Differential and Integral Equations Volume 12, Number 4, July 1999, 453–469.

EXISTENCE OF GLOBAL SOLUTIONS AND ENERGY DECAY FOR THE CARRIER EQUATION WITH DISSIPATIVE TERM

Cícero Lopes Frota and Alfredo Tadeu Cousin
Universidade Estadual de Maringá, Departamento de Matemática
Av. Colombo, 5790 - CEP: 87020-900 Maringá, Pr., Brasil

Nickolai A. Lar′kin[1]
The Institute of Theoretical and Applied Mechanics
Novossibirsk - 90, 630090, Russia

(Submitted by: J.A. Goldstein)

Abstract. We prove the existence and uniqueness of global solutions to the mixed problem for the Carrier equation

$$u_{tt} - M(\int_\Omega u^2\,dx)\Delta u + g(u_t) = f,$$

where $g'(s) \geq 0$, $0 < m_0 \leq M(\lambda)$ and no "smallness" conditions are imposed on the initial data. Moreover, the algebraic and exponential decays of the energy are proved.

1. Introduction. Our goal is to establish global solvability and decay of the energy of the damped Carrier equation

$$u_{tt} - M\Big(\int_\Omega u^2\,dx\Big)\Delta u + \alpha u_t + \mid u_t \mid^\rho u_t = f, \tag{1.1}$$

with the Dirichlet boundary condition

$$u_{|\Sigma} = 0 \tag{1.2}$$

Received for publication November 1997.
[1]Supported by CNPq-Brasil as a Visiting Professor at the State University of Maringá.
AMS Subject Classifications: 35L05, 35L15, 35L70

and the initial data

$$u(x,0) = u_0(x)\,, \quad u_t(x,0) = u_1(x)\,. \tag{1.3}$$

Here, $x \in \Omega \subset \mathbf{R}^n, \Omega$ is a bounded domain with the smooth boundary Γ, $\Sigma = \Gamma \times (0,T)$, $\rho \in [1,\infty)$ and α is a nonnegative constant.

Unlike the Kirchhoff equation

$$u_{tt} - M\Big(\int_\Omega \mid \nabla u \mid^2 \, dx\Big)\Delta u = f, \tag{1.4}$$

which was derived in [9] to model vibrations of an elastic string with fixed ends and have been studied by many authors, the Carrier equation

$$u_{tt} - M\Big(\int_\Omega u^2 dx\Big)\Delta u = f, \tag{1.5}$$

derived in [3] to model vibrations of an elastic string with fixed ends when the changes in tension are not small, does not have a long history.

Concerning equation (1.4), the greatest part of papers is devoted to establishing of global existence results. In [1] was proved global solvability of the mixed problem in a class of analytic functions. Later, this result was generalized in [18] for a class of operator equations. Global solvability in Sobolev functional spaces was proved in [19] for the very special class of functions $M(\lambda)$, see also [7] when $M'(\lambda)$ is sufficiently small and [16] when the solutions are close to equilibrium solutions.

To obtain global solvability for the Kirchhoff model, many authors considered damped equation when the function $g(u_t)$ is such that $g'(s) \geq \tau_0 > 0$. In this case, it is possible to obtain global existence results if the initial data and the function $f(x,t)$ are sufficiently small in some sense. In this direction we can mention [2], [5], [8], [10], [13], [14], and [20]. In the last paper, linear damping was considered at the boundary of Ω.

In connection with equation (1.5), we can mention [4] where an operator equation, which includes Kirchhoff and Carrier cases, was considered. Global solvability in a class of analytic functions and local solvability in Sobolev spaces were proved. Unilateral problems for the Kirchhoff and Carrier equations were studied in [6].

In the present paper we use a nonlinear damping and technique from [11] to prove global solvability without "smallness" conditions for the initial data. Moreover, the algebraic and the exponential decays are obtained.

2. Global solutions. Throughout the paper, the norm and the inner product in the Hilbert spaces $L^2(\Omega)$ and $H_0^1(\Omega)$ are denoted, respectively, by

$$\|u\| \;=\Big(\int_\Omega \mid u(x)\mid^2\,dx\Big)^{\frac{1}{2}}\;,\;\;(u,v)=\int_\Omega u(x)\,v(x)\,dx$$

$$\|u\|_{H_0^1}=\Big(\sum_{i=1}^{n}\int_\Omega \mid \frac{\partial u}{\partial x_i}(x)\mid^2\,dx\Big)^{\frac{1}{2}}\;,\;\;((u,v))=\sum_{i=1}^{n}\int_\Omega \frac{\partial u}{\partial x_i}(x)\,\frac{\partial v}{\partial x_i}(x)\,dx\,;$$

and $\|u\|_p$ denotes the $L^p(\Omega)-$norm given by

$$\|u\|_p=\Big(\int_\Omega \mid u(x)\mid^p\;dx\Big)^{\frac{1}{p}}.$$

For the functional spaces we employ the usual notations (see [12]). To prove the energy decay, we use the following lemma of M. Nakao, [15] and [16].

Lemma 2.1 (M. Nakao)**.** *Let $E(t)$ be a bounded nonnegative function defined on $\mathbf{R}^+$, satisfying the inequality*

$$\sup_{t\le s\le (t+1)} E(s)^{1+r}\;\le\;C_0\,[E(t)-E(t+1)]\,,\qquad \textit{for all } t\in\mathbf{R}^+,$$

where C_0 is a positive constant and r a nonnegative constant.
i) *If $r>0$, then there exists a constant $C>0$ such that*

$$E(t)\;\le\;C(1+t)^{-\frac{1}{r}},\qquad \textit{for all } t\ge 0.$$

ii) *If $r=0$, then there are positive constants K and θ such that*

$$E(t)\;\le\;Ke^{-\theta t}\,,\quad \textit{for all } t\ge 1.$$

In this section we prove the following theorem.

Theorem 2.1. *Let T be a finite positive number, $\rho>1$ and $M:[0,\infty)\to\mathbf{R}$ be a function satisfying*

$$M\in C^1([0,\infty))\;\;\textit{with}\;\;0<m_0\le M(\lambda),\;\;\textit{for all}\;\lambda\ge 0;\tag{2.1}$$

$$\frac{\mid M'(\lambda)\,\lambda^{\frac{1}{2}}\mid}{M(\lambda)}\;\le k_0,\quad \textit{for all}\;\lambda\ge 0,\tag{2.2}$$

where k_0 is a positive constant. Then for each $u_0 \in H_0^1(\Omega) \cap H^2(\Omega), u_1 \in H_0^1(\Omega) \cap L^{2\rho+2}(\Omega)$ and $f, f' \in L^2(0,T;L^2(\Omega))$, there exists a unique function $u(x,t)$ in the class $u \in L^\infty(0,T;H_0^1(\Omega) \cap H^2(\Omega))$, $u' \in L^\infty(0,T;H_0^1(\Omega))$, $u'' \in L^\infty(0,T;L^2(\Omega))$ such that

$$u'' - M(\int_\Omega u^2 dx)\Delta u + \mid u' \mid^\rho u' = f \quad a.e. \ in \ Q = \Omega \times (0,T)\,; \tag{2.3}$$

$$u = 0 \quad on \ \Sigma = \Gamma \times (0,T)\,; \tag{2.4}$$

$$u(0) = u_0\,, \qquad u'(0) = u_1\,. \tag{2.5}$$

Moreover, if $\rho = 1$ and k_0 satisfies

$$k_0 \ \mu_\Omega^{\frac{1}{2}} < 1\,, \tag{2.6}$$

(here μ_Ω denotes the Lebesgue measure of Ω), then for each $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $u_1 \in H_0^1(\Omega) \cap L^4(\Omega)$ and $f, f' \in L^2(0,T;L^2(\Omega))$, there exists a unique solution $u(x,t)$ to the problem (2.3)-(2.5).

Proof. Let $\{\omega_j\}_{j\in\mathbf{N}}$ be eigenfunctions of the following problem

$$-\Delta\omega_j = \lambda_j\omega_j, \qquad \omega_{j_{|\Gamma}} = 0, \quad j = 1,2,3,.... \tag{2.7}$$

For each $m \in \mathbf{N}$ we construct the function

$$u_m(x,t) = \sum_{j=1}^m g_{jm}(t)\,\omega_j(x), \quad x \in \Omega \ \text{and} \ t \in [0,T_m], \tag{2.8}$$

which is a solution to the following problem

$$(u_m''(t),\omega_j) - M(\|u_m(t)\|^2)(\Delta u_m(t),\omega_j) + (\mid u_m'(t) \mid^\rho u_m'(t),\omega_j) = (f(t),\omega_j)\,, \quad 1 \le j \le m,$$

$$u_m(0) = u_{0m} = \sum_{j=1}^m (u_0,\omega_j)\,\omega_j, \qquad u_m'(0) = u_{1m} = \sum_{j=1}^m (u_1,\omega_j)\,\omega_j.$$

Taking into account (2.1), we can rewrite this problem in the form

$$\frac{(u_m''(t),v)}{M(\|u_m(t)\|^2)} - (\Delta u_m(t),v) + \frac{(\mid u_m'(t) \mid^\rho u_m'(t),v)}{M(\|u_m(t)\|^2)} = \frac{(f(t),v)}{M(\|u_m(t)\|^2)}, \tag{2.9}$$

for all $v \in V_m$.

The first estimate. Putting $v = 2u_m'(t)$ in (2.9), we obtain

$$\frac{d}{dt}\left[\frac{\|u_m'(t)\|^2}{M(\|u_m(t)\|^2)} + \|u_m(t)\|^2_{H_0^1}\right] + 2\,\frac{\|u_m'(t)\|^{\rho+2}_{\rho+2}}{M(\|u_m(t)\|^2)} \tag{2.10}$$
$$= -2\,\frac{M'(\|u_m(t)\|^2)\,(u_m'(t), u_m(t))}{\left(M(\|u_m(t)\|^2)\right)^2}\|u_m'(t)\|^2 + 2\frac{(f(t), u_m'(t))}{M(\|u_m(t)\|^2)}.$$

By (2.2), we have

$$2\frac{|\,M'(\|u_m(t)\|^2)(u_m'(t), u_m(t))\,|}{\left(M(\|u_m(t)\|^2)\right)^2}\|u_m'(t)\|^2 \le \frac{2K_0\|u_m'(t)\|^3}{M(\|u_m(t)\|^2)}$$
$$\le 2C_1\frac{\|u_m'(t)\|^3_{\rho+2}}{M(\|u_m(t)\|^2)}.$$

Here we have used $L^{\rho+2}(\Omega) \hookrightarrow L^2(\Omega)$; the positive constant C_1 depends on K_0 and Ω. Since $\rho > 1$ using Young's inequality for all $\epsilon > 0$, we find

$$2C_1\|u_m'(t)\|^3_{\rho+2} \;\le\; \frac{(\rho-1)(2C_1)^{\frac{\rho+2}{\rho-1}}}{\epsilon^{\frac{3}{\rho-1}}(\rho+2)} \;+\; \frac{3\epsilon\|u_m'(t)\|^{\rho+2}_{\rho+2}}{(\rho+2)}.$$

From the above inequalities and (2.10), we obtain

$$\frac{d}{dt}\left[\frac{\|u_m'(t)\|^2}{M(\|u_m(t)\|^2)} + \|u_m(t)\|^2_{H_0^1}\right] + (2-\epsilon C_2)\frac{\|u_m'(t)\|^{\rho+2}_{\rho+2}}{M(\|u_m(t)\|^2)}$$
$$\le \frac{C_3(\epsilon)}{M(\|u_m(t)\|^2)} + \frac{2\;(f(t), u_m'(t))}{M(\|u_m(t)\|^2)}.$$

Choosing $\epsilon > 0$ sufficiently small, we have

$$\frac{d}{dt}\left[\frac{\|u_m'(t)\|^2}{M(\|u_m(t)\|^2)} + \|u_m(t)\|^2_{H_0^1}\right] + C_4\frac{\|u_m'(t)\|^{\rho+2}_{\rho+2}}{M(\|u_m(t)\|^2)}$$
$$\le C_5 + 2\frac{|\,f(t)\,|\;\;|\,u_m'(t)\,|}{M(\|u_m(t)\|^2)},$$

where C_4, C_5 are positive constants independent of m, t. Taking into account that $L^{\rho+2}(\Omega) \hookrightarrow L^2(\Omega)$ and applying Young's inequality, we get

$$\frac{d}{dt}\left[\frac{\|u_m'(t)\|^2}{M(\|u_m(t)\|^2)} + \|u_m(t)\|^2_{H_0^1}\right] + C_6\frac{\|u_m'(t)\|^{\rho+2}_{\rho+2}}{M(\|u_m(t)\|^2)} \le C_7 + C_8\|f(t)\|^{\frac{\rho+2}{\rho+1}},$$

where C_6, C_7 and C_8 are positive constants which do not depend on t and m. Integrating the above inequality over (0,t), we obtain the following estimate

$$\frac{\|u_m'(t)\|^2}{M(\|u_m(t)\|^2)} + \|u_m(t)\|^2_{H_0^1} + C_6 \int_0^t \frac{\|u_m'(\tau)\|^{\rho+2}_{\rho+2}}{M(\|u_m(\tau)\|^2)} \;\le C_9. \tag{2.11}$$

Using this and Poincaré′s inequality, we find

$$\|u_m(t)\|^2 + \|u_m(t)\|^2_{H_0^1} \;\le\; C_{10}, \qquad \text{for all } t \in [0, T_m]. \tag{2.12}$$

This estimate allows us to extend solutions u_m to the whole interval $[0,T]$. Moreover, since M is a C^1 function, we have

$$0 < m_0 \le M(\|u_m(t)\|^2) \le \max\{M(\lambda);\; 0 \le \lambda \le C_{10}\} = M_0. \tag{2.13}$$

Therefore, (2.11) gives

$$\|u_m'(t)\|^2 + \|u_m(t)\|^2_{H_0^1} + \int_0^T \|u_m'(t)\|^{\rho+2}_{\rho+2}\, dt \;\le C_{11}, \text{ for all } t \in [0,T]. \tag{2.14}$$

The second estimate. First, from equation (2.9), we have

$$\|u_m''(0)\| \;\le C_{12}(\|\Delta u_{0m}\| + \|u_{1m}\|^{\rho+1}_{2\rho+2} + \|f(0)\|)$$

which implies

$$\|u_m''(0)\|^2 \;\le\; C_{13}. \tag{2.15}$$

Differentiating (2.9) with respect to t and taking $v = 2u_m''(t)$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{\|u_m''(t)\|^2}{M(\|u_m(t)\|^2)} + \|u_m'(t)\|^2_{H_0^1}\right] + \frac{2(\rho+1)}{M(\|u_m(t)\|^2)}(|\, u_m'(t)\,|^\rho, (u_m''(t))^2) \\
&= \frac{4M'(\|u_m(t)\|^2)}{(M(\|u_m(t)\|^2))^2}\,(u_m'(t), u_m(t))(|\, u_m'(t)\,|^\rho\; u_m'(t), u_m''(t)) \\
&+ \frac{2M'(\|u_m(t)\|^2)}{(M(\|u_m(t)\|^2))^2}(u_m'(t), u_m(t))\|u_m''(t)\|^2 + \frac{2}{M(\|u_m(t)\|^2)}(f'(t), u_m''(t)) \\
&- \frac{4M'(\|u_m(t)\|^2)}{(M(\|u_m(t)\|^2))^2}(u_m'(t), u_m(t))(f(t), u_m''(t)).
\end{aligned} \tag{2.16}$$

Using (2.12) and (2.14), we get for all $\epsilon > 0$

$$\begin{aligned}&\frac{4M'(\|u_m(t)\|^2)}{(M(\|u_m(t)\|^2))^2}(u'_m(t), u_m(t))(|\ u'_m(t)\ |^\rho\ u'_m(t), u''_m(t))\\&\le C_{14}\ |\ (|\ u'_m(t)\ |^\rho\ u'_m(t), u''_m(t))\ |\\&\le \frac{\epsilon C_{14}}{2}(|\ u'_m(t)\ |^\rho, u''_m(t))^2) + \frac{C_{14}}{2\epsilon}\|u'_m(t)\|^{\rho+2}_{\rho+2}.\end{aligned}$$

This inequality and (2.16) give

$$\begin{aligned}&\frac{d}{dt}\left[\frac{\|u''_m(t)\|^2}{M(\|u_m(t)\|^2)} + \|u'_m(t)\|^2_{H^1_0}\right] + \frac{2(\rho+1)}{M_0}(|\ u'_m(t)\ |^\rho, (u''_m(t))^2)\\&\le \epsilon C_{15}(|\ u'_m(t)\ |^\rho, (u''_m(t))^2) + C_{16}(\epsilon)\|u'_m(t)\|^{\rho+2}_{\rho+2}\\&+ C_{17}(\|f(t)\|^2 + \|f'(t)\|^2 + \|u''_m(t)\|^2).\end{aligned}$$

Choosing an appropriate $\epsilon > 0$, we have

$$\begin{aligned}&\frac{d}{dt}\left[\frac{\|u''_m(t)\|^2}{M(\|u_m(t)\|^2)} + \|u'_m(t)\|^2_{H^1_0}\right] + C_{18}(|\ u'_m(t)\ |^\rho, (u''_m(t))^2)\\&\le C_{19}\left(\|u'_m(t)\|^{\rho+2}_{\rho+2} + \|f(t)\|^2 + \|f'(t)\|^2 + \|u''_m(t)\|^2\right).\end{aligned}$$

Integrating over (0,t) and taking into account (2.13), we obtain

$$\|u''_m(t)\|^2+\|u'_m(t)\|^2_{H^1_0}+\int_0^t(|\ u'_m(\tau)\ |^\rho, (u''_m(\tau))^2)d\tau \le C_{20}(1+\int_0^t\|u''_m(\tau)\|^2 d\tau),$$

whence,

$$\|u''_m(t)\|^2 + \|u'_m(t)\|^2_{H^1_0} \le C_{21}. \tag{2.17}$$

The third estimate. Putting $v = -2\Delta u'_m(t)$ in (2.9), we have

$$\begin{aligned}&\frac{d}{dt}[\frac{\|u'_m(t)\|^2_{H^1_0}}{M(\|u_m(t)\|^2)} + \|\Delta u_m(t)\|^2] + \frac{2(\rho+1)}{M(\|u_m(t)\|^2)}\sum_{i=1}^n\int_\Omega |\ u'_m\ |^\rho\ (\frac{\partial u'_m}{\partial x_i})^2 dx\\&= -2\frac{M'(\|u_m(t)\|^2)}{(M(\|u_m(t)\|^2))^2}(u'_m(t), u_m(t))\|u'_m(t)\|^2_{H^1_0} + 2(f(t), -\Delta u'_m(t)).\end{aligned}$$

Integration over $(0,t)$ yields

$$\|\Delta u_m(t)\|^2 \le C_{22}. \tag{2.18}$$

Estimates (2.14), (2.17) and (2.18) allow us to pass to the limit as $m \to \infty$ and get a solution of (2.3)–(2.5). One can prove uniqueness in a standard way. It remains to prove solvability of $(2.3)-(2.5)$ for $\rho = 1$. Taking $v = 2u_m'(t)$ in (2.9) with $\rho = 1$, we find

$$\frac{d}{dt}\left[\frac{\|u_m'(t)\|^2}{M(\|u_m(t)\|^2)} + \|u_m(t)\|^2_{H_0^1}\right] + 2\frac{\|u_m'(t)\|_3^3}{M(\|u_m(t)\|^2)}$$
$$= -2\frac{M'(\|u_m(t)\|^2)(u_m'(t),u_m(t))}{(M(\|u_m(t)\|^2))^2}\|u_m'(t)\|^2 + 2\frac{(f(t),u_m'(t))}{M(\|u_m(t)\|^2)}.$$

Since $L^3(\Omega) \hookrightarrow L^2(\Omega)$ and M satisfies (2.2), we get

$$\frac{d}{dt}\left[\frac{\|u_m'(t)\|^2}{M(\|u_m(t)\|^2)} + \|u_m(t)\|^2_{H_0^1}\right] + 2(1-\mu_\Omega^{\frac{1}{2}}k_0)\frac{\|u_m'(t)\|_3^3}{M(\|u_m(t)\|^2)}$$
$$\le \frac{2}{M(\|u_m(t)\|^2)}\|f(t)\|\|u_m'(t)\|.$$

Hence, by assumption (2.6) and Young′s inequality, we have

$$\frac{d}{dt}\left[\frac{\|u_m'(t)\|^2}{M(\|u_m(t)\|^2)} + \|u_m(t)\|^2_{H_0^1}\right] + C_{23}\frac{\|u_m'(t)\|_3^3}{M(\|u_m(t)\|^2)} \le C_{24}\|f(t)\|^{\frac{3}{2}}.$$

Integration over (0,t) yields

$$\frac{\|u_m'(t)\|^2}{M(\|u_m(t)\|^2)} + \|u_m(t)\|^2_{H_0^1} + C_{23}\int_0^t \frac{\|u_m'(\tau)\|_3^3}{M(\|u_m(\tau)\|^2)} \le C_{25}. \tag{2.19}$$

Now, we can proceed as before and conclude the proof of Theorem 2.1.

Remark 2.1. If we consider equation (1.1) with $\alpha > 0$ instead of (2.3), the assertions of Theorem 2.1 remain valid.

3. Energy decay. In the present section we shall investigate the energy decay for the problem (1.1)–(1.3). First, we consider the case $\alpha = 0$ and $\rho = 1$

$$u'' - M(\int_\Omega u^2 dx)\Delta u + \mid u' \mid u' = 0 \quad \text{a.e. in } \Omega \times (0,\infty); \tag{3.1}$$

$$u = 0 \quad \text{on } \Sigma = \Gamma \times (0,\infty); \tag{3.2}$$

$$u(0) = u_0, \qquad u'(0) = u_1. \tag{3.3}$$

If u is a solution of (3.1) $-$ (3.3), that is, u belongs to the class

$$u \in L^\infty_{loc}(0,\infty; H^1_0(\Omega) \cap H^2(\Omega)), \qquad (3.4)$$
$$u' \in L^\infty_{loc}(0,\infty; H^1_0(\Omega)), \qquad u'' \in L^\infty_{loc}(0,\infty; L^2(\Omega));$$

satisfies equation (3.1) almost everywhere on $\Omega \times (0,\infty)$ and the initial conditions (3.3), then the energy is defined by

$$E(t) = E_u(t) = \|u'(t)\|^2 + \|u(t)\|^2_{H^1_0}, \qquad t \geq 0. \qquad (3.5)$$

For the existence and uniqueness of such solution we apply Theorem 2.1. Our decay result is the following.

Theorem 3.1. *Let $u(x,t)$ be a solution to problem* (3.1)-(3.3). *If $n \leq 6$, then there exists a positive constant k such that*

$$E(t) \leq k(1+t)^{-\frac{1}{2}}, \qquad \text{for all } \ t \geq 0. \qquad (3.6)$$

Proof. Multiplying (3.1) by $2u'(x,t)$, integrating the result over Ω, and using (2.6), we obtain

$$\Psi'(t) + C_1 \frac{\|u'(t)\|_3^3}{M(\|u(t)\|^2)} \ \leq 0, \qquad (3.7)$$

where

$$\Psi(t) = \frac{\|u'(t)\|^2}{M(\|u(t)\|^2)} + \|u(t)\|^2_{H^1_0}, \qquad t \geq 0. \qquad (3.8)$$

Integrating (3.7) over $(0,t)$, we get

$$\Psi(t) + C_1 \int_0^t \frac{\|u'(\tau)\|_3^3}{M(\|u(\tau)\|^2)} d\tau \ \leq \ \Psi(0). \qquad (3.9)$$

By (3.7) and (3.9), we see that Ψ is a decreasing and bounded function. From (3.8) and the immersion $H^1_0(\Omega) \hookrightarrow L^2(\Omega)$, it follows that

$$m_0 \leq M(\|u(t)\|^2) \leq \sup\{M(\|u(t)\|^2); 0 \leq t < \infty\} = M_0. \qquad (3.10)$$

Using (3.10) and integrating (3.7) over (t , t+1), we find

$$\int_t^{t+1} \|u'(\tau)\|_3^3 d\tau \leq C_2 I^3(t), \qquad (3.11)$$

where $I^3(t) = [\Psi(t) - \Psi(t+1)]$, $t \geq 0$. By the Mean Value Theorem, there exist two points $t_1 \in [t, t + \frac{1}{4}]$ and $t_2 \in [t + \frac{3}{4}, t+1]$ such that

$$\|u'(t_i)\| \leq C_3 I(t), \qquad i = 1, 2. \tag{3.12}$$

Recalling the definition of $\Psi(t)$, we have

$$\|u(t)\| \leq C_4 \Psi^{\frac{1}{2}}(t), \qquad \text{for all } \; t \geq 0; \tag{3.13}$$

and, using Holder′s inequality and (3.11), we get

$$\int_{t_1}^{t_2} \|u'(\tau)\|^2 d\tau \leq C_5 I^2(t). \tag{3.14}$$

Since $n \leq 6$, it follows from Sobolev′s imbedding theorem that $H_0^1(\Omega) \hookrightarrow L^3(\Omega)$. Then

$$\int_{t_1}^{t_2} \mid (\mid u'(\tau) \mid u'(\tau), u(\tau)) \mid d\tau \leq C_6 I^2(t). \tag{3.15}$$

Now, multiplying the equation (3.1) by $u(x,t)$ and integrating the result over Ω, we obtain

$$M(\|u(t)\|^2)\|u(t)\|^2_{H_0^1} = -\frac{d}{dt}(u'(t), u(t)) + \|u(t)\|^2 - (\mid u'(t) \mid u'(t), u(t)).$$

Integrating this equality from t_1 to t_2 and taking into account (3.12)-(3.15), we have

$$\int_{t_1}^{t_2} M(\|u(\tau)\|^2)\|u(\tau)\|^2_{H_0^1} d\tau \leq C_7 I(t) \Psi^{\frac{1}{2}}(t) + C_8 I^2(t). \tag{3.16}$$

From (3.14) and (3.16), we can see that

$$\int_{t_1}^{t_2} \Psi(\tau) d\tau \leq C_9 (I^2(t) + I(t)\Psi^{\frac{1}{2}}(t)), \tag{3.17}$$

hence, there exists a point $t^* \in [t_1, t_2]$ such that

$$\Psi(t^*) \leq C_{10}(I^2(t) + I(t)\Psi^{\frac{1}{2}}(t)). \tag{3.18}$$

Multiplying the equation (3.1) by $2u'$ and integrating from t^* to t, we get

$$\Psi(t) = \Psi(t^*) + 2\int_t^{t^*} \frac{\|u'(\tau)\|_3^3}{M(\|u(\tau)\|)} d\tau + 2\int_t^{t^*} \frac{M'(\|u(\tau)\|^2)}{(M(\|u(\tau)\|^2))^2}(u'(\tau), u(\tau))\|u'(\tau)\|^2 d\tau.$$

Therefore, $\Psi(t) \leq C_{11} I^2(t)(1 + I(t))$. Since $I(t)$ is bounded, we can see that $\Psi^{\frac{3}{2}}(t) \leq C_{12} I^3(t)$ which implies

$$\sup_{t \leq s \leq t+1} \Psi^{\frac{3}{2}}(s) \leq C_{12}[\Psi(t) - \Psi(t+1)]. \tag{3.19}$$

Nakao′s Lemma (cf. Lemma 2.1) allows us to conclude the proof of Theorem 3.1.

Example. Assume that $n \leq 6, u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ and $u_1 \in H_0^1(\Omega) \cap L^4(\Omega)$. If δ is a positive constant such that

$$\delta \mu_\Omega < \sqrt{2}, \tag{3.20}$$

then there exists a unique function $u(x,t)$ in the class $u \in L^\infty_{loc}(0,\infty; H_0^1(\Omega) \cap H^2(\Omega))$, $u' \in L^\infty_{loc}(0,\infty; H_0^1(\Omega))$, $u'' \in L^\infty_{loc}(0,\infty; L^2(\Omega))$ such that

$$\begin{aligned} &u'' - (1 + \delta \int_\Omega u^2 dx)\Delta u + \mid u' \mid u' = 0 \quad \text{a.e. in } \Omega \times (0,\infty); \\ &u = 0 \quad \text{on } \Sigma = \Gamma \times (0,\infty); \\ &u(0) = u_0, \qquad u'(0) = u_1. \end{aligned}$$

Moreover,

$$\|u'(t)\|^2 + \|u(t)\|^2_{H_0^1} \leq k(1+t)^{-\frac{1}{2}}, \qquad \text{for all } t \geq 0,$$

where k is a positive constant. In this case we have

$$M(\lambda) = 1 + \delta\lambda, \qquad \lambda \geq 0, \tag{3.21}$$

$$h(\lambda) = \frac{\mid M'(\lambda)\lambda^{\frac{1}{2}} \mid}{M(\lambda)} = \frac{\delta \lambda^{\frac{1}{2}}}{(1+\delta\lambda)}, \quad \text{for all } \lambda \geq 0$$

which implies that $\lim_{\lambda\to 0^+} h(\lambda) = \lim_{\lambda\to\infty} h(\lambda) = 0$. It can be seen that the function h has the maximum value $k_0 = \frac{\delta^{\frac{1}{2}}}{2}$ at the point $\lambda_0 = \frac{1}{\delta}$. By (3.20), we can see that the function M, given by (3.21), satisfies (2.1), (2.2) and (2.6). □

Next, we consider $\alpha > 0$ and $\rho > 1$. What concerns the existence result, due to Remark 2.1, we can see that for $\alpha > 0, \rho > 1$ and under assumptions (2.1), (2.2), for each $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ and $u_1 \in H_0^1(\Omega) \cap L^4(\Omega)$ there exists a unique function $u(x,t)$ from the class (3.4) such that

$$u'' - M(\int_\Omega u^2 dx)\Delta u + \alpha u' + \mid u' \mid^\rho u' = 0 \quad \text{a.e. in } \Omega \times (0,\infty); \tag{3.22}$$

$$u_{|\Sigma} = 0, \qquad \Sigma = \Gamma \times (0,\infty); \tag{3.23}$$

$$u(0) = u_0, \qquad u'(0) = u_1. \tag{3.24}$$

Now, we are able to claim our exponential decay result.

Theorem 3.2. *Let $\alpha > 0$ satisfy the following inequality*

$$\alpha > \max\Big\{ \frac{k_0^{\frac{\rho}{\rho-1}} \mu_\Omega^{\frac{\rho}{2(\rho-1)}} (\rho - 1)}{\rho(\frac{\rho}{2})^{\frac{1}{(\rho-1)}}} \quad , \quad k_0(M_0\Psi(0))^{\frac{1}{2}} \Big\}, \tag{3.25}$$

where $\Psi(0)$ and M_0 were given by (3.9), (3.10); *and let $u(x,t)$ be a solution to the problem* (3.22)-(3.24). *Then there exist positive constants C and K such that*

$$\|u'(t)\|^2 + \|u(t)\|^2_{H_0^1} \le Ce^{-Kt}, \qquad \text{for all } \ t \ \ge 0; \tag{3.26}$$

for

$$n = 1,2,3,4 \quad \textit{and} \quad \rho > 1; \tag{3.27}$$

$$n = 5 \quad \textit{and} \quad 1 < \rho < 4; \tag{3.28}$$

$$n = 6 \quad \textit{and} \quad 1 < \rho < 2; \tag{3.29}$$

$$n = 7 \quad \textit{and} \quad 1 < \rho < \tfrac{4}{3}. \tag{3.30}$$

Proof. First, we prove (3.26) for the cases

$$n = 1,2 \quad \text{and} \quad \rho > 1; \tag{3.31}$$

$$n \ge 3 \quad \text{and} \quad 1 < \rho \le \tfrac{4}{n-2}. \tag{3.32}$$

We see that under the condition (3.32) we have the cases: $n = 3$ and $1 < \rho \le 4$, $n = 4$ and $1 < \rho \le 2$, $n = 5$ and $1 < \rho \le \frac{4}{3}$. Multiplying (3.22) by $2u'(x,t)$, integrating over Ω and taking into account (2.1), (2.2), we have

$$\frac{d}{dt}\left[\frac{\|u'(t)\|^2}{M(\|u(t)\|^2)} + \|u(t)\|^2_{H^1_0}\right] + \frac{2\|u(t)\|^{\rho+2}_{\rho+2}}{M(\|u(t)\|^2)} + \frac{2\alpha\|u'(t)\|^2}{M(\|u(t)\|^2)} \le \frac{2k_0\|u'(t)\|^3}{M(\|u(t)\|^2)}. \tag{3.33}$$

Applying Young's inequality for all $\epsilon > 0$, we obtain

$$\begin{aligned}\frac{2k_0\|u'(t)\|^3}{M(\|u(t)\|^2)} &\le \frac{2\epsilon\|u'(t)\|^{\rho+2}}{\rho M(\|u(t)\|^2)} + \frac{2(\rho-1)k_0^{\frac{\rho}{\rho-1}}\|u'(t)\|^2}{\rho\epsilon^{\frac{1}{\rho-1}}} \\ &\le \frac{2(\rho-1)k_0^{\frac{\rho}{\rho-1}}\|u'(t)\|^2}{\rho\epsilon^{\frac{1}{\rho-1}}M(\|u(t)\|^2)} + \frac{2\epsilon\mu_\Omega^{\frac{\rho}{2}}\|u'(t)\|^{\rho+2}_{\rho+2}}{\rho M(\|u(t)\|^2)}.\end{aligned} \tag{3.34}$$

Choosing an appropriate ϵ and considering $\Psi(t)$ as in (3.8), we get from (3.25)

$$\Psi'(t) + \frac{\|u'(t)\|^{\rho+2}_{\rho+2}}{M(\|u(t)\|^2)} + C_1\frac{\|u'(t)\|^2}{M(\|u(t)\|^2)} \le 0, \tag{3.35}$$

where the positive constant C_1 depends on α, k_0, ρ, and μ_Ω.

After integration, this implies

$$\Psi(t) + \int_0^t \frac{\|u'(\tau)\|^{\rho+2}_{\rho+2}}{M(\|u(\tau)\|^2)}d\tau + C_1\int_0^t \frac{\|u'(\tau)\|^2}{M(\|u(\tau)\|^2)}d\tau \le \Psi(0), \tag{3.36}$$

whence,

$$\int_t^{t+1} \|u'(\tau)\|^{\rho+2}_{\rho+2}d\tau + \int_0^{t+1} \|u'(\tau)\|^2 d\tau \le C_2F^2(t), \tag{3.37}$$

where C_2 is a positive constant depending on $\alpha, k_0, \rho, \mu_\Omega$ and M_0; M_0 is equal to M_0 in (3.10), and

$$F^2(t) = \Psi(t) - \Psi(t+1), \qquad \text{for all } \ t \ge 0. \tag{3.38}$$

By the Mean Value Theorem, there exist points $t_1 \in [t, t+\frac{1}{4}]$ and $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$\|u'(t_i)\| \le C_3F(t), \qquad i = 1, 2. \tag{3.39}$$

It is easy to see that

$$\|u(t)\| \leq C_4\Psi^{\frac{1}{2}}(t), \qquad \text{for all } \ t \geq 0. \tag{3.40}$$

Consequently, multiplying (3.22) by $u(x,t)$, integrating over $\Omega \times (t_1, t_2)$, taking into account (3.37)-(3.40) and using Young's inequality, we find for $\epsilon > 0$

$$\begin{aligned} &\int_{t_1}^{t_2} M(\|u(\tau)\|^2)\|u(\tau)\|^2_{H_0^1} d\tau \qquad (3.41)\\ &\leq C_5\epsilon(\sup_{t\leq\tau\leq t+1} \Psi(\tau)) + C_6(\epsilon)F^2(t) + \int_{t_1}^{t_2} \mid (\mid u'(\tau) \mid^\rho u'(\tau), u(\tau)) \mid d\tau. \end{aligned}$$

Now we are going to estimate the last term of (3.41). In the cases (3.31) or (3.32), from Sobolev′s imbedding theorem we know that $H_0^1(\Omega) \hookrightarrow Ł^{\rho+2}(\Omega)$. Thus,

$$\begin{aligned} &\int_{t_1}^{t_2} \mid (\mid u'(\tau) \mid^\rho u'(\tau), u(\tau)) \mid d\tau \leq \int_{t_1}^{t_2} \|u'(\tau)\|^{\rho+1}_{\rho+2}\|u(\tau)\|_{\rho+2} d\tau \\ &\leq C_7\big(\sup_{t\leq\tau\leq t+1} \|u(\tau)\|_{H_0^1}\big) \int_{t_1}^{t_2} \|u'(\tau)\|^{\rho+1}_{\rho+2} d\tau \qquad (3.42)\\ &\leq C_8\big(\sup_{t\leq\tau\leq t+1} \|u(\tau)\|_{H_0^1}\big) F^{\frac{2(\rho+1)}{(\rho+2)}}(t) \leq \epsilon\big(\sup_{t\leq\tau\leq t+1} \Psi(\tau)\big) + C_9(\epsilon)F^{\frac{4(\rho+1)}{(\rho+2)}}(t). \end{aligned}$$

From (3.41) and (3.42), we have

$$\begin{aligned} &\int_{t_1}^{t_2} M(\|u(\tau)\|^2)\|u(\tau)\|^2_{H_0^1} d\tau \qquad (3.43)\\ &\quad \leq C_{10}\epsilon\big(\sup_{t\leq\tau\leq t+1} \Psi(\tau)\big) + C_{11}(\epsilon)\big(F^2(t) + F^{\frac{4(\rho+1)}{(\rho+2)}}(t)\big). \end{aligned}$$

From (3.37) and (3.43), we can see that

$$\int_{t_1}^{t_2} \Psi(\tau) d\tau \leq C_{12}\epsilon\big(\sup_{t\leq\tau\leq t+1} \Psi(\tau)\big) + C_{13}(\epsilon)\big(F^2(t) + F^{\frac{4(\rho+1)}{(\rho+2)}}(t)\big), \tag{3.44}$$

and by the Mean Value Theorem,

$$\Psi(t^*) \leq 2C_{12}\epsilon(\sup_{t\leq\tau\leq t+1} \Psi(\tau)) + 2C_{13}(\epsilon)(F^2(t) + F^{\frac{4(\rho+1)}{(\rho+2)}}(t)) \tag{3.45}$$

for some $t^* \in [t_1, t_2]$. Therefore, acting as by the proof of Theorem 3.1, we obtain

$$\Psi(t) \le C_{14}\epsilon(\sup_{t\le\tau\le t+1} \Psi(\tau)) + C_{15}(\epsilon)(F^2(t) + F^{\frac{4(\rho+1)}{(\rho+2)}}(t)).$$

Choosing $\epsilon > 0$, sufficiently small, we can find a positive constant C_{16} such that

$$\sup_{t\le\tau\le t+1} \Psi(\tau) \le C_{16}F^2(t)(1 + F^{\frac{2\rho}{(\rho+2)}}(t)). \tag{3.46}$$

Then by Nakao′s Lemma (Lemma 2.1), (3.26) holds. To complete the proof of Theorem 3.2, we must prove (3.26) for the case

$$2 + \tfrac{4}{\rho} < n < 4 + \tfrac{4}{\rho} \qquad \text{and} \qquad \rho > 1. \tag{3.47}$$

Note that under the assumption (3.47) we have the following cases: $n = 3$ and $\rho > 4$, $n = 4$ and $\rho > 2$, $n = 5$ and $\frac{4}{3} < \rho < 4$, $n = 6$ and $1 < \rho < 2$, $n = 7$ and $1 < \rho < \frac{4}{3}$. We employ the following lemma:

Lemma 3.1 (Gagliardo-Nirenberg)**.** *Let $u \in W^{m,p}(\Omega)$. Then $u \in W^{k,q}(\Omega)$ and*

$$\|u\|_{W^{k,q}(\Omega)} \le C\|u\|^{\theta}_{W^{m,p}(\Omega)}\|u\|^{1-\theta}_{L^r(\Omega)},$$

provided that $1 \le p \le q$, $1 \le r \le q$, $0 < \theta \le 1$ *and* $k - \frac{n}{q} \le \theta(m - \frac{n}{q}) - (1-\theta)\frac{n}{r}$.

We put in this Lemma $k_0 = 0$, $q = \rho + 2$, $m = 2$, $p = 2$, and $r = \frac{2n}{n-2}$. From (3.32), we have

$$\|u\|_{\rho+2} \le C\|u\|^{\theta}_{H^2(\Omega)}\|u\|^{1-\theta}_{\frac{2n}{n-2}} \tag{3.48}$$

with $\theta = \frac{n-2}{2} - \frac{n}{\rho+2}$.

On the other hand, by Sobolev′s imbedding theorem, we obtain

$$\|u\|_{\frac{2n}{n-2}} \le C\|u\|_{H_0^1}, \tag{3.49}$$

provided that $n \ge 3$. Now, from (3.48) and (3.49) we can estimate the last term of (3.26) as follows

$$\begin{aligned}
&\int_{t_1}^{t_2} |\,(|\, u'(\tau)\,|^{\rho}\, u'(\tau), u(\tau))\,|\, d\tau \le \int_{t_1}^{t_2} \|u'(\tau)\|^{\rho+1}_{\rho+2}\|u(\tau)\|_{\rho+2}d\tau \\
&\le C_{17}\int_{t_1}^{t_2} \|u'(\tau)\|^{\rho+1}_{\rho+2}\|\Delta u(\tau)\|^{\theta}\|u(\tau)\|^{1-\theta}_{H_0^1}d\tau \\
&\le C_{18}\left(\sup_{t\le\tau\le t+1}\|\Delta u(\tau)\|^{\theta}\right)\left(\sup_{t\le\tau\le t+1}\|u(\tau)\|^{1-\theta}_{H_0^1}\right)F^{\frac{2(\rho+1)}{(\rho+2)}}(t).
\end{aligned}$$

Let us estimate $\|\Delta u(t)\|$ uniformly on $[0,\infty)$. Multiplying equation (3.22) by $-2\Delta u'(x,t)$ and integrating over Ω, we have

$$\frac{d}{dt}\Big[\frac{\|u'(t)\|^2_{H^1_0}}{M(\|u(t)\|^2)}+\|\Delta u(t)\|^2\Big]+\frac{2(\rho+1)}{M(\|u(t)\|^2)}\sum_{i=1}^{n}\int_\Omega \mid u' \mid^\rho (\frac{\partial u'}{\partial x_i})^2 dx$$
$$+\frac{2\alpha}{M(\|u(t)\|^2)}\|u'(t)\|^2_{H^1_0}=\frac{-2M'(\|u(t)\|^2)}{(M(\|u(t)\|^2))^2}(u'(t),u(t))\|u'(t)\|^2_{H^1_0}$$
$$\leq \frac{2K_0}{M(\|u(t)\|^2)}\|u'(t)\| \ \|u'(t)\|^2_{H^1_0}\leq 2K_0(M_0\Psi(0))^{\frac{1}{2}}\frac{\|u'(t)\|^2_{H^1_0}}{M(\|u(t)\|^2)}.$$

This inequality and (3.25), after integration over (0,t), give $\|\Delta u(t)\|^2\leq C_{19}$, where C_{19} is independent of $t\in[0,\infty]$. Then, for $\epsilon>0$, we have

$$\int_{t_1}^{t_2}|(\mid u'\tau \mid^\rho u'(\tau),u(\tau))\mid d\tau \leq C_{19}\big(\sup_{t\leq\tau\leq t+1}\|u(\tau)\|^{1-\theta}_{H^1_0}\big)F^{\frac{2(\rho+1)}{(\rho+2)}}(t)$$
$$\leq \epsilon \sup_{t\leq\tau\leq t+1}\Psi(\tau)+C_{20}(\epsilon)F^{\frac{4(\rho+1)}{(\rho+2)(\theta+1)}}(t). \tag{3.50}$$

Now, we can proceed as by considering the cases (3.31), (3.32), and complete the proof of Theorem 3.2. □

REFERENCES

[1] S. Bernstein, *Sur une classe d'equations fonctionelles aux derivées partielles,* Isv. Acad. Nauk. SSSR Ser. Math., 4 (1940), 17-26.

[2] E. H. Brito, *The damped elastic stretched string equation generalized: existence, uniqueness, regularity and stability,* Applicable Anal., 13 (1982), 219-233.

[3] G. F. Carrier, *On the non-linear vibration problem of the elastic string,* Quart. Appl. Math., 3 (1945), 157-165.

[4] A.T. Cousin, C.L. Frota, N.A. Lar′kin, and L.A. Medeiros, *On the abstract model of the Kirchhoff-Carrier equation,* Communications in Applied Analysis, 1 (1997), 389-404.

[5] P. D′ancona and S. Spagnolo, *Nonlinear pertubations of the Kirchhoff equation,* Comm. Pure Appl. Math., 47 (1994), 1005-1029.

[6] C. L. Frota and N. A. Lar'kin, *On global existence and uniqueness for the unilateral problem associated to the degenerated Kirchhoff equation,* Nonlinear Analysis, Theory, Methods & Applications, 28 (1997), 443-452.

[7] J. M. Greenberg and S. C. Hu, *The initial-value problem for a stretched string,* Quart. Appl. Math., (1980), 289-311.

[8] R. Ikehata, *A note on the global solvability of solutions to some nonlinear wave equations with dissipative terms,* Differential and Integral Equations, 8 (1995), 607-616.

[9] G. Kirchhoff, Vorlesungen über Mechanik, *Tauber Leipzig* (1883).
[10] S. Kouémou-Patcheu, *Existence globale et décroissance exponentielle de l′énergie d′une équation quasilinéaire,* C. R. Acad. Sci. Paris, 322, Série I (1996), 631-634.
[11] N. A. Lar′kin, *Solvability in the large of boundary value problems for a class of quasilinear hyperbolic equations,* Siberian Math. Journal, 22 (1981), 82-88.
[12] J. L. Lions, "Quelques méthodes de résolution des problèmes aux limites non linéaires," Dunod-Gauthier-Villars, Paris (1968).
[13] T. Matsuyama, *Singular limit of some quasilinear wave equations with damping and restoring terms,* Tokyo J. Math., 19 (1996), 197-210.
[14] L. A. Medeiros and M. Milla Miranda, *On a nonlinear wave equation with damping,* Revista Matematica de la Universidad Complutense de Madrid, 3 (1990), 213-231.
[15] M. Nakao, *Asymptotic stability of the bounded or almost periodic solution of the wave equation with nonlinear dissipative term,* J. Math. Anal. Appl., 58 (1977), 336-344.
[16] M. Nakao, *On the decay of solutions of some nonlinear dissipative wave equations in higher dimensions,* Mathematishe Zeitschrift, 193 (1986), 227-234
[17] W. G. Newman, *Global solution of a nonlinear string quation,* J. Math. Anal. Appl., 192 (1995), 689-704.
[18] S. Pohozaev, *On a class of quasilinear hyperbolic equations,* Math. Sbornik, 96 (1974), 152-166.
[19] S. Pohozaev, *The Kirchhoff quasilinear hyperbolic equation,* Differential Equations, 21 (1985), 101-108.
[20] M. Tucsnak, *Boundary stabilization for the stretched string equation,* Differential and Integral Equations, 6 (1993), 925-935.