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ON NONCONVEX DIFFERENTIAL INCLUSIONS WHOSE STATE IS CONSTRAINED IN THE CLOSURE OF AN OPEN SET. APPLICATIONS TO DYNAMIC PROGRAMMING

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Abstract. Results of the type of Filippov's Theorem are proved for nonconvex differential inclusions whose state variable is constrained in the closure of an open subset of $\mathbb{R}^n$. An application is provided to dynamic programming for optimal control problems with state constraints and a control set depending on the time and the state.

1. Introduction. For a Lipschitz nonconvex multifunction $F : [0,T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ the celebrated Filippov's Theorem states that, given a trajectory $y_1(\cdot)$ of

$$\begin{cases} \dot{y} \in F(t,y) \quad t \in [0,T] \\ y(0) = y_1 \end{cases} \tag{1.1}$$

and an initial condition $y_2 \neq y_1$, one can find a trajectory $y_2(\cdot)$ of (1.1) with $y_2(0) = y_2$ and such that

$$|y_1(t) - y_2(t)| \leq e^{Lt}|y_1 - y_2| \tag{1.2}$$

$$|\dot{y}_1(t) - \dot{y}_2(t)| \leq Le^{Lt}|y_1 - y_2|, \tag{1.3}$$

where L is the Lipschitz constant of F. This result extends to differential inclusions a well–known consequence of Gronwall's Lemma.

Of course, as soon as a state constraint of the form

$$y(t) \in \bar{\theta}, \tag{1.4}$$

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where $\theta \subseteq \mathbb{R}^n$ is an open subset, is imposed to the system, Filippov's Theorem is no longer valid. To begin with, it is not even guaranteed that a solution of (1.1) verifying (1.2), (1.3) exists.

In this paper we address the problem of finding results *à la Filippov* for a system (1.1), (1.4), when suitable *constraint qualifications* are assumed on the pair (F, θ). These are extensions of a condition originally introduced by H.M. Soner in the case of a control system, and, roughly speaking, involve a selection of F strictly *pointing inward* θ. We present two main results. In the former the boundary of θ is locally Lipschitz, F is Lipschitz in (t, x) and an inequality like (1.2) is proved. In the latter, F is allowed to be merely measurable in t, while, as a counterpart, the boundary $\partial\theta$ has to verify a smoothness hypothesis (see assumption **(Sm)** below). Despite the diminished regularity assumed on F, in this case both (1.2) and (1.3) can be proved, while most of the akin results for (the special case of) control systems (see [12], [14], [20]) involve just an estimate on the trajectories. Only a recent paper by Arisawa and Lions presents an estimate —in the case of a regular, autonomous control system and a smooth θ— on the distance of the controls. Yet, though this result is valid even for $T = +\infty$, it does not provide an explicit construction of $y_2(\cdot)$.

Both the results we prove rely on the possibility of constructing admissible trajectories whose distance (in the C^0 or the $W^{1,1}$ topology, respectively) from a given (generally not admissible) trajectory is proportional to the maximal distance of the latter from the constraint. These kinds of approximability properties display also an intrinsic interest and have been applied (in the version involving derivatives) in [19] to prove the nondegeneracy of necessary conditions for (nonconvex) minimum problems with state constraints.

The last section of the paper is devoted to an application to dynamic programming for minimum problems involving state constraints. We remark that the absence of convexity assumption on F makes the problem genuinely more general than the analogous problem for control systems (while, — see [17]—, if F is convex valued and sufficiently regular, (1.1) can be written as a suitable control system).

2. Preliminaries. If Z is a metric space with distance d, for every subset $W \subset Z$ and every $\rho \geq 0$ we set $B\,[W;\rho] \doteq \{\, z \in Z : d(z, W) \leq \rho \,\}$. If $w \in Z$, we shall write $B\,[w;\rho]$ instead of $B\,[\{w\}\,;\rho]$.

Let $F : X \rightsquigarrow Y$ be a multivalued map from a metric space X to a metric space Y. Let us recall the concept of Lipschitz map.

Definition 2.1. The map F is called *Lipschitz* if there is $L \geq 0$ such that

$$\forall\, x, x' \in X, \qquad F(x) \subseteq B[F(x'); Ld(x,x')].$$

If $X \subseteq \mathbb{R}^n$, $Y = \mathbb{R}^m$ and the values of F are nonempty and compact, then the above definition is equivalent to say that

$$\forall\, x, x' \in X, \qquad \mathcal{D}(F(x), F(x')) \leq L|x - x'|.$$

In the above formula, $\mathcal{D}$ denotes the Hausdorff distance on the set of compact, nonempty subsets of $\mathbb{R}^n$, which is defined by

$$\mathcal{D}(H,K) \doteq \max\left(\max_{x\in H} d(x,K), \max_{y \in K} d(y,H) \right).$$

Let $F : \mathbb{R} \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ be a multivalued map, let $\theta \subseteq \mathbb{R}^n$ be an open subset, and, for $\bar{t}$ less than a given T and $x \in \bar{\theta}$ (where $\bar{\theta}$ denotes the closure of θ), let us consider the initial value problem

$$\begin{cases} \dot{x}(t) \in F(t, x(t)) \\ x(\bar{t}) = \bar{x}, \quad x(t) \in \bar{\theta}. \end{cases} \tag{2.1}$$

By a *trajectory* of (2.1) we mean an absolutely continuous map $x : [\bar{t}, T] \to \mathbb{R}^n$, with $x(\bar{t}) = \bar{x}$, which verifies both the differential inclusion and the state constraint for almost every $t \in [\bar{t}, T]$ (in fact, this implies that $x(t) \in \bar{\theta}$ for every $t \in [\bar{t}, T]$).

Corresponding to hypotheses **(NSm)** and **(Sm)** below on the boundary $\partial\theta$ we shall consider the following two Lipschitz–type hypotheses **(L)** and **(M–L)** on the multifunction F, respectively.

Hypothesis (L).

(i) $\forall\, (t, y) \in \mathbb{R} \times \mathbb{R}^n$, $F(t, y)$ is compact and nonempty;

(ii) F is Lipschitz on the compact sets of $\mathbb{R} \times \mathbb{R}^n$, i.e., for every compact $K \subset \mathbb{R} \times \mathbb{R}^n$ there is $L > 0$ such that

$$\mathcal{D}(F(t,x), F(s,y)) \leq L|(t,x) - (s,y)|, \tag{2.2}$$

for all (t, x), $(s, y) \in K$;

(iii) there is a $c > 0$ such that

$$\max\{\, |v| \mid v \in F(t,y)\,\} \leq c(1 + |y|)$$

for all $(t, y) \in \mathbb{R} \times \mathbb{R}^n$.

Hypothesis (M–L).

(i) $\forall\,(t, y) \in \mathbb{R} \times \mathbb{R}^n$, $F(t, y)$ is compact and nonempty;

(ii) $\forall\, x \in \mathbb{R}^n$, $F(\cdot, x)$ is measurable, i.e., for each Borel set $B \subset \mathbb{R}^n$,

$$F^{-1}(\cdot, x)(B) \doteq \{\, t \in \mathbb{R} : F(t, x) \cap B \neq \varnothing \,\}$$

is Lebesgue measurable;

(iii) for almost every $t \in \mathbb{R}$ the map $F(t, \cdot)$ is $\lambda(t)$–Lipschitz (i.e., Lipschitz with constant $\lambda(t)$, for a suitable $\lambda(\cdot) \in L^1(\mathbb{R})$);

(iv) there exists $c > 0$ such that

$$\max \{\, |v| : v \in F(t, y) \,\} \leq c(1 + |y|)$$

for all $(t, y) \in \mathbb{R} \times \mathbb{R}^n$.

Of course **(M–L)** is weaker than **(L)**. We will be able to apply the former whenever $\partial\theta$ is sufficiently smooth while **(L)** will be enough for the case when $\partial\theta$ is locally Lipschitz.

For unconstrained differential inclusions, i.e., $\theta = \mathbb{R}^n$, Filippov's Theorem gives a way of approximating any absolute continuous curve with a trajectory of the differential inclusion issuing from a different initial point. For the reader's convenience, let us recall it, in a form (see e.g. [3]) adapted to the general hypothesis **(M–L)**.

Theorem 2.1 (Filippov Theorem)**.** *Let us assume* $\mathbf{(M\text{–}L)_{i}}$—$\mathbf{(M\text{–}L)_{iii}}$ *and let* $y : [\bar{t}, T] \to \mathbb{R}^n$ *be any absolute continuous map. Let* $\delta > 0$ *and define*

$$\gamma(t) \doteq d\left(\dot{y}(t), F(t, y(t))\right), \qquad m(t) \doteq \exp\left(\int_{\bar{t}}^{t} \lambda(\sigma)\, d\sigma\right)$$

$$\eta(t) \doteq m(t)\left(\delta + \int_{\bar{t}}^{t} \gamma(\sigma)\, d\sigma\right).$$

Then, for every x_0 *such that* $|x_0 - y(\bar{t})| = \delta$ *there exists a solution* $x(\cdot)$ *on* $[\bar{t}, T]$ *of*

$$\dot{x}(t) \in F(t, x(t)), \qquad x(\bar{t}) = x_0$$

such that

$$|x(t) - y(t)| \leq \eta(t) \qquad \forall\, t \in [\bar{t}, T]$$

and

$$|\dot{x}(t) - \dot{y}(t)| \leq \lambda(t)\eta(t) + \gamma(t) \qquad \textit{for a.e. } t \in [\bar{t}, T].$$

In particular, when the curve $y(\cdot)$ is a solution of the differential inclusion we have that both the C^0 distance between $y(t)$ and $x(t)$ and the L^1 distance between their derivatives are proportional to the distance $|x(\bar{t}) - y(\bar{t})|$ between the initial conditions.

3. Approximability of trajectories in the presence of state constraints. It is obvious that a result like Theorem 2.1 cannot hold when an actual constraint set $\theta \neq \mathbb{R}^n$ is considered, unless further assumptions are made on F and θ. In this section we shall present Filippov type results in the presence of state constraints under one of the constraint qualifications **(NSm)** and **(Sm)** below. The constraint qualification **(NSm)** refers to the case where $\partial\theta$ is non smooth and the field F is regular, i.e., it satisfies **(L)**. Conversely, assumption **(Sm)** involves regularity of the boundary $\partial\theta$ and allows for a field F which merely satisfies **(ML)**

Let us set, for any $\alpha \geq 0$, $C_\alpha \doteq \bar{\theta} \cap B[\partial\theta; \alpha]$.

(NSm)(*Non smooth boundary*).

i. F verifies **(L)**;

ii. there exists positive constants η, r, q, and a continuous function f

$$\begin{aligned} f : [\bar{t}, T] \times C_\eta &\longrightarrow \mathbb{R}^n \\ (t, y) &\longmapsto f(t, y) \in F(t, y) \end{aligned}$$

such that

$$B\left[y + hf(t, y); hr\right] \subset \theta$$

for all $(t, y) \in [\bar{t}, T] \times C_\eta$ and $h \in]0, q]$.

Assumption **(NSm)** allows for non smooth boundaries $\partial\theta$. On the other hand it implies that $\partial\theta$ is locally Lipschitz (see e.g. the appendix in [4]).

In order to consider the case where F verifies the weaker assumption **(M–L)** we shall assume the following constraint qualification, which is weaker in the selection requirement but stronger in the regularity assumption on $\partial\theta$.

(Sm)(*Smooth boundary*).

i. F verifies **(M–L)**;

ii. the boundary $\partial\theta$ is such that signed distance function

$$\tilde{d}(x) \doteq \begin{cases} d(x, \partial\theta) & x \in \bar{\theta} \\ -d(x, \partial\theta) & x \in \mathbb{R}^n \backslash \bar{\theta} \end{cases}$$

is locally of class $C^{1,1}$ (i.e., differentiable, with a locally Lipschitz continuous gradient, see Remark 3.3 below) on a neighbourhood $B[\partial\theta;\alpha]$ for some $\alpha > 0$;

iii. there exist positive constants η, r and a selection

$$\begin{aligned} f : [\bar{t}, T] \times C_\eta &\longrightarrow \mathbb{R}^n \\ (t, y) &\longmapsto f(t, y) \in F(t, y) \end{aligned}$$

measurable in t and continuous in y, uniformly with respect to t, such that

$$\langle f(t, y) \cdot \nabla\tilde{d}(y)\rangle \geq r$$

for all $(t, y) \in [\bar{t}, T] \times C_\eta$, where $\nabla\tilde{d}$ denotes the differential of $\tilde{d}$.

Remark 3.1. Assuming that the map $d(x, \partial\theta)$ is of class $C^{1,1}$ on $B[\partial\theta;\alpha]$ for some α is quite usual in this kind of problems (see [6] and [1]) and in elliptic boundary value problems (see e.g. [11]).
We refer to the Appendix in [6] and to [8] for characterizations of this property in terms of the gradient or the proximal subgradient of d.

A sufficient condition for $d(x, \partial\theta)$ to be locally of class $C^{1,1}$ is that $\partial\theta$ is a C^2 surface (see e.g. the Appendix in [11]). Actually, in this case $d(x, \partial\theta)$ is of class C^2 in $(B[\partial\theta;\alpha]\backslash\partial\theta) \cap B[0;R]$, for every R, with $\alpha > 0$ depending on R. It is straightforward to check that in this case $\tilde{d}$ is of class C^2 in $B[\partial\theta;\alpha] \cap B[0, R]$.

We present two kind of results. The former (see Theorem 3.1 below), which is valid in the case when $\partial\theta$ is merely locally Lipschitz, (i.e., assumption **(NSm)** is in force), provides an approximation estimate involving only the trajectories. Instead, the latter (see Theorem 3.2 below), which is proved under the constraint qualification **(Sm)** , provides an approximation estimate in L^1 for the derivatives of the trajectories as well.

Actually, Theorems 3.1 and 3.2 are easy consequences of Theorems 4.1 and 4.2 below, respectively. Roughly speaking, Theorems 4.1 and 4.2 state the possibility of modifying a given trajectory violating the constraint into a new trajectory $x(\cdot)$ which remains in $\bar{\theta}$ for all $t \in [\bar{t}, T]$ and keeps a distance from $y(\cdot)$ –with respect to the C^0 and $W^{1,1}$ norm, respectively–controlled by the maximum distance of $y(t)$ from $\bar{\theta}$.

Theorem 3.1. *Let us assume hypothesis* **(NSm)**. *Then for every compact subset $Q \subseteq \bar{\theta}$ there exists a positive constant C such that for all x_1, $x_2 \in Q$*

and for each absolutely continuous map $y_1 : [\bar{t}, T] \to \bar{\theta}$, $y_1(\bar{t}) = x_1$, *there exists a solution,* $y_2(\cdot)$, *of*

$$\begin{cases} \dot{y}(t) \in F(t, y(t)) \\ y(\bar{t}) = x_2 \\ y(t) \in \bar{\theta} \qquad \forall t \in [\bar{t}, T] \end{cases}$$

such that

$$|y_1(t) - y_2(t)| \le C\big(|x_1 - x_2| + \int_{\bar{t}}^{T} \gamma(t)\, dt\big) \tag{3.1}$$

for all $t \in [\bar{t}, T]$, *where* γ *is defined almost everywhere by*

$$\gamma(t) \doteq d\big(\dot{y}_1(t), F(t, y_1(t))\big).$$

In the case when $\partial\theta$ is smooth we have the following result:

Theorem 3.2. *Let us assume hypothesis* **(Sm)**. *Then for every compact subset* $Q \subseteq \bar{\theta}$ *there exists a positive constant* C *such that for all* $x_1, x_2 \in Q$ *and for each absolutely continuous map* $y_1 : [\bar{t}, T] \to \bar{\theta}$, $y_1(\bar{t}) = x_1$, *there exists a solution,* $y_2(\cdot)$, *of*

$$\begin{cases} \dot{y}(t) \in F(t, y(t)) \\ y(\bar{t}) = x_2 \\ y(t) \in \bar{\theta} \qquad \forall t \in [\bar{t}, T] \end{cases}$$

such that

$$\sup_{t \in [\bar{t}, T]} |y_1(t) - y_2(t)| + \int_{\bar{t}}^{T} |\dot{y}_1(t) - \dot{y}_2(t)|\, dt \le C\big(|x_1 - x_2| + \int_{\bar{t}}^{T} \gamma(t)\, dt\big) \tag{3.2}$$

for all $t \in [\bar{t}, T]$, *where* γ *is as in Theorem* 3.1.

The proofs of Theorems 3.1 and 3.2 are sketched at the end of Section 4.

Remark 3.2. These two theorems can be considered as generalizations of Filippov's Theorem. What is remarkable in Theorem 3.2 is that an estimate is provided on derivatives as well, just like in the unconstrained case. If one considers the case when y_1 is a solution, Theorem 3.1 can be expressed by saying that the *solution map*, i.e., the map $\mathcal{S} : \bar{\theta} \rightsquigarrow C^0([\bar{t}, T], \bar{\theta})$ which to every $z \in \bar{\theta}$ associates the set of solutions of the differential inclusion which issue from z and remain in $\bar{\theta}$, is lower semicontinuous when the range is endowed with the sup–norm. And Theorem 3.2 states that under hypothesis **(Sm)** the solution map is lower semicontinuous when the range is endowed with the $W^{1,1}$ topology (the latter being equivalent to the topology induced by the norm $\|x\|_{AC} = \|x\|_\infty + \|\dot{x}\|_1$). Actually, in both cases the solution map $\mathcal{S}$ is locally Lipschitz. Notice that, unless convexity hypotheses are assumed, $\mathcal{S}$ is not upper semicontinuous.

Remark 3.3. Theorem 3.1 and Theorem 4.1 below, from which the former is deduced, are generalizations of analogous results obtained by H.M. Soner [20], P. Loreti–M.E. Tessitore [14], and I. Ishii–S. Koike [12] for control systems. It is known that, unless convexity is assumed on the values of F (see [17]), a differential inclusion cannot be expressed as a control system. On the other hand the estimation on the derivatives, provided by Theorem 3.2, –that in the case of a control system implies an estimation on the controls–, was not given in [20] and [14], where the boundary was assumed to be smooth.

Remark 3.4. Concerning the infinite horizon problem for control problems and a smooth compact set θ, some results can be found in [1]. Roughly speaking, when $F(t,x) = F(x) = f(x,U)$, the authors prove prove that given a control u_1, two initial conditions $\bar{x}_1$, $\bar{x}_2$, and $\epsilon > 0$, if $x_1(\cdot)$ is the solution of $\dot{x}(t) = f(x(t), u_1(t))$, $x(\bar{t}) = \bar{x}_1$ on $[\bar{t}, +\infty)$, there exists a control u_2 such that the solution $x_2(\cdot)$ of $\dot{x}(t) = f(x(t), u_2(t))$, $x(\bar{t}) = \bar{x}_2$ satisfies

$$|x_1(t) - x_2(t)| + \int_{\bar{t}}^{t} |u_1(s) - u_2(s)|\, ds \le Me^{Lt}(|\bar{x}_1 - \bar{x}_2| + \epsilon),$$

where M and L are constants depending only on f and θ. There is not an explicit construction of such a u_2 in [1], for the proof is based on a dynamic programming argument applied to a differential game introduced ad hoc.

Remark 3.5. The requirement, shared by both **(NSm)** and **(Sm)**, that the selection $f \in F$ has to be continuous in x is actually quite strong. However the following example seems to suggest that this continuity hypothesis can be hardly weakened.

Example 3.1. Consider the subset of $\mathbb{R}^2$

$$\theta = \mathbb{R}^+ \times \mathbb{R} \backslash \{(x,y) \in \mathbb{R}^2 \,|x \ge 2|y| + 1\}$$

and the constant multifunction

$$F(x, y) = \{(1, 1),\ (1, -1)\}$$

It is straightforward to check that **(NSm)** is verified except for the requirement of the continuity of the selection f at the point $(1, 0) \in \partial\theta$. In fact the thesis of Theorem 3.2 does not hold true: the trajectory $t \mapsto (1, 0) + (1, -1)t$ which starts from $(x, y) = (1, 0)$ at $\bar{t} = 0$ cannot be approximated by trajectories issuing from the points $(x_n, y_n) = (1 + 1/n, n/2) \in \partial\theta$, while $(x_n, y_n) \to (x, y)$ as n tends to infinity.

Observe that the situation does not get improved if we take the convex hull of F instead of F.

Remark 3.6. The fact that the vector field f points *strictly* inside θ is also crucial. For example, take $\theta = \mathbb{R}^2 \backslash B[(0, 0); 1]$ and

$$F(x, y) = \{(1, 0), (y, -x)\}.$$

There is a continuous selection, namely $f(x, y) = (y, -x)$, tangential to the boundary $\partial\theta = \{(x, y) || (x, y)| = 1\}$.

However the trajectory $t \mapsto (0, -1) + (1, 0)t$ starting from $(\bar{x}, \bar{y}) = (0, -1)$ at $t = 0$ cannot be approximated by trajectories starting from the points $(\bar{x}_n, \bar{y}_n) = (-\frac{1}{n}, -\sqrt{1 - \frac{1}{n^2}}) \in \partial\theta$.

Remark 3.7. Similarly to the case with no constraints, where Filippov's Theorem incidentally provides an existence result, Theorems 3.1 and 3.2 imply the existence of at least one trajectory remaining in $\bar{\theta}$ on $[\bar{t}, T]$. It can be observed that this almost obvious fact does not follow from the Viability Theorem (see e.g. [2]) where the convexity of the values of F is required. (see also [18] and [22]).

4. Construction of neighboring admissible trajectories and proofs of Theorems 3.1 and 3.2. We prove now two theorems stating the possibility of modifying a given trajectory $y(\cdot)$ (which in general violates the constraint) into a new trajectory $\tilde{y}(\cdot)$ such that: i) $\tilde{y}(\cdot)$ has the same initial point as $y(\cdot)$; ii) it verifies $\tilde{y}(t) \in \bar{\theta}\ \forall\, t \in [\bar{t}, T]$; iii) $\tilde{y}(\cdot)$ keeps a distance from $y(\cdot)$ (in the C^0 or $W^{1,1}$ norm, under hypotheses **(NSm)** and **(Sm)**, respectively), which is controlled by the maximal distance of $y(t)$ from θ. Theorems 3.1 and 3.2 will follow straightforwardly.

Theorem 4.1. *Let us assume hypothesis* **(NSm)**. *Then for every subset $Q \subseteq \bar{\theta}$ there exists a positive constant C such that for every $z \in Q$ and every solution $y(\cdot)$ of*

$$\begin{cases} \dot{y} \in F(t, y) \\ y(\bar{t}) = z \qquad t \in [\bar{t}, T] \end{cases} \tag{4.1}$$

there exists a solution $\hat{y}(\cdot)$ of (4.1) such that

$$\hat{y}(t) \in \bar{\theta} \qquad \forall\, t \in [\bar{t}, T] \tag{4.2}$$

and

$$\sup_{t \in [\bar{t}, T]} |\hat{y}(t) - y(t)| \leq C \sup\{d(y(t), \theta) \,|\, t \in [\bar{t}, T]\}. \tag{4.3}$$

Proof. For notational convenience we assume, without loss of generality, that $\bar{t} = 0$. By Gronwall's lemma there exist two constants M, M' such that all trajectories $y(\cdot)$ of (4.1) issuing from points $z \in Q$ verify

$$\sup_{t \in [0,T]} |y(t) - z| \leq M',$$
$$|y(t) - y(t')| \leq M|t - t'|, \tag{4.4}$$

for all $t, t' \in [0, T]$. For later purposes we assume that $M > 1$. Let us set

$$Q' \doteq \{y \,|\, d(y, Q) \leq 2M'\}.$$

Set $K \doteq C_\eta \cap Q'$) (C_η defined as in Section 3) and let $\tilde{K} \subseteq \mathbb{R}^n$ be a compact subset containing a neighborhood of K. Since f is continuous on $[0, T] \times K$ one can construct a uniformly continuous function $\tilde{f}$ on $[0, T] \times \mathbb{R}^n$ such that $\tilde{f}(t, x) = f(t, x)$ if $(t, x) \in [0, T] \times K$ and $\tilde{f}(t, x) = 0$ if $(t, x) \in [0, T] \times (\mathbb{R}^n \backslash \tilde{K})$. Then there exists $\delta_1 > 0$ such that for all (t, x), $(t', x') \in [0, T] \times \mathbb{R}^n$ verifying $|(t, x) - (t', x')| \leq 3\delta_1$ one has

$$|\tilde{f}(t, x) - \tilde{f}(t', x')| \leq r/6, \tag{4.5}$$

where r is the constant involved in hypothesis **(NSm)**. Without loss of generality we can assume $M > 1$.

Let us choose a $t^* \in [0, T]$ verifying

$$t^* \leq \min\left\{\frac{\delta_1}{M}, \frac{qr}{6M}, \frac{\eta}{M}\right\} \text{ and } \frac{6}{r}\,(e^{Lt^*} - 1)[M + Lt^*] < 1. \tag{4.6}$$

Let us begin by proving that a trajectory of (4.1) on $[\bar{t}, T]$ exists satisfying (4.2) and (4.3) with T replaced by t^*.

By (4.4) and (4.6), for any $t, t' \in [0, t^*]$ and for any trajectory y with initial point in Q one has

$$|y_(t) - y(t')| \leq \delta_1 \qquad |t - t'| \leq \delta_1. \tag{4.7}$$

Let us fix $z \in Q$ and a solution $y(\cdot)$ of (4.1) starting at z. Whenever $y(t) \notin \bar{\theta}$ for some $t \in [0, t^*]$, let us set

$$\hat{t} \doteq \inf \left\{ t \in [0, t^*] \mid y(t) \in \mathbb{R}^n \setminus \bar{\theta} \right\}.$$

If $y(t) \in \bar{\theta}$ for every $t \in [0, t^*]$, we mean that $\hat{t} = t^*$.

If $\hat{t} < t^*$ let $\bar{y}(\cdot)$ be a solution of

$$\begin{cases} \dot{z} = \tilde{f}(t, z) \\ z(\hat{t}) = y(\hat{t}) \doteq \hat{y} \quad t \in [\hat{t}\,, \hat{t} + k\delta], \end{cases} \tag{4.8}$$

where

$$\delta \doteq \sup \{ d(y(t), \theta) \,|\, t \in [0, t^*] \}$$

and

$$k \doteq \min \left(\frac{6}{r}, \frac{t^* - \hat{t}}{\delta} \right).$$

Since $\tilde{f}$ is bounded, such a solution exists in whole interval $[\hat{t}\,, \hat{t} + k\delta]$.

By (4.4) and (4.6) we have that

$$|\bar{y}(t) - \hat{y}| \leq \eta \qquad \forall\, t \in [\hat{t}\,, \hat{t} + k\delta]. \tag{4.9}$$

Let us show that $\bar{y}(t) \in \bar{\theta}$ for all $t \in [\hat{t}\,, \hat{t} + k\delta]$. Since $\delta \leq Mt^*$, by (4.6) one has $k\delta \leq q$. In particular by hypothesis $\mathbf{(NSm)_{ii}}$ we obtain

$$B \left[\hat{y} + (t - \hat{t}) \tilde{f}(\hat{t}, \hat{y}); (t - \hat{t}) r \right] \subseteq \theta \tag{4.10}$$

for all $t \in (\hat{t}, \hat{t} + k\delta]$ $(\tilde{f}(\hat{t}, \hat{y}) = f(\hat{t}, \hat{y}))$.

Moreover, by (4.5), which is verified because of (4.7), we obtain

$$|\bar{y}(t) - (\hat{y} + (t - \hat{t}) \tilde{f}(\hat{t}, \hat{y}))| \leq \int_{\hat{t}}^{t} |\tilde{f}(s, \bar{y}(s)) - \tilde{f}(\hat{t}, \hat{y})| \, ds \leq r(t - \hat{t})/2. \tag{4.11}$$

Therefore we have

$$B\left[\bar{y}(t);(t-\hat{t})r/2\right]\subseteq\theta\quad\forall\, t\in(\hat{t},\hat{t}+k\delta],\tag{4.12}$$

thence $\bar{y}(t)\in\bar{\theta}$ for all $t\in[\hat{t},\hat{t}+k\delta]$. In particular in (4.8) $\tilde{f}$ can be replaced by f, which implies that $\bar{y}(\cdot)$ is indeed a trajectory of our differential inclusion.

Moreover, setting

$$y_1(t)=\begin{cases} y(t) & t\in[0,\hat{t}] \\ \bar{y}(t) & t\in[\hat{t},\hat{t}+k\delta],\end{cases}$$

one has

$$|y_1(t)-y(t)|\leq 2Mk\delta\leq\frac{12}{r}M\delta.$$

If $\hat{t}+\frac{6}{r}\delta\geq\hat{t}+k\delta=t^*$ we are done. Alternatively, let us consider the case when $\hat{t}+k\delta=\hat{t}+\frac{6}{r}\delta\ <t^*$.

By (4.10) and (4.12) we have

$$B\left[\hat{y}+k\delta f(\hat{t},\hat{y});6\delta\right]\subseteq\theta,\quad B\left[y_1(\hat{t}+k\delta);3\delta\right]\subseteq\theta,\tag{4.13}$$

respectively.

For any $t\in[\hat{t}+k\delta,T]$ let us set

$$z_0(t)\doteq y(t-k\delta).$$

Filippov's Theorem yields the existence of a solution $z(\cdot)$ to

$$\begin{cases}\dot{z}\in F(t,z) \\ z(\hat{t}+k\delta)=y_1(\hat{t}+k\delta) \qquad t\in[\hat{t}+k\delta,T]\end{cases}\tag{4.14}$$

such that

$$\begin{aligned}|z(t)-z_0(t)|\leq& e^{L[t-(\hat{t}+k\delta)]}(|y_1(\hat{t}+k\delta)-\hat{y}| \\ &+\int_{\hat{t}+k\delta}^{t}d(\dot{y}(s-k\delta),F(s,y(s-k\delta)))\,ds)\end{aligned}\tag{4.15}$$

$$\begin{aligned}|\dot{z}(t)-\dot{z}_0(t)|\leq& Le^{L[t-(\hat{t}+k\delta)]}(|y_1(\hat{t}+k\delta)-\hat{y}| \\ &+\int_{\hat{t}+k\delta}^{t}d(\dot{y}(s-k\delta),F(s,y(s-k\delta)))\,ds)\end{aligned}\tag{4.16}$$

for almost every $t \in [\hat{t} + k\delta, T]$.

Let us extend the definition of $y_1(\cdot)$ by setting

$$y_1(t) = z(t) \qquad \forall\, t \in [\hat{t} + k\delta, T]. \tag{4.17}$$

We claim that $y_1(t) \in \bar{\theta}$ for all $t \in [\hat{t} + k\delta, t^*]$.

Indeed, for every $t \in [\hat{t} + k\delta, t^*]$ one has

$$y_1(t) = \big[y_1(\hat{t} + k\delta) - \hat{y}\big] + y(t - k\delta) + X(t), \tag{4.18}$$

where

$$X(t) \doteq \int_{\hat{t}+k\delta}^{t} \left(\dot{y}_1(s) - \dot{y}(s - k\delta)\right)\, ds. \tag{4.19}$$

Now, by (4.16) and the Lipschitz continuity of F we obtain:

$$\begin{aligned}
|X(t)| = \int_{\hat{t}+k\delta}^{t} |\dot{z}(s) - \dot{z}_0(s)|\, ds &\le \int_{\hat{t}+k\delta}^{t} Le^{Ls}\Big[|y_1(\hat{t} + k\delta) - \hat{y}| \\
&\quad + \int_{\hat{t}+k\delta}^{s} d(\dot{y}(\sigma - k\delta), F(\sigma, y(\sigma - k\delta)))\, d\sigma\Big] ds \\
\le \left(e^{Lt} - 1\right)\Big[Mk\delta + \int_{\hat{t}+k\delta}^{t} &\mathcal{D}(F(\sigma - k\delta, y(\sigma - k\delta)), F(\sigma, y(\sigma - k\delta)))\, d\sigma\Big] \\
&\le \left(e^{Lt} - 1\right)[Mk\delta + h\delta Lt] \le \frac{6\delta}{r}\left(e^{Lt} - 1\right)[M + Lt].
\end{aligned}$$

In view of (4.6) one obtains

$$|X(t)| \le \delta \quad \forall\, t \in (\hat{t} + k\delta, t^*]. \tag{4.20}$$

Observe that, thanks to the choice of δ_1, for every $t \in [\hat{t}, t^*]$ one can choose a point $y_\pi(t) \in \partial\theta \cap B[\hat{y}\,; \delta_1] \cap Q'$ such that

$$|y(t) - y_\pi(t)| = d(y(t), \partial\theta).$$

Let us write, for every $t \in [\hat{t} + k\delta, t^*]$,

$$\begin{aligned}
&y(t - k\delta) + [y_1(\hat{t} + k\delta) - \hat{y}] \\
&\quad = [y(t - k\delta) - y_\pi(t - k\delta)] + y_\pi(t - k\delta) + \int_{\hat{t}}^{\hat{t}+k\delta} \tilde{f}(s, y_1(s))ds.
\end{aligned} \tag{4.21}$$

Now,

$$y_\pi(t-k\delta)+\int_{\hat t}^{\hat t+k\delta}\tilde f(s,y_1(s))\,ds$$
$$=x_t(\hat t+k\delta)+\int_{\hat t}^{\hat t+k\delta}(\tilde f(s,y_1(s))-\tilde f(s,x_t(s)))\,ds,$$

where $x_t(\cdot)$ is a solution, in $[\hat t,\hat t+k\delta]$, of

$$\begin{cases}\dot x=\tilde f(s,x(s))\\ x(\hat t)=y_\pi(t-k\delta).\end{cases}$$

With the same argument used to prove (4.13) one easily checks that

$$B[x_t(\hat t+k\delta);3\delta]\subseteq\theta.$$

Moreover, by (4.5) one obtains

$$\int_{\hat t}^{\hat t+k\delta}|\tilde f(s,\bar y(s))-\tilde f(s,x_t(s))|\,ds<\delta.$$

Putting together the previous estimates one obtains

$$B\left[y(t-k\delta)+\bar y(\hat t+k\delta)-\hat y;\delta\right]\subseteq\theta, \tag{4.22}$$

which, together with (4.20), allows one to conclude that

$$y_1(t)\in\bar\theta\qquad\forall\,t\in[\hat t+k\delta,t^*].$$

Moreover, one has, for each $t\in[\hat t+k\delta,t^*]$,

$$|y_1(t)-y(t)|\le|y_1(t+k\delta)-\hat y|+|y(t-k\delta)-y(t)|+|X(t)|.$$

Hence the map $y_1(\cdot)$ (is a trajectory of the differential inclusion and) satisfies

$$y_1(0)=y(0),\qquad y_1(t)\in\bar\theta\quad\forall\,t\in[0,t^*]$$
$$|y_1(t)-y(t)|\le\frac{16}{r}\max(M,Lt^*)e^{Lt^*}\delta.$$

If $t^* = T$, the theorem is proved, with $\hat{y}(\cdot) = y_1(\cdot)$ and

$$C = \frac{16}{r} \max(M, Lt^*) e^{Lt^*}.$$

Otherwise one can apply similar arguments to the map $y_1(\cdot)$ on the interval $[0, 2t^*]$. It is clear that in at most N steps, with $N > \frac{T}{t^*} + 1$, one is able to prove the theorem in the whole interval $[0, T]$ with a constant C which is a polynomial function of degree N of e^{LT}. $\square$

We now examine the case where the smooth boundary boundary of θ is smooth, namely when hypothesis **(Sm)** is assumed. Let us recall that under this hypothesis the map F is allowed to be merely measurable in t (see hypothesis **(M–L)**. More remarkably, we can approximate an admissible trajectory that approximates the given one *in the* $W^{1,1}$ *topology.*

Theorem 4.2. *Let us assume hypothesis* **(Sm)**. *Then for every compact subset* $Q \subseteq \bar{\theta}$ *there exists a positive constant* C *such that for every* $z \in Q$ *and every solution* $y(\cdot)$ *of*

$$\begin{cases} \dot{y} \in F(t, y) \\ y(\bar{t}) = z \qquad t \in [\bar{t}, T] \end{cases} \tag{4.23}$$

one can find a solution $\hat{y}(\cdot)$ *of* (4.23) *verifying*

$$\hat{y}(t) \in \bar{\theta} \qquad \forall\, t \in [\bar{t}, T] \tag{4.24}$$

and

$$\sup_{t \in [\bar{t}, T]} |\hat{y}(t) - y(t)| + \int_{\bar{t}}^{T} |\dot{\hat{y}}(t) - \dot{y}(t)|\, dt \le C \sup \{ d(y(t), \theta) \mid t \in [\bar{t}, T] \}. \tag{4.25}$$

Proof. As in the previous theorem we can assume $\bar{t} = 0$ and find two constants M, M' such that all trajectories $y(\cdot)$ of (4.23) issuing from points $z \in Q$ verify

$$\sup_{t \in [0, T]} |y(t) - z| \le M', \qquad |y(t) - y(t')| \le M|t - t'|, \tag{4.26}$$

for all $t, t' \in [0, T]$. Let us set

$$Q' \doteq \{ y \mid d(y, Q) \le M' \}.$$

Let us choose a $t^* \in [0,T]$ verifying

$$t^* \leq \min\left\{ \frac{\delta_1}{M}, \frac{\eta}{M}, \frac{\rho}{M} \right\}, \tag{4.27}$$

where η is the same as in hypothesis **(Sm)**, and δ_1, ρ are constants which will be determined below. Let us begin by proving that a trajectory of (4.23) on $[\bar{t}, T]$ exists satisfying (4.24) and (4.25) with T replaced by t^*.

Let us fix $z \in Q$ and a solution $y(\cdot)$ of (4.23). Whenever $y(t) \notin \bar{\theta}$ for some $t \in [0, t^*]$, let us set

$$\hat{t} \doteq \inf\left\{ t \in [0,t^*] \mid y(t) \in \mathbb{R}^n \setminus \bar{\theta} \right\},$$

otherwise, if $y(t) \in \bar{\theta}$ for any $t \in [0,t^*]$, let us set $\hat{t} = t^*$.

If $\hat{t} < t^*$ we set $\hat{y} \doteq y(\hat{t})$. Thanks to Proposition A.2 in [4] (whose hypotheses are verified because **(Sm)** implies the more general hypothesis **(NSm)**), one can find $\beta, \rho > 0$, $n \in \mathbb{R}^n$ with $|n| = 1$, and a continuous Lipschitz function $g : V \doteq \{ v \mid |v| \leq \beta \quad v \cdot n = 0 \} \to \mathbb{R}$ such that

$$\bar{\theta} \cap B[\hat{y}; 2\rho] = \cup_{v \in V} \{ \hat{y} + qn + v \mid q \geq g(v) \} \cap B[\hat{y}; 2\rho],$$

where β, ρ and the Lipschitz constant of g, L, can be chosen independently of the particular point $\hat{y} \in \partial\theta$ considered.

Let K be the rectangular neighborhood of $\hat{y}$ defined by

$$K \doteq \left\{ z = \hat{y} + qn + v \mid |v| \leq \sqrt{2}\rho,\ |q| \leq \sqrt{2}\rho \right\}$$

and let us define the vector field $\check{f} : [0,T] \times K \to \mathbb{R}^n$ by setting

$$\check{f}(t,x) = \begin{cases} f(t,x) & \text{if } x = \hat{y} + qn + v \text{ with } q \geq g(v) \\ f(t, \hat{y} + g(v)n + v) & \text{if } x = \hat{y} + qn + v \text{ with } q < g(v). \end{cases}$$

It's easy to check that $\check{f}(t,x)$ is still measurable in t and continuous in x, uniformly with respect to t. Now let us extend $\check{f}$ out of K setting $\check{f}(t,x) = 0$ if $x \in \mathbb{R}^n \setminus K$. Let $\tilde{K} \doteq \{ z = \hat{y} + qn + v \mid |v| \leq \rho,\ |q| \leq \rho \}$ and let $\psi \in C^\infty(\mathbb{R}^n)$ verify $\psi(x) = 1$ if $x \in \tilde{K}$, $\psi(x) = 0$ if $x \in \mathbb{R}^n \setminus K$.

The map

$$\tilde{f}(t,x) \doteq \check{f}(t,x)\psi(x) \quad \forall\, (t,x) \in [0,T] \times \mathbb{R}^n$$

is still measurable in t, continuous in x, uniformly with respect to t, and $\tilde{f}(t,x) = f(t,x)$ for all $(t,x) \in [0,T] \times \tilde{K} \cap \bar{\theta}$. Then there exists $\delta_1 > 0$ such that for all $x, x' \in \tilde{K}$ satisfying $|x - x'| \le \delta_1$ one has

$$|\tilde{f}(t,x) - \tilde{f}(t,x')| \le \frac{r}{6}, \tag{4.28}$$

where r is the constant involved in hypothesis **(Sm)**, with δ_1 depending only on f and L.

Let $\bar{y}(\cdot)$ be a solution of

$$\dot{z} = \tilde{f}(t,z), \quad z(\hat{t}) = y(\hat{t}) \doteq \hat{y}, \quad t \in [\hat{t}\,, \hat{t} + k\delta], \tag{4.29}$$

where $\delta \doteq \sup\{d(y(t),\theta) \,|\, t \in [0,t^*]\}$ and $k \doteq \min\left(\frac{6}{r}, \frac{t^*-\hat{t}}{\delta}\right)$. Since $\tilde{f}$ is bounded, such a solution exists in the whole interval $[\hat{t}\,, \hat{t} + k\delta]$.

We claim that $\bar{y}(t) \in \bar{\theta}$ for all $t \in [\hat{t}\,, \hat{t} + k\delta]$. Indeed, by $\mathbf{(Sm)_{iii}}$ and (4.28)—the latter being verified because of (4.27)—, we have

$$\begin{aligned}
\tilde{d}(\bar{y}(t)) &= \int_{\hat{t}}^{t} \nabla\tilde{d}(\bar{y}(s)) \cdot \tilde{f}(s,\bar{y}(s))\,ds = \int_{\hat{t}}^{t} \nabla\tilde{d}(\bar{y}(\hat{t})) \cdot \tilde{f}(s,\bar{y}(\hat{t}))\,ds \\
&\quad + \int_{\hat{t}}^{t} \Big[\nabla\tilde{d}(\bar{y}(s)) \cdot \tilde{f}(s,\bar{y}(s)) - \nabla\tilde{d}(\bar{y}(\hat{t})) \cdot \tilde{f}(s,\bar{y}(\hat{t}))\Big]\,ds \\
&\ge \int_{\hat{t}}^{t} \Big[\nabla\tilde{d}(\bar{y}(\hat{t})) \cdot \tilde{f}(s,\bar{y}(\hat{t})) - \Big|\nabla\tilde{d}(\bar{y}(\hat{t})) \cdot \Big(\tilde{f}(s,\bar{y}(\hat{t})) - \tilde{f}(s,\bar{y}(s))\Big)\Big| \\
&\quad - \Big|\Big(\nabla\tilde{d}(\bar{y}(\hat{t})) - \nabla\tilde{d}(\bar{y}(s))\Big) \cdot \tilde{f}(s,\bar{y}(s))\Big|\Big]\,ds \\
&\ge \int_{\hat{t}}^{t} \Big[r - \frac{r}{6} - M\tilde{L}\Big|\bar{y}(\hat{t}) - \bar{y}(s)\Big|\Big]\,ds,
\end{aligned}$$

where $\tilde{L}$ is the Lipschitz constant for $\nabla\tilde{d}$ in Q'.

Then, possibly reducing t^* in order that $t^* \le \frac{r}{3M^2\tilde{L}}$, we obtain

$$\tilde{d}(\bar{y}(t)) \ge (t - \hat{t})\frac{r}{2}. \tag{4.30}$$

Therefore $\bar{y}(t) \in \bar{\theta}$ for all $t \in [\hat{t}, \hat{t} + k\delta]$ and in particular in (4.29) $\tilde{f}$ can be replaced by f, which implies that $\bar{y}(\cdot)$ is a trajectory of the original differential inclusion.

Setting

$$y_1(t) = \begin{cases} y(t) & t \in [0, \hat{t}] \\ \bar{y}(t) & t \in [\hat{t}, \hat{t} + k\delta], \end{cases}$$

we have

$$\int_0^{\hat{t}+k\delta} |\dot{y}_1(s) - \dot{y}(s)|\, ds \le 2Mk\delta \le \frac{12M\delta}{r}, \tag{4.31}$$

which in turn implies

$$|y_1(t) - y(t)| \le \frac{12M\delta}{r}. \tag{4.32}$$

If $t^* = \hat{t} + k\delta (\le \hat{t} + \frac{6}{r}\delta)$ the proof is concluded (relatively to the interval $[0, t^*]$).

If, on the contrary, $\hat{t} + k\delta = \hat{t} + \frac{6}{r}\delta < t^*$, (4.30) implies that

$$\tilde{d}(y_1(\hat{t} + k\delta)) \ge 3\delta. \tag{4.33}$$

Let $z(\cdot)$ be a solution of

$$\begin{cases} \dot{z} \in F(t, z) \\ z(\hat{t} + k\delta) = y_1(\hat{t} + k\delta) \quad t \in [\hat{t} + k\delta, T] \end{cases}$$

such that

$$|z(t) - y(t)| \le e^{\int_{\hat{t}+k\delta}^{t} \lambda(s)\, ds}(|y_1(\hat{t} + k\delta) - y(\hat{t} + k\delta)|) \tag{4.34}$$

$$|\dot{z}(t) - \dot{y}(t)| \le \lambda(t) e^{\int_{\hat{t}+k\delta}^{t} \lambda(s)\, ds}(|y_1(\hat{t} + k\delta) - y(\hat{t} + k\delta)|). \tag{4.35}$$

The existence of such a z is guaranteed by Filippov's Theorem.

Let us extend $y_1(\cdot)$ to the domain $[0, T]$ by setting

$$y_1(t) = z(t) \qquad \forall\, t \in [\hat{t} + k\delta, T].$$

Let us possibly reduce t^* by assuming that $p(t^*) \le \frac{r}{6M}$, where

$$p(t^*) = \left[1 + \tilde{L}Mt^*\right] e^{\int_0^{t^*} \lambda(t)\, dt} - 1.$$

We claim that $y_1(t) \in \bar{\theta}$ for all $t \in [\hat{t}+k\delta, t^*]$. Indeed, for every $t \in [\hat{t}+k\delta, t^*]$ by (4.33), (4.35), we have

$$\begin{aligned}\tilde{d}(y_1(t)) &= \tilde{d}(y_1(\hat{t}+k\delta)) + \int_{\hat{t}+k\delta}^{t} \nabla\tilde{d}(y_1(s)) \cdot \dot{y}_1(s)\, ds \\ &= \tilde{d}(y_1(\hat{t}+k\delta)) + \int_{\hat{t}+k\delta}^{t} \nabla\tilde{d}(y(s)) \cdot \dot{y}(s)\, ds \\ &\quad + \int_{\hat{t}+k\delta}^{t} \Big(\nabla\tilde{d}(y_1(s)) - \nabla\tilde{d}(y(s))\Big) \cdot \dot{y}(s)\, ds \\ &\quad + \int_{\hat{t}+k\delta}^{t} \nabla\tilde{d}(y_1(s)) \cdot (\dot{y}_1(s) - \dot{y}(s))\, ds \\ &\geq 3\delta - \delta - \tilde{L}M \int_{\hat{t}+k\delta}^{t} e^{\int_{\hat{t}+k\delta}^{s} \lambda(\sigma)\, d\sigma} |y_1(\hat{t}+k\delta) - y(\hat{t}+k\delta)|\, ds \\ &\quad - \int_{\hat{t}+k\delta}^{t} \lambda(s) e^{\int_{\hat{t}+k\delta}^{s} \lambda(\sigma)\, d\sigma} |y_1(\hat{t}+k\delta) - y(\hat{t}+k\delta)|\, ds \\ &\geq 2\delta - \frac{12}{r} M p(t^*)\delta.\end{aligned}$$

In view of the choice of t^* we obtain that

$$\tilde{d}(y_1(t)) \geq 0 \qquad \forall t \in [\hat{t}+k\delta, t^*]$$

i.e., $y_1(t) \in \bar{\theta}\ \forall t \in [0, t^*]$.

Furthermore, collecting (4.31)–(4.35), we obtain

$$|y_1(t) - y(t)| + \int_0^t |\dot{y}_1(s) - \dot{y}(s)|\, ds \leq C\delta, \tag{4.36}$$

where

$$C = \frac{12M}{r}\left(2e^{\int_0^{t^*} \lambda(s)\, ds} - 1\right).$$

Hence the theorem is proved when $t^* = T$, with the constant C defined above. To conclude the proof one applies a recursive argument as in the conclusion of the proof of Theorem 4.1.

Proofs of Theorems 3.1 and 3.2. Theorems 3.1 and 3.2 follow straightforwardly from Theorems 4.1 and 4.2, respectively. For example, to prove Theorem 3.2 let us take a solution $\tilde{y}_2(\cdot)$ of

$$\begin{cases} \dot{y} \in F(t, y) \\ y(\bar{t}) = x_2 \end{cases} \tag{4.37}$$

such that

$$|y_1(t)-\tilde{y}_2(t)|+\int_{\bar{t}}^{t}\left|\dot{y}_1(s)-\dot{\tilde{y}}_2(s)\right|\,ds\le C_1\left(|x_1-x_2|+\int_{\bar{t}}^{t}\gamma(s)\,ds\right) \quad (4.38)$$

with $C_1=2e^{\int_{\bar{t}}^{T}\lambda(s)\,ds}$. Such a $\tilde{y}_2(\cdot)$ exists, by Filippov's Theorem. Of course in general $\tilde{y}_2(\cdot)$ violates the constraints. Let us apply Theorem 4.2, which provides a trajectory $y_2(\cdot)$ of (4.37) such that $y_2(t)\in\bar{\theta}\ \forall\, t\in[\bar{t},T]$ and

$$|y_2(t)-\tilde{y}_2(t)|+\int_{\bar{t}}^{T}\left|\dot{y}_2(s)-\dot{\tilde{y}}_2(s)\right|\,ds\le C_2\sup\left\{\,d(\tilde{y}_2(t),\theta)\ \middle|\ t\in[\bar{t},T]\right\}. \quad (4.39)$$

Since

$$d(\tilde{y}_2(t),\theta)\le|y_1(t)-\tilde{y}_2(t)|\qquad\forall\, t\in[\bar{t},T],$$

by (4.38) and (4.39) one obtains the thesis of Theorem 3.2, with $C=C_1(1+C_2)$.

The proof of Theorem 3.1 is obtained in the same way by using Theorem 4.1 instead of Theorem 4.2.

5. The value function of a Boltz problem with state constraints where the control set is (t,x)-dependent. As an application of Theorems 3.1, 3.2 we prove some continuity properties of the value function for a Boltz problem with state constraints and a control set depending (on time and) on the state. (Further applications can be found in [10], where a relaxation problem is studied; moreover, in [19] Theorems 4.1 and 4.2 are applied as well).

Let us consider the control system

$$\begin{cases}\dot{x}(t)=f(t,x(t),u(t))\\ u(t)\in U(t,x(t))\\ x(t)\in\bar{\theta}\\ x(\bar{t})=\bar{x}.\end{cases} \quad (5.1)$$

A solution to (5.1) is an absolutely continuous map $x:[\bar{t},T]\to\bar{\theta}$ such that $x(\bar{t})=\bar{x}$ and there exists a measurable control $u(t)$ such that the $(5.1)_{\mathrm{i-ii}}$ are verified for almost every $t\in[\bar{t},T]$ and $(5.1)_{\mathrm{iii}}$ holds on $[\bar{t},T]$. Such a control is called *admissible* and we shall denote the class of admissible controls for the initial condition $(\bar{x},\bar{t})$ by $\mathcal{U}(\bar{t},\bar{x})$.

We associate a Boltz functional

$$J(\bar{t},\bar{x},u) \doteq \Psi\left(y_{\bar{t},\bar{x}}[u,T]\right) + \int_{\bar{t}}^{T} l(t, y_{\bar{t},\bar{x}}[u,t], u(t))\,dt \tag{5.2}$$

to the dynamics (5.1), where $u \in \mathcal{U}(\bar{t},\bar{x})$, and $y_{\bar{t},\bar{x}}[u,\cdot]$ denotes the corresponding solution of (5.1). The maps Ψ and l are usually called the final cost and the current cost, respectively.

The *value function* $V : [0,T] \times \bar{\theta}$ is defined by

$$V(\bar{t},\bar{x}) \doteq \inf_{u\in\mathcal{U}(\bar{t},\bar{x})} J(\bar{t},\bar{x},u). \tag{5.3}$$

For the infinite horizon problem the value function in the presence of state constraints has been extensively investigated in the case where the control set is constant (see [1], [7], [12], [13], [14], [15], [16], [20], [21]). Under this latter assumption the finite horizon problem does not present substantial new problems. Actually, the point here is not the fact that $T < +\infty$, but rather the $(t,x)-$dependence of the control set. We limit our attention to the finite horizon problem just for the sake of brevity.

Let us posit the assumptions on the data of the problem.

i. The map:$f : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is t–measurable, x–Lipschitzean on compact sets and Lipschitz continuous in u.

Moreover there exists $c > 0$ such that

$$|f(t,x,u)| \le c(1 + |x| + |u|)$$

for all $(t,x,u) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^m$.

ii. The multifunction

$$U : [0,T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$$

has nonempty compact values and is measurable in t. Moreover U is $\lambda(t)$–Lipschitzean in x with $\lambda(t) \in L^1$ and satisfies a sublinear growth condition in x, i.e.,

$$\mathcal{D}(U(t,x), U(t,y)) \le \lambda(t)|x-y|,$$
$$\sup\{\, |v| \mid v \in U(t,x)\,\} \le c_1(1 + |x|),$$

for almost all $t \in [0,T]$ and for all $x, y \in \mathbb{R}^n$.

Under these conditions the multivalued map

$$F(t,x) \doteq f(t,x,U(t,x))$$

verifies hypothesis **(M–L)**.

Remark 5.1. It is rather common to deal with a map $U(t,x)$ defined by

$$U(t,x) \doteq \{u \in U \,|\, h_i(t,x,u) \leq 0,\; g_j(t,x,u) = 0;\; i = 1,\ldots,r, j = 1,\ldots,s\},$$

where $U \subseteq \mathbb{R}^m$, and h_i, g_j are maps on $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$. In fact $U(t,x)$ is Lipschitz provided that (h,g) satisfies a Mangasarian–Fromowitz type hypothesis (see e.g. [10]).

As a well–known consequence of the Measurable Selection Theorem (see e.g. [2] and [3]) one has that under the above assumptions a map $x(\cdot)$ is a solution of (5.1) if and only if it is a solution of

$$\begin{cases} \dot{x} \in F(t,x) & x(t) \in \bar{\theta} \\ x(\bar{t}) = \bar{x} & \forall t \in [\bar{t}, T]. \end{cases} \tag{5.4}$$

Hence the results of the previous sections can be used in the study of V.

The hypothesis on Ψ and l are as follows:

i. $\Psi : \bar{\theta} \to \mathbb{R}$ is continuous;

ii. $l : [0,T] \times \bar{\theta} \times \mathbb{R}^m \to \mathbb{R}$ is measurable in the t variable, continuous in (x,u), uniformly with respect to t, and bounded.

Finally, on θ we assume the regularity hypothesis $\mathbf{(Sm)_{ii}}$:

the signed function $\tilde{d}$ is locally of class $C^{1,1}$ on $B[\partial\theta;\alpha]$, for some $\alpha > 0$.

The constraint qualification $\mathbf{(Sm)_{iii}}$ now reads:

there exist positive constants η, r and a selection

$$\begin{aligned} \nu : [0,T] \times C_\eta &\longrightarrow \mathbb{R}^m \\ (t,x) &\longmapsto \nu(t,x) \in U(t,x) \end{aligned}$$

measurable in t and continuous in x, uniformly with respect to t, such that

$$\langle f(t,x,\nu(t,x)), \nabla d(x)\rangle \geq r$$

for all $(t,x) \in [0,T] \times C_\eta$.

Remark 5.2. The choice of a smooth $\partial\theta$ is not crucial and a result similar to Theorem 5.1 below can be proved when $\partial\theta$ is merely locally Lipschitz.

However we assume the regularity of $\partial\theta$ to emphasize how the stronger conclusion of Theorem 3.2 allows one to weaken the hypotheses on the current cost l. In fact, an estimate on the derivatives as the one in Theorem 3.2 makes it possible to prove the continuity of V when l is just *continuous in* (x,u) (while, if l were Lipschitz continuous, one could prove the continuity of V by means of the sole estimate on the state variable provided by Theorem 3.1 and by exploiting the standard trick of considering the integral of l as a new state variable).

Theorem 5.1. *Under the above hypotheses the value function V is continuous. If l is Lipschitz and Ψ is locally Lipschitz, then V is locally Lipschitz as well.*

This theorem is a corollary, via obvious (and standard) arguments, of the following proposition.

Proposition 5.1. *Under the above hypotheses for every compact subset $Q \subseteq \bar{\theta}$ there exist a positive constant C and a modulus $\rho(\cdot)$ such that for all $t_1, t_2 \in [0,T]$, $x_1, x_2 \in Q$ and $u_1 \in \mathcal{U}(t_1, x_1)$ there exists a control $u_2 \in \mathcal{U}(t_2, x_2)$ verifying*

$$\sup_{t\in[t_1\vee t_2,T]} |y_1(t) - y_2(t)| + \int_{t_1\vee t_2}^{T} |u_1(t) - u_2(t)|\, dt \le C(|x_1 - x_2| + |t_1 - t_2|) \tag{5.5}$$

$$|J(t_1, x_1, u_1) - J(t_2, x_2, u_2)| \le \rho(|x_1 - x_2| + |t_1 - t_2|), \tag{5.6}$$

where $y_i(\cdot) \doteq y_{t_i,x_i}[u_i, \cdot]$. Moreover, if l is locally Lipschitz continuous in (x,u), uniformly with respect to t and Ψ is locally Lipschitz continuous then ρ can be chosen linear, i.e., $\rho(s) = ks$ for some $k > 0$.

Proof (sketch). In order to obtain (5.5) when $t_1 = t_2$ it is sufficient to apply Theorem 3.2 to the differential inclusion

$$\begin{cases} \dot{z} \in \tilde{F}(t,z) \\ z(t) \in \bar{\theta} \times \mathbb{R}^m, \quad t \in [t_i, T] \\ z(t_i) = (x_i, 0), \end{cases}$$

where $z = (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^m$ and

$$\tilde{F}(t,z) \doteq \{(f(t,x,u), u) \mid u \in U(t,x)\}.$$

The case when $t_1 \neq t_2$ follows straightforwardly.

Let us prove (5.6) when $t_1 = t_2$, the general case being a trivial consequence of the former in view of the fact that l is bounded. Actually (5.6) for $t_1 = t_2$ follows from (5.5): in fact, there exists a compact set $\tilde{Q} \supseteq Q$ such that $y_1(t) \in \tilde{Q}\ \forall\, t \in [t_1, T]$ and $y_2(t) \in \tilde{Q}\ \forall t \in [t_2, T]$. Hence the controls u_1 and u_2 take values inside a ball $B[0; R]$ where R is such that

$$\max_{x \in \tilde{Q}} c_1(1 + |x|) \leq R,$$

where c_1 is the constant appearing in the hypotheses on U made at the beginning of this section. Thus (5.6) follows from (5.5), for Ψ is uniformly continuous on $\tilde{Q}$ and $f(t, \cdot, \cdot)$ is uniformly continuous on $\tilde{Q} \times B[0; R]$, uniformly with respect to t. The part of the statement concerning the choice of a linear ρ is straightforward. □

Finally, thanks to the above regularity result one can prove that V is the unique solution (in a sense specified below) of a suitable boundary value problem involving the Bellman equation

$$0 = H(t, x, D_t v, D_x v) \doteq \max_{u \in U(t,x)} \left\{ -D_t v - f(t, x, u) \cdot D_x v - l(t, x, u) \right\}. \tag{HJB}$$

Let us state the result, giving just some hints of its proofs.

We assume in what follows some more regularity in the dynamics. Precisely we add to the previous hypotheses the following one:

Hypothesis (LL). *The maps f and U are Lipschitz continuous on compact sets.*

Let us begin by recalling the definition of viscosity solution of a first order partial differential equation (see e.g. [9]). Let K be a subset of $\mathbb{R}^{n+1}$ and let $C^1(\mathbb{R}^{n+1})$, $C(K)$ denote the set of continuous differentiable functions on $\mathbb{R}^{n+1}$ and the set of continuous functions on K, respectively.

Definition 5.1. We say that $v \in C(K)$ is a *viscosity subsolution* of (HJB) on K if for all $(t_0, x_0) \in K$ and for all $\phi \in C^1(\mathbb{R}^{n+1})$ such that $v - \phi$ has a maximum relative to K at (t_0, x_0) one has

$$H(t_0, x_0, D_t\phi(t_0, x_0), D_x\phi(t_0, x_0)) \leq 0.$$

We say that $v \in C(K)$ is a *viscosity supersolution* of (HJB) on K if for all $(t_0, x_0) \in K$ and for all $\psi \in C^1(\mathbb{R}^{n+1})$ such that $v - \psi$ has a minimum relative to K at (t_0, x_0) one has

$$H(t_0, x_0, D_t\psi(t_0, x_0), D_x\psi(t_0, x_0)) \geq 0.$$

If v is both a subsolution and a supersolution, then v is called a *viscosity solution* of (HJB) on K.

In [20] H.M. Soner introduced the notion of *constrained viscosity solution* for the infinite horizon problem. Let us slightly modify this definition in order to obtain a concept of constrained viscosity solution for the finite horizon problem as well.

Definition 5.2. A function $v \in C([0,T] \times \bar{\theta})$ is called a *constrained viscosity solution* of (HJB) on $[0,T] \times \bar{\theta}$ if v is a subsolution on $[0,T[\times\theta$, v is a supersolution on $[0,T[\times\bar{\theta}$. Moreover, we say that v is a solution of *the Cauchy Problem* (briefly, v *is a solution of* (CP)) if v is a constrained viscosity solution of (HJB) and $v(T,x) = \Psi(x)\ \forall\, x \in \bar{\theta}$.

Theorem 5.2. *Under hypothesis* **(LL)** *the value function V is a solution of* (CP). *Moreover, if Ψ is bounded, then V is bounded as well and is the unique solution of*(CP).

Proof (sketch). First of all one proves for V a Dynamic Programming Principle. This does not involve any new argument with respect to the unconstrained case. As a consequence, the proof that V is a supersolution of (HJB) on $[0,T[\times\bar{\theta}$ is standard. In order to prove that V is a subsolution at any point $(\bar{t},\bar{x}) \in [0,T[\times\theta$ the only nontrivial step which makes the difference with the unconstrained case concerns the possibility of constructing an admissible trajectory which is C^1 at the initial time $\bar{t}$ and has a given (admissible) initial velocity. This can be easily done in two steps. First one finds a (generally non admissible) C^1 trajectory with the given initial velocity, whose existence is stated in Lemma 5.1 below (for an analogous goal, this lemma was also exploited in [5]). Secondly, one modifies this trajectory into a neighboring admissible trajectory in view of Theorem 4.2. Both the Cauchy condition at $t = T$ and the fact that V is bounded (provided Ψ is bounded) are trivial.

To prove uniqueness, let us introduce the map

$$W(t,x) \doteq \frac{1}{N} \log\Big[\frac{V(t,x)+C}{T'-t}\Big], \tag{5.7}$$

where C and T' are positive numbers such that $C > \max_{(t,x)\in[0,T]\times\bar{\theta}} |V(t,x)|$, $T' > T$ and $N \geq \max_{(t,x,u)\in[0,T]\times\bar{\theta}\times\mathbb{R}^m} |l(t,x,u)| + 1$.

Then W is a constrained viscosity solutions of

$$\begin{aligned} 0 =& u + \mathcal{H}(t,x,u,D_t u, D_x u) \doteq \\ & u + \max_{u\in U(t,x)} \big\{ -N(T'-t)D_t u - N(T'-t)D_x u \cdot f(t,x,u) \\ & \qquad + (N-1)u - \frac{l(t,x,u)}{e^{N(T'-t)u}} \big\} \end{aligned} \tag{HJB$'$}$$

verifying

$$W(T,x) = \Phi(x) \doteq \frac{1}{N} \log \Big[\frac{\Psi(x)+C}{T'-T} \Big]. \tag{BC}$$

It is simple to check that $\mathcal{H}$ verifies the hypotheses of Theorem 1.1 in [4], from which we obtain that W is the unique constrained viscosity solution of (HJB)$'$ that verifies condition (BC).

Hence, by simply exploiting the change of (independent) variable provided by the inversion of (5.7), we recover that V is the unique solution of (CP).

Lemma 5.1. *Under hypothesis* **(LL)**, *for all* $\bar{u} \in U(\bar{t},\bar{x})$ *there exist a continuously differentiable function* $y_{\bar{t},\bar{x}}(\cdot)$ *and a continuous function* $u(\cdot)$ *solutions of*

$$\begin{cases} \dot{x}(t) \in f(t,x,U(t,x)) \quad u(t) \in U(t,x) \\ x(\bar{t}) = \bar{x} \\ \dot{x}(\bar{t}) = f(\bar{t},\bar{x},\bar{u}). \end{cases}$$

The proof of this Lemma can be found e.g. in [2].

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