Differential and Integral Equations Volume 12, Number 4, July 1999, 499–520.

HIGHER REGULARITY OF SOLUTIONS OF FREE DISCONTINUITY PROBLEMS

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(Submitted by: J.A. Goldstein)

Abstract. In this paper we continue the analysis, started in [6], [7] of the regularity of solutions of free discontinuity problems. We choose as a model problem the minimization of the Mumford-Shah functional. Assuming that in some region the optimal discontinuity set Γ is the graph of a $C^{1,\rho}$ function, we look for conditions ensuring the higher regularity of Γ. Our results are optimal in the two dimensional case. As an application, we prove that in the case of the Mumford-Shah functional and in similar problems the Lavrentiev phenomenon does not occur.

1. Introduction. Free discontinuity problems are variational problems where the functional to be minimized depends on a pair (u, K), where K is a hypersurface, or more generally a closed set, and u is usually smooth outside K. The model problem is the Mumford-Shah functional, proposed in [27] as a variational model of image segmentation:

$$F(u,K) = \int_{\Omega\setminus K} \left[|\nabla u|^2 + \alpha|u-g|^2\right] dx + \beta\mathcal{H}^{n-1}(\Omega\cap K); \qquad (1.1)$$

Accepted for publication October 1997.
AMS Subject Classifications: 49N60, 49Q20, 35J25.

here, $\Omega \subset \mathbf{R}^n$ is an open set, $\alpha, \beta > 0$, $g \in L^\infty(\Omega) \cap L^2(\Omega)$ and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. We refer to [27], [26] for a comprehensive discussion of this model.

The existence of a minimizing pair (u, K), with K closed and $u \in C^1(\Omega \setminus K)$, has been proved by De Giorgi, Carriero and Leaci in [21] (see also [16]) using the theory of SBV functions introduced in [20]. But, while the smoothness of the function u in $\Omega \setminus K$ can be easily investigated using classical arguments based on first variation and elliptic equations techniques, as regards the regularity of K the existence theory ensures only that K is rectifiable – a very mild regularity property.

Recently, we faced the problem of the partial regularity of the discontinuity set K: our main results are presented in [6], [7] (see also [8] for a brief survey) and are based on a double blow-up technique, suitable to exploit the interaction between the volume term and the surface term in functionals like F in (1.1). Our results, which hold for a class of free discontinuity problems including the Mumford-Shah functional, can be briefly summarized by stating that if (u, K) is a minimizing pair then K is locally a $C^{1,\rho}$ hypersurface up to a $\mathcal{H}^{n-1}$ negligible singular set S. Some other results, valid only for the Mumford-Shah functional in two space dimensions, have been obtained with different methods in [18], [17], [11].

The aim of this paper is to improve the regularity results on the discontinuity set K: in particular, we focus our attention on the Mumford-Shah functional and show the following result, which is an immediate consequence of Theorem 4.1:

Theorem 1.1. *Let* (u, K) *be a minimizer of* (1.1)*; then, up to a* $\mathcal{H}^{n-1}$*-negligible singular set* S*, the minimizing set* K *is* (*locally*) *the graph of a* C^1 *function* ϕ*; moreover, in two space dimensions the gradient of* ϕ *is* $C^{0,1}$*, and in the general case* ϕ *is* H^2 *with gradient in* $C^{0,\rho}$ *for any* $\rho < 1$*.*

Moreover (see Theorem 4.2), if the datum g is assumed to belong to $C^{k,\rho}$ with $k \geq 1$ and $\rho \in (0, 1]$, then (for any $n \geq 2$) $\phi \in C^{k+2,s}$ for some $s \in (0, 1]$. The basic idea can be explained as follows: given a minimizing pair (u, K), by the first variation formula, see [13], [27], it is known that the function u solves the equation $\Delta u = u - g$ in $\Omega \setminus K$ with homogeneous Neumann boundary conditions on K, and moreover, under regularity assumptions, the mean curvature of K equals the jump of $|\nabla u|^2 + \alpha(u - g)^2$ across it. It follows that ϕ is a weak solution of a mean curvature equation, and this can be exploited to get further regularity. The first step is the proof of the regularity of ∇u up to (both sides of) K: this amounts to prove a boundary regularity result for elliptic problems in $C^{1,\rho}$ domains which seems to be interesting by itself. This is achieved in §2, proving that ∇u has Hölder continuous traces on both sides of K. In §3 we prove a regularity result for weak solutions of mean curvature equations which will be used in §4 to prove the announced regularity theorem on the discontinuity set K. In §5, coupling an approximation result for SBV functions (see also [22]) to our regularity results, we prove that in Mumford-Shah problem (and even in more general situations, see Remark 5.5 below) the so called Lavrentiev phenomenon does not occur, *i.e.* the minimum of F among all the pairs (u, K) with K closed and $u \in C^1(\Omega \setminus K)$ agrees with the infimum of F among all the pairs (u, K) with K piecewise smooth and $u \in C^\infty(\Omega \setminus K)$.

Finally, we remark that all the results of this paper are independent of [6], [7], and of the existence of minimizers; we simply assume that a minimizing pair (u, K) exists and that, in some region A, K is the graph of a $C^{1,\rho}$ function. It has been proved in [6], [7] that $\mathcal{H}^{n-1}$-all points of a minimizer K have a neighbourhood A where this property holds. This paper, except for the last section, is independent also of the theory of SBV functions, whose main properties will be recalled at the beginning of the last section.

2. Regularity of Neumann problems in $C^{1,\rho}$ domains. Let us recall the definition of the Morrey and Campanato spaces in a regular domain $\Omega \subset \mathbf{R}^n$. For more information we refer to [23]. The space $L^{2,\lambda}(\Omega)$, with $0 \le \lambda \le n$, is the Morrey space of the $L^2(\Omega)$ functions u satisfying

$$\sup_{x\in\Omega,\, \varrho>0} \varrho^{-\lambda} \int_{B_\varrho(x)\cap\Omega} |u|^2 \, dy < +\infty;$$

for $0 \le \lambda \le n+2$, $\mathcal{L}^{2,\lambda}(\Omega)$ denotes the Campanato space of the $L^2(\Omega)$ functions satisfying

$$\sup_{x\in\Omega,\, \varrho>0} \varrho^{-\lambda} \int_{B_\varrho(x)\cap\Omega} |u - u_{x,\varrho}|^2 \, dy < +\infty,$$

where $u_{x,\varrho}$ is the mean value of u in $B_\varrho(x) \cap \Omega$. Finally, notice that for $\lambda > n$, $\mathcal{L}^{2,\lambda}(\Omega)$ is isomorphic to $C^{0,\gamma}(\overline{\Omega})$ for $\gamma = (\lambda - n)/2$.

Let $u \in H^1(\Omega)$ be satisfying, in the distribution sense

$$-\Delta u = f \quad \text{in } \Omega, \qquad\qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma \tag{2.2}$$

where $\Gamma \subset \partial\Omega$ is a $C^{1,\rho}$ hypersurface, relatively open in $\partial\Omega$ and $f \in L^\infty(\Omega)$.

In this section we investigate on the regularity of ∇u near Γ. In the two dimensional case the regularity of ∇u can be easily achieved using conformal mappings and Shwartz's reflection principle:

Theorem 2.1. *Assume $n = 2$. Then ∇u has an extension to $\Omega \cup \Gamma$ which is locally $C^{0,\rho}$ if $\rho < 1$, or $C^{0,s}$ for any $s < \rho$ if $\rho = 1$.*

Proof. It is not restrictive to assume $\rho < 1$. For any $x_0 \in \Gamma$ and for $\varrho > 0$ sufficiently small there is a $C^{1,\rho}$ conformal mapping Φ between $B_\varrho(x_0) \cap \Omega$ and a half ball B^+ such that $B_\varrho(x_0) \cap \partial\Omega = B_\varrho(x_0) \cap \Gamma$ is mapped onto the flat part H of ∂B^+. Setting

$$v(y) = u(\Phi^{-1}(y)), \qquad g(y) = f(\Phi^{-1}(y))|\mathrm{det} J\Phi^{-1}(y)|$$

the function v solves $\Delta v = g$ in B^+ with homogeneous Neumann boundary conditions on H. By the boundary regularity of conformal mappings between $C^{1,\rho}$ domains (see for instance [28], Theorem 3.6) g is bounded near H. By an easy reflection argument the functions v and g can be extended to the whole ball B in such a way that $\Delta v = g$ in B. In particular, $v \in C^{1,s}_{\mathrm{loc}}(B)$ for any $s < 1$. Since $u = v \circ \Phi$, again the $C^{1,\rho}$ regularity of Φ implies our statement. □

In higher dimensions, we still use a change of domain and a reflection argument. However, the change of variables changes the equation hence the proof is much more technical and our result is not sharp as in dimension 2.

Theorem 2.2. *Under the assumptions above, the function ∇u has a locally Hölder continuous extension to $\Omega \cup \Gamma$.*

Equations (2.2) are equivalent to

$$\int_\Omega \langle \nabla u(x), \nabla\varphi(x)\rangle \, dx = \int_\Omega f(x)\varphi(x)\, dx \tag{2.3}$$

for any $\varphi \in C^1_0(\mathbf{R}^n)$ with $\varphi = 0$ on $\partial\Omega\backslash\Gamma$. We know that for any $x_0 \in \Gamma$ there is a $C^{1,\rho}$ diffeomorphism Φ of a neighbourhood U of x_0 and the unit cube Q mapping $U \cap \Omega$ into $Q^+ = Q \cap \{x_n > 0\}$ and $U \cap \partial\Omega$ into $H = Q \cap \{x_n = 0\}$. Then, assuming that U is sufficiently small to have the inclusion $U \cap \partial\Omega \subset \Gamma$, condition (2.3) implies

$$\int_{Q^+} A_{ij} \frac{\partial v}{\partial y_i} \frac{\partial \phi}{\partial y_j} \, dy = \int_{Q^+} g(y)\phi(y)\, dy \tag{2.4}$$

for any $\phi \in C_0^1(\mathbf{R}^n)$ with $\phi = 0$ on $\partial Q^+ \setminus H$, and with $v(y) = u\big(\Phi^{-1}(y)\big)$, $g(y) = f\big(\Phi^{-1}(y)\big)\big|\det J\Phi^{-1}(y)\big|$ and

$$A_{ij}(y) = \sum_{k=1}^{n} \frac{\partial \Phi^i}{\partial x_k}\big(\Phi^{-1}(y)\big)\frac{\partial \Phi^j}{\partial x_k}\big(\Phi^{-1}(y)\big)\big|\det J\Phi^{-1}(y)\big|. \qquad (2.5)$$

The functions A_{ij} are Hölder continuous and g is bounded. Moreover, we may assume that $|\det J\Phi^{-1}|$ is bounded away from 0 in U and therefore the matrix A is uniformly elliptic, *i.e.*,

$$A_{ij}(y)\xi_i\xi_j \geq \theta|\xi|^2 \qquad \forall \xi \in \mathbf{R}^n,\ \forall y \in Q \qquad (2.6)$$

for some constant $\theta > 0$. Then, since $x_0 \in \Gamma$ is arbitrary, Theorem 2.2 is a straightforward consequence of the following result:

Theorem 2.3. *Let $v \in H^1(Q^+)$ be satisfying* (2.4) *with the matrix A ρ-Hölder continuous, uniformly elliptic in Q^+ and $g \in L^\infty(Q^+)$. Then, ∇v has a locally Hölder continuous extension to $Q^+ \cup H$.*

The proof of Theorem 2.3 will be achieved by a reflection argument. For any $x = (x', x_n) \in Q^-$ we define $v(x) = v(x', -x_n)$, $g(x) = g(x', -x_n)$ and

$$A_{ij}(x', x_n) = \begin{cases} A_{ij}(x', -x_n) & \text{if either } 1 \leq i,\ j < n \text{ or } i = j = n; \\ -A_{ij}(x', -x_n) & \text{otherwise.} \end{cases}$$

In this way the coefficients $A_{in}(x)$, $A_{ni}(x)$ are (for $i < n$) odd with respect to x_n, while v, g and all the other coefficients are even with respect to the same variable. We point out that

(a) $A \in C^{0,\rho}(Q^+) \cap C^{0,\rho}(Q^-)$;

(b) A is uniformly elliptic.

This extension of v, g and A to Q is motivated by the equation

$$\int_Q A_{ij}\frac{\partial v}{\partial y_i}\frac{\partial \phi}{\partial y_j}\,dy = \int_\Omega g(y)\phi(y)\,dy \qquad \forall \phi \in C_0^1(Q) \qquad (2.7)$$

that immediately follows by (2.4). Then, Theorem 2.3 is a direct consequence of the following interior regularity result:

Theorem 2.4. *Let $v \in H^1(Q)$ be satisfying* (2.7) *and assume that A fulfills* (a), (b). *Let*

$$D_\tau v = \big(\nabla_1 v, \ldots, \nabla_{n-1} v\big), \qquad D_{\mathrm{C}} v = \sum_{i=1}^{n} A_{in}\nabla_i v.$$

Then, $D_\tau v$ and $D_{\rm C} v$ are locally Hölder continuous in Q.

Proof. In the following, c stands for a generic (computable) constant depending only on n, ρ, the ellipticity constant θ, the Hölder norm of A in Q^+ and in Q^- and the L^∞ norm of g in Q.

Let $B_R(x_0)$ be a ball contained in Q. We denote by A^+ and A^- the averages of A on $B_R(x_0)\cap Q^+$ and $B_R(x_0)\cap Q^-$ respectively, for $x \in B_R(x_0)$ we set

$$\bar A(x) = \begin{cases} A^+ & \text{if } x_n > 0; \\ A^- & \text{if } x_n < 0 \end{cases}$$

and we denote by $w \in H^1(B_R)$ the solution of the problem

$$-\mathrm{div}\big(\bar A(x-x_0)\nabla w(x)\big) = 0, \qquad w(x) = v(x+x_0) \quad \text{on } \partial B_R. \tag{2.8}$$

The Hölder continuity of A implies

$$|A(x) - \bar A(x)| \le cR^\rho \qquad \forall x \in B_R(x_0). \tag{2.9}$$

Setting

$$\bar D_{\rm C} w(x) = \sum_{i=1}^n \bar A_{in}(x)\nabla_i w(x)$$

the plan of the proof is the following: in the first step we estimate $D_\tau w$; in the second step we estimate $\bar D_{\rm C} w$; in the last ones we get a $L^{2,\lambda}$ estimate on ∇v and eventually $\mathcal{L}^{2,\mu}$ estimates which lead to the Hölder continuity of $D_\tau w$ and $D_{\rm C} w$.

Step 1. Using the fact that $\bar A$ is constant in the tangential directions, a standard application of Nirenberg's difference quotient method gives that $D_\tau w \in H^1_{\rm loc}(B_R)$ and the Caccioppoli estimate

$$\int_{B_\varrho} |\nabla(D_\tau w)|^2\, dx \le c\varrho^{-2} \int_{B_{2\varrho}} |D_\tau w - (D_\tau w)_{2\varrho}|^2\, dx \tag{2.10}$$

for $0 < \varrho \le R/2$, where as usual $(h)_\varrho$ denotes the average of the function h in B_ϱ. Moreover, $D_\tau w$ satisfies

$$\int_{B_R} \bar A_{ij} \frac{\partial D_\tau w}{\partial x_i}\frac{\partial \phi}{\partial x_j}\, dx = 0 \qquad \forall \phi \in C^1_0(B_R).$$

The regularity theory for solutions u of elliptic equations with measurable coefficients (see for instance [23], page 80 and page 82) yields the estimates

$$\varrho^n \max_{B_\varrho} |u|^2 \leq c \int_{B_{2\varrho}} |u^2(x)|\, dx \qquad 0 < \varrho \leq R/2$$

$$\int_{B_\varrho} |\nabla u|^2\, dx \leq c\left(\frac{\varrho}{R}\right)^{n-2+2\gamma} R^{-2} \int_{B_R} |u - (u)_R|^2\, dx \qquad 0 < \varrho \leq R.$$

Applying these estimates to $u = D_\tau w$ we infer

$$\varrho^n \sup_{B_\varrho} |D_\tau w|^2 \leq c \int_{B_{2\varrho}} |D_\tau w|^2\, dx, \qquad 0 < \varrho \leq R/2 \tag{2.11}$$

$$\int_{B_\varrho} |D_\tau w - (D_\tau w)_\varrho|^2 dx \leq c\big(\frac{\varrho}{R}\big)^{n+2\gamma} \int_{B_R} |D_\tau w - (D_\tau w)_R|^2 dx \quad 0 < \varrho \leq R \tag{2.12}$$

for some $\gamma > 0$ depending only on n, the ellipticity constant θ and the L^∞ norm of A.

Step 2. Since $D_\tau(\bar{D}_{\mathrm{C}} w) = \bar{D}_{\mathrm{C}}(D_\tau w)$, we have that $\bar{D}_{\mathrm{C}} w$ has tangential distributional derivatives in $L^2_{\mathrm{loc}}(B_R)$. To estimate the normal derivative of $\bar{D}_{\mathrm{C}} w$ we use equation (2.8). Indeed, the equality

$$\nabla_n(\bar{D}_{\mathrm{C}} w) = -\sum_{i=1}^{n-1} \sum_{j=1}^{n} \bar{A}_{ij} \nabla^2_{ij} w$$

implies that also $\nabla_n(\bar{D}_{\mathrm{C}} w)$ belongs to $L^2_{\mathrm{loc}}(B_R)$. Moreover, we have the estimate

$$|\nabla(\bar{D}_{\mathrm{C}} w)| \leq c|\nabla(D_\tau w)|. \tag{2.13}$$

Using Poincaré inequality, (2.13) and (2.10) we get

$$\begin{aligned}\int_{B_\varrho} |\bar{D}_{\mathrm{C}} w - (\bar{D}_{\mathrm{C}} w)_\varrho|^2\, dx &\leq c\varrho^2 \int_{B_\varrho} |\nabla(\bar{D}_{\mathrm{C}} w)|^2\, dx \leq c\varrho^2 \int_{B_\varrho} |\nabla(D_\tau w)|^2\, dx \\ &\leq c \int_{B_{2\varrho}} |D_\tau w - (D_\tau w)_{2\varrho}|^2\, dx,\end{aligned}$$

hence (2.12) yields

$$\int_{B_\varrho} |\bar{D}_{\mathrm{C}} w - (\bar{D}_{\mathrm{C}} w)_\varrho|^2\, dx \leq c\left(\frac{\varrho}{R}\right)^{n+2\gamma} \int_{B_R} |D_\tau w - (D_\tau w)_R|^2\, dx. \tag{2.14}$$

The same estimates are valid in balls not centered at the origin, hence the isomorphism between the Campanato space $\mathcal{L}^{2,n+2\gamma}$ and $C^{0,\gamma}$ implies that $\bar{D}_{\mathbb{C}}w \in C^{0,\gamma}_{loc}(B_R)$ and that in particular

$$R^n \sup_{B_{R/2}} |\bar{D}_{\mathbb{C}}w|^2 \le c \int_{B_R} |D_\tau w - (D_\tau w)_R|^2 \, dx,$$

hence,

$$R^n \sup_{B_{R/2}} |\bar{D}_{\mathbb{C}}w|^2 \le c \int_{B_R} |\nabla w|^2 \, dx. \tag{2.15}$$

Step 3. Let $\hat{v}(x) = w(x - x_0)$. By (2.7) and (2.8) we infer

$$\begin{aligned}&\int_{B_R(x_0)} \bar{A}_{ij}(x) \left(\frac{\partial v}{\partial x_i} - \frac{\partial \hat{v}}{\partial x_i} \right) \frac{\partial \phi}{\partial x_j} \, dx \\ &= \int_{B_R(x_0)} \left[\bar{A}_{ij}(x) - A_{ij}(x) \right] \frac{\partial v}{\partial x_i} \frac{\partial \phi}{\partial x_j} \, dx + \int_{B_R(x_0)} g(x)\phi(x) \, dx\end{aligned}$$

for any $\phi \in H^1_0(B_R(x_0))$. Choosing $\phi = v - \hat{v}$, the ellipticity of $\bar{A}$, (2.9) and Poincaré inequality yield

$$\int_{B_R(x_0)} |\nabla v - \nabla \hat{v}|^2 \, dx \le cR^{2\rho} \int_{B_R(x_0)} |\nabla v|^2 \, dx + cR^{n+2}. \tag{2.16}$$

To estimate the Dirichlet integral we notice that (we use (2.11), (2.15) and (2.16)) for $0 < \varrho \le R/2$ we have

$$\begin{aligned}&\int_{B_\varrho(x_0)} |\nabla v|^2 \, dx \le 2 \int_{B_\varrho(x_0)} |\nabla \hat{v}|^2 \, dx + 2 \int_{B_\varrho(x_0)} |\nabla v - \nabla \hat{v}|^2 \, dx \\ &\le c\varrho^n \sup_{B_{R/2}(x_0)} |\nabla \hat{v}|^2 + cR^{2\rho} \int_{B_R(x_0)} |\nabla v|^2 \, dx + cR^{n+2} \\ &\le c \left(\frac{\varrho}{R} \right)^n \int_{B_R(x_0)} |\nabla \hat{v}|^2 \, dx + cR^{2\rho} \int_{B_R(x_0)} |\nabla v|^2 \, dx + cR^{n+2} \\ &\le c \left(\frac{\varrho}{R} \right)^n \int_{B_R(x_0)} |\nabla v|^2 \, dx + cR^{2\rho} \int_{B_R(x_0)} |\nabla v|^2 \, dx + cR^{n+2}\end{aligned}$$

hence, by applying Lemma 2.5 below, with $\alpha = n$, $\beta < n$, $s = 2\rho$, to

$$f(\varrho) = \int_{B_\varrho(x_0)} |\nabla v|^2 \, dx$$

we infer

$$|\nabla v| \in L^{2,\lambda}_{\mathrm{loc}}(Q) \qquad \forall \lambda < n. \tag{2.17}$$

In the following we will use (2.17) with λ sufficiently close to n, such that $n - \lambda < \rho$.

Step 4. Let Q' be a cube strictly contained in Q and assume that $2R < \delta$, δ being the distance of Q' from ∂Q. For $x_0 \in Q'$, $0 < \varrho \le R$ and $d \in \mathbf{R}$, using (2.9), (2.16), (2.17), we have

$$\int_{B_\varrho(x_0)} |D_{\mathrm{C}} v - d|^2\,dx \le 2\int_{B_\varrho(x_0)} |\bar{D}_{\mathrm{C}} v - d|^2\,dx + c\varrho^{2\rho}\int_{B_\varrho(x_0)} |\nabla v|^2\,dx$$
$$\le c\int_{B_\varrho} |\bar{D}_{\mathrm{C}} w - d|^2\,dx + cR^{n+\rho}.$$

Using $d = (\bar{D}_{\mathrm{C}} w)_\varrho$ in the inequality above and also (2.14) we get

$$\begin{aligned}&\int_{B_\varrho(x_0)} |D_{\mathrm{C}} v - (D_{\mathrm{C}} v)_{x_0,\varrho}|^2\,dx\\ \le\;& c\Big(\frac{\varrho}{R}\Big)^{n+2\gamma}\int_{B_R(x_0)} |D_\tau v - (D_\tau v)_{x_0,R}|^2\,dx + cR^{n+\rho}.\end{aligned} \tag{2.18}$$

By a similar argument, based on (2.12), we get

$$\begin{aligned}&\int_{B_\varrho(x_0)} |D_\tau v - (D_\tau v)_{x_0,\varrho}|^2\,dx\\ \le\;& c\Big(\frac{\varrho}{R}\Big)^{n+2\gamma}\int_{B_R(x_0)} |D_\tau v - (D_\tau v)_{x_0,R}|^2\,dx + cR^{n+\rho}.\end{aligned} \tag{2.19}$$

Inequalities (2.18), (2.19) together with Lemma 2.5, where we take $\alpha = n+2\gamma$, $\beta < \min\{n+\rho, \alpha\}$ imply that $D_{\mathrm{C}} v$ and $D_\tau v$ belong to the Campanato space $\mathcal{L}^{2,n+\sigma}(Q')$, with any $\sigma < \min\{\rho, 2\gamma\}$, hence they are Hölder continuous in Q'. □

Lemma 2.5. *Let $f : (0, R_0] \to [0, +\infty)$ be a nondecreasing function and assume that*

$$f(\varrho) \le A\Big[\Big(\frac{\varrho}{R}\Big)^\alpha + R^s\Big] f(R) + BR^\beta, \qquad 0 < \varrho \le R/2 < R \le R_0$$

for some constants A, $B \ge 0 < \beta < \alpha$, $s > 0$. Then, the supremum of $f(\varrho)/\varrho^\beta$ in $(0, R_0]$ is bounded by a constant depending only on A, B, α, β, s, R_0 and $f(R_0)$.

Proof. Let us fix $\beta < \gamma < \alpha$ and $0 < \tau \le 1/2$ such that $2A\tau^\alpha \le \tau^\gamma$. If we choose $R_1 \le R_0$ such that $R_1^s \le \tau^\alpha$ from the assumption we get for any $i = 1, 2, \ldots$

$$f(\tau^{i+1}R_1) \le \tau^\gamma f(\tau^i R_1) + B\tau^{i\beta}R_1^\beta.$$

Therefore, iterating this estimate, we obtain for any positive integer k

$$f(\tau^{k+1}R_1) \le \tau^{\gamma k} f(\tau R_1) + B\tau^{k\beta}R_1^\beta \sum_{i=1}^{k-1} \tau^{i(\gamma-\beta)} \le c\tau^{k\beta}\left[f(R_1) + BR_1^\beta\right].$$

If $\varrho \le R_1/2$, choosing k such that $\tau^{k+1}R_1 < \varrho \le \tau^k R_1$, we then get

$$\frac{f(\varrho)}{\varrho^\beta} \le c\Big[\frac{f(R_1)}{R_1^\beta} + B\Big] \le c\,[f(R_0) + B]$$

and from this inequality the result immediately follows. □

3. Regularity of solutions of the mean curvature equation. In this section we prove an interior regularity result for solutions $\phi \in C^1(D)$ of the prescribed mean curvature equation

$$-\mathrm{div}\Big(\frac{\nabla\phi}{\sqrt{1+|\nabla\phi|^2}}\Big) = H, \qquad \textit{in } D. \tag{3.20}$$

Here $D \subset \mathbf{R}^m$ is an open set; coupling the results of this section (with $m = (n-1)$) with the results of the previous one, the higher regularity of the discontinuity set will be achieved in the forthcoming section.

Our result is the following:

Theorem 3.1. *Assume that* (3.20) *holds with $H \in L^\infty(D)$ and $\phi \in C^1(D)$. Then, $\phi \in C^{1,1}_{\mathrm{loc}}(D)$ if $m = 1$ and $\phi \in H^2_{\mathrm{loc}}(D)$, $\nabla\phi \in C^{0,\rho}_{\mathrm{loc}}(D; \mathbf{R}^m)$ for any $\rho < 1$ if $m > 1$.*

Proof. In the case $m = 1$, the equation trivially implies that $\phi'/\sqrt{1+|\phi'|^2}$ is locally Lipschitz continuous in D, hence ϕ is locally $C^{1,1}$ in D. In the case $m > 1$, we notice that ϕ satisfies an equation of the form

$$-\mathrm{div}\big(F(\nabla\phi)\big) = H \in L^\infty(D) \tag{3.21}$$

with $F(z) = z/(1+|z|^2)^{1/2}$. Being F the gradient of $\sqrt{1+|z|^2}$, F is locally strictly monotone. Then, the H^2 regularity of ϕ follows by Proposition 3.2

below; the Hölder continuity of $\nabla\phi$ follows by writing the equation in non divergence form

$$\sum_{i,j=1}^{m} A_{ij}(x)\frac{\partial^2\phi}{\partial x_i \partial x_j}(x) = -H(x)$$

with

$$A_{ij}(x) = \frac{\delta_{ij} - \nu_i(x)\nu_j(x)}{(1 + |\nabla\phi(x)|^2)^{1/2}}, \qquad \nu_i(x) = \frac{\frac{\partial\phi}{\partial x_i}(x)}{(1 + |\nabla\phi(x)|^2)^{1/2}}$$

and applying the regularity theory for equations in non-divergence form with continuous coefficients (see for instance [24], Theorem 9.15). □

Proposition 3.2. *Let $\phi \in C^1(D)$ be a solution, in the distribution sense, of equation* (3.21)*, where $F \in C^1(\mathbf{R}^m)$ is a locally strictly monotone function,* i.e.*, for any compact set $K \subset \mathbf{R}^m$, we have*

$$(F(z) - F(z'), z - z') \geq c_K |z - z'|^2 \qquad \forall z,\, z' \in K \tag{3.22}$$

for a suitable constant $c_K > 0$. Then, $\phi \in H^2_{\mathrm{loc}}(D)$.

Proof. Possibly reducing D we can assume that the gradient of ϕ is bounded by a constant M on D. We also assume, to simplify some integral estimates, that D is a ball of radius less than 1. We denote by c the constant in (3.22) corresponding to the compact set $K = \overline{B}_M(0)$ and by L the Lipschitz constant of F on K.

Given an integer s in $[1, m]$, let $\Delta_h v(x) = [v(x + he_s) - v(x)]/h$ be the difference quotient of a generic function v in the direction e_s. We will use the following inequality

$$\int_{\mathbf{R}^m} \psi^2(x)|\Delta_{-h}v(x)|\, dx \leq \int_{\mathbf{R}^m} \chi_h(x)|\nabla_s v(x)|\, dx \tag{3.23}$$

with

$$\chi_h(x) = \int_0^1 \psi^2(x + the_s)\, dt,$$

which is true provided ψ is continuous with compact support, and v is C^1 in the h-neigbourhood of the support of ψ. The proof is standard, being based on the integration of

$$\Delta_{-h}v(x) = \int_0^1 \nabla_s v(x - the_s)\, dt$$

and Fubini's theorem, so we omit it.

Let $\eta \in C_0^1(D)$; applying (3.21) (for h sufficiently small, depending on the support of η) with test functions $\eta(x)$ and $\eta(x-he_s)$ and subtracting both sides we get

$$\frac{1}{h}\int_D [F_i(\nabla\phi(x+he_s)) - F_i(\nabla\phi(x))]\nabla_i\eta(x)\,dx = -\int_D H(x)\Delta_{-h}\eta(x)\,dx. \tag{3.24}$$

Now we choose $\psi \in C_0^1(D)$ such that $0 \le \psi \le 1$ and apply (3.24) with $\eta = \psi^2\Delta_h\phi$; the integral containing H is equal to

$$\int_D H\Delta_{-h}\psi^2\Delta_{-h}\phi\,dx + \int_D H\psi^2\Delta_{-h}\Delta_h\phi\,dx$$

and can be estimated from above, using (3.23) with $v = \Delta_h\phi$, by

$$\|H\|_\infty\Big[2M\|\nabla\psi\|_\infty + \int_D \chi_h|\Delta_h\nabla_s\phi|\,dx\Big].$$

Adding and subtracting ψ^2, and using $\|\chi_h - \psi^2\|_\infty \le 2h\|\nabla\psi\|_\infty$, this term can be estimated from above by

$$\|H\|_\infty\Big[6M\|\nabla\psi\|_\infty + \int_D \psi^2|\Delta_h\nabla_s\phi|\,dx\Big].$$

Now we estimate from below the first term in (3.24). Since $\nabla\eta = \nabla\psi^2\Delta_h\phi + \psi^2\Delta_h\nabla\phi$, using (3.22) we obtain the estimate

$$c\int_D \psi^2|\Delta_h\nabla\phi|^2\,dx - 2LM\|\nabla\psi\|_\infty\int_D \psi|\Delta_h\nabla\phi|\,dx.$$

Taking into account these estimates, we obtain that

$$\int_D \psi^2|\Delta_h\nabla\phi|^2\,dx$$

is bounded as $h \to 0$. Hence, $\nabla_s\phi \in H^1$ in the interior of the support of ψ. Since ψ is arbitrary, the statement follows. □

4. Higher regularity of the discontinuity set. Let $A \subset \mathbf{R}^n$ be an open set, $g \in L^\infty(A)$, and let (u, Γ) be an admissible pair for the Mumford-Shah functional F, minimizing F with respect to relatively compact perturbations in A of u and Γ. In this section we assume that $\Gamma \cap A$ is a $C^{1,\rho}$ hypersurface, we assume for simplicity $\alpha = \beta = 1$ in (1.1) and we investigate the higher regularity of Γ.

There is no loss of generality if we assume that $A = D \times (-1, 1)$ for some open set $D \subset \mathbf{R}^{n-1}$ and that $\Gamma \cap A$ is the graph of a $C^{1,\rho}$ function ϕ defined on D such that $\|\phi\|_\infty = \varrho < 1$. We denote by $A^+ \subset A$ and $A^- \subset A$ respectively the epigraph and the hypograph of ϕ.

By the regularity results of the preceding sections (see Theorem 2.2 or, in the case $n = 2$, Theorem 2.1) applied with $\Omega = A^\pm$, we can also assume that u and ∇u are Hölder continuous up to Γ. We will denote by (z, y) the generic point in A, with $z \in D$ and $y \in (-1, 1)$. We first see what can be said without extra regularity assumptions on g:

Theorem 4.1. *Under the assumptions above, ϕ is a solution in the distribution sense of the prescribed mean curvature equation* (3.20) *for some $H \in L^\infty(D)$. In particular, by Theorem* 3.1, *$\phi \in C^{1,1}_{\mathrm{loc}}(D)$ if $n = 2$ and $\phi \in H^2_{\mathrm{loc}}(D)$, $\nabla\phi \in C^{0,\rho}_{\mathrm{loc}}(D; \mathbf{R}^{n-1})$ for any $\rho < 1$ if $n > 2$.*

Proof. We will first show a minimality condition satisfied by ϕ; then, a suitable choice of the comparison function will allow to conclude.

Step 1. Let us prove, by a deformation argument, that ϕ satisfies the following minimality condition:

$$\int_D \sqrt{1 + |\nabla\phi|^2}\, dz \le \int_D \sqrt{1 + |\nabla(\phi + \psi)|^2}\, dz + \lambda \int_D |\psi(z)|\, dz \qquad (4.25)$$

for any function $\psi \in C^1_0(D)$ such that $\Lambda\|\psi\|_{C^1} \le 1$. The constants λ, Λ depend only on ϕ but not on ψ.

Indeed, let Γ' be the graph of $\phi + \psi$ and assume that Λ is so large that $\Lambda\|\psi\|_{C^1} \le 1$ implies $\|\phi\|_\infty + 2\|\psi\|_\infty \le 1$ and $\|\nabla\psi\|_\infty \le 1$. Let $x = (z, y)$ and let $\Phi : A \to A$ be the map defined by $\Phi(z, y) = (z, L_z(y))$ where $L_z(y) : [-1, 1] \to [-1, 1]$ is a piecewise linear bijection, $1/2 \le L'_z \le 3/2$, and for any $z \in D$ satisfies the properties

$$L_z(\phi(z)) = \phi(z) + \psi(z), \qquad L_z(y) = y \quad \text{if } |y - \phi(z)| \ge 2|\psi(z)|.$$

It is easy to check that the restriction of Φ to ∂A is the identity map, and that Φ is invertible. Moreover, since the coefficients of $L_z(y)$ are linear functions of $\phi(z)$, $\psi(z)$, the Lipschitz constant of Φ^{-1} can be estimated with the Lipschitz constants of ϕ, $\phi+\psi$. In addition,

$$\mathcal{L}^n\big(\{x\in A:\Phi(x)\neq x\}\big)\le 4\int_D|\psi(z)|\,dz. \tag{4.26}$$

The function $v=u\circ\Phi^{-1}$ has the same trace of u on ∂A and its discontinuity set is contained in Γ'. Taking into account the boundedness of u and ∇u, comparing the energy of (u,Γ) with the energy of (v,Γ') and taking into account

$$\mathcal{L}^n\big(\{x\in A:u(x)\neq v(x)\}\big)\le 4\int_D|\psi(z)|\,dz.$$

we obtain

$$\mathcal{H}^{n-1}(\Gamma)\le\mathcal{H}^{n-1}(\Gamma')+4\lambda\int_D|\psi(z)|\,dz$$

for λ large enough (depending on the bounds on ∇u and on $\nabla\Phi^{-1}$), as claimed.

Step 2. Let $\varphi\in C^1_0(D)$ and $\varepsilon>0$. Taking $\psi=\varepsilon\varphi$ in the minimality condition (4.25), differentiation in ε gives

$$-\int_D\frac{\langle\nabla\phi,\nabla\varphi\rangle}{\sqrt{1+|\nabla\phi|^2}}\,dz\le 4\lambda\|\varphi\|_1.$$

Since φ is arbitrary, equation (3.20) is satisfied in the distribution sense for some function H whose L^∞ norm does not exceed 4λ. □

Now we make additional regularity assumptions on g. We need at least C^1 regularity to start the bootstrap argument.

Theorem 4.2. *If $g\in C^{k,\rho}_{\rm loc}(A)$ for some integer $k\ge1$ and $\rho\in(0,1]$, then there exists $s\in(0,1]$ depending on n, ρ such that ϕ belongs to $C^{k+2,s}_{\rm loc}(D)$. In particular, the smoothness of g implies the smoothness of the free discontinuity set Γ.*

Proof. We first assume $k=1$. We know that u satisfies the equation

$$\Delta u=(u-g)\quad\text{in } A^{\pm},\qquad \frac{\partial u}{\partial\nu}=0\quad\text{in }\Gamma. \tag{4.27}$$

We recall that, by Theorem 2.2, u and ∇u are Hölder continuous up to Γ, on both sides. Orienting with the inner normal to A^+

$$\nu=\frac{(-\nabla\phi,1)}{\sqrt{1+|\nabla\phi|^2}}$$

we will prove by a deformation argument that the scalar mean curvature of Γ at x is given by the Hölder continuous function

$$H(x) = \big[|\nabla u|^2 + (u-g)^2\big](x), \tag{4.28}$$

where $[\,\cdot\,]$ denotes the jump across Γ. Indeed, if $\eta \in C_0^\infty(A)$ is given, setting for ε small $\tau_\varepsilon(x) = x + \varepsilon\eta(x)$ and $u_\varepsilon(\tau_\varepsilon(x)) = u(x)$, computing the first variation along u_ε leads to the following equation:

$$\begin{aligned}&\int_A \big[\big(|\nabla u|^2 + (u-g)^2\big)\mathrm{div}\eta - 2(u-g)\langle\nabla g, \eta\rangle - 2\langle \nabla u, \nabla u \nabla\eta\rangle\big]\,dx\\ &= -\int_\Gamma \delta_i\eta_i d\mathcal{H}^{n-1}\end{aligned} \tag{4.29}$$

where δ stands for the tangential differential operator. Fix a cut-off function $\chi \in C_0^1(-1,1)$ such that $\chi \equiv 1$ in $[-\varrho, \varrho]$ (recall that $\varrho = \|\phi\|_\infty$) and define $\eta_i(x) = 0$ for $i < n$ and $\eta_n(x_1, \dots, x_n) = \varphi(x_1, \dots, x_{n-1})\chi(x_n)$ with $\varphi \in C_0^1(D)$. An easy computation shows that

$$\int_\Gamma \delta_i\eta_i d\mathcal{H}^{n-1} = \int_D \frac{\langle\nabla\phi, \nabla\varphi\rangle}{\sqrt{1+|\nabla\phi|^2}}\,dz. \tag{4.30}$$

Integrating by parts on A^+ and using the equation $\Delta u = (u-g)$ and the homogeneous Neumann boundary conditions we obtain

$$\begin{aligned}&\int_{A^+} \big[\big(|\nabla u|^2 + (u-g)^2\big)\mathrm{div}\eta\big]\,dx - 2\int_{A^+}\langle\nabla u, \nabla u\nabla\eta\rangle\,dx\\ =\;& 2\int_{A^+} \Delta u\langle\eta, \nabla u\rangle + \langle\eta, \nabla u \nabla^2 u\rangle\,dx - \int_{A^+}\big\langle\eta, \nabla\big(|\nabla u|^2 + (u-g)^2\big)\big\rangle\,dx\\ &-\int_\Gamma \big[|\nabla u|^2 + (u-g)^2\big]^+\langle\eta,\nu\rangle\,d\mathcal{H}^{n-1}\\ =\;& -\int_\Gamma \big[|\nabla u|^2 + (u-g)^2\big]^+\langle\eta,\nu\rangle\,d\mathcal{H}^{n-1} + 2\int_{A^+}(u-g)\langle\eta, \nabla g\rangle\,dx.\end{aligned}$$

By a similar argument on A^- we get

$$\begin{aligned}&\int_{A^-} \big[\big(|\nabla u|^2 + (u-g)^2\big)\mathrm{div}\eta\big]\,dx - 2\int_{A^-}\langle\nabla u, \nabla u\nabla\eta\rangle\,dx \\ =\;& \int_\Gamma \big[|\nabla u|^2 + (u-g)^2\big]^-\langle\eta,\nu\rangle\,d\mathcal{H}^{n-1} + 2\int_{A^-}(u-g)\langle\eta, \nabla g\rangle\,dx.\end{aligned} \tag{4.31}$$

The above argument works if u is of class H^2 up to Γ. However, in our case u is only $H^2_{\rm loc}$ in $A^{\pm}$. This difficulty can be overcome by lifting up or pulling down the graph of ϕ, so getting integration by parts formulas similar to the ones above, with extra terms which disappear in the limit because of the (classical) Neumann boundary condition satisfied by u.

Hence, (4.28) follows by (4.29), (4.30), (4.31). By Theorem 3.1, $\phi \in H^2_{\rm loc}(D)$; expanding the divergence in (3.20) we can use the Schauder estimates for equations in non-divergence form (see for instance [23], Theorem 3.6) to conclude that $\phi \in C^{2,s}_{\rm loc}(D)$ for some $s \in (0,1)$.

At this point, since $\phi \in C^{2,s}_{\rm loc}(D)$, the classical Schauder estimates for the Neumann problem imply that u has the same regularity up to Γ. Coming back to (3.20), (4.28) we get the $C^{3,s}_{\rm loc}$ regularity of ϕ.

If $k > 1$, the higher regularity of u up to Γ and of ϕ follows by the usual bootstrap argument, based on (4.27), (3.20) and (4.28). □

5. Density of functions with regular jump set. In this section we show that in Mumford-Shah problem the Lavrentiev phenomenon does not occur, *i.e.*, that the infimum of the functional F among all pairs (Γ, u) with Γ piecewise smooth (according to Definition 5.1) and $u \in C^\infty(\Omega \setminus \Gamma)$ coincides with the minimum of the weak form of F. This property is particularly useful in connection with the numerical minimization of the Mumford-Shah functional (see for instance [15], [9]).

Our strategy is to prove a more general approximation theorem of SBV functions by smooth functions with discontinuities contained in piecewise smooth sets. A byproduct of this theorem is the absence of the Lavrentiev phenomenon for general classes of functionals, including the Mumford-Shah functional (see Remark 5.5). The proof of the approximation property is achieved in two steps: first, using the partial regularity results of [6], [7] and Section 4, we prove in Proposition 5.3 that the approximation property holds for minimizers of the Mumford-Shah functional. Successively we see that a more general class of SBV functions can be approximated by minimizers.

Let us start by recalling the definition and the main properties of the SBV functions; for more details we refer to [20], [3], [5], [2]. A function $u \in L^1(\Omega)$ is said to be in $SBV(\Omega)$ if its distributional gradient Du is a Radon vector measure with finite total variation (*i.e.* $u \in BV(\Omega)$) and moreover $Du = \nabla u \, dx + Ju$, where Ju is the restriction of Du to the complement S_u of the Lebesgue set of u and Ju is absolutely continuous with respect to

$\mathcal{H}^{n-1} \llcorner S_u$. By ∇u we denote the density of Du with respect to the Lebesgue measure; it can be interpreted as an approximate differential of u. The weak form of the Mumford-Shah functional in $SBV(\Omega)$ can then be formulated as follows:

$$G(u) = \int_\Omega \big[|\nabla u|^2 + \alpha |u - g|^2 \big] \, dx + \beta \mathcal{H}^{n-1}(S_u)$$

and it can be proved that a minimizer $u \in SBV(\Omega)$ exists (see [2]).

Let us now define the class of piecewise smooth manifolds that will be considered.

Definition 5.1. Let $K \subset \mathbf{R}^n$ be compact. We say that K is piecewise smooth if there exist finitely many C^∞ hypersurfaces $\Gamma_1, \ldots, \Gamma_N$ with C^∞ boundaries $\partial\Gamma_1, \ldots, \partial\Gamma_N$ such that $K = \bigcup_{i=1}^N \Gamma_i$ and $\big(\Gamma_i \backslash \partial\Gamma_i\big) \cap \big(\Gamma_j \backslash \partial\Gamma_j\big) = \emptyset$ whenever $i \neq j$. We denote by $\mathcal{K}$ the class of these sets.

The main result of this section is the following theorem, which is an immediate consequence of Theorem 5.4.

Theorem 5.2. *Let $\Omega \subset \mathbf{R}^n$ be a bounded open set with Lipschitz boundary, α, β positive and $g \in L^\infty(\Omega)$; then*

$$\inf\big\{F(u, K) : \ K \in \mathcal{K}, \ u \in C^\infty(\Omega \setminus K)\big\} = \min\big\{G(u) : \ u \in SBV(\Omega)\big\},$$

where F is defined in (1.1).

Proposition 5.3. *Let Ω, $A \subset \mathbf{R}^n$ be bounded open sets with $\Omega \subset\subset A$, let $w \in C^\infty(A)$ and let (v, Γ) be a minimizer of the Mumford-Shah functional*

$$(v, \Gamma) \mapsto \int_A \big[|\nabla v|^2 + \alpha |v - w|^2 \big] dx + \beta \mathcal{H}^{n-1}(\Gamma \cap A).$$

Then, for every $\varepsilon > 0$ we can find a compact set $K \in \mathcal{K}$ and $u \in C^\infty(\overline{\Omega} \setminus K)$ such that

$$\begin{cases} \displaystyle\int_\Omega \big[\alpha |v - u|^2 + |\nabla v - \nabla u|^2 \big] \, dx < \varepsilon \\ \\ \beta \big| \mathcal{H}^{n-1}(K \cap \overline{\Omega}) - \mathcal{H}^{n-1}(\Gamma \cap \overline{\Omega}) \big| < \varepsilon. \end{cases} \tag{5.32}$$

Proof. Let $S \subset \Gamma$ be the singular set of Γ, *i.e.*, the set of points $x \in \Gamma$ such that Γ is not a $C^{1,\rho}$ hypersurface in any neighbourhood of x. We know from Theorem 3.1 and Remark 3.2 of [7] that $\mathcal{H}^{n-1}(S) = 0$; moreover, Theorem 4.2 implies that $\Gamma \setminus S$ is locally a C^∞ hypersurface.

For any $\delta \in (0,1)$ we can find a finite family of balls $B_{r_i}(x_i)$ centered at points $x_i \in \overline{\Omega}$ whose union contains $S \cap \overline{\Omega}$, such that $r_i < \delta$ for any $i = 1, \dots, N$ and

$$\sum_{i=1}^{N} r_i^{n-1} < \delta.$$

Let $\chi \in C_0^\infty(A)$ such that $\chi \equiv 1$ on the δ neighbourhood of $\overline{\Omega}$. Denoting by U the union of the balls $B_{\tau r_i}(x_i)$, we define the function

$$u(x) = \begin{cases} v(x) & \text{if } x \notin U \\ 0 & \text{if } x \in U, \end{cases}$$

the compact set

$$K = \big(\Gamma \setminus (U \cup \{1 + \chi \geq \tau\})\big) \cup \bigcup_{i=1}^{N} \partial B_{\tau r_i}(x_i)$$

and we check that for a suitable choice of $\tau \in (1,2)$ and for δ sufficiently small, (u, K) satisfy all the stated properties. Indeed, since $r_i < \delta < 1$, we have

$$\mathcal{L}^n(U) \leq \sum_{i=1}^{N} \omega_n (2r_i)^n < 2^n \omega_n \delta,$$

so that

$$\int_\Omega \big[\alpha |v - u|^2 + |\nabla v - \nabla u|^2\big]\, dx = \int_U \alpha |v|^2 + |\nabla v|^2 \, dx < \varepsilon$$

for δ small enough. Since the measure of the symmetric difference of K and Γ in $\overline{\Omega}$ does not exceed the sum of the perimeter of the balls, also the second condition in (5.32) is fulfilled for δ small enough.

Finally, by Sard's theorem, the balls $B_{\tau r_i}(x_i)$ and the smooth level set $\{\chi = \tau - 1\}$ meet transversally Γ for a.e. τ, hence a choice of $\tau \in (1,2)$ is possible ensuring that $K \in \mathcal{K}$. □

Theorem 5.4. *Let $\Omega \subset \mathbf{R}^n$ be a bounded open set with Lipschitz boundary. Then, for every $u \in SBV(\Omega) \cap L^\infty(\Omega)$ satisfying*

$$\int_\Omega |\nabla u|^2 \, dx + \mathcal{H}^{n-1}(S_u) < +\infty$$

we can find sets $K_h \in \mathcal{K}$ *and functions* $u_h \in C^\infty(\overline{\Omega} \setminus K_h)$ *such that*

$$\begin{cases} \lim_{h\to+\infty} \int_\Omega \left[|u_h - u|^2 + |\nabla u_h - \nabla u|^2\right] dx = 0, \\ \lim_{h\to+\infty} \mathcal{H}^{n-1}(K_h \cap \overline{\Omega}) = \mathcal{H}^{n-1}(S_u \cap \Omega). \end{cases}$$

Proof. As a first step, we consider a bounded open set $A \subset \mathbf{R}^n$ such that $\overline{\Omega} \subset A$ and a function $w \in SBV(A)$ such that $w = u$ on Ω, $\mathcal{H}^{n-1}(S_w \cap \partial\Omega) = 0$ and

$$\int_A |\nabla w|^2\, dx + \mathcal{H}^{n-1}(S_w \cap A) < +\infty.$$

An extension with these properties can be easily obtained if Ω is a rectangle, by a reflection argument. In the general case the extension can be obtained using partitions of unity and a deformation argument.

Let $(w_h) \subset C^\infty(A)$ be such that $h\|w_h - w\|_{L^2(A)} \to 0$ as $h \to +\infty$, and let $v_h \in SBV(A)$ be minimizers of the functionals

$$v \mapsto \mathcal{F}_h(v) = \int_A \left[|\nabla v|^2 + h(v - w_h)^2\right] dx + \mathcal{H}^{n-1}(S_v \cap A).$$

Since $\mathcal{F}_h(v_h) \le \mathcal{F}_h(w)$, we obtain

$$\begin{aligned} &\limsup_{h\to+\infty} \left[h\|v_h - w_h\|^2_{L^2(A)} + \int_A |\nabla v_h|^2\, dx + \mathcal{H}^{n-1}(S_{v_h} \cap A)\right] \qquad (5.33) \\ \le\ &\int_A |\nabla w|^2\, dx + \mathcal{H}^{n-1}(S_w \cap A). \end{aligned}$$

In particular, $\|v_h - w_h\|_{L^2(A)} \to 0$, hence v_h converges to w in $L^2(A)$. Moreover, the SBV compactness theorem (see [1], [4]) implies

$$\begin{cases} \liminf_{h\to+\infty} \int_A |\nabla v_h|^2\, dx \ge \int_A |\nabla w|^2\, dx \\ \liminf_{h\to+\infty} \mathcal{H}^{n-1}(S_{v_h} \cap A) \ge \mathcal{H}^{n-1}(S_w \cap A). \end{cases}$$

Then, these inequalities and (5.33) imply

$$\begin{cases} \lim_{h\to+\infty} \int_A |\nabla v_h|^2\, dx = \int_A |\nabla w|^2\, dx \\ \lim_{h\to+\infty} \mathcal{H}^{n-1}(S_{v_h} \cap A) = \mathcal{H}^{n-1}(S_w \cap A). \end{cases}$$

In particular, ∇v_h converges strongly to ∇w as $h \to +\infty$ and $\mathcal{H}^{n-1}(S_{v_h} \cap A)$ converges to $\mathcal{H}^{n-1}(S_w \cap A)$. Again, the SBV compactness theorem gives

$$\begin{cases} \liminf_{h\to+\infty} \mathcal{H}^{n-1}(S_{v_h} \cap \Omega) \geq \mathcal{H}^{n-1}(S_w \cap \Omega) \\ \liminf_{h\to+\infty} \mathcal{H}^{n-1}\big(S_{v_h} \cap (A \setminus \overline{\Omega})\big) \geq \mathcal{H}^{n-1}\big(S_w \cap (A \setminus \overline{\Omega})\big). \end{cases}$$

Since $\mathcal{H}^{n-1}(S_w \cap \partial\Omega) = 0$, these inequalities imply that both liminf above are limits and that equalities hold. In particular, $\mathcal{H}^{n-1}(S_{v_h} \cap \Omega)$ converges to $\mathcal{H}^{n-1}(S_u \cap \Omega)$ and

$$\lim_{h\to+\infty} \mathcal{H}^{n-1}(S_{v_h} \cap \partial\Omega) = 0. \tag{5.34}$$

Let $\Gamma_h = \overline{S}_{v_h}$. By the minimality of v_h and the smoothness of w_h, the functions v_h are C^∞ in $A \setminus \Gamma_h$; moreover, (5.34) and (see [6], Proposition 2.8)

$$\mathcal{H}^{n-1}(A \cap \Gamma_h \setminus S_{v_h}) = 0$$

imply that $\mathcal{H}^{n-1}(\Gamma_h \cap \overline{\Omega})$ converges to $\mathcal{H}^{n-1}(S_u \cap \Omega)$.

However, the sets Γ_h do not necessarily belong to $\mathcal{K}$; using Proposition 5.3 we can modify (v_h, Γ_h) to obtain pairs (u_h, K_h) satisfying the statement. □

Remark 5.5. As a consequence of the approximation Theorem 5.4, we obtain that the Lavrentiev phenomenon does not occur for minimum problems in SBV of the form

$$\min\left\{\int_\Omega f(x, u, \nabla u)\, dx + \mathcal{H}^{n-1}(S_u)\right\}$$

where f is any Carathéodory function satisfying quadratic growth conditions. Using Lemma 4.3 in [5], more general surface energy densities can also be considered.

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