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REGULARITY OF MINIMIZING SEQUENCES FOR FUNCTIONALS OF THE CALCULUS OF VARIATIONS VIA THE EKELAND PRINCIPLE

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1. Introduction. In this paper we are going to consider some properties of the minimizing sequences for functionals of the type

$$J(u) = \int_\Omega f(x, u, \nabla u)\, dx, \tag{1}$$

with f a Carathéodory function satisfying the classical (see [9]) hypotheses that guarantee the existence of a minimum for J on the Sobolev space $W_0^{1,p}(\Omega)$, and Ω is an open, bounded subset of $\mathbf{R}^N$, $N \geq 2$. Various regularity results are known for the minima of J, depending on the regularity of the integrand f with respect to x, u and ∇u. We are going to show that if a minimum u of J has a certain regularity (say, u belongs to $L^\infty(\Omega)$, or to

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$L^s(\Omega)$), and $\bar{u}_n$ is a given minimizing sequence of J converging to u, then it is possible to construct another minimizing sequence u_n which is close to $\bar{u}_n$ and is bounded in the space where u belongs. The main tool of the proof will be Ekeland's ε-variational principle.

Results of this kind are, for example, contained in [10], where it is proved that it is possible to choose the sequence u_n locally bounded in $W^{1,p+\sigma}(\Omega)$, for some $\sigma > 0$. Another related result can be found in [4] (see also [5]), where, in the scalar-valued case, the increased regularity of the sequences converging to a critical point is used in order to prove that the critical points (not only local minima, but also of mountain-pass type) of some functional are also critical points of the same functional, restricted to the smaller space.

We recall that the regularity of the minima depends only on the growth assumptions on the integrand $f(x,s,\xi)$ with respect to (s,ξ).

On the other hand, the semicontinuity of the functional J depends on the geometrical assumptions on the function $f(x,s,\xi)$ with respect to ξ: the convexity is a necessary and sufficient condition for the semicontinuity. Our construction of the sequence u_n depends only on the growth of the function f. Thus one can conclude the existence of a minimizing sequence compact in $L^s(\Omega)$ or in $L^\infty(\Omega)$ or in $C^{0,\alpha}(\Omega)$ also for functionals which do not need to have a minimum, without using the integral representation of the relaxed functional J^*.

Moreover, in some simple cases, we shall prove that $\|u_n - u\|_{L^\infty(\Omega)}$ converges to zero as n tends to infinity, even if the minimum u does not belong to $L^\infty(\Omega)$.

2. Assumptions and preliminary results. Let us state the main hypotheses on the terms of (1). In the following, Ω will be an open, bounded subset of $\mathbf{R}^N$, $N \geq 2$, and p will be a real number, with $1 < p < N$. We will denote with $p^* = \frac{Np}{N-p}$, the Sobolev exponent of p. If k is a positive real number, we will define

$$T_k(s) = \max(-k, \min(k,s)), \qquad G_k(s) = s - T_k(s).$$

Let $f : \Omega \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}$ be a Carathéodory function (i.e., measurable with respect to x for every $(s,\xi) \in \mathbf{R} \times \mathbf{R}^N$, and continuous with respect to (s,ξ) for almost every $x \in \Omega$) such that

$$f(x,s,\xi) \geq \alpha\,|\xi|^p - \varphi_2(x)\,|s|^{\gamma_2}, \tag{2}$$

$$f(x,s,0) \leq \varphi_0(x) + \varphi_1(x)\,|s|^{\gamma_1}, \tag{3}$$

for almost every $x \in \Omega$, for every $(s, \xi) \in \mathbf{R} \times \mathbf{R}^N$, where α is a positive real number, φ_0, φ_1, φ_2 are measurable functions, and γ_1 and γ_2 are real numbers such that

$$\begin{cases} \varphi_0 \in L^{r_0}(\Omega) & r_0 > 1, \\ \varphi_1 \in L^{r_1}(\Omega), & 0 \le \gamma_1 < p^* \frac{r_1 - 1}{r_1}, \\ \varphi_2 \in L^{r_2}(\Omega), & 0 \le \gamma_2 < p^* \frac{r_2 - 1}{r_2}. \end{cases} \tag{4}$$

Let $J : W_0^{1,p}(\Omega) \to \mathbf{R}$ be defined by

$$J(u) = \int_\Omega f(x, u, \nabla u)\, dx, \qquad u \in W_0^{1,p}(\Omega). \tag{5}$$

Under the hypotheses on f, J is well defined. Moreover, about the regularity of the minima, we have the following results.

Theorem 2.1. *Let u be a minimum of J on $W_0^{1,p}(\Omega)$.*

(i) *If $1 < r_0 < \frac{N}{p}$, $r_i > \frac{N}{p}$, $i = 1, 2$, then u belongs to $L^\sigma(\Omega)$, $\sigma = (pr_0)^*$;*
(ii) *If $r_i > \frac{N}{p}$, $i = 0, 1, 2$, then u belongs to $L^\infty(\Omega)$.*

Proof. See [1], Lemmas 2.1 and 2.2 for the $L^\sigma(\Omega)$ case, and [9], Theorem 3.2, for the $L^\infty(\Omega)$ case. $\square$

Let us recall the Ekeland ε-variational principle.

Lemma 2.2. *Let (V, d) be a complete metric space, and let $\mathcal{F} : V \to (-\infty, +\infty]$ be a lower semicontinuous function such that $\inf_V \mathcal{F}$ is finite. Let $\varepsilon > 0$ and $u \in V$ be such that*

$$\mathcal{F}(u) \le \inf_{v \in V} \mathcal{F}(v) + \varepsilon.$$

Then there exists $v \in V$ such that

(i) *$d(u, v) \le 1$;*
(ii) *$\mathcal{F}(v) \le \mathcal{F}(u)$;*
(iii) *v minimizes the functional $\mathcal{G}(w) = \mathcal{F}(w) + \varepsilon\, d(v, w)$.*

Proof. See [3], [8]. $\square$

We now give two technical lemmas, yielding summability properties for a $W_0^{1,p}(\Omega)$ function u satisfying some inequalities. The proof of the first one follows very closely that of Lemma 2.2 in [1].

Lemma 2.3. *Let u be a function in $W_0^{1,p}(\Omega)$, and let ψ_0, ψ_1 and ψ_2 be nonnegative measurable functions, and γ_1, γ_2 be real numbers such that*

$$\begin{cases} \psi_0 \in L^{r_0}(\Omega) & & 1 < r_0 < \frac{N}{p}, \\ \psi_1 \in L^{r_1}(\Omega) & r_1 > \frac{N}{p}, & 0 \le \gamma_1 < p^* \frac{r_1 - 1}{r_1}, \\ \psi_2 \in L^{r_2}(\Omega) & r_2 > N, & 0 \le \gamma_2 < \frac{N}{N-1} \frac{r_2 - 1}{r_2}. \end{cases} \tag{6}$$

Suppose that, for every $k > 0$,

$$\int_{\{|u| \ge k\}} |\nabla u|^p \, dx \le \int_{\{|u| \ge k\}} [\psi_0 + \psi_1 \, |u|^{\gamma_1} + \psi_2 \, |u|^{\gamma_2}] \, dx. \tag{7}$$

Then there exists a positive constant c, depending on the various parameters and on the $W_0^{1,p}(\Omega)$ norm of u, such that

$$\|u\|_{L^\sigma(\Omega)} \le c \qquad \sigma = (p \, r_0)^*. \tag{8}$$

Remark 2.4. Observe the stronger hypotheses (with respect to (4)) made on γ_2 and r_2 in (6). This restriction is due to the fact that this result will be applied in Theorem 3.1 below, where lower semicontinuity on $W_0^{1,1}(\Omega)$ will be needed.

Proof of Lemma 2.3. Let us observe that the integrals in (7) can be split in the sum for $j \ge k$ of the integrals of the same functions on the sets $\{j \le |u| < j+1\}$. Hence, if m is a positive real number to be chosen later,

$$\begin{aligned} &\sum_{k=0}^{\infty} (k+1)^{pm-1} \sum_{j=k}^{\infty} \int_{\{j \le |u| < j+1\}} |\nabla u|^p \, dx \\ &\quad \le \sum_{k=0}^{\infty} (k+1)^{pm-1} \sum_{j=k}^{\infty} \int_{\{j \le |u| < j+1\}} [\psi_0 + \psi_1 \, |u|^{\gamma_1} + \psi_2 \, |u|^{\gamma_2}] \, dx. \end{aligned}$$

Exchanging the summation order, we get

$$\begin{aligned} &\sum_{j=0}^{\infty} \Big(\sum_{k=0}^{j} (k+1)^{pm-1} \Big) \int_{\{j \le |u| < j+1\}} |\nabla u|^p \, dx \\ &\le \sum_{j=0}^{\infty} \Big(\sum_{k=0}^{j} (k+1)^{pm-1} \Big) \int_{\{j \le |u| < j+1\}} [\psi_0 + \psi_1 \, |u|^{\gamma_1} + \psi_2 \, |u|^{\gamma_2}] \, dx. \end{aligned}$$

Since there exist two positive constants c_1 and c_2 (independent on j) such that

$$c_1(j+1)^{pm} \le \sum_{k=0}^{j} (k+1)^{pm-1} \le c_2(j+1)^{pm},$$

the previous inequality yields

$$\int_\Omega |\nabla u|^p |u|^{pm}\,dx$$
$$\le c(m,p)\int_\Omega [\psi_0 + \psi_1(|u|+1)^{\gamma_1} + \psi_2(|u|+1)^{\gamma_2}](|u|+1)^{pm}\,dx.$$

Using Sobolev and Hölder inequalities, we obtain

$$\Big(\int_\Omega |u|^{(m+1)p^*}\,dx\Big)^{\frac{p}{p^*}} \le c\Big[\Big(\int_\Omega |u|^{pmr_0'}\,dx\Big)^{\frac{1}{r_0'}} + \Big(\int_\Omega |u|^{(pm+\gamma_1)r_1'}\,dx\Big)^{\frac{1}{r_1'}} + \Big(\int_\Omega |u|^{(pm+\gamma_2)r_2'}\,dx\Big)^{\frac{1}{r_2'}} + 1\Big].$$

We choose m so that $(m+1)p^* = pmr_0'$, that is, $m = \frac{N(r_0-1)}{N-r_0p}$; then $m>0$ if and only if $1<r_0<\frac{N}{p}$; this choice of m yields $(m+1)p^* = (pr_0)^*$. Moreover, by the hypotheses on r_1, r_2, γ_1 and γ_2, we have $(pm+\gamma_1)r_1' < (m+1)p^*$, $(pm+\gamma_2)r_2' < (m+1)p^*$. Thus, if $(pm+\gamma_1)r_1' \le p^*$ (we carry on the calculations for r_1 and γ_1, but the same holds for r_2 and γ_2) then the integral

$$\Big(\int_\Omega |u|^{(pm+\gamma_1)r_1'}\,dx\Big)^{\frac{1}{r_1'}} \tag{9}$$

is bounded from above by a constant times the norm of u in $W_0^{1,p}(\Omega)$; otherwise, we can interpolate between $L^{p^*}(\Omega)$ and $L^{(m+1)p^*}(\Omega)$ to obtain that, for some $\theta \in (0,1)$,

$$\Big(\int_\Omega |u|^{(pm+\gamma_1)r_1'}\,dx\Big)^{\frac{1}{r_1'}} \le \|u\|_{L^{p^*}(\Omega)}^{(pm+\gamma_1)\theta}\,\|u\|_{L^{(m+1)p^*}(\Omega)}^{(pm+\gamma_1)(1-\theta)},$$

and so the integral in (9) is bounded from above by a constant (depending on the norm of u in $W_0^{1,p}(\Omega)$) times the norm of u in $L^{(m+1)p^*}(\Omega)$ (raised to a suitable power which is smaller than $\frac{p}{p^*}$). Summing up, we obtain, for some positive real numbers $A > B$,

$$\|u\|^A_{L^{(pr_0)^*}(\Omega)} \leq c(1 + \|u\|^B_{L^{(pr_0)^*}(\Omega)}),$$

with c depending on the norm of u in $W^{1,p}_0(\Omega)$, and this yields the desired result. □

Remark 2.5. We remark that, using the same technique of [1], Lemma 2.1, it is possible to find the same result of Lemma 2.3, under the hypothesis $r_1 < \frac{N}{p}$. Note that in this case the bound $\gamma_1 < p^* \frac{r_1 - 1}{r_1}$ implies $\gamma_1 < p$.

Define, for $r > 0$ and $x_0 \in \mathbf{R}^N$, $B_r(x_0) = \{x \in \mathbf{R}^N : |x - x_0| < r\}$. In the following we shall say that a bounded open set Ω is *regular* if there exist $R_0 > 0$ and $0 < \theta_0 < 1$ such that for any $x_0 \in \partial\Omega$, and for any $0 < R < R_0$,

$$\text{meas}(B_R(x_0) \backslash \bar{\Omega}) \geq \theta_0 \, \omega_N R^N,$$

where $\omega_N = \text{meas}(B_1)$.

Lemma 2.6. *Let u be a function in $W^{1,p}_0(\Omega)$, where Ω is a regular bounded open set, and let ψ_0, ψ_1, ψ_2 be nonnegative measurable functions, γ_1, γ_2 be real numbers such that* (6) *holds. Let us assume that there exists $Q \geq 1$ such that for every $\phi \in W^{1,p}_0(\Omega)$,*

$$\begin{aligned} \int_{\text{supp}\phi} |\nabla u|^p \, dx \leq & Q \int_{\text{supp}\phi} \big\{ |\nabla u + \nabla \phi|^p + \psi_0 \\ & + \psi_1(|u| + |\phi|)^{\gamma_1} + \psi_2(|u| + |\phi|)^{\gamma_2} \big\} \, dx. \end{aligned} \tag{10}$$

Then there exist $\delta > 0$, $C > 0$, depending on the various parameters and on $\|u\|_{W^{1,p}_0(\Omega)}$, such that $\nabla u \in L^{p+\delta}(\Omega)$, and

$$\begin{aligned} & \|\nabla u\|_{L^{p+\delta}(\Omega)} \\ & \leq C\big(\|\nabla u\|_{L^p(\Omega)} + \|\psi_1\|_{L^{r_1}(\Omega)} + \|\psi_2\|_{L^{r_2}(\Omega)} + \|\psi_0\|_{L^{r_0}(\Omega)} + 1\big). \end{aligned}$$

Proof. Let us extend u outside Ω with value 0. We split the proof in an interior and a boundary estimate.

Let us assume that $B_R(x_0) \subset B_{2R}(x_0) \subset \Omega$. If $R < s < t < 2R$, and η is a function in $C_0^1(B_t(x_0))$ such that $\eta = 1$ on $B_s(x_0)$, $0 \le \eta \le 1$, $|\nabla \eta| \le 2/(t-s)$, supp$\eta = B_t(x_0)$. Then, choosing in (10) $\phi = \eta(u_t - u)$, where $u_t = \fint_{B_t(x_0)} u\,dx$ is the average of u on $B_t(x_0)$, we get

$$\begin{aligned}
&\int_{B_t(x_0)} |\nabla u|^p dx \\
&\le Q \int_{B_t(x_0)} \big\{|\nabla u + \nabla\phi|^p + \psi_0 + \psi_1(|u| + |\phi|)^{\gamma_1} + \psi_2(|u| + |\phi|)^{\gamma_2}\big\} dx \\
&\quad \le c \int_{B_t(x_0)} \big\{|\nabla u|^p(1-\eta)^p + |u - u_t|^p|\nabla\eta|^p + \psi_0 + (|u| + |\phi|)^{p^*-\sigma} \\
&\quad + \psi_1^{\frac{p^*-\sigma}{p^*-\sigma-\gamma_1}} + (|u| + |\phi|)^{1^*} + \psi_2^{\frac{1^*}{1^*-\gamma_2}}\big\} dx,
\end{aligned}$$

where $\sigma > 0$ is such that $p^* - \sigma > 1^*$ and $\frac{p^*-\sigma}{p^*-\sigma-\gamma_1} < r_1$. We have, by the Young inequality,

$$\begin{aligned}
\int_{B_t(x_0)} |\phi|^{p^*-\sigma} dx &\le c\big(\epsilon \int_{B_t(x_0)} |u - u_t|^{p*} dx + c_\epsilon R^N\big) \\
&\le c\big(\epsilon \int_{B_t(x_0)} |\nabla u|^p dx + c_\epsilon R^N\big),
\end{aligned}$$

with c depending on diam(Ω) and $\|\nabla u\|_{L^p(\Omega)}$; hence, taking ϵ small enough, and recalling that $p^* - \sigma > 1^*$, we get

$$\begin{aligned}
\int_{B_t(x_0)} |\nabla u|^p\,dx \le c\Big\{&\int_{B_t(x_0)\setminus B_s(x_0)} |\nabla u|^p\,dx + \int_{B_t(x_0)} \frac{|u-u_t|^p}{(t-s)^p}\,dx \\
&+ \int_{B_{2R}(x_0)} \big[\psi_0 + |u|^{p^*-\sigma} + \psi_1^{\frac{p^*-\sigma}{p^*-\sigma-\gamma_1}} + \psi_2^{\frac{1^*}{1^*-\gamma_2}} + 1\big]\,dx\Big\}.
\end{aligned}$$

Define

$$g(t) = \int_{B_t(x_0)} |u - u_t|^p\,dx.$$

Then g is continuous on $[R,2R]$, as easy calculations show; hence, there exists $\bar{t}\in[R,2R]$ such that $g(\bar{t})=\max_{t\in[R,2R]} g(t)$. Thus,

$$\int_{B_t(x_0)}|\nabla u|^p\,dx \le c\Big\{\int_{B_t(x_0)\setminus B_s(x_0)}|\nabla u|^p\,dx+\frac{g(\bar{t})}{(t-s)^p} + \int_{B_{2R}(x_0)}\big[1+\psi_0+\psi_1^{\frac{p^*-\sigma}{p^*-\sigma-\gamma_1}}+\psi_2^{\frac{1^*}{1^*-\gamma_2}}+|u|^{p^*-\sigma}\big]\,dx\Big\}.$$

A "hole-filling" argument (see [7]) then implies

$$\int_{B_R(x_0)}|\nabla u|^p\,dx \le c\int_{B_{2R}(x_0)} b(x)\,dx+\int_{B_{\bar{t}}(x_0)}\frac{|u-u_{\bar{t}}|^p}{R^p}\,dx, \tag{11}$$

where

$$b(x)=1+|u|^{p^*-\sigma}+\psi_0+\psi_1^{\frac{p^*-\sigma}{p^*-\sigma-\gamma_1}}+\psi_2^{\frac{1^*}{1^*-\gamma_2}}.$$

Notice that $b\in L^{s_1}(\Omega)$ where $1<s_1/leqr_0$ and $s_1\le\frac{p^*}{p^*-\sigma}$, $(\frac{p^*-\sigma}{p^*-\sigma-\gamma_1})s_1\le r_1$, $\frac{1^*}{1^*-\gamma_2}s_1\le r_2$. Setting

$$p_*=\begin{cases}1 & \text{if } 1<p\le\frac{N}{N-1},\\ \frac{Np}{N+p} & \text{if } p\ge\frac{N}{N-1},\end{cases}$$

we then obtain from (11) using the Sobolev-Poincaré inequality

$$\begin{aligned}\int_{B_R(x_0)}|\nabla u|^p\,dx &\le \frac{c}{R^p}\Big(\int_{B_{\bar{t}}(x_0)}|\nabla u|^{p_*}\,dx\Big)^{\frac{p}{p_*}}+\int_{B_{2R}(x_0)}b(x)\,dx\\ &\le \frac{c}{R^p}\Big(\int_{B_{2R}(x_0)}|\nabla u|^{p_*}\,dx\Big)^{\frac{p}{p_*}}+\int_{B_{2R}(x_0)}b(x)\,dx,\end{aligned}$$

and so

$$\fint_{B_R(x_0)}|\nabla u|^p\,dx\le c\big\{\Big(\fint_{B_{2R}(x_0)}|\nabla u|^{p_*}\,dx\Big)^{\frac{p}{p_*}}+\fint_{B_{2R}(x_0)}b(x)\,dx\big\}. \tag{12}$$

Let us now take $x_0\in\partial\Omega$, $R<\frac{R_0}{2}$, (where R_0 is the given real number such that Ω is regular), s, t, η as before, but $\phi=-u\eta$. With the same argument used before we get

$$\int_{B_R(x_0)\cap\Omega}|\nabla u|^p\,dx\le c\int_{B_{2R}(x_0)\cap\Omega}\Big[\frac{|u|^p}{R^p}+b(x)\Big]\,dx.$$

¿From this estimate it follows, using again the Sobolev-Poincaré inequality and the regularity assumption on Ω, that if $x_0 \in \partial\Omega$, $2R < R_0$,

$$\fint_{B_R(x_0)\cap\Omega} |\nabla u|^p \, dx \le c\big\{ \big(\fint_{B_{2R}(x_0)\cap\Omega} |\nabla u|^{p_*} \, dx\big)^{\frac{p}{p_*}} + \fint_{B_{2R}(x_0)\cap\Omega} b(x) \, dx \big\}. \tag{13}$$

Finally, let us consider $B_R(x_0) \subset \mathbf{R}^N$, with $R < \frac{R_0}{8}$, and such that $B_R(x_0) \cap \Omega \neq \emptyset$. If $B_{2R}(x_0) \subset \Omega$ then (12) holds. Otherwise, if $B_{2R}(x_0) \setminus \Omega \neq \emptyset$, we can take $x_1 \in B_{2R}(x_0) \cap \partial\Omega$ and from (13) we get

$$\begin{aligned}
\frac{1}{R^N} \int_{B_R(x_0)\cap\Omega} |\nabla u|^p dx &\le \frac{1}{R^N} \int_{B_{4R}(x_1)\cap\Omega} |\nabla u|^p dx \\
&\le C\left\{ \left(\frac{1}{R^N} \int_{B_{8R}(x_1)\cap\Omega} |\nabla u|^{p_*} dx \right)^{\frac{p}{p_*}} + \frac{1}{R^N} \int_{B_{8R}(x_1)\cap\Omega} b(x) dx \right\} \\
&\le C\left\{ \left(\frac{1}{R^N} \int_{B_{10R}(x_0)\cap\Omega} |\nabla u|^{p_*} dx \right)^{\frac{p}{p_*}} + \frac{1}{R^N} \int_{B_{10R}(x_0)\cap\Omega} b(x) dx \right\}.
\end{aligned}$$

In conclusion, for any $B_R(x_0) \subset \mathbf{R}^N$, with $R < \frac{R_0}{8}$,

$$\fint_{B_R(x_0)\cap\Omega} |\nabla u|^p \, dx \le C\left\{ \left(\fint_{B_{10R}(x_0)\cap\Omega} |\nabla u|^{p_*} \, dx \right)^{\frac{p}{p_*}} + \fint_{B_{10R}(x_0)\cap\Omega} b(x) dx \right\}.$$

Using this inequality, and applying the version of Gehring Lemma proved in [7], we get that there exists $1 < s_0 < s_1$ such that

$$\begin{aligned}
\int_\Omega |\nabla u|^{ps_0} \, dx &\le C\left\{ \left(\int_\Omega |\nabla u|^p \, dx \right)^{s_0} + \int_\Omega b(x)^{s_0} \, dx \right\} \\
&\le C\left\{ \left(\int_\Omega |\nabla u|^p \, dx \right)^{s_0} + \left(\int_\Omega |u|^{p^*} \, dx \right)^{\frac{s_0}{p^*}} \right. \\
&\quad \left. + \|\psi_0\|^{s_0}_{L^{r_0}(\Omega)} + \|\psi_1\|^{s_0}_{L^{r_1}(\Omega)} + \|\psi_2\|^{s_0}_{L^{r_2}(\Omega)} \right\},
\end{aligned}$$

and the result then follows from this inequality. $\square$

3. Regularity results in Lebesgue spaces. Our first regularity result is the following.

Theorem 3.1. *Let J be as in* (5)*, with f satisfying hypotheses* (2) *and* (3)*. Suppose that r_i, $i = 0, 1, 2$ and γ_i, $i = 1, 2$ are such that $1 < r_0 < \frac{N}{p}$, $r_1 > \frac{N}{p}$, $\gamma_1 < p^* \frac{r_1 - 1}{r_1}$, $r_2 > N$, $\gamma_2 < \min\left(p, \frac{N}{N-1}\frac{r_2-1}{r_2}\right)$. Then there exists a minimizing sequence $\{u_n\}$ that is bounded in $L^\sigma(\Omega)$, $\sigma = (pr_0)^*$.*

Remark 3.2. The request $\gamma_2 < p$ has been made in order to ensure that the functional J is bounded from below, a fact which may not be true in general. As an example, consider the functional

$$I(v) = \int_\Omega |\nabla v|^p \, dx - \int_\Omega |v|^m \, dx,$$

defined on $W_0^{1,p}(\Omega)$, with p and m such that $1 < p < m < \frac{N}{N-1}$. It is then easy to see that I is not bounded from below. However, it is possible to have $\gamma_2 > p$ and the functional bounded from below. Again, consider the functional

$$I(v) = \int_\Omega |\nabla v|^p \, dx + \int_\Omega |v|^q \, dx - \lambda \int_\Omega |v|^m \, dx,$$

defined on $W_0^{1,p}(\Omega)$, with $1 < p < m < \frac{N}{N-1} < q < p^*$, and λ large enough.

Proof of Theorem 3.1. Let ε_n be a sequence of positive real numbers, converging to zero, and let $\bar{u}_n$ be such that, for every $n \in \mathbf{N}$,

$$J(\bar{u}_n) \le \inf_{u \in W_0^{1,p}(\Omega)} J(u) + \varepsilon_n.$$

Thanks to the hypotheses on r_2 and γ_2, J is lower semicontinuous on $W_0^{1,1}(\Omega)$ (we observe that the lower semicontinuity of J on $W_0^{1,1}(\Omega)$ depends only on the control of the function from below) endowed with the distance

$$d_n(u, v) = \frac{1}{\sqrt{\varepsilon_n}} \int_\Omega |\nabla u - \nabla v| \, dx.$$

Moreover, $\inf_{u \in W_0^{1,p}(\Omega)} J(u) = \inf_{u \in W_0^{1,1}(\Omega)} J(u)$. Thus, in view of Lemma 2.2, there exists a sequence $\{u_n\}$ in $W_0^{1,1}(\Omega)$ such that

$$\int_\Omega |\nabla u_n - \nabla \bar{u}_n| \, dx \le \sqrt{\varepsilon_n}, \qquad J(u_n) \le J(\bar{u}_n),$$

$$J(u_n) \le J(w) + \sqrt{\varepsilon_n} \int_\Omega |\nabla u_n - \nabla w| \, dx, \quad \forall w \in W_0^{1,1}(\Omega).$$

By the growth properties of J we derive that u_n is bounded in $W_0^{1,p}(\Omega)$; actually, by (2),

$$\int_\Omega |\nabla u_n|^p\,dx \le J(u_n) + \int_\Omega \varphi_2\,|u_n|^{\gamma_2}\,dx \le c,$$

since $\{u_n\}$ is bounded in $L^{\frac{N}{N-1}}(\Omega)$ (this is true by Sobolev inequality since $\{u_n\}$ is bounded in $W_0^{1,1}(\Omega)$ being "close" in $W_0^{1,1}(\Omega)$ to the minimizing sequence $\{\bar{u}_n\}$, which is bounded in $W_0^{1,p}(\Omega)$ by the hypotheses on γ_2); hence, u_n converges to u weakly in $W_0^{1,p}(\Omega)$. Moreover, recalling the definition of J, we have

$$\int_\Omega f(x,u_n,\nabla u_n)\,dx \le \int_\Omega f(x,w,\nabla w)\,dx + \sqrt{\varepsilon_n}\int_\Omega |\nabla u_n - \nabla w|\,dx, \quad (14)$$

for every w in $W_0^{1,1}(\Omega)$. We choose $w = T_k(u_n)$, where k is a positive real number. Then

$$\begin{aligned}&\int_{\{|u_n|\ge k\}} f(x,u_n,\nabla u_n)\,dx\\ &\le \int_{\{|u_n|\ge k\}} f(x,k\operatorname{sgn} u_n,0)\,dx + \sqrt{\varepsilon_n}\int_{\{|u_n|\ge k\}} |\nabla u_n|\,dx.\end{aligned}$$

Recalling (2) and (3), and using Hölder inequality, we derive

$$\begin{aligned}\int_{\{|u_n|\ge k\}} |\nabla u_n|^p\,dx - \int_{\{|u_n|\ge k\}} \varphi_2|u_n|^{\gamma_2}\,dx &\le \int_{\{|u_n|\ge k\}} [\varphi_0 + \varphi_1|k|^{\gamma_1}]\,dx\\ &\quad + c\sqrt{\varepsilon_n}\int_{\{|u_n|\ge k\}} [|\nabla u_n|^p + 1]\,dx.\end{aligned}$$

Choosing n large enough (so that $c\sqrt{\varepsilon_n} < \frac{1}{2}$), we obtain

$$\int_{\{|u_n|\ge k\}} |\nabla u_n|^p\,dx \le c\int_{\{|u_n|\ge k\}} [\varphi_0 + 1 + \varphi_1|u_n|^{\gamma_1} + \varphi_2|u_n|^{\gamma_2}]\,dx,$$

that yields the result by Lemma 2.3 and by the fact that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. □

Remark 3.3. Starting from the *a priori* estimate of Theorem 3.1, and reasoning as in [1], it is easy to prove that the sequence $\{u_n\}$ is not only bounded in $L^\sigma(\Omega)$, but also compact in the same space. Moreover, if $\{\bar{u}_n\}$ is any minimizing sequence, it is clear from the proof of Theorem 3.1 that the sequence $\{u_n\}$ built after $\{\bar{u}_n\}$ as in the proof of Theorem 3.1 is such that $\|u_n - \bar{u}_n\|_{W_0^{1,q}(\Omega)}$ converges to zero as n tends to infinity, for any $q < p$.

The convergence result of Theorem 3.1 can be improved.

Theorem 3.4. *Let us assume that Ω is regular. Under the same assumptions of Theorem* 3.1*, if furthermore f satisfies the bound from above*

$$f(x,s,\xi) \leq L(|\xi|^p + \varphi_0 + \varphi_1|s|^{\gamma_1}), \tag{15}$$

for some positive constant L, we have also that $\{u_n\}$ is bounded in $W_0^{1,\bar{q}}(\Omega)$ for some $\bar{q} > p$.

Proof. Notice that from (15) and the assumptions on f we get that, if ϕ belongs to $W_0^{1,p}(\Omega)$,

$$\begin{aligned} &\int_{\mathrm{supp}\phi} \left[\alpha|\nabla u|^p - \varphi_2|u|^{\gamma_2}\right] dx \\ &\leq L\int_{\mathrm{supp}\phi} \left[|\nabla u + \nabla\phi|^p + \varphi_0 + \varphi_1|u+\phi|^{\gamma_1}\right] dx + \sqrt{\varepsilon_n}\int_{\mathrm{supp}\phi} |\nabla\phi|\, dx. \end{aligned}$$

Since

$$\sqrt{\varepsilon_n}\int_{\mathrm{supp}\phi} |\nabla\phi|\, dx \leq c\int_{\mathrm{supp}\phi} |\nabla u + \nabla\phi|^p\, dx + c\sqrt{\varepsilon_n}\int_{\mathrm{supp}\phi} |\nabla u|^p\, dx,$$

we get (10) for n large enough, and the result then follows from Lemma 2.6 (observe that u_n is bounded in $W_0^{1,p}(\Omega)$, so that δ, given by Lemma 2.6, can be chosen independent on n). $\square$

The next result is about the boundedness of minimizing sequences.

Theorem 3.5. *Let J be as in* (5)*, with f satisfying hypotheses* (2) *and* (3)*. Suppose that r_i, $i = 0,1,2$ and γ_i, $i = 1,2$ are such that $r_0 > \frac{N}{p}$, $r_1 > \frac{N}{p}$, $\gamma_1 < p^* \frac{r_1-1}{r_1}$, $r_2 > N$, $\gamma_2 < \min(p, \frac{N}{N-1}\frac{r_2-1}{r_2})$. Then there exists a minimizing sequence $\{u_n\}$ that is bounded in $L^\infty(\Omega)$.*

Proof. Reasoning as in the proof of Theorem 3.1 we obtain, for $k > 0$,

$$\int_{\{|u_n|\geq k\}} |\nabla u_n|^p \, dx \leq c \int_{\{|u_n|\geq k\}} [\varphi_0 + 1 + \varphi_1 |u_n|^{\gamma_1} + \varphi_2 |u_n|^{\gamma_2}] \, dx. \quad (16)$$

Let ρ be a positive real number, and let $A_{k,n} = \{|u_n| \geq k\}$. Using Hölder inequality, we have

$$\int_{\{|u_n|\geq k\}} [\varphi_0 + 1 + \varphi_1 |u_n|^{\gamma_1} + \varphi_2 |u_n|^{\gamma_2}] \, dx \leq$$

$$\|\varphi_0 + 1\|_{L^{r_0}(\Omega)} \text{meas}\,(A_{k,n})^{\frac{1}{r_0'}} + \|\varphi_1\|_{L^{r_1}(\Omega)} \|u_n\|^{\gamma_1}_{L^{\rho\gamma_1}(\Omega)} \text{meas}\,(A_{k,n})^{\frac{1}{r_1'} - \frac{1}{\rho}}$$

$$+ \|\varphi_2\|_{L^{r_2}(\Omega)} \|u_n\|^{\gamma_2}_{L^{\rho\gamma_2}(\Omega)} \text{meas}\,(A_{k,n})^{\frac{1}{r_2'} - \frac{1}{\rho}}.$$

By Theorem 3.1, and by the fact that $(pr)^*$ can be taken as any positive real number larger that p^* if r belongs to $(1, \frac{N}{p})$, we have that $\{u_n\}$ is bounded in $L^q(\Omega)$ for every q. Thus, the previous inequality becomes

$$\int_{\{|u_n|\geq k\}} [\varphi_0 + 1 + \varphi_1 |u_n|^{\gamma_1} + \varphi_2 |u_n|^{\gamma_2}] \, dx$$
$$\leq c \Big[\text{meas}\,(A_{k,n})^{\frac{1}{r_0'}} + \text{meas}\,(A_{k,n})^{\frac{1}{r_1'} - \frac{1}{\rho}} + \text{meas}\,(A_{k,n})^{\frac{1}{r_2'} - \frac{1}{\rho}} \Big].$$

Recalling the definition of $G_k(s)$, the left hand side of (16) can be written as

$$\int_{\{|u_n|\geq k\}} |\nabla u_n|^p \, dx = \int_\Omega |\nabla G_k(u_n)|^p \, dx.$$

Using Sobolev inequality and choosing $h > k$, we have

$$\int_\Omega |\nabla G_k(u_n)|^p \, dx \geq c \Big(\int_\Omega |G_k(u_n)|^{p^*} \, dx \Big)^{\frac{p}{p^*}} \geq (h-k)^p \, \text{meas}\,(A_{h,n})^{\frac{p}{p^*}},$$

so that (16) becomes

$$(h-k)^p \text{meas}\,(A_{h,n})^{\frac{p}{p^*}}$$
$$\leq c(\text{meas}\,(A_{k,n})^{\frac{1}{r_0'}} + \text{meas}\,(A_{k,n})^{\frac{1}{r_1'} - \frac{1}{\rho}} + \text{meas}\,(A_{k,n})^{\frac{1}{r_1'} - \frac{1}{\rho}}).$$

The hypotheses on r_0, r_1, r_2 imply that there exists $\rho > 0$ such that $\frac{p}{p^*} < \max(\frac{1}{r_0'}, \frac{1}{r_1'} - \frac{1}{\rho}, \frac{1}{r_1'} - \frac{1}{\rho})$. A well known technical result due to Stampacchia (see [11], Lemme 4.1) then yields that $\{u_n\}$ is bounded in $L^\infty(\Omega)$, hence concluding the proof. □

We conclude this section with another result about the uniform convergence of minimizing sequences.

Theorem 3.6. *Let us assume that Ω is a Lipschitz bounded open set. If the integrand f satisfies*

$$\alpha|\xi|^p - \varphi_1|s|^{\gamma_1} - \varphi_0 \le f(x,s,\xi) \le L\,(|\xi|^p + \varphi_1|s|^{\gamma_1} + \varphi_0),$$

where φ_0, φ_1, γ_1 are as in Theorem 3.5*, and L is a positive real number, then there exists a minimizing sequence $\{u_n\}$ converging strongly in $C^{0,\beta}(\bar\Omega)$ for some $\beta > 0$ and bounded in $W_0^{1,q}(\Omega)$ for some $q > p$.*

Proof. From (10) it is clear that $\{u_n\}$ is a sequence of Q-minima of the functional $J(u)$ (see [8], Chapter 6). Therefore, applying Theorem 7.8 of [8], we get that $\{u_n\}$ is bounded in $C^{0,\alpha}(\bar\Omega)$ for some $\alpha > 0$. The result then follows from the fact that $\{u_n\}$ is also bounded in $W_0^{1,q}(\Omega)$ for some $q > p$ (see Theorem 3.4). □

The results proved in this section allow us to improve some well known relaxation results for scalar integrals (see, e.g., [2], [6], [10]). Let $f(x,s,\xi)$ be such that

$$\alpha|\xi|^p - c_1|s|^{\gamma_1} - \varphi_0 \le f(x,s,\xi) \le L\,(|\xi|^p + |s|^{\gamma_1} + \varphi_0), \tag{17}$$

where $\varphi_0 \in L^r(\Omega)$, $r > N/p$, $1 \le \gamma_1 < p^*$ and α, c_1, L are positive constants. If (x,s) belong to $\Omega \times \mathbf{R}$, we denote by $f^{**}(x,s,\cdot)$ the convex envelope of $f(x,s,\cdot)$; then it is well known that $\inf_{u\in W_0^{1,p}(\Omega)} J(u) = \min_{u\in W_0^{1,p}(\Omega)} J^*(u)$, where

$$J^*(u) = \int_\Omega f^{**}(x,u,\nabla u)\,dx.$$

Then we have the following result.

Theorem 3.7. *Let us assume that Ω is a Lipschitz bounded open set and $f(x,s,\xi)$ satisfies condition* (17)*. Then:*

(i) *if u is a minimizer of functional J^*, there exists a minimizing sequence $\{u_n\}$ for the functional J such that $u_n \to u$ in $C^{0,\alpha}(\bar\Omega)$, for some $\alpha > 0$, and weakly in $W_0^{1,\bar p}(\Omega)$, for some $\bar p > p$;*

(ii) *conversely, given a minimizing sequence* $\{\bar{u}_n\}$ *for the functional* J*, there exists another minimizing sequence* $\{u_n\}$ *such that the norm of* $\bar{u}_n - u_n$ *in* $W_0^{1,q}(\Omega)$ *converges to zero for every* $q < p$*, and* $\{u_n\}$ *has a subsequence converging to a minimizer of* J^* *in* $C^{0,\alpha}(\bar{\Omega})$*, for some* $\alpha > 0$*, and weakly in* $W_0^{1,\bar{p}}(\Omega)$*, for some* $\bar{p} > p$*.*

As a final remark, we are going to prove that if we add some hypotheses on the functional J (that is, we suppose that J is differentiable on $W_0^{1,p}(\Omega)$, and has a strongly monotone derivative), then any minimizing sequence $\{\bar{u}_n\}$ allows to construct (via the Ekeland lemma) a new minimizing sequence $\{u_n\}$ which is close to $\bar{u}_n$ in $W_0^{1,p}(\Omega)$, and that converges uniformly to the minimum u. This fact, that is not surprising in itself if the minimum is already known to belong to $L^\infty(\Omega)$, turns out to be rather striking if we place ourselves in general hypotheses. In some sense, the sequence built by the Ekeland lemma "inherits" all the irregularities of the minimum u.

Before stating and proving the result, let us give the precise hypotheses on J.

Let p be a real number such that $2 \leq p < N$. Let $f : \Omega \times \mathbf{R}^N$ be a Carathéodory function such that, for almost every $x \in \Omega$, and for every $\xi \in \mathbf{R}^N$,

$$\alpha\,|\xi|^p - \varphi(x) \leq f(x,\xi) \leq \beta\,|\xi|^p + \varphi(x) \quad \varphi \in L^1(\Omega), \tag{18}$$

with α and β being positive real numbers. Let ϕ_0 be a measurable function such that

$$\phi_0 \in L^r(\Omega) \quad r \geq \frac{N\,p}{N+p}. \tag{19}$$

Let us define

$$J(u) = \int_\Omega f(x,\nabla u)\,dx - \int_\Omega \phi_0\,u\,dx \qquad \forall u \in W_0^{1,p}(\Omega).$$

Under the hypotheses on f, J turns out to be well defined on $W_0^{1,p}(\Omega)$; we will further assume that J is differentiable and that its derivative J' satisfies

$$\langle J'(u) - J'(v), u - v\rangle \geq \alpha\,\|u-v\|^p_{W_0^{1,p}(\Omega)} \qquad \forall u,v \in W_0^{1,p}(\Omega). \tag{20}$$

Observe that the hypothesis above, which implies that $f(x,\xi)$ is strongly convex with respect to ξ, is a request only on the higher-order term of J, since the lower order term of the derivative (the one containing ϕ_0) "cancels out" with the difference. Examples of functions f such that (20) holds true are $f(x,\xi) = a(x)\,|\xi|^p$, with a a measurable function such that $\alpha \leq a(x) \leq \beta$ almost everywhere in Ω.

Theorem 3.8. *Let J be defined as above, with f satisfying* (18) *and ϕ_0 satisfying* (19)*, and suppose that J' satisfies* (20)*. Let u be the minimum of J on $W_0^{1,p}(\Omega)$, and let $\{\bar{u}_n\}$ be any minimizing sequence for J. Then the minimizing sequence $\{u_n\}$ built after $\{\bar{u}_n\}$ using the ε-variational principle satisfies*

$$\lim_{n\to\infty} \|u_n - u\|_{W_0^{1,p}(\Omega)} = 0, \tag{21}$$

and

$$\lim_{n\to\infty} \|u_n - u\|_{L^\infty(\Omega)} = 0. \tag{22}$$

Proof. Working as in the proof of Theorem 3.1, we obtain a minimizing sequence $\{u_n\}$, bounded in $W_0^{1,p}(\Omega)$, such that

$$J(u_n) \le J(w) + \sqrt{\varepsilon_n} \int_\Omega |\nabla u_n - \nabla w|\, dx \quad \forall w \in W_0^{1,1}(\Omega).$$

Choosing $w = u_n - t\,\psi$, where t is a positive real number and ψ is a function in $W_0^{1,p}(\Omega)$, we obtain

$$J(u_n - t\,\psi) - J(u_n) + \sqrt{\varepsilon_n}\, t \int_\Omega |\nabla\psi|\, dx \ge 0.$$

Dividing by t, and letting t tend to zero, we get, since J is differentiable,

$$-\langle J'(u_n), \psi\rangle + \sqrt{\varepsilon_n} \int_\Omega |\nabla\psi|\, dx \ge 0,$$

so that

$$\langle J'(u_n), \psi\rangle \le \sqrt{\varepsilon_n} \int_\Omega |\nabla\psi|\, dx.$$

Recalling that $J'(u) = 0$ since u is a minimum, we have

$$\langle J'(u_n) - J'(u), \psi\rangle \le \sqrt{\varepsilon_n} \int_\Omega |\nabla\psi|\, dx,$$

for every ψ in $W_0^{1,p}(\Omega)$. Choosing $\psi = u_n - u$, and using (20), we immediately obtain (21). In order to prove (22), we choose $\psi = G_k(u_n - u)$, thus obtaining, by (20),

$$\alpha \int_\Omega |\nabla G_k(u_n - u)|^p\, dx \le \sqrt{\varepsilon_n} \int_\Omega |\nabla G_k(u_n - u)|\, dx,$$

which in turn yields, by Hölder inequality

$$\alpha\Big(\int_\Omega |\nabla G_k(u_n-u)|^p\,dx\Big)^{\frac{1}{p'}} \le \sqrt{\varepsilon_n}\,\mathrm{meas}\,(A_{k,n})^{\frac{1}{p'}},$$

where we have defined $A_{k,n}=\{|u_n-u|\ge k\}$. Using Sobolev inequality, and choosing $h>k$ we arrive at

$$(h-k)^p\,\mathrm{meas}\,(A_{h,n})^{\frac{p}{p^*}} \le c\,\varepsilon_n^{\frac{p'}{2}}\,\mathrm{meas}\,(A_{k,n}),$$

that yields, again by the result of Stampacchia (see [11], Lemme 4.1),

$$\|u_n-u\|_{L^\infty(\Omega)} \le c\,\varepsilon_n^A,$$

for some positive constant A depending only on p and N. Recalling that ε_n converges to zero, we have the result. □

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