}$ is called an *integral submanifold* for Δ if, for every $z\in\mathcal{N}$, one has $\Delta_z=T_z\mathcal{N}$.

A distribution Δ on $\mathcal{M}$ is said to be *integrable* if there exists an integral submanifold for Δ through every point of $\mathcal{M}$. A condition equivalent to the integrability for a distribution is given by the Frobenius Theorem, see [14] for reference.

Definition 3.2. A Lorentzian manifold $\mathcal{M}$ of dimension $n+1$ is said to have a *local space–like decomposition* if there exists an n-dimensional, space–like, integrable distribution around every point of $\mathcal{M}$.

Remark 3.3. One-dimensional distributions are always integrable, as they are locally generated by one vector field, whose integral curves are integral submanifolds of the distribution. In general, k-dimensional distributions, with $k \geq 2$, are not integrable. For example, if $\mathcal{M} = \mathbb{R}^3$ and Δ is the 2-dimensional distribution such that $\Delta_{(x_1,x_2,x_3)}$ is generated by the vector fields $v_1(x_1,x_2,x_3) = (1,0,x_2)$ and $v_2(x_1,x_2,x_3) = (0,1,0)$, then, by Frobenius theorem, Δ is not integrable.

Example 3.4. If $\mathcal{M}$ is (locally) isometric to an orthogonal splitting $\mathcal{M}_0 \times \mathbb{R}$, then $\mathcal{M}$ has a (local) space–like decomposition. Namely, for every $p = (x_0,t_0) \in \mathcal{M}$, the submanifold $\mathcal{M}_0 \times \{t_0\}$ is space–like. If (U_0,x) is a chart of $\mathcal{M}_0$ around x_0, the tangent spaces of these submanifolds are generated on $U_0 \times \mathbb{R}$ by the vector fields $v_i = (\frac{\partial}{\partial x_i}, 0)$, $i = 1,\ldots,n$.

These examples basically exhaust the category of all the locally space–time orthogonal Lorentzian manifolds, as explained in the following Proposition.

Recall that if U is an open subset of $\mathcal{M}$ and Y is a smooth vector field on U, Y is said *integrable* on U if there exist smooth functions f and Φ on U such that $Y(z) = \Phi(z)\nabla f(z)$ for every z in U. Using the Implicit Function Theorem, it is easy to see that if $Y(z) \neq 0$ for every z, then Y is integrable if and only if the distribution $\Delta_z = Y(z)^{\perp}$ is integrable.

Proposition 3.5. *For a Lorentzian manifold $\mathcal{M}$, the following conditions are equivalent:*

(1) *$\mathcal{M}$ has a local space–like decomposition,*
(2) *$\mathcal{M}$ is locally isometric to an orthogonal splitting,*
(3) *for every $p \in \mathcal{M}$, there exists an open neighborhood $U^{(p)}$ of p and an integrable time–like smooth vector field $Y^{(p)}$ on $U^{(p)}$,*
(4) *for every point $p \in \mathcal{M}$ there exists a time function $f^{(p)}$ defined around p.*

Proof. We have already seen the implication (2) $\Longrightarrow$ (1) (Example 3.4). We prove the implications (1) $\Longrightarrow$ (3) and (1) $\Longrightarrow$ (4) at the same time. Let $\mathcal{M}$ have a local space–like decomposition, $p \in \mathcal{M}$, $\widetilde{U}^{(p)}$ an open neighborhood of p, and $v_1, v_2, \dots, v_n$ smooth space–like vector fields on $\widetilde{U}^{(p)}$ that generate in each point $q \in \widetilde{U}^{(p)}$ the tangent space of a space–like submanifold $\mathcal{N}^{(q)}$ of $\mathcal{M}$. The n–dimensional distribution $\Delta^{(p)}$ on $\widetilde{U}^{(p)}$ generated by the v_i's is integrable by definition, and that implies the existence of a local chart $(U^{(p)}, x^{(p)})$, with $p \in U^{(p)} \subseteq \widetilde{U}^{(p)}$, such that

$$\mathcal{N}^{(p)} \cap U^{(p)} = \{q \in U^{(p)} : x^{(p)}_{n+1}(q) = 0\},$$

(see [14], ch. 6, Theorem 5 for reference). We set $Y^{(p)} = \nabla x^{(p)}_{n+1}$, which is a smooth, integrable vector field on $U^{(p)}$. We claim that $Y^{(p)}$ is orthogonal to the v_i's. Namely, if γ_i is a curve on $\mathcal{N}^{(p)}$ such that $\dot\gamma_i(s) = v_i(\gamma(s))$, then $x^{(p)}_{n+1}(\gamma_i(s)) \equiv 0$, so that

$$0 = \frac{\mathrm{d}}{\mathrm{d}s} x^{(p)}_{n+1}(\gamma_i(s)) = \langle \nabla x^{(p)}_{n+1}, \dot\gamma \rangle_{\gamma(s)} = \langle Y^{(p)}, v_i \rangle_{\gamma(s)},$$

and therefore $Y^{(p)}$ is a vector field orthogonal to the Riemannian manifolds $\mathcal{N}^{(q)}$, for all $q \in U^{(p)}$. Since Riemannian manifolds can be orthogonalized (pointwise), for every $q \in U^{(p)}$, it is possible to make a choice (not necessarily smooth) of space–like vectors $w_1(q), w_2(q), \dots, w_n(q)$ such that the family $\{Y^{(p)}(q), w_1(q), \dots, w_n(q)\}$ is an orthogonal basis for $T_q\mathcal{M}$. By Lemma 2.3, it follows then that $Y^{(p)}$ is time–like and (3) is proven. Of course, (4) is also proven by taking $f^{(p)} = x^{(p)}_{n+1}$.

The proof of the equivalence (4) $\Longleftrightarrow$ (3) is straightforward, by taking $Y^{(p)} = \nabla f^{(p)}$ and we omit the details.

We need to prove now the arrow (4) $\Longrightarrow$ (2). Let $p \in \mathcal{M}$, $\widetilde{U}^{(p)}$ an open neighborhood of p and $f^{(p)}\colon \widetilde{U}^{(p)} \longrightarrow \mathbb{R}$ be a time function. Since $\nabla f^{(p)}$ is a time–like vector (non–zero), f is regular around p, and therefore the equation

$$f^{(p)}(z) \equiv f^{(p)}(p)$$

defines a smooth submanifold $\widetilde{\mathcal{N}}^{(p)}$ of $\mathcal{M}$ around p. Since the time–like vector field $\nabla f^{(p)}$ is orthogonal to $\mathcal{N}^{(p)}$, it follows that $T_q\mathcal{N}^{(p)}$ is a space–like subspace of $T_q\mathcal{M}$, and therefore $\mathcal{N}^{(p)}$ is a Riemannian submanifold of

$\mathcal{M}$. We denote by $V^{(p)}(z)$ the smooth vector field given by:

$$V^{(p)}(z) = \frac{\nabla f^{(p)}(z)}{\langle \nabla f^{(p)}(z) \, , \nabla f^{(p)}(z) \rangle}, \tag{3.5.1}$$

so that $V^{(p)}$ is time–like and $\langle V^{(p)}(z) \, , \nabla f^{(p)}(z) \rangle \equiv 1$. It is a very well known result in Dynamical Systems (see for instance Theorem 1.4 of [4]) that we can find a number $\delta^{(p)} > 0$, an open neighborhood $U^{(p)} \subseteq \widetilde{U}^{(p)}$ of p and a smooth map: $\Psi: U^{(p)} \times (-\delta^{(p)}, \delta^{(p)}) \longmapsto \mathcal{M}$ such that

(i) $\Psi(z, 0) = z$ for all $z \in U^{(p)}$;

(ii) for each z fixed in $U^{(p)}$, the curve $\gamma_z(t) = \Psi(z, t)$ in $\mathcal{M}$ is an integral curve for the vector field $V^{(p)}$, i.e.,

$$\dot{\gamma}_z(s) = V^{(p)}(\gamma_z(s)),$$

for all $s \in (-\delta^{(p)}, \delta^{(p)})$;

(iii) the map $t \longmapsto \Phi(\cdot, t)$ is a one–parameter family of local diffeomorphism of $\mathcal{M}$.

If we denote by $\mathcal{N}^{(p)} = \widetilde{\mathcal{N}}^{(p)} \cap U^{(p)}$, we can define a smooth map:

$$I: \mathcal{N}^{(p)} \times (-\delta^{(p)}, \delta^{(p)}) \longmapsto \mathcal{M} \tag{3.5.2}$$

as the restriction to $\mathcal{N}^{(p)} \times (-\delta^{(p)}, \delta^{(p)})$ of Ψ. Clearly, $I(p, 0) = p$; what we would like to do next is to study the differential of I in $(p, 0)$ and prove that it is non–singular. Notice that $\mathcal{N}^{(p)} \times (-\delta^{(p)}, \delta^{(p)})$ is an $(n+1)$–dimensional manifold.

We will finish the proof of Proposition 3.5 by collecting the rest of the calculations into a separate Lemma:

Lemma 3.6. *With the above notations, the map I of* (3.5.2) *is a local diffeomorphism. The resulting Lorentzian structure induced on $\mathcal{N}^{(p)} \times (-\delta^{(p)}, \delta^{(p)})$ by $\mathrm{d}I^{-1}$ is an orthogonal splitting.*

Proof of Lemma 3.6. As it was shown in the first part of the proof of Proposition 3.5, it is possible to choose an orthogonal family of $n+1$ vector fields around p, $v_1(q), v_2(q), \ldots, v_n(q), v_{n+1}(q)$ such that $v_1(q), v_2(q), \ldots, v_n(q) \in T_q\mathcal{N}^{(p)}$ for every $q \in \mathcal{N}^{(p)}$, and $v_{n+1}(q) = V^{(p)}(q)$ for every $q \in U^{(p)}$. Using the identification with $T_q\mathcal{N}^{(p)} \times \mathbb{R}$, we choose as a basis for $T_{(q,t)}\left(\mathcal{N}^{(p)} \times (-\delta^{(p)}, \delta^{(p)})\right)$ the vector fields $w_i(q, t) = (v_i(q), 0)$, $i = 1, \ldots, n$;

and $w_{n+1}(q,t) = (0_n, 1)$. Using the basis $\{v_i\}$ and $\{w_i\}$, the differential $\mathrm{d}I(q,t)$ is seen as a smooth matrix field $A(q,t) = (a_{i,j}(q,t))_{i,j=1}^{n+1}$, where $a_{i,j}(q,t)$ is the component of $\mathrm{d}I(q,t)(w_i(q,t))$ along $v_i(I(q,t))$. Since the v_i's form an orthogonal basis, then

$$a_{i,j} = \frac{\langle \mathrm{d}I(w_i)\,,v_j\rangle}{\langle v_j\,,v_j\rangle},$$

where the variables (q,t) have been temporarily suppressed for convenience. Recalling the definition of I, since the map Ψ is the identity on $U^{(p)}$ for $t=0$, then one has $\mathrm{d}I(q,0)(w_i) = v_i$ for $i = 1,\dots,n$. Moreover, since for q fixed the map $\Psi(q,t)$ is a curve with tangent vector $V^{(p)}(q) = v_{n+1}(q)$, then $\mathrm{d}I(q,t)(w_{n+1}) = v_{n+1}$. We have established that for every $q \in U^{(p)}$, $a_{i,j}(q,0) = \delta_{i,j}$, and $a_{n+1,j}(q,t) = \delta_{n+1,j}$, for $i,j \in \{1,2,\dots,n{+}1\}$. It follows that $A(p,0)$ is the identity matrix, and by the Implicit Function Theorem I is a local diffeomorphism. Now, for $t = t_0$ fixed, the map $q \longmapsto \Phi(q,t_0)$ sends the submanifold $\mathcal{N}^{(p)}$ into the submanifold

$$\{z \in \mathcal{M}:\ f^{(p)}(z) = f^{(p)}(p) + t_0\} \cap U^{(p)} = \mathcal{N}^{(\Phi(p,t_0))}.$$

Namely, for every $q \in \mathcal{N}^{(p)}$, since $\Phi(q,0) = q$ one has

$$\begin{aligned} f^{(p)}(\Phi(q,t_0)) - f^{(p)}(\Phi(q,0)) &= \int_0^{t_0} \frac{\mathrm{d}}{\mathrm{d}s} f^{(p)}(\Phi(q,s))\mathrm{d}s \qquad (3.5.1)\\ &= \int_0^{t_0} \frac{\mathrm{d}}{\mathrm{d}s} f^{(p)}(\gamma_q(s))\mathrm{d}s = \int_0^{t_0} \langle \nabla f^{(p)}(\gamma_q(s))\,, \dot\gamma_q(s)\rangle\ \mathrm{d}s\\ &= \int_0^{t_0} \langle \nabla f^{(p)}(\gamma_q(s))\,, V^{(p)}(\gamma_q(s))\rangle \mathrm{d}s = \int_0^{t_0} 1\mathrm{d}s = t_0. \end{aligned}$$

It follows that, for $i = 1,\dots,n$, $\mathrm{d}I(q,t_0)(w_i)$ is a vector in the tangent plane of $\mathcal{N}^{(\Phi(p,t_0))}$, therefore it is orthogonal to $v_{n+1}(I(q,t_0))$, which means $a_{i,n+1}(q,t) = 0$, for all $i = 1,2,\dots,n$, whereas the $n \times n$ matrix $A'(q,t) = (a_{i,j}(q,t))_{i,j=1}^n$ is invertible for every (q,t). For every $q \in \mathcal{N}^{(p)}$ and $t \in (-\delta^{(p)}, \delta^{(p)})$, there exists a unique positive automorphism $\alpha(q,t)$ of $T_q\mathcal{N}^{(p)}$ such that $\langle \alpha(q,t)v_i(q)\,, v_j(q)\rangle = \langle w_i(q,t)\,, w_j(q,t)\rangle$; which is given, in the basis $\{v_1.v_2,\dots,v_n\}$, by the matrix $A'(q,t)^*A'(q,t)$, where * denotes the transpose. By construction, $\mathcal{M}$ is isometric to the orthogonal splitting $\mathcal{N}^{(p)} \times (-\delta^{(p)}, \delta^{(p)})$ with Lorentzian structure given by:

$$\langle (v_1,\tau_1)\,,(v_2,\tau_2)\rangle_{(q,t)} = \langle \alpha(q,t)v_1\,, v_2\rangle_q - \beta(q,t)\tau_1\tau_2, \qquad (3.6.1)$$

where

$$\beta(q,t) = -\langle \nabla T(I(q,t))\,, \nabla T(I(q,t))\rangle^{-1} > 0. \tag{3.6.2}$$

Finally, observe that (3.5.1) says that the coordinate function t may be chosen to coincide with the time function $f^{(p)}$. □

Corollary 3.7. *The following types of Lorentzian manifolds admit a space–like decomposition:*

(1) *Two-dimensional manifolds;*
(2) *Conformally stationary manifolds;*
(3) *Globally Hyperbolic manifolds;*
(4) *Manifolds with a Universal Time function.*

Proof. (1) depends on the fact that any non–zero planar vector field is locally integrable. (2), (3) and (4) are examples of manifolds with a globally defined time function. A detailed study of the splitting properties of a globally hyperbolic manifold, given in terms of Cauchy surfaces, can be found in [3]. □

If $\mathcal{M}$ has a universal time function T, we have already observed in the proof of Lemma 3.6 that the local *time* coordinate can be chosen to coincide with T. We state this formally:

Corollary 3.8. *Let $\mathcal{M}$ be a Lorentzian manifold of dimension $n+1 \geq 2$, endowed with a time function T. Then, around every point p of $\mathcal{M}$ it can be chosen a coordinate system of the form $\{x_1, \dots, x_n, T\}$, such that the metric of $\mathcal{M}$ is written in terms of these coordinates as:*

$$\mathrm{d}s^2 = \sum_{i,j} \alpha_{i,j}\, \mathrm{d}x_i \otimes \mathrm{d}x_j - \beta\, \mathrm{d}T^2,$$

for some smooth functions α_i, j and β.

If $\mathcal{M}$ is a Lorentzian manifold with a time function T and $z\colon [a,b] \longmapsto \mathcal{M}$ is a smooth curve, then for every $s_0 \in [a,b]$ there exists a coordinate system around $z(s_0)$ such that, in a neighborhood of $z(s_0)$, z can be written as $z(s) = (x^{s_0}(s), t(s))$, with $x\colon [a_1,b_1] \subseteq [a,b] \longmapsto \mathcal{M}_0^{(s_0)}$, for some Riemannian manifold $\mathcal{M}_0^{(s_0)}$, and $t(s) = T(z(s))$.In particular, the time coordinate can be chosen in a canonical way, not dependent on the coordinate system. Even though the spatial coordinate $x^{(s_0)}(s)$ depends on the coordinate system chosen around $z(s_0)$, the *tangent vector* $\dot{x}^{(s_0)}(s)$ is defined intrinsically as the projection of $\dot{z}(s)$ onto the orthogonal subspace of $Y(z)$:

$$\dot{x}^{(s_0)}(s) = \dot{x}(s) = \dot{z}(s) - \frac{\langle \dot{z}(s)\,, Y(z(s))\rangle}{\langle Y(z(s))\,, Y(z(s))\rangle} Y(z(s)).$$

If z is a light–like curve, then for every s_0 one has

$$\dot{t}(s_0)^2 = \langle \alpha^{(s_0)}(z(s_0))\dot{x}(s_0)\,, \dot{x}(s_0)\rangle,$$

and since the left hand side does not depend on the local coordinate system, it follows that the quantity $\langle \alpha^{(s_0)}(z(s_0))\dot{x}(s_0)\,, \dot{x}(s_0)\rangle$ is defined intrinsically.

In some of the results that are about to be discussed in the paper, it will be needed to *rescale* the time function T, in order to get some extra properties. Our next goal is to show that rescaling is an operation that does not affect our setup, as far as the c- compactness is concerned. We start with a definition:

Definition 3.9. If T_1 is a time function on $\mathcal{M}$, then T_1 is said to be a *rescaling* of T if there exists a smooth function $\varphi\colon \mathbb{R} \longmapsto \mathbb{R}$, with $\varphi' > 0$ and $\varphi(\mathbb{R}) = \mathbb{R}$, such that $T_1 = \varphi \circ T$.

The rescaling is an equivalence relation in the set of all the time functions defined on M. Obviously, if $T_1 = \varphi \circ T$ is a rescaling of T, then φ is invertible, and $T = \varphi^{-1} \circ T_1$, with $(\varphi^{-1})' = (\varphi')^{-1} > 0$.

Our definition of c- precompactness depends on the time function. Therefore, when dealing with more than one time function, it is necessary to specify which particular time function the concept of c- precompactness is relative to. Nevertheless, we are able to prove that the precompactness does not depend on the scaling of the time function, in the sense explained in the following:

Lemma 3.10. *Let T_1 be a rescaling of T. Then, $\mathcal{L}^+_{p,\gamma}$ is c-precompact for every $c \in \mathbb{R}$ with respect to T if and only if it is c-precompact for every $c \in \mathbb{R}$ with respect to T_1.*

Proof. We denote by F and F_1 the functionals defined in (1.0.1) using the time functions T and T_1 respectively; we also denote by $\varphi\colon \mathbb{R} \longmapsto \mathbb{R}$ the rescaling function, i.e., it satisfies $T_1 = \varphi \circ T$.

If $c \in \mathbb{R}$ and $z \in F^c \cap \mathcal{L}^+_{p,\gamma}$, since φ is increasing, then

$$F_1(z) = \varphi(T(z(1)) - \varphi(T(p)) = \varphi(F(z) + T(p)) - \varphi(p) \le \varphi(c + T(p)) - \varphi(T(p)),$$

so that $z \in F_1^d \cap \mathcal{L}^+_{p,\gamma}$, with $d = \varphi(c + T(p)) - \varphi(T(p))$.

This proves that the d– precompactness with respect to T_1 implies the c–precompactness with respect to T. The converse is also proven, considering that $T = \psi \circ T_1$, with $\psi = \varphi^{-1}$. □

Remark 3.11. It is a simple observation that also the concept of time–like vertical line (Definition 1.4) is not affected passing from a given time function T to a rescaling. Namely, if $T_1 = \varphi \circ T$ is a rescaling of T, then γ is a maximal solution of the equation $\dot\gamma = \nabla T(\gamma)$ if and only if $\gamma_1 = \gamma \circ \varphi$ is a maximal solution of $\dot\gamma_1 = \nabla T_1(\gamma_1)$. Clearly, in this situation $\mathrm{supp}\big(\gamma\big) = \mathrm{supp}\big(\gamma_1\big)$. Furthermore, it is obvious that the local space–like decomposition of $\mathcal{M}$ given by T as in Proposition 3.5 is unchanged after a rescaling of T. The only change occurs to the factor β in (3.6.1) and (3.6.2), that becomes $(\varphi')^2\beta$.

We conclude this section with the introduction of a Riemannian structure on $\mathcal{M}$, that will be used in the next sections for making computations. We state the following result for a general time oriented Lorentzian manifold.

Proposition 3.12. *If $(\mathcal{M}, \langle\cdot\,,\cdot\rangle)$ is a Lorentzian manifold which is time oriented by the vector field Y, then there exists a* natural *Riemannian structure on $\mathcal{M}$ associated to Y, given by the form:*

$$\langle\zeta\,,\zeta\rangle_z^{(\mathrm{R})} = \langle\zeta\,,\zeta\rangle_z - 2\frac{\langle\zeta\,,Y(z)\rangle_z^2}{\langle Y(z)\,,Y(z)\rangle_z}, \tag{3.12.1}$$

for all $z \in \mathcal{M}$, $\zeta \in T_z\mathcal{M}$.

Proof. It suffices to show that (3.12.1) defines a positive definite quadratic form on $T_z\mathcal{M}$. Since $Y(z)$ is time–like, from Lemma 2.1 it follows that:

$$2\langle\zeta\,,Y(z)\rangle^2 \geq \langle\zeta\,,Y(z)\rangle^2 \geq \langle\zeta\,,\zeta\rangle\cdot\langle Y(z)\,,Y(z)\rangle,$$

which gives $\langle\zeta\,,\zeta\rangle^{(\mathrm{R})} \geq 0$. Moreover, in the above inequalities, the equal sign holds if and only if

(1) $\langle\zeta\,,Y(z)\rangle = 0$,
(2) ζ is parallel to $Y(z)$.

Since Y is time–like, this clearly implies that $\zeta = 0$ and we are done. □

Remark 3.13. The adjective *natural* in the statement of Proposition 3.12 refers to the fact that $\langle\cdot\,,\cdot\rangle^{(\mathrm{R})}$ is defined algebraically by means only of the Lorentzian structure of $\mathcal{M}$ and its time orientation Y.

Observe that we can write (3.12.1) as:

$$\langle \zeta , \zeta \rangle_z^{(\mathrm{R})} = \langle \zeta^+ , \zeta^+ \rangle_z - \langle \zeta^- , \zeta^- \rangle_z,$$

where ζ^+ and ζ^- are respectively the positive and the negative part of ζ, given by the formulae:

$$\zeta^+ = \zeta - \frac{\langle \zeta , Y(z) \rangle_z}{\langle Y(z) , Y(z) \rangle_z} Y(z), \quad \zeta^- = \frac{\langle \zeta , Y(z) \rangle_z}{\langle Y(z) , Y(z) \rangle_z} Y(z);$$

they also satisfy $\zeta = \zeta^+ + \zeta^-$, $\langle \zeta^+ , \zeta^- \rangle = 0$. The Riemannian structure induced on $\mathcal{M}$ by $\langle \cdot , \cdot \rangle^{(\mathrm{R})}$ is *compatible* with the Lorentzian structure of $\mathcal{M}$, in the sense that, for each $z \in \mathcal{M}$, the space $T_z\mathcal{M}$ admits a basis which is orthogonal (orthonormal) with respect to both structures. Namely, every basis of $T_z\mathcal{M}$ that contains a multiple of $Y(z)$ is orthogonal with respect to $\langle \cdot , \cdot \rangle$ if and only if it is orthogonal with respect to $\langle \cdot , \cdot \rangle^{(\mathrm{R})}$.

Finally, we observe that if Y and Y_1 are linearly dependent smooth, time–like vector field on $\mathcal{M}$, then, the Riemannian structures on $\mathcal{M}$ associated with Y and with Y_1 coincide, as it can be easily checked from (3.12.1). In particular, if T is a time function on $\mathcal{M}$ and T_1 is a rescaling of T, then the Riemannian structures associated with ∇T and ∇T_1 coincide.

4. The variational framework. We will assume in the rest of the paper that $(\mathcal{M}, \langle \cdot , \cdot \rangle)$ is a Lorentzian manifold, endowed with a time function T, and that γ is a time–like vertical line.

Given the strong analogy of the theory in the case of future pointing and past pointing curves, we will only look at the case of future pointing curves, and for the sake of simplicity we will omit the superscript $+$ in the symbol $\mathcal{L}^+_{p,\gamma}$.

We will denote by $\langle \cdot , \cdot \rangle^{(\mathrm{R})}$ the Riemannian metric associated with the time–like vector field ∇T as in Proposition 3.12; by the Nash Embedding Theorem, we will assume that the $\mathcal{M}$ is embedded in some Euclidean space $\mathbb{R}^N$ isometrically with respect to $\langle \cdot , \cdot \rangle^{(\mathrm{R})}$. The symbol $\| \cdot \|$ will denote the Euclidean norm on $\mathbb{R}^N$, and $\| \cdot \|_r$ will denote the norm in the spaces $L^r\left([0,1], \mathbb{R}^N\right)$, where $r \in [1, +\infty]$. We define the following spaces:

$$H^{1,2}\left([0,1], \mathcal{M}\right) = \{ z \in H^{1,2}\left([0,1], \mathbb{R}^N\right) \, : \, z(s) \in \mathcal{M} \;\; \forall \, s \},$$

which is a Hilbert submanifold of $H^{1,2}\left([0,1], \mathbb{R}^N\right)$,

$$\Lambda_{p,\gamma} = \{ z \in H^{1,2}\left([0,1], \mathcal{M}\right) \, \big| \, z(0) = p, \; z(1) \in \; \mathrm{supp}\big(\gamma\big) \},$$

which is a submanifold of $H^{1,2}([0,1],\mathcal{M})$ with the same regularity of γ.

For every $z \in \Lambda_{p,\gamma}$ we define s_z to be the real number such that $z(1) = \gamma(s_z)$; if γ is parametrized with the universal time, then $s_z = T(z(1)) - T(p) = F(z)$. The tangent spaces of $H^{1,2}$ and $\Lambda_{p,\gamma}$ can be identified respectively with

$$T_z H^{1,2} = \{\zeta \in H^{1,2}\left([0,1],\mathbb{R}^N\right) \,\big|\, \zeta(s) \in T_{z(s)}\mathcal{M},\ \forall\, s\,\},$$

and

$$T_z\Lambda_{p,\gamma} = \{\zeta \in T_z H^{1,2} \,\big|\, \zeta(0) = 0,\ \zeta(1) \parallel \dot\gamma(s_z)\}$$

We will also consider the space:

$$\hat{\mathcal{L}}_{p,\gamma} = \{z \in H^{1,2}([0,1],\mathcal{M}) \,:\, z(0) = p,\ z(1) \in \ \mathrm{supp}\big(\gamma\big), \\ \langle \dot z(s)\,, Y(z(s))\rangle \geq 0,\ \langle \dot z(s)\,, \dot z(s)\rangle = 0 \text{ a.e.}\}$$

Remark 4.1. Observe that in general $\hat{\mathcal{L}}_{p,\gamma}$ is *not* a smooth manifold. For instance, in the orthogonal splitting case, $\hat{\mathcal{L}}_{p,\gamma}$ can be thought as the graph of the map $x \longmapsto t_x$, where x is in a suitable Sobolev space, and t_x is the solution of the Cauchy problem:

$$\dot t = \sqrt{\langle \alpha(x,t)\dot x\,, \dot x\rangle}, \qquad t(0) = 0. \tag{4.1.1}$$

The smoothness of $\hat{\mathcal{L}}_{p,\gamma}$ fails at the points $z = (x,t)$ for which $\langle \alpha(x,t)\dot x\,, \dot x\rangle = 0$ somewhere.

Remark 4.2. The spaces $\mathcal{L}_{p,\gamma}$ and $\hat{\mathcal{L}}_{p,\gamma}$ are homotopically equivalent. Indeed, the inclusion $\iota\colon \mathcal{L}_{p,\gamma} \longmapsto \hat{\mathcal{L}}_{p,\gamma}$ has a homotopical inverse constructed with convolution–like regularizing operators (see [Pa] for reference). In particular, their Ljusternik–Schnirelman category is the same:

$$\operatorname*{cat}_{\mathcal{L}_{p,\gamma}}\big(\mathcal{L}_{p,\gamma}\big) = \operatorname*{cat}_{\hat{\mathcal{L}}_{p,\gamma}}\big(\hat{\mathcal{L}}_{p,\gamma}\big). \tag{4.2.1}$$

In the proof of most of our analytical results of the paper, we will make a heavy use of the following Lemma. It provides a *canonical* way of choosing space-time orthogonal coordinates around curves in $H^{1,2}([0,1],\mathcal{M})$:

Localization Lemma 4.3. *Let z be a curve in $H^{1,2}([0,1],\mathcal{M})$. Then, there exists a positive number δ, a positive integer $n \in \mathbb{N}$, a partition of the interval $[0,1]$ given by a finite sequence $0 = a_0 < a_1 < a_2 < \cdots < a_n = 1$, a collection of open subsets $\{\mathcal{U}_i\}_{i=1}^n$ of $\mathcal{M}$ and local coordinates $\{x_1^{(i)}, \ldots, x_n^{(i)}, T\}$ on each $\mathcal{U}_i$, such that, for every $i = 1, \cdots, n$, the following conditions hold:*

(1) *$(\mathcal{U}_i, \langle\cdot\,,\cdot\rangle)$ is isometric to an orthogonal splitting and the metric is written in terms of the coordinates as in Corollary 3.8;*
(2) *for every $w \in H^{1,2}([0,1],\mathcal{M})$ with $\| w - z \|_\infty < \delta$ one has:*

$$w([a_{i-1}, a_i]) \subset \mathcal{U}_i.$$

In particular, if z_k is a sequence which is weakly convergent in $H^{1,2}([0,1],\mathcal{M})$, then there exists a collection of open sets $\{\mathcal{U}_i\}$ as in (1), such that for every $i = 1, \ldots, n$, one has $z_k([a_{i-1}, a_i]) \subset \mathcal{U}_i$ eventually.

Proof. We denote by $\mathrm{d_R}$ the Riemannian metric on $\mathcal{M}$ induced by $\langle\cdot\,,\cdot\rangle^{(\mathrm{R})}$, which coincides with the Euclidean metric in $\mathbb{R}^N$. Since $\mathcal{M}$ is locally space–time orthogonal and $\mathrm{supp}(z)$ is compact, then there exists a positive number μ such that for every $s \in [0,1]$ and every $q \in \mathcal{M}$ with $\mathrm{d_R}(q, z(s)) < \mu$, it is possible to find a local chart $\mathcal{U}$ that contains both q and $z(s)$, with local coordinates $\{x_1, \ldots, x_n, T\}$ as in Corollary 3.8. Let $0 = a_0 < a_1 < a_2 < \cdots < a_n = 1$ be a partition of the interval $[0,1]$, and $\{\mathcal{U}_i\}_{i=1}^n$ a family of local charts, with local coordinates $\{x_1^{(i)}, \ldots, x_n^{(i)}, T\}$, such that for every $i = 1, 2, \cdots n$ one has:

(a) $\int_{a_{i-1}}^{a_i} \sqrt{\langle \dot z(s)\,, \dot z(s)\rangle^{(\mathrm{R})}}\,\mathrm{d}s < \mu$,
(b) the coordinates $\{x_1^{(i)}, \ldots, x_n^{(i)}, T\}$ on $\mathcal{U}_i$ are as in Corollary 3.8,
(c) $z([a_{i-1}, a_i]) \subset \mathcal{U}_i$.

For every $i = 1, \cdots, n$, we set $K_i = z([a_{i-1}, a_i]) \subset \mathcal{U}_i$, $\nu_i = \mathrm{d_R}(K_i, \mathcal{U}_i^c) > 0$, and $\delta = \min_i \nu_i > 0$. Then, for every $i = 1, 2, \cdots, n$ and every $s \in [a_{i-1}, a_i]$, one has:

$$\mathrm{d_R}(w(s), z(s)) \le \delta \le \nu_i \quad \Longrightarrow \quad w(s) \in \mathcal{U}_i,$$

which proves (1) and (2).

For the second part of the proof, it suffices to observe that the weak convergence in $H^{1,2}([0,1],\mathcal{M})$ implies the uniform convergence and we can apply (2). □

Since $\hat{\mathcal{L}}_{p,\gamma}$ is not a smooth manifold, we will use an approximation method, implemented by a family of smooth manifolds $\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$, depending on a positive parameter ε, and defined by:

$$\hat{\mathcal{L}}_{p,\gamma,\varepsilon} = \{z \in H^{1,2}\left([0,1],\mathcal{M}\right) \,:\, z(0)=p,\ z(1)\in \ \mathrm{supp}\big(\gamma\big), \\ \langle \dot z(s)\,,Y(z(s))\rangle \geq 0,\ \langle \dot z(s)\,,\dot z(s)\rangle = -\varepsilon^2 \text{ a.e.}\}.$$

The space $\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$ consists of time–like curves, with constant energy. As in the orthogonal splitting case, we show that $\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$ is a family of smooth manifolds:

Lemma 4.4. *For every $\varepsilon > 0$, $\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$ is a smooth submanifold of $H^{1,2}([0,1],\mathcal{M})$.*

Proof. $\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$ is the inverse image $\Psi^{-1}(0)$, where Ψ is the map:

$$\Psi : H^{1,2}\big([0,1],\mathcal{M}\big) \longmapsto L^2\left([0,1],\mathcal{M}\right),$$

$$\Psi(z) = \sqrt{\varepsilon^2 + \langle \dot z\,,\dot z\rangle - \frac{\langle \dot z\,,Y(z)\rangle^2}{\langle Y(z)\,,Y(z)\rangle}} + \frac{\langle \dot z\,,Y(z)\rangle}{\sqrt{\big|\langle Y(z)\,,Y(z)\rangle\big|}}. \tag{4.4.1}$$

Passing in local coordinates, it is possible to show that ψ is of class C^1. Moreover, its differential Ψ' is surjective on $\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$, so that the Lemma follows from the Implicit Function Theorem. □

Remark 4.5. The functionals F and Q can be extended to the whole manifold $H^{1,2}\left([0,1],\mathcal{M}\right)$, and in particular to the spaces $\hat{\mathcal{L}}_{p,\gamma}$ and $\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$, for every $\varepsilon > 0$, formally by the same formulas of (1.0.1) and (1.0.2). We will keep the same notation to denote their extensions.

If $\mathcal{M}$ is isometric to an orthogonal splitting $\mathcal{M}_0 \times \mathbb{R}$, with $\beta(x,t) \equiv 1$, and $Y(z) = Y(x,t) = (0,1)$, then (1.0.1) and (1.0.2) become

$$F(z) = F(x,t) = \int_0^1 \sqrt{\langle \alpha(x,t)\,\dot x\,,\dot x\rangle}\,\mathrm{d}s,$$

and

$$Q(z) = F(x,t) = \int_0^1 \langle \alpha(x,t)\,\dot x\,,\dot x\rangle\,\mathrm{d}s.$$

For every $c \in \mathbb{R}$, we denote by F^c and Q^c the c-sublevels of F and Q in $H^{1,2}\left([0,1],\mathcal{M}\right)$: $F^c = \{z \in H^{1,2}\left([0,1],\mathcal{M}\right) \bigm| F(z) \leq c\}$, $Q^c = \{z \in H^{1,2}\left([0,1],\mathcal{M}\right) \bigm| Q(z) \leq c\}$. Observe that, by Schwartz's inequality, for every $z \in H^{1,2}\left([0,1],\mathcal{M}\right)$ one has $F(z) \leq \sqrt{Q(z)}$, so that, for every $c \in \mathbb{R}$, the following inclusion yields: $Q^{c^2} \subseteq F^c$ Also, the concept of c-precompactness can be extended to these manifolds in the obvious sense.

Lemma 4.6. *For every $\varepsilon > 0$, the functionals F and Q are of class C^1 on $\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$.*

Proof. Q is clearly of class C^1 on $H^{1,2}\big([0,1],\mathbb{R}^N\big)$, and the thesis follows from Lemma 4.4. As far as F is concerned, it is a Lipschitz continuous functional, and its regularity fails at those points z for which $\langle \dot z\,,\dot z\rangle$ is zero somewhere. Therefore, F is smooth on $\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$. □

In [1], it is found the following explicit formula for the Fréchet differential of Q at the regular points z in $\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$ ($\hat{\mathcal{L}}_{p,\gamma}$):

$$Q'(z)[\zeta] = 2\int_0^1 \langle \dot z(s)\,,\nabla T(z(s))\rangle \frac{\mathrm{d}}{\mathrm{d}s}\langle \nabla T(z(s))\,,\zeta(s)\rangle\,\mathrm{d}s,$$

where $\zeta \in T_z\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$ ($\zeta \in T_z\hat{\mathcal{L}}_{p,\gamma}$).

We come now to a discussion of the property of c-pre compactness in the spaces $\mathcal{L}_{p,\gamma}$, $\hat{\mathcal{L}}_{p,\gamma}$ and $\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$, defined in 1.3. First of all, from the continuity of Q and the density of $\mathcal{L}_{p,\gamma}$ in $\hat{\mathcal{L}}_{p,\gamma}$, we have the following immediate result:

Lemma 4.7. *Let $c \in \mathbb{R}$. Then, $\mathcal{L}_{p,\gamma}$ is c-precompact if and only if $\hat{\mathcal{L}}_{p,\gamma}$ is c-precompact.*

We now show an important characterization of the c-pre compactness, that shows its purely topological nature:

Proposition 4.8. *$\hat{\mathcal{L}}_{p,\gamma}$ ($\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$) is c-precompact if and only if there exists a compact subset K of $\mathcal{M}$ such that $\operatorname{supp}\big(z\big) \subset K$ for every $z \in \mathcal{L}_{p,\gamma}$ ($\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$) with $F(z) \le c$.*

Proof. Clearly, if $\mathcal{L}_{p,\gamma}$ ($\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$) is c-precompact such a compact set K exists. Conversely, suppose that there exists a compact set K such that $\operatorname{supp}\big(z\big) \subset K$ for every $z \in \mathcal{L}_{p,\gamma}$ ($\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$) with $F(z) \le c$, with $c > 0$. Since we are in a compact set, we can assume that the Riemannian metric induced by $\langle\cdot\,,\cdot\rangle^{(\mathrm{R})}$ is complete. Let z_n be a sequence in $\mathcal{L}_{p,\gamma}$, with $F(z_n) \le c$.

We need to show that z_n has a uniformly convergent subsequence, up to a reparametrization. By hypothesis, z_n is equibounded. We reparametrize each of the z_n with the parameter σ_n defined by

$$\sigma_n(s) = -F(z_n)^{-1}\int_0^s \langle \dot z_n(r)\,, Y(z_n(r))\rangle\,\mathrm{d}r, \tag{4.8.1}$$

so that, denoting by $\dot z_n$ the derivative with respect to σ_n, one has

$$\langle \dot z_n(\sigma_n)\,, Y(z_n(\sigma_n))\rangle \equiv F(z_n).$$

Observe that, using this parametrization for z_n, one has $Q(z_n) = F(z_n)^2$. We compute the Riemannian inner product $\langle \dot z_n\,, \dot z_n\rangle^{(\mathrm{R})}$ as follows:

$$\langle \dot z_n\,, \dot z_n\rangle^{(\mathrm{R})} = \langle \dot z_n\,, \dot z_n\rangle - 2\frac{\langle \dot z_n\,, Y(z_n)\rangle^2}{\langle Y(z_n)\,, Y(z_n)\rangle} = (-\varepsilon^2) - 2\frac{F(z_n)^2}{\langle Y(z_n)\,, Y(z_n)\rangle}.$$

Since $F(z_n)$ is bounded, and $\langle Y(z_n)\,, Y(z_n)\rangle$ is bounded away from 0 on the compact set K, it follows that $\langle \dot z_n\,, \dot z_n\rangle^{(\mathrm{R})}$ is also bounded. Hence, the sequence z_n is equicontinuous, and since K is compact, by the Ascoli–Arzelà theorem z_n has a uniformly convergent subsequence. □

We are now able to prove that, for $c \in \mathbb{R}$, the c-pre compactness implies the completeness of the c-sublevels of the functional Q:

Proposition 4.9. *If $\hat{\mathcal{L}}_{p,\gamma}$ ($\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$) is c-precompact, then the sublevel $Q^c \cap \hat{\mathcal{L}}_{p,\gamma}$ ($Q^c \cap \hat{\mathcal{L}}_{p,\gamma,\varepsilon}$) is a complete metric subspace of $H^{1,2}([0,1],\mathcal{M})$.*

Proof. Let $\hat{\mathcal{L}}_{p,\gamma}$ ($\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$) be c-precompact and $\{z_n\}_{n\in\mathbb{N}} \subset Q^c \cap \hat{\mathcal{L}}_{p,\gamma}$ ($Q^c \cap \hat{\mathcal{L}}_{p,\gamma,\varepsilon}$) be a Cauchy sequence in $H^{1,2}([0,1],\mathcal{M})$. Then, z_n converges to z in $H^{1,2}\left([0,1],\mathbb{R}^N\right)$, and from the continuity of Q it follows that $F(z_n) \le Q(z)^{\frac12} \le \sqrt{c}$. Hence, the $\sqrt{c}$–precompactness implies that z_n has a subsequence converging uniformly to z in $\mathcal{M}$, so that $z(s) \in \mathcal{M}$ for every $s \in [0,1]$. Also, up to passing to a subsequence, we can also assume that $\dot z_n$ is pointwise convergent to $\dot z$ almost everywhere, which implies that $z \in \hat{\mathcal{L}}_{p,\gamma}$ ($\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$). □

5. Rescaling the time function. The main goal of this section is that the time function T can be *rescaled* (see Definition 3.9) in such a way that a negativity condition on its Hessian is satisfied on a relevant region of the space–time. In Corollary 5.4 we will translate in local coordinates this negativity condition of the Hessian together with a conformal change of the metric, obtaining a generalization of [5, Proposition 2.5] and [16, Lemma 2.3]. For $d \in \mathbb{R}$, we introduce the following subsets of $\mathcal{M}$:

$$\Gamma(p,\gamma,d) = \{q \in \mathcal{M} \mid \exists\, z \in \hat{\mathcal{L}}_{p,\gamma} \text{ with } T(z(1)) \le d \text{ and such that } q \in \operatorname{supp} z\},$$

and

$$\Gamma(p,\gamma) = \bigcup_{d\in\mathbb{R}} \Gamma(p,\gamma,d).$$

We recall that the Hessian $H^\theta(z)$ of a smooth function θ on $\mathcal{M}$ at the point z, is the bilinear form on $T_z\mathcal{M}$ defined by

$$H^\theta(z)[\zeta,\zeta] = \frac{\mathrm{d}^2}{\mathrm{d}s^2}\Big|_{s=0} \theta\left(\gamma_\zeta(s)\right), \tag{5.0.1}$$

where $\gamma_\zeta(s)$ is the (unique) geodesic in $\mathcal{M}$ satisfying $\gamma_\zeta(0) = z$ and $\dot\gamma_\zeta(0) = \zeta$, $\zeta \in T_z\mathcal{M}$. We show now that the time function T can be rescaled in such a way that its Hessian satisfies a negativity condition on $\Gamma(p,\gamma)$:

Proposition 5.1. *Suppose that $\hat{\mathcal{L}}_{p,\gamma}$ is c-precompact for every $c \in \mathbb{R}$. Then, there exists a rescaling $T_1 = \phi \circ T$ of the time function T satisfying*

$$H^{T_1}(z)[\zeta,\zeta] \le 0,$$

for every $z \in \Gamma(p,\gamma)$ and every light–like vector $\zeta \in T_z\mathcal{M}$.

Proof. Let $d \in \mathbb{R}$ and $z \in \hat{\mathcal{L}}_{p,\gamma}$ be such that $T(z(1)) = F(z) + F(p) \le d$. We can reparametrize z as in (4.8.1) obtaining the curve $\tilde z$ satisfying

$$Q(\tilde z) = F(\tilde z)^2 = F(z)^2 \le (d - F(p))^2.$$

Clearly, z and $\tilde z$ have the same support, and, since $\hat{\mathcal{L}}_{p,\gamma}$ is $(d-F(p))^2$-precompact, Proposition 4.8 says that the set $\Gamma(p,\gamma,d)$ is precompact in $\mathcal{M}$. Consider the function

$$G(q,\zeta) = \frac{H^T(q)[\zeta,\zeta]}{\langle \nabla T(q)\,,\zeta\rangle^2},$$

for $q \in \mathcal{M}$ and $\zeta \in T_q\mathcal{M}$, $\zeta \ne 0$ light–like. This is a continuous function, which is homogeneous of degree 0 in ζ: $G(q,\rho\zeta) = G(q,\zeta)$, $\forall\ \rho \ne 0$, so that, for every $d \in \mathbb{R}$, the pre compactness of $\Gamma(p,\gamma,d)$ implies $\sup G(q,\zeta) < +\infty$ for $q \in \Gamma(p,\gamma,d)$, $\zeta \in T_qM$, $\zeta \ne 0$ light–like. In particular, there exists a continuous function $\mu : \mathbb{R}^+ \longmapsto \mathbb{R}$ such that $G(q,\zeta) \le \mu(d)$, $\forall\, q \in \Gamma(p,\gamma,d)$, $\forall\, \zeta \in T_q\mathcal{M}$, $\zeta \ne 0$ causal. Let ϕ be the smooth function defined by

$$\phi(t) = \int_0^t \exp\Big(- \int_0^s \mu(\sigma)\,\mathrm{d}\sigma\Big)\,\mathrm{d}s;$$

it satisfies

$$\ddot\phi(\tau) + \dot\phi(\tau)\mu(\tau) = 0.$$

We define $T_1 = \phi \circ T$; clearly $\phi' > 0$ and T_1 is a rescaling of T. Let $\zeta \in T_q\mathcal{M}$ be a non zero causal vector, and let γ_ζ be a geodesic satisfying $\gamma_\zeta(0) = q$ and $\dot\gamma_\zeta(0) = \zeta$. We compute the Hessian H^{T_1} as follows:

$$\begin{aligned} H^{T_1}(q)[\zeta,\zeta] &= \frac{\mathrm{d}^2}{\mathrm{d}s^2}\Big|_{s=0} \phi(T(\gamma_\zeta(s)) \\ &= \ddot\phi(T(\gamma_\zeta(0))) + \dot\phi(T(\gamma_\zeta(0)))\frac{H^T(\gamma_\zeta(0))[\zeta,\zeta]}{\langle\nabla T(\gamma_\zeta(0))\,,\zeta\rangle^2} = \\ &= \ddot\phi(T(q)) + \dot\phi(T(q))\frac{H^T(q)[\zeta,\zeta]}{\langle\nabla T(q)\,,\zeta\rangle^2} \le \ddot\phi(T(q)) + \dot\phi(T(q))\mu(T(q)) \\ &= 0, \end{aligned}$$

which concludes the proof. □

The Hessian H^T is related to the expression in local coordinates of the metric, as explained in the following:

Lemma 5.2. *Let $q \in \mathcal{M}$ and T be a time function around q and let (x,T) orthogonally split coordinates around q, as in Corollary* 3.8. *Then, for every $\zeta = (\xi,\tau) \in T_q\mathcal{M}$, we have*

$$\langle\frac{\partial\alpha}{\partial T}(q)\,\xi\,,\xi\rangle = \frac{\partial\beta}{\partial T}(q)\,\tau^2 - 2\langle\frac{\partial\beta}{\partial x}(q)\,,\xi\rangle\tau - 2\frac{\partial\beta}{\partial T}(q)\tau^2 - 2\beta(q)H^T(q)[\zeta,\zeta].$$

Proof. Let $\zeta = (\xi,\tau) \in T_q\mathcal{M}$ and $\gamma_\zeta(s) = (x(s),T(s))$ be a geodesic in $\mathcal{M}$ such that $\gamma_\zeta(0) = q$ and $\dot\gamma_\zeta(0) = \zeta$. The curve γ_ζ is a critical point of the action functional f, which is written in local coordinates as:

$$2f(x,T) = \int_a^b \Big(\langle\alpha(x,T)\,\dot x\,,\dot x\rangle - \beta(x,T)\,\dot T^2\Big)\,\mathrm{d}r. \tag{5.2.1}$$

Then, $f'(\gamma_\zeta) = 0$, and in particular $\frac{\partial}{\partial T}f(\gamma_\zeta)[\tilde\tau] = 0$, for every $\tilde\tau \in H^{1,2}_o([a,b],\mathbb{R})$. We differentiate (5.2.1) with respect to T and, keeping in mind that γ_ζ is smooth so that we can use integration by parts, we get:

$$\begin{aligned} 2\frac{\partial f}{\partial T}(\gamma_\zeta)[\tilde\tau] &= \int_a^b \Big(\langle\frac{\partial\alpha}{\partial T}(x,T)\,\dot x\,,\dot x\rangle\tilde\tau - \frac{\partial\beta}{\partial T}(x,T)\,\dot T^2\tilde\tau - 2\beta(x,T)\,\dot T\dot{\tilde\tau}\Big)\,\mathrm{d}r \\ &= \int_a^b \Big(\langle\frac{\partial\alpha}{\partial T}(x,T)\,\dot x\,,\dot x\rangle - \frac{\partial\beta}{\partial T}(x,T)\,\dot T^2 + 2\langle\frac{\partial\beta}{\partial x}(x,T)\,,\dot x\rangle\,\dot T \\ &\quad + 2\frac{\partial\beta}{\partial T}(x,T)\,\dot T^2 + 2\beta(x,T)\,\ddot T\Big)\tilde\tau\,\mathrm{d}r = 0. \end{aligned} \tag{5.2.2}$$

for every $\tilde{\tau} \in H_o^{1,2}([a,b],\mathbb{R})$. We pass from a weak to a strong equality in (5.2.2), and recalling the definition of Hessian of (5.0.1), we get

$$\begin{aligned}\langle\frac{\partial\alpha}{\partial T}(x(r),T(r))\dot{x}(r)\,,\dot{x}(r)\rangle - \frac{\partial\beta}{\partial T}(x(r),T(r))T(r)^2 \\ + 2\langle\frac{\partial\beta}{\partial x}(x(r),T(r))\,,\dot{x}(r)\rangle\dot{T}(r) \\ + 2\frac{\partial\beta}{\partial T}(x(r),T(r))\,\dot{T}(r) + 2H^T(x(r),T(r))[\dot{\gamma}_\zeta(r),\dot{\gamma}_\zeta(r)] = 0,\end{aligned}$$

for every $r \in [a,b]$. Evaluating at $r = 0$ we obtain the thesis. □

Corollary 5.3. *Let $(\mathcal{M},g)$ be a Lorentzian manifold, T a time function on $\mathcal{M}$, $p \in \mathcal{M}$ and γ a time–like vertical curve such that $\hat{\mathcal{L}}_{p,\gamma}$ is not empty and c-precompact, for every $c \in \mathbb{R}$. Then, there exists a rescaling T_1 of T and a Lorentzian metric metric g_1 on $\mathcal{M}$ which is conformally equivalent to g, such that*

- i) $g_1(z)[\nabla T_1(z),\nabla T_1(z)] \equiv 1$ *for every* $z \in \mathcal{M}$
- ii) *if* $(x,T_1) = (x_1,\dots,x_n,T_1)$ *is a coordinate system as in Corollary* 3.8, *the metric g_1 is written in terms of these coordinates as:*

$$\mathrm{d}s^2 = \sum_{i,j=1}^{n} \alpha_{i,j}(x,T_1)\,\mathrm{d}x_i \otimes \mathrm{d}x_j - \mathrm{d}T_1^2, \tag{5.3.1}$$

where the operator $\alpha(x,T_1) = (\alpha_{i,j}(x,T_1))_{i,j}$ satisfies:

$$\frac{\partial\alpha}{\partial T_1}(z) \geq 0, \tag{5.3.2}$$

for every $z \in \Gamma(p,\gamma)$.

Proof. From Lemma 5.2, it suffices to show that there exists a time function T_1 on $\mathcal{M}$ and a Lorentzian metric g_1 conformally equivalent to g, such that, denoting by $\nabla_1 T_1$ and $H_1^{T_1}$ respectively the gradient and the Hessian of T_1 in the metric g_1, the following two conditions hold:

- a) $g_1(z)[\nabla_1 T_1(z),\nabla_1 T_1(z)] = -1$, for every $z \in \mathcal{M}$;
- b) $H_1^{T_1}(z)[\zeta,\zeta] \leq 0$, for every $z \in \Gamma(p,\gamma)$, and every light–like vector $\zeta \in T_z\mathcal{M}$.

Up by normalizing the metric g, we can assume without loss of generality that T is a time function with normalized gradient, i.e., for every $z \in \mathcal{M}$

$$g(z)[\nabla T(z), \nabla T(z)] = -1 \tag{5.3.3}$$

In order to obtain a) and b), we set $T_1(z) = \phi(T(z))$, $g_1(z) = \psi(T(z))\, g(z)$, where ϕ and ψ are smooth, real functions, to be determined in such a way that a) and b) hold. To avoid trivialities, the maps ϕ and ψ should also satisfy

c) $\phi'(s) > 0$ for every $s \in \mathbb{R}$;
d) $\phi(\mathbb{R}) = \mathbb{R}$;
e) $\psi(s) > 0$ for every $s \in \mathbb{R}$.

Easy, but awkward, calculations show that a) and b) are equivalent to

a') $\psi(s) = \phi'(s)^2$, for every $s \in \mathbb{R}$;
b') $\phi''(T(z)) - \phi'(T(z))\dfrac{H^T(z)[\zeta,\zeta]}{g(z)[\nabla T(z),\zeta]^2} \geq 0$, for every $z \in \Gamma(p,\gamma)$ and every light–like vector $\zeta \in T_z\mathcal{M}$.

Arguing as in Proposition 5.1, it can be proven that a function ϕ satisfying b') can be found as the solution of a suitable second order differential equation. From a') then, also ψ is determined, and the Corollary is proven. $\square$

We conclude the section with the observation that, since light-like geodesics do not depend, up to a reparametrization, on conformal changes of the metric, and also our setup does not change by rescaling the time function, we will be allowed to assume, whenever needed, that $\mathcal{M}$ is endowed with a time function T satisfying (5.3.1) and (5.3.2).

6. The approximation scheme. In this section we will study the method of approximation of the non smooth manifold $\hat{\mathcal{L}}_{p,\gamma}$ with the smooth manifolds $\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$, $\varepsilon > 0$, and the corresponding variation of the functional Q.

Proposition 6.1. *Suppose that $\hat{\mathcal{L}}_{p,\gamma}$ is c-precompact for some $c > \inf_{\hat{\mathcal{L}}_{p,\gamma}} Q$. Then, there exists a positive number $\varepsilon_0 = \varepsilon_0(c) > 0$ and for every $\varepsilon \in (0, \varepsilon_0]$ there are two injective maps: $\phi_\varepsilon \colon Q^c \cap \hat{\mathcal{L}}_{p,\gamma} \longmapsto \hat{\mathcal{L}}_{p,\gamma,\varepsilon}$, $\psi_\varepsilon \colon \hat{\mathcal{L}}_{p,\gamma,\varepsilon} \longmapsto \hat{\mathcal{L}}_{p,\gamma}$, such that*

(1) *ϕ_ε and ψ_ε are continuous;*
(2) *for every $z \in \hat{\mathcal{L}}_{p,\gamma,\varepsilon}$ such that $Q(\psi_\varepsilon(z)) < c$, then $\phi_\varepsilon(\psi_\varepsilon(z)) = z$;*
(3) *for every $z \in Q^c \cap \hat{\mathcal{L}}_{p,\gamma}$ it is $\psi_\varepsilon(\phi_\varepsilon(z)) = z$;*

(4) $\lim_{\varepsilon\to 0}\phi_\varepsilon(z) = z$ *in* $H^{1,2}([0,1],\mathcal{M})$ *for every* $z \in Q^c \cap \hat{\mathcal{L}}_{p,\gamma}$;

(5) *if* $z_\varepsilon \in \hat{\mathcal{L}}_{p,\gamma,\varepsilon}$ *satisfies* $\lim_{\varepsilon\downarrow 0} z_\varepsilon = z$ *in* $H^{1,2}([0,1],\mathcal{M})$, *then* $\lim_{\varepsilon\to 0}\psi_\varepsilon(z_\varepsilon) = z$ *in* $H^{1,2}([0,1],\mathcal{M})$.

Proof. We fix c and we find a compact subset K of $\mathcal{M}$ such that the support of every $z \in Q^c \cap \hat{\mathcal{L}}_{p,\gamma}$ lies in its interior K°. Let δ be a positive number such that the flow $\Phi(q,\sigma)$ of the vector field ∇T is defined on $K \times [-\delta,\delta]$.

For $z \in Q^c \cap \hat{\mathcal{L}}_{p,\gamma}$, we define $z_\varepsilon(s) = \phi_\varepsilon(z)(s) = \Phi(z(s), \tau_{z,\varepsilon}(s))$, for some function $\tau_{z,\varepsilon}(s) = \tau(s)$ on $[0,1]$ and with values in $[0,\delta)$, to be determined in such a way that $\tau(0) = 0$ $(z_\varepsilon(0) = p)$, $\langle \dot z_\varepsilon\,, \nabla T(z_\varepsilon)\rangle \geq 0$ and that $\langle \dot z_\varepsilon\,, \dot z_\varepsilon\rangle \equiv -\varepsilon^2$. Observe that any such curve automatically satisfies $z_\varepsilon(1) \in \operatorname{supp}(\gamma)$, since γ is an integral curve of ∇T and $\Phi(z(1),0) = z(1) \in \operatorname{supp}(\gamma)$. We compute $\dot z_\varepsilon$ as follows

$$\dot z_\varepsilon = \Phi_q(z,\tau)[\,\dot z] + \Phi_\sigma(z,\tau)[\dot\tau] = \Phi_q(z,\tau)[\dot z] + \nabla T(z_\varepsilon)\,\dot\tau,$$

which gives

$$\begin{aligned}\langle \dot z_\varepsilon\,, \dot z_\varepsilon\rangle &= \langle \Phi_q(z,\tau)[\dot z]\,, \Phi_q(z,\tau)[\dot z]\rangle \\ &\quad + 2\langle \Phi_q(z,\tau)[\dot z]\,, \nabla T(z_\varepsilon)\rangle\,\dot\tau + \langle \nabla T(z_\varepsilon)\,, \nabla T(z_\varepsilon)\rangle\,\dot\tau^2.\end{aligned}$$

Notice that the discriminant of this second order equation is strictly positive by Schwartz's inequality of Lemma 2.1. The equation

$$\langle \dot z_\varepsilon\,, \dot z_\varepsilon\rangle + \varepsilon^2 = 0 \tag{6.1.1}$$

gives a second degree equation on $\dot\tau$, whose coefficients clearly depend continuously on ε. Using local coordinates as in Corollary 3.8, and solving the above equation as in [5, Section 3], we find τ as the solution of a first order Cauchy problem (see also Remark 6.2). Such a solution is defined on the whole interval $[0,1]$ if ε is sufficiently small.

For $\varepsilon \in (0,\varepsilon_0]$, we have built the map ϕ_ε, whose continuity is given by the continuous dependence on the data of the solutions of a Cauchy problem. To prove the injectivity we argue as follows. Suppose that z_1 and z_2 are in $Q^c \cap \hat{\mathcal{L}}_{p,\gamma}$, and $\varepsilon > 0$ is such that $\phi_\varepsilon(z_1) = \phi_\varepsilon(z_2)$. By construction, this implies in particular that for every s, the points $z_1(s)$ and $z_2(s)$ belong to the same integral curve for ∇T, hence the support of z_2 lies on some two-dimensional time–like surface of the form $\Sigma = \{\Phi(z_1(s),\sigma)\,\big|\, s \in [0,1],\, \sigma \in (-a,a)\}$;

moreover $z_1(0) = z_2(0) = p$. But in two-dimensional Lorentzian geometry, there is only one light–like, future pointing curve through every point, up to a reparametrization. It follows that z_2 can only be a reparametrization of z_1. For every $s \in [0,1]$, the points $z_1(s)$ and $z_2(s)$ belong to the same integral curve of ∇T, and they also belong to a curve which is at every point transversal to ∇T. But in dimension two, this happens only if $z_1(s) = z_2(s)$, and the injectivity of ϕ_ε is proven.

The map ψ_ε is constructed in a similar fashion, by letting $\Psi(q,\sigma)$ denote the flow of the vector field $-\nabla T$ and by defining $\psi_\varepsilon(z_\varepsilon)(s) = z_0(s) = \Phi(z_\varepsilon(s), \tau(s))$, where τ is a suitable function to be determined with the conditions that $\tau(0) = 0$ and $\langle \dot z_0 \,, \dot z_0 \rangle = 0$. The argument is actually easier than in the previous case, because the Cauchy problem that defines τ, using standard comparison theorems for the solutions of Cauchy problems, admits a solution on the entire interval $[0,1]$ regardless of z. This corresponds to the fact that the maps ψ_ε are defined on the whole manifolds $\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$. We omit the details, addressing the reader to [5] for a complete exposition of the orthogonally split case. The only difference that is worth noting in the two cases is the fact that switching between a light–like curve and a time–like curve via the maps ϕ_ε and ψ_ε, the arrival time on γ changes in the obvious way $s_{\phi_\varepsilon(z)} \geq s_z$ and $s_{\psi_\varepsilon(z)} \leq s_z$. Part (2) and (3) of the thesis follow from the uniqueness of the solution of the Cauchy problems used to define ϕ_ε and ψ_ε. Part (4) follows from the continuous dependence on ε and part (5) from the continuous dependence on z and $\dot z$ in the above mentioned Cauchy problem. □

Remark 6.2. In orthogonally split coordinates (x,t), the maps ϕ_ε and ψ_ε can be described as follows. Since the gradient of the time function is the vector field $(0,1)$, then its flow lines are simply the vertical lines $s \longmapsto (x,s)$; in the notation of Proposition 6.1 one has $\Phi_q \equiv 1$. It follows that if $z = (x,t)$ is in $\hat{\mathcal{L}}_{p,\gamma}$, then $\phi_\varepsilon(z)$ is the curve with components (x, t_ε), where t_ε satisfies the Cauchy problem

$$\dot t_\varepsilon = \sqrt{\langle \alpha(x,t_\varepsilon)\dot x \,, \dot x\rangle + \varepsilon^2}; \quad t_\varepsilon(0) = 0. \tag{6.2.1}$$

Similarly, if $z = (x,\tau)$ is in $\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$, then $\psi_\varepsilon(z)$ is the curve with components (x, τ_ε), where τ_ε satisfies the Cauchy problem

$$\dot \tau_\varepsilon = \sqrt{\langle \alpha(x,\tau_\varepsilon)\dot x \,, \dot x\rangle}; \quad \tau_\varepsilon(0) = 0. \tag{6.2.2}$$

Using comparisons theorem for ordinary differential equations, it is easy to see that if x is such that (6.2.1) admits a global solution on the interval $[0,1]$, then so does (6.2.2). In our terminology, this is the analog of the fact that the maps ψ_ε are defined on the entire manifold $\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$.

Observe that the right hand sides of (6.2.1) and (6.2.2) are simply L^2-functions. A detailed discussion of local and global existence and uniqueness of the solution for this special kind of problems can be found in [5].

We can prove now the following:

Corollary 6.3. *Suppose that $\mathcal{M}$ has a time function T satisfying* (5.3.1) *and* (5.3.2). *If $\hat{\mathcal{L}}_{p,\gamma}$ is c-precompact for some $c > \inf_{\hat{\mathcal{L}}_{p,\gamma}} Q$, then there exists $\varepsilon_0 = \varepsilon_0(c) > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0]$, $\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$ is c-precompact.*

Proof. Let c be a real number; let us assume that $\hat{\mathcal{L}}_{p,\gamma}$ is c-precompact, and let $\varepsilon_0 = \varepsilon_0(c)$ as in Proposition 6.1. Let $\varepsilon \in (0, \varepsilon_0]$ and $\{z_n\}_n$ be a sequence of curves in $F^c \cap \hat{\mathcal{L}}_{p,\gamma,\varepsilon}$. We need to show that z_n converges uniformly up to a reparametrization.

We choose local charts as in Corollary 6.3, we write z_n locally as (x_n, t_n) and $\psi_\varepsilon(z_n) \in \hat{\mathcal{L}}_{p,\gamma}$ as (x_n, τ_n). Comparisons theorem for ordinary differential equations give immediately that $0 \le \tau_n \le t_n$ for every n. Then, $F(\psi_\varepsilon(z_n)) \le c$, and $\psi_\varepsilon(z_n)$ has a uniformly convergent subsequence, up to a reparametrization. By Proposition 6.1, $\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$ is c-precompact. □

Remark 6.4. Suppose that $\mathcal{M}$ has a time function T satisfying (5.3.1) and (5.3.2). Since $\alpha_t \ge 0$, in the same notations of Corollary 6.3, it follows

$$\langle \dot z_n\,, \nabla T(z_n)\rangle^2 = \dot t_n^2 \ge \dot\tau_n^2 = \langle \dot\psi_\varepsilon(z_n)\,, \nabla T(\psi_\varepsilon(z_n))\rangle^2. \tag{6.4.1}$$

Integrating on $[0,1]$, we have $Q(\psi_\varepsilon(z)) \ \le\ Q(z)$, and in particular

$$\psi_\varepsilon(Q^c \cap \hat{\mathcal{L}}_{p,\gamma,\varepsilon}) \subseteq Q^c \cap \hat{\mathcal{L}}_{p,\gamma}.$$

We also have a uniform version of the c-pre compactness, as explained in the following:

Corollary 6.5. *Suppose that $\mathcal{M}$ has a time function T satisfying* (5.3.1) *and* (5.3.2). *If $\hat{\mathcal{L}}_{p,\gamma}$ is c-precompact for some $c > \inf_{\hat{\mathcal{L}}_{p,\gamma}} Q$, then there exists $\varepsilon = \varepsilon_0(c)$ and a compact subset $K = K(c)$ of $\mathcal{M}$, such that every $\varepsilon \in (0, \varepsilon_0]$ and every $z \in Q^c \cap \hat{\mathcal{L}}_{p,\gamma,\varepsilon}$, it follows $\mathrm{supp}(z) \subset K$.*

Proof. Let $\varepsilon_0 = \varepsilon_0(c)$ be as in Corollary 6.3 and K_1 be a compact subset of $\mathcal{M}$ containing the support of every curve in $F^c \cap \hat{\mathcal{L}}_{p,\gamma}$, given by Proposition 4.8. If $z \in \hat{\mathcal{L}}_{p,\gamma,\varepsilon}$, with $\varepsilon \in (0, \varepsilon_0]$, then $z_\varepsilon = \psi_\varepsilon(z) \in (F^c \cap \hat{\mathcal{L}}_{p,\gamma})$ has support in K_1. Since $z(0) = z_\varepsilon(0) = p$, by construction of the map ψ_ε, from (6.3.2) it follows:

$$\begin{aligned}\|z(s) - z_\varepsilon(s)\| &= \left\|\int_0^s (\dot{z}(\sigma) - \dot{z}_\varepsilon(\sigma))\,\mathrm{d}\sigma\right\| \\ &= \left|\int_0^s \frac{\mathrm{d}}{\mathrm{d}\sigma}\big(T(z(\sigma)) - T(z_\varepsilon(\sigma))\big)\,\mathrm{d}\sigma\right| \le \sqrt{c}.\end{aligned}$$

We can therefore take $K = \{q \in \mathcal{M} \mid \min_{r \in K_1} \|\, q - r\| \le \sqrt{c}\}$ and the proof is concluded. □

The maps ϕ_ε and ψ_ε provide a way of passing from $\hat{\mathcal{L}}_{p,\gamma}$ to $\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$ and vice versa.

Next result shows that, under the hypothesis of c-pre compactness, the variation of Q is continuous with respect to ε, uniformly on its c-sublevel:

Proposition 6.6. *Suppose that $\mathcal{M}$ has a time function T satisfying* (5.3.1) *and* (5.3.2). *Let $c > \inf_{\hat{\mathcal{L}}_{p,\gamma}} Q$, and $\varepsilon_0 = \varepsilon_0(c)$ as in Proposition* 6.1. *Then there exists a positive constant $\lambda = \lambda(c)$ such that for every $z \in Q^c \cap \hat{\mathcal{L}}_{p,\gamma}$ and every $\varepsilon \in (0, \varepsilon_0]$, we have*

$$|Q(\phi_\varepsilon(z)) - Q(z)\,| \le \lambda(c) \cdot \varepsilon.$$

Proof. From the c–pre compactness and the Localization Lemma we can assume that $\mathcal{M}$ is (globally) isometric to an orthogonal splitting. The proof in this case is the same as [5, Proposition 3.11]. Alternatively, a direct proof can be easily obtained using a Lipschitz property of Q, the continuity of ϕ_ε with respect to ε, proven in part (4) of Proposition 6.1, and the compactness property of the c-sublevels of Q. □

7. Existence of light rays. In this section, under the compactness condition of Definition 1.3, we will prove the existence of at least one critical point for the functional Q on $\mathcal{L}_{p,\gamma}$.

In this and in the next section, we will assume, as observed at the end of Section 5, that $\mathcal{M}$ is endowed with a time function T satisfying (5.3.1) and (5.3.2). For the study of the critical points for the functional Q, we will need the following compactness condition, called the *Palais–Smale condition*:

Definition 7.1. Let X be a Hilbert manifold, Λ an open subset of X, $R\colon\Lambda\to\mathbb{R}$ a C^1–functional, and c a real number. We say that R satisfies the (P.S.)–condition at the level c on Λ, if for every sequence $\{x_n\}_{n\in\mathbb{N}}$ in Λ satisfying:

(1) $\lim_{n\to\infty} R(x_n)=c$;
(2) $\lim_{n\to\infty} R'(x_n)=0$,

there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ converging in Λ. A sequence $\{x_n\}$ in Λ satisfying (1) and (2) is called a Palais–Smale sequence at the sublevel c.

Proposition 7.2. *For every $c\in\mathbb{R}$ there exists $\varepsilon_0=\varepsilon_0(c)>0$ such that Q satisfies the Palais–Smale condition at the level c on $\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$, for every $\varepsilon\in(0,\varepsilon_0]$.*

Proof. For $c\in\mathbb{R}$, let $\varepsilon_0>0$ be such that for every $\varepsilon\in(0,\varepsilon_0]$ the manifold $\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$ is c-precompact (see Proposition 4.8). We will show that the c-pre compactness implies the Palais–Smale condition at the level c.

In order to do this, let $\{z_n\}$ be a Palais–Smale sequence at the sublevel c in $\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$. From the Localization Lemma 4.3 and the corresponding results in on orthogonally split manifolds, proven by Giannoni and Masiello in [5, Proposition 4.4], it will suffice to prove that z_n is uniformly convergent to an element $z\in H^{1,2}([0,1],\mathcal{M})$.

From the c-pre compactness of $\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$, it follows that there exists a compact subset K of $\mathcal{M}$ such that $\operatorname{supp}\big(z_n\big)\subseteq K$ for every $n\in\mathbb{N}$. Hence, we can assume that $\mathcal{M}$ is complete with respect to the Riemannian metric (3.12.1). Recalling (4.8.1), we compute the euclidean norm of $\dot z_n$ as follows:

$$\|\dot z_n(s)\|^2=\langle\dot z_n(s)\,,\dot z_n(s)\rangle^{(\mathrm{R})}=-\varepsilon^2+2\frac{\langle\dot z_n(s)\,,\nabla T(z_n(s))\rangle^2}{|\langle\nabla T(z_n(s))\,,\nabla T(z_n(s))\rangle|}.$$

We set $m=\min_{p\in K}|\langle\nabla T(p)\,,\nabla T(p)\rangle|$, and we get

$$\|\dot z_n(s)\|^2\le-\varepsilon^2+\frac{2}{m}\langle\dot z_n(s)\,,\nabla T(z_n(s))\rangle^2;$$

finally, integrating on $[0,1]$, we have:

$$\begin{aligned}\|\dot z_n\|_2^2&=\int_0^1\|\dot z_n(s)\|^2\,\mathrm{d}s\le-\varepsilon^2+\frac{2}{m}\int_0^1\langle\dot z_n(s)\,,\nabla T(z_n(s))\rangle^2\,\mathrm{d}s\\&=-\varepsilon^2+\frac{2}{m}Q(z_n)\le-\varepsilon^2+\frac{2}{m}c.\end{aligned}\tag{7.2.1}$$

This says that $\dot{z}_n$ is bounded in $L^2([0,1],\mathcal{M})$, and since $z_n(0) \equiv p$ for every $n \in \mathbb{N}$, this implies that z_n is a bounded sequence in $H^{1,2}([0,1],\mathcal{M})$. Hence, up to passing to a subsequence, we can assume that z_n is weakly convergent to an element z in $H^{1,2}([0,1],\mathcal{M})$. In particular, z_n tends to z uniformly, and the proof is finished. $\square$

Since the manifold $\hat{\mathcal{L}}_{p,\gamma}$ is not smooth (see Remark 4.2), we cannot apply global variational methods to study the critical points of Q on $\hat{\mathcal{L}}_{p,\gamma}$. On the other hand, it is not even clear the definition of critical points for functionals on a non-smooth manifold. The link between the critical points of Q on $\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$ and the light–like geodesics in $\mathcal{M}$ is explained in the following Proposition, in which it is basically proven that the (PS)–condition of Q on $\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$ holds uniformly with respect to ε.

Proposition 7.3. *Let ε_n be a sequence of positive real numbers and $z_n \in \hat{\mathcal{L}}_{p,\gamma,\varepsilon_n}$ a sequence of curves in $\mathcal{M}$. Suppose that*

(1) $\lim_{n\to\infty} \varepsilon_n = 0$;

(2) $\lim_{n\to\infty} Q(z_n) = c \in \mathbb{R}^+$;

(3) $\lim_{n\to\infty} Q'(z_n) = 0$.

Then, there exists a subsequence $z_{n_k} \in \mathcal{L}_{p,\gamma,\varepsilon_{n_k}}$ that converges in $H^{1,2}([0,1],\mathcal{M})$ to an element $z \in \mathcal{L}_{p,\gamma}$, which is a pre–geodesic in $\mathcal{M}$.

Proof. As in the proof of Proposition 7.2, from the Localization Lemma 4.3 and the corresponding results on orthogonally split manifolds ([5, Proposition 5.1]), it suffices to show that a sequence $z_n \in \hat{\mathcal{L}}_{p,\gamma,\varepsilon_n}$ satisfying (1), (2) and (3) is uniformly convergent to an element $z \in H^{1,2}([0,1],\mathcal{M})$.

Let $z_n \in \hat{\mathcal{L}}_{p,\gamma,\varepsilon_n}$ be a sequence satisfying (1) and (2). Fix $c' > c$, let $\varepsilon_0 = \varepsilon_0(c')$ and $K = K(c') \subset \mathcal{M}$ be as in Corollary 6.3. Thus, if n is large enough so that $\varepsilon_n \le \varepsilon_0$, we have $\operatorname{supp}\big(z_n\big) \subset K$. We can use the same argument as in the proof of Proposition 7.2 to get to the inequality (7.2.1), which implies that z_n is a bounded sequence in $H^{1,2}([0,1],\mathcal{M})$.

Therefore, we can find a subsequence z_{n_k} that is weakly (and uniformly) convergent to $z \in H^{1,2}([0,1],\mathcal{M})$, and the proof is done. $\square$

Using Proposition 7.3, we define the critical points of Q on $\hat{\mathcal{L}}_{p,\gamma}$:

Definition 7.4. A curve $z \in \hat{\mathcal{L}}_{p,\gamma}$ is a *critical point* for Q on $\hat{\mathcal{L}}_{p,\gamma}$ if there exists a sequence ε_n of positive numbers and a sequence $z_n \in \hat{\mathcal{L}}_{p,\gamma,\varepsilon_n}$ satisfying (1), (2) and (3) of Proposition 7.1, that converges to z in $H^{1,2}([0,1],\mathcal{M})$.

We denote by $\mathcal{K}_Q$ the set of critical points of Q in $\hat{\mathcal{L}}_{p,\gamma}$. A real number μ is called a *critical value* of Q in $\hat{\mathcal{L}}_{p,\gamma}$ if there exists $z \in \mathcal{K}_Q$ with $Q(z) = \mu$.

We are ready to state and prove the main result of the paper concerning the existence of at least one future pointing, light-like geodesic joining p and γ:

Proposition 7.5. *There is at least one critical point of Q on $\hat{\mathcal{L}}_{p,\gamma}$.*

Proof. From Definition 7.4, we will need to show that there exists a sequence of positive numbers $\varepsilon_n \downarrow 0$ and a sequence $z_n \in \hat{\mathcal{L}}_{p,\gamma,\varepsilon_n}$ of critical points for Q, with $\lim_{n\to\infty} Q(z_n) \in \mathbb{R}$. Let $I = \inf_{\hat{\mathcal{L}}_{p,\gamma}} Q(z)$ and $c \geq I$ be such that $\hat{\mathcal{L}}_{p,\gamma}$ is c-precompact. The first observation to make is that, for ε small enough, the sublevel $Q^c \cap \hat{\mathcal{L}}_{p,\gamma,\varepsilon}$ is non empty. Indeed, let $z \in \hat{\mathcal{L}}_{p,\gamma}$ be such that $Q(z) < c$, and let ε_0 be such that $\varepsilon_0 < (c - Q(z))\,\lambda(c)^{-1}$, where $\lambda(c)$ is the constant of Proposition 6.6. Then, for every $\varepsilon \in (0, \varepsilon_0]$, it follows $Q(\phi_\varepsilon(z)) \leq Q(z) + \lambda(c) \cdot \varepsilon < c$, which implies that $Q^c \cap \hat{\mathcal{L}}_{p,\gamma,\varepsilon} \neq \emptyset$.

We have proven that Q is bounded below ($Q \geq 0$) smooth functional on $\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$ for every $\varepsilon \in (0, \varepsilon_0]$, and from Proposition 7.2, it satisfies the Palais–Smale condition at some non empty sublevel. A standard argument in critical point theory shows that Q attains its minimum on $\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$

Therefore, we have a family $\{z_\varepsilon\}_{0\leq\varepsilon\leq\varepsilon_0}$ of minimal points for Q on $\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$. Clearly, $Q'(z_\varepsilon) = 0$ for every ε. If ε_n is any decreasing sequence of reals tending to 0, then $Q(z_{\varepsilon_n})$ is decreasing, so the limit $\lim\limits_{n\to\infty} Q(z_{\varepsilon_n})$ exists. From Proposition 7.3, the sequence z_{ε_n} tends to $z \in \hat{\mathcal{L}}_{p,\gamma}$, which is a critical point for Q, and we are done. $\square$

Putting together the results of this section and the Fermat Principle of Theorem 1.2, we have the following:

Corollary (Theorem 1.5). *There is at least one light–like geodesic in $\mathcal{M}$ joining p and γ.*

Proof. From Proposition 7.5, there exists at least one critical point z of Q in $\hat{\mathcal{L}}_{p,\gamma}$. From Proposition 7.3, z is smooth and it is a critical point for Q on $\mathcal{L}_{p,\gamma}$. By Theorem 1.2, z is a light–like pre-geodesic. Choosing an appropriate reparametrization of z we have a geodesic joining p and γ. $\square$

8. Multiplicity of light rays. In this section, we will assume that $\mathcal{M}$ is a Lorentzian manifold, T a time function on $\mathcal{M}$, $p \in \mathcal{M}$, and γ a time–like vertical line in $\mathcal{M}$ such that $\hat{\mathcal{L}}_{p,\gamma}$ is non empty and c-precompact for every $c \in \mathbb{R}$. We develop a non smooth Ljusternik–Schnirelman theory for the functional Q on $\hat{\mathcal{L}}_{p,\gamma}$, using the approximation with $\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$ and the uniformity property of the Palais–Smale condition proven in Proposition 7.3.

We start with some results concerning the deformation of the sublevels of Q in $\hat{\mathcal{L}}_{p,\gamma}$ near its regular points. The first result is a consequence of the uniform Palais–Smale condition:

Lemma 8.1. *If c is not a critical value for Q on $\hat{\mathcal{L}}_{p,\gamma}$, then there exists $\delta > 0$ and $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0]$, Q does not have critical values in the interval $(c-\delta, c+\delta)$ on $\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$.*

Proof. By contradiction, if $\varepsilon_n \in \mathbb{R}$ with $\lim_{n\to\infty} \varepsilon_n = 0$ and and $z_n \in \hat{\mathcal{L}}_{p,\gamma,\varepsilon_n}$ is a sequence of critical points of Q, with $\lim_{n\to\infty} Q(z_n) = c$, then, by Proposition 7.3, c would be a critical value for Q on $\hat{\mathcal{L}}_{p,\gamma}$ and the Lemma is proven. □

Proposition 8.2. *If c is not a critical value for Q on $\hat{\mathcal{L}}_{p,\gamma}$, then there exist $\delta > 0$ and a continuous map $\eta\colon [0,1] \times \hat{\mathcal{L}}_{p,\gamma} \longmapsto \hat{\mathcal{L}}_{p,\gamma}$, such that*

(1) *$\eta\,(0,z) = z$ for every $z \in \hat{\mathcal{L}}_{p,\gamma}$;*
(2) *$\eta\,(1, Q^{c+\delta} \cap \hat{\mathcal{L}}_{p,\gamma}) \subseteq Q^{c-\delta} \cap \hat{\mathcal{L}}_{p,\gamma}$.*

Proof. By Proposition 6.1, there exists ε_1 such that the map $\phi_\varepsilon : (Q^{c+1} \cap \hat{\mathcal{L}}_{p,\gamma}) \longmapsto \hat{\mathcal{L}}_{p,\gamma,\varepsilon}$ is defined for every $\varepsilon \in (0, \varepsilon_1]$. By Lemma 8.1, there exist $\delta_1, \varepsilon_2 > 0$ such that Q does not have critical values in the interval $(c-\delta_1, c+\delta_1)$ on $\hat{\mathcal{L}}_{p,\gamma,\varepsilon}$, for every $\varepsilon \in (0, \varepsilon_2]$. Moreover, by Proposition 6.6, there exists $\lambda > 0$ such that for every $z \in Q^{c+1} \cap \hat{\mathcal{L}}_{p,\gamma}$ and for every $\varepsilon \in (0, \varepsilon_1]$, we have $Q(\phi_\varepsilon(z)) - \lambda\varepsilon \le Q(z) \le Q(\phi_\varepsilon(z))$. Let $0 < \delta < \frac{\delta_1}{2}$ and $\overline{\varepsilon} = \min\{\varepsilon_1, \varepsilon_2, \frac{\delta}{\lambda}\}$. The functional Q is smooth on $\hat{\mathcal{L}}_{p,\gamma,\overline{\varepsilon}}$, it satisfies the Palais–Smale condition and its sublevels are complete metric spaces, so that, by a well known deformation theorem for C^1 functionals (see [12]), there exists a continuous map $\eta_{\overline{\varepsilon}} : [0,1] \times \hat{\mathcal{L}}_{p,\gamma,\overline{\varepsilon}} \longmapsto \hat{\mathcal{L}}_{p,\gamma,\overline{\varepsilon}}$ satisfying:

i) $\eta_{\overline{\varepsilon}}(0,z) = z$, for every $z \in \hat{\mathcal{L}}_{p,\gamma,\overline{\varepsilon}}$;
ii) $\eta_{\overline{\varepsilon}}(t,z) = z$, for every $t \in [0,1]$ and every $z \in \hat{\mathcal{L}}_{p,\gamma,\overline{\varepsilon}}$ such that $Q(z) \notin [c-\delta_1, c+\delta_1]$;
iii) $\eta_{\overline{\varepsilon}}(1, Q^{c+2\delta} \cap \hat{\mathcal{L}}_{p,\gamma,\overline{\varepsilon}}) \subset Q^{c-2\delta} \cap \hat{\mathcal{L}}_{p,\gamma,\overline{\varepsilon}} \subset Q^{c-\delta} \cap \hat{\mathcal{L}}_{p,\gamma,\overline{\varepsilon}}$.

We define $\eta : [0,1] \times \hat{\mathcal{L}}_{p,\gamma} \longmapsto \hat{\mathcal{L}}_{p,\gamma}$ by

$$\eta(t,z) = \begin{cases} z, & \text{if } Q(z) > c + \delta_1; \\ \psi_{\overline{\varepsilon}}\left(\eta_{\overline{\varepsilon}}(t, \phi_{\overline{\varepsilon}}(z))\right), & \text{otherwise.} \end{cases}$$

The map η is well defined because $\phi_{\overline{\varepsilon}}$ is defined on $Q^{c+\delta_1} \cap \hat{\mathcal{L}}_{p,\gamma}$, and by ii), since $Q(\phi_{\overline{\varepsilon}}(z)) \geq Q(z)$, it is a continuous map. Clearly, by (3) of Proposition 6.1, $\eta(0,z) = z$ for every $z \in \hat{\mathcal{L}}_{p,\gamma}$. Moreover, if $z \in Q^{c+\delta} \cap \hat{\mathcal{L}}_{p,\gamma}$, one has

$$Q(\phi_{\overline{\varepsilon}}(z)) \leq Q(z) + \lambda\overline{\varepsilon} \leq c + 2\delta,$$

so that, by iii) $\eta_{\overline{\varepsilon}}(1, \phi_{\overline{\varepsilon}}(z)) \in Q^{c-\delta} \cap \hat{\mathcal{L}}_{p,\gamma,\overline{\varepsilon}}$. Then, for $z \in Q^{c+\delta} \cap \hat{\mathcal{L}}_{p,\gamma}$, it follows $Q(\eta(1,z)) = Q(\psi_{\hat{\varepsilon}}(\eta_{\overline{\varepsilon}}(1, \phi_{\overline{\varepsilon}}(z)))) \leq Q(\eta_{\overline{\varepsilon}}(1, \phi_{\overline{\varepsilon}}(z))) \leq c - \delta$, which shows that $\eta(1, Q^{c+\delta} \cap \hat{\mathcal{L}}_{p,\gamma}) \subset Q^{c-\delta} \cap \hat{\mathcal{L}}_{p,\gamma}$ and concludes the proof. □

If the set of critical points of Q on $\hat{\mathcal{L}}_{p,\gamma}$ is bounded, then the whole space $\hat{\mathcal{L}}_{p,\gamma}$ can be continuously contracted to a sublevel of Q, as shown in the following:

Proposition 8.3. *Let d be a (positive) real number. Suppose that Q does not have critical values on $\hat{\mathcal{L}}_{p,\gamma}$ in the half line $[d, +\infty)$, then there exists a continuous map $\eta\colon [0,1] \times \hat{\mathcal{L}}_{p,\gamma} \longmapsto \hat{\mathcal{L}}_{p,\gamma}$, such that*

(1) *$\eta\,(0, z) = z$ for every $z \in \hat{\mathcal{L}}_{p,\gamma}$;*
(2) *$\eta\,(1, \hat{\mathcal{L}}_{p,\gamma}) \subseteq Q^{d+1} \cap \hat{\mathcal{L}}_{p,\gamma}$.*

Proof. With small modifications to the arguments used in Lemma 4.4 and Proposition 6.1, it is not difficult to prove the existence of a C^1 function μ: $\mu : H^{1,2}\,([0,1], \mathcal{M}) \longmapsto \mathbb{R}^+$, and an associated C^1 map Ψ_μ:

$$\Psi_\mu : H^{1,2}\,([0,1], \mathcal{M}) \longmapsto L^2\,([0,1], \mathcal{M})$$

$$\Psi_\mu(z) = \sqrt{\mu(z)^2 + \langle \dot{z}\,, \dot{z}\rangle - \frac{\langle \dot{z}\,, Y(z)\rangle^2}{\langle Y(z)\,, Y(z)\rangle}} + \frac{\langle \dot{z}\,, Y(z)\rangle}{\sqrt{\left|\langle Y(z)\,, Y(z)\rangle\right|}}$$

that satisfy the following properties:

(1) $\mu(z) \leq \varepsilon_0(Q(z))$ for every $z \in \hat{\mathcal{L}}_{p,\gamma}$, where $\varepsilon_0(\cdot)$ is the map defined in Proposition 6.1;
(2) the set $\hat{\mathcal{L}}_{p,\gamma,\mu} = \Psi_\mu^{-1}(0)$ is a C^1 submanifold of $H^{1,2}\,([0,1], \mathcal{M})$;
(3) Q does not have critical values in the half line $[d, +\infty)$ on $\hat{\mathcal{L}}_{p,\gamma,\mu}$.

Observe that (1) says that, for every $z \in \hat{\mathcal{L}}_{p,\gamma,\mu}$, it makes sense to compute $\phi_{\mu(z)}(z)$ and $\psi_{\mu(z)}(z)$, where ϕ and ψ are the maps of Proposition 6.1.

Thus, we can build a flow η_μ on $\hat{\mathcal{L}}_{p,\gamma,\mu}$ (see [12]) $\eta_\mu : [0,1] \times \hat{\mathcal{L}}_{p,\gamma,\mu} \longmapsto \hat{\mathcal{L}}_{p,\gamma,\mu}$ such that

(4) $\eta_\mu(0,z) = z$ for every $z \in \hat{\mathcal{L}}_{p,\gamma,\mu}$;
(5) $\eta_\mu(1,\hat{\mathcal{L}}_{p,\gamma,\mu}) \subset Q^d \cap \hat{\mathcal{L}}_{p,\gamma,\mu}$.

Then, using the same argument as Proposition 8.2, the required flow η on $\hat{\mathcal{L}}_{p,\gamma}$ is defined by $\eta(t,z) = \psi_{\mu(z)}(\eta_\mu(t,\phi_{\mu(z)}))$, and the proof is complete. □

Corollary 8.4. *Under the hypothesis of Lemma* 8.3, *we have*

$$\operatorname*{cat}_{\hat{\mathcal{L}}_{p,\gamma}}\big(\hat{\mathcal{L}}_{p,\gamma}\big) = \operatorname*{cat}_{\hat{\mathcal{L}}_{p,\gamma}}\big(Q^{d+1} \cap \hat{\mathcal{L}}_{p,\gamma}\big).$$

Proof. The Ljusternik–Schnirelman category is invariant by homotopy and monotone by inclusion, so that from the previous Lemma it follows:

$$\operatorname*{cat}_{\hat{\mathcal{L}}_{p,\gamma}}\big(\hat{\mathcal{L}}_{p,\gamma}\big) \geq \operatorname*{cat}_{\hat{\mathcal{L}}_{p,\gamma}}\big(Q^{d+1} \cap \hat{\mathcal{L}}_{p,\gamma}\big) \geq \operatorname*{cat}_{\hat{\mathcal{L}}_{p,\gamma}}\big(\eta(1,\hat{\mathcal{L}}_{p,\gamma})\big) = \operatorname*{cat}_{\hat{\mathcal{L}}_{p,\gamma}}\big(\hat{\mathcal{L}}_{p,\gamma}\big). \qquad \square$$

We have an analogous of Proposition 8.2 in the case when c is a critical value of Q. If c is a critical value, we denote by $\mathcal{K}_Q(c)$ the set of critical points $z \in \hat{\mathcal{L}}_{p,\gamma}$ with $Q(z) = c$.

Proposition 8.5. *Let c be a critical value of Q on $\hat{\mathcal{L}}_{p,\gamma}$ and let U be any neighborhood of $\mathcal{K}_Q(c)$. Then, there exists a positive number $\delta = \delta(c,U) \in (0,\frac{1}{2}]$ and a continuous map $\eta : [0,1] \times \hat{\mathcal{L}}_{p,\gamma} \longmapsto \hat{\mathcal{L}}_{p,\gamma}$ such that*

(1) $\eta(0,z) = z$ *for every* $z \in \hat{\mathcal{L}}_{p,\gamma}$;
(2) $\eta(1,(Q^{c+\delta} \cap \hat{\mathcal{L}}_{p,\gamma}) \setminus U) \subseteq Q^{c-\delta} \cap \hat{\mathcal{L}}_{p,\gamma}$.

Proof. In order to apply the classical proof of this result, that can be found in [12], the only thing we need to show is the compactness of $\mathcal{K}_Q(c)$. This follows from the uniform Palais–Smale condition proven in Proposition 7.3. Namely, suppose that $z_m \in \hat{\mathcal{L}}_{p,\gamma}$ is a sequence of critical points of Q, with $Q(z_m) \equiv c$. Then, by definition, there exists a sequence $\varepsilon_n \downarrow 0$ and a double sequence $\tilde{z}_{n,m} \in \hat{\mathcal{L}}_{p,\gamma,\varepsilon_n}$ such that, for every fixed m, $\tilde{z}_{n,m}$ satisfies (1), (2) and (3) of Proposition 7.3, and $\tilde{z}_{n,m} \longmapsto z_m$ as $n \longmapsto \infty$ in $H^{1,2}([0,1],\mathcal{M})$. For every $m \in \mathbb{N}$, let n_m be such that

$$\|\tilde{z}_{n_m,m} - z_m\|_{1,2} \leq \frac{1}{m}. \tag{8.5.1}$$

Proposition 7.3 applied to the sequence $w_m = \tilde{z}_{n_m,m}$ gives the existence of a subsequence w_{m_k} converging to an element $z \in \mathcal{K}_Q(c)$, and, from (8.5.1), also z_{m_k} converges to z. This says that $\mathcal{K}_Q(c)$ is compact and concludes the proof. $\square$

We can now prove that the sublevels of Q have finite Ljusternik–Schnirelman category:

Lemma 8.6. *For every $c \in \mathbb{R}$, $\mathrm{cat}_{\hat{\mathcal{L}}_{p,\gamma}}\big(Q^c \cap \hat{\mathcal{L}}_{p,\gamma}\big) < +\infty$.*

Proof. By contradiction, suppose that there exists a number $c \in \mathbb{R}$ such that $\mathrm{cat}_{\hat{\mathcal{L}}_{p,\gamma}}\big(Q^c \cap \hat{\mathcal{L}}_{p,\gamma}\big) = +\infty$, and let $\overline{c} \geq 0$ be defined by

$$\overline{c} = \inf\{c \in \mathbb{R} \mid \underset{\hat{\mathcal{L}}_{p,\gamma}}{\mathrm{cat}}\big(Q^c \cap \hat{\mathcal{L}}_{p,\gamma}\big) = +\infty\}.$$

It is shown in Proposition 8.5 that $\mathcal{K}_Q(\overline{c})$ is compact, so that it can be covered by a finite number of open contractible balls B_i, $i = 1, \dots, k$. Let $U = \bigcup_{i=1}^{k} B_i$; Proposition 8.5 implies that there exists $\delta > 0$ such that

$$\underset{\hat{\mathcal{L}}_{p,\gamma}}{\mathrm{cat}}\big((Q^{\overline{c}+\delta} \cap \hat{\mathcal{L}}_{p,\gamma}) \setminus U\big) \leq \underset{\hat{\mathcal{L}}_{p,\gamma}}{\mathrm{cat}}\big(Q^{\overline{c}-\delta} \cap \hat{\mathcal{L}}_{p,\gamma}\big). \tag{8.6.1}$$

But by definition of $\overline{c}$, we have

$$\begin{aligned}\underset{\hat{\mathcal{L}}_{p,\gamma}}{\mathrm{cat}}\big((Q^{\overline{c}+\delta} \cap \hat{\mathcal{L}}_{p,\gamma}) \setminus U\big) &\geq \underset{\hat{\mathcal{L}}_{p,\gamma}}{\mathrm{cat}}\big(Q^{\overline{c}+\delta} \cap \hat{\mathcal{L}}_{p,\gamma}\big) - \underset{\hat{\mathcal{L}}_{p,\gamma}}{\mathrm{cat}}\big(U\big)\\ &\geq \underset{\hat{\mathcal{L}}_{p,\gamma}}{\mathrm{cat}}\big(Q^{\overline{c}+\delta} \cap \hat{\mathcal{L}}_{p,\gamma}\big) - k = +\infty,\end{aligned}$$

and $\mathrm{cat}_{\hat{\mathcal{L}}_{p,\gamma}}\big(Q^{\overline{c}-\delta} \cap \hat{\mathcal{L}}_{p,\gamma}\big) < +\infty$, which contradicts (8.6.1) and proves the Lemma. $\square$

In the next Proposition we present a non smooth version of the classical min-max argument of the Ljusternik–Schnirelman theory:

Proposition 8.7. *There exist at least $\mathrm{cat}_{\hat{\mathcal{L}}_{p,\gamma}}\big(\hat{\mathcal{L}}_{p,\gamma}\big)$ critical points of Q on $\hat{\mathcal{L}}_{p,\gamma}$.*

Proof. If Q has infinitely many critical points on $\hat{\mathcal{L}}_{p,\gamma}$, then the proof is done. Assume that $\mathcal{K}_Q$ is finite and define $\overline{c} = \max_{z \in \mathcal{K}_Q} Q(z)$. By Corollary 8.4, there exists $d > c$ such that

$$\underset{\hat{\mathcal{L}}_{p,\gamma}}{\mathrm{cat}}\big(\hat{\mathcal{L}}_{p,\gamma}\big) = \underset{\hat{\mathcal{L}}_{p,\gamma}}{\mathrm{cat}}\big(Q^d \cap \hat{\mathcal{L}}_{p,\gamma}\big) = m < +\infty.$$

Clearly, we can assume $M > 1$. For every $k \in \{1, 2, \ldots, m\}$, let

$$\Gamma_k = \{B \subset (Q^c \cap \hat{\mathcal{L}}_{p,\gamma}) \,\big|\, \operatorname*{cat}_{\hat{\mathcal{L}}_{p,\gamma}}(B) \geq k\},$$

and define $c_k = \inf_{B \in \Gamma_k} \sup_{z \in B} Q(z)$. Clearly, the c_k's are well defined, since $\inf_{\hat{\mathcal{L}}_{p,\gamma}} Q \leq c_k \leq d$. The c_k's are critical values for Q on $\hat{\mathcal{L}}_{p,\gamma}$. To prove this, assume by contradiction that, for some k, c_k is not a critical value for Q on $\hat{\mathcal{L}}_{p,\gamma}$. Then, by Proposition 8.2, there would exist $\delta > 0$, with $c_k + \delta < d$, and a homotopy $\eta : [0,1] \times \hat{\mathcal{L}}_{p,\gamma} \longmapsto \hat{\mathcal{L}}_{p,\gamma}$, such that

$$\eta(0, Q^{c_k+\delta} \cap \hat{\mathcal{L}}_{p,\gamma}) = Q^{c_k+\delta} \cap \hat{\mathcal{L}}_{p,\gamma} \tag{8.7.1}$$

and

$$\eta(1, Q^{c_k+\delta} \cap \hat{\mathcal{L}}_{p,\gamma}) = Q^{c_k-\delta} \cap \hat{\mathcal{L}}_{p,\gamma}. \tag{8.7.2}$$

Moreover, by definition of c_k, there would exist a $B \in \Gamma_k$ such that

$$\sup_{z \in B} Q(z) \leq c_k + \delta,$$

so that $B \subseteq Q^{c_k+\delta} \cap \hat{\mathcal{L}}_{p,\gamma}$. Denoting by B' the set $\eta(1, B)$, from (8.7.1) and (8.7.2) it follows that $\operatorname{cat}_{\hat{\mathcal{L}}_{p,\gamma}}(B') = \operatorname{cat}_{\hat{\mathcal{L}}_{p,\gamma}}(B) \geq k$, so that $B' \in \Gamma_k$. Moreover, since $B' \subseteq Q^{c_k-\delta} \cap \hat{\mathcal{L}}_{p,\gamma}$, one has $\sup_{z \in B'} Q(z) \leq c_k - \delta$, which contradicts the minimality of c_k.

If the c_k's are distinct, then we have at least m critical points of Q on $\hat{\mathcal{L}}_{p,\gamma}$, and we are done. If for some $k \in \{1, \ldots, m-1\}$ it is $c_k = c_{k+1}$, then Proposition 8.2 and a classical argument in critical point theory (see [13]) show that there are infinitely many critical points at the level c_k, and the Proposition is proven. $\square$

The next two Corollaries together prove the multiplicity result of Theorem 1.7 and conclude the paper.

Corollary 8.8. *There are at least* $\operatorname{cat}_{\hat{\mathcal{L}}_{p,\gamma}}(\hat{\mathcal{L}}_{p,\gamma})$ *future pointing, light-like geodesics in* $\mathcal{M}$ *joining* p *and* γ.

Proof. From Theorem 1.2, critical points of Q on $\hat{\mathcal{L}}_{p,\gamma}$ correspond to light–like geodesics, up to a reparametrization. The proof is concluded with the observation that this correspondence is one-to-one, since the reparametrizations needed for passing from a critical point of Q to a geodesic and vice versa are uniquely determined. $\square$

Corollary 8.9. *If* $\operatorname{cat}_{\hat{\mathcal{L}}_{p,\gamma}}(\hat{\mathcal{L}}_{p,\gamma}) = +\infty$*, then there exists a sequence of future pointing, light–like geodesics in* $\mathcal{L}_{p,\gamma}$ *with* $\lim_{n\to\infty} F(z_n) = +\infty$.

Proof. If $\operatorname{cat}_{\hat{\mathcal{L}}_{p,\gamma}}(\hat{\mathcal{L}}_{p,\gamma}) = +\infty$, then Q has arbitrarily large critical values on $\hat{\mathcal{L}}_{p,\gamma}$. For, if Q did not have critical values in the half line $[d, +\infty)$, then by Corollary 8.4 and Lemma 8.6, it would be

$$\operatorname*{cat}_{\hat{\mathcal{L}}_{p,\gamma}}(\hat{\mathcal{L}}_{p,\gamma}) = \operatorname*{cat}_{\hat{\mathcal{L}}_{p,\gamma}}(\hat{\mathcal{L}}_{p,\gamma} \cap Q^{d+1}) < +\infty,$$

which is a contradiction.

It follows that there exists a sequence $\lambda_n \geq 0$ of critical values of Q on $\hat{\mathcal{L}}_{p,\gamma}$ such that $\lim_{n\to\infty} \lambda_n = +\infty$, and a sequence $\tilde{z}_n \in \hat{\mathcal{L}}_{p,\gamma}$ of critical points of Q, with $Q(\tilde{z}_n) = \lambda_n$. From Theorem 1.2, for every $n \in \mathbb{N}$ there exists a reparametrization z_n of $\tilde{z}_n$ which is a light–like geodesic in $\mathcal{M}$. Since F is invariant by reparametrization, it follows that

$$F(z_n) = F(\tilde{z}_n) = \sqrt{Q(\tilde{z}_n)} = \sqrt{\lambda_n} \longmapsto +\infty$$

as $n \longmapsto \infty$, which concludes the proof. $\square$

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