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SHARP REGULARITY OF A COUPLED SYSTEM OF A WAVE AND A KIRCHOFF EQUATION WITH POINT CONTROL ARISING IN NOISE REDUCTION*

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Abstract. We consider a mathematical model of the noise reduction problem, which couples two *hyperbolic* equations: the wave equation in the interior ("chamber")—which describes the unwanted acoustic waves—and a (hyperbolic) Kirchoff equation —which models the vibrations of the elastic wall. In past models, the elastic wall was modeled by an Euler-Bernoulli equation with Kelvin-Voight damping (*parabolic* model). Our main result is a *sharp regularity result*, in two dual versions, of the resulting system of two coupled hyperbolic P.D.E.'s. With this regularity result established, one can then invoke a wealth of abstract results from [14], [15], [16], [19], etc. on optimal control problems, min-max game theory (and H^∞-problems), etc. The proof of the main result is based on combining technical results from [18] and [11].

1. Introduction. Statement of problem for $\dim \Omega = 2$**. Main results.** We consider a mathematical model which arises in the problem of noise reduction in a chamber, with piezo-ceramic patches attached to a moving wall. This model couples a wave equation, which describes the acoustic waves in the chamber, with a (hyperbolic) Kirchoff equation on the moving wall, which describes its elastic displacement. Such a wall is subject to the action of, say, a piezo-ceramic patch (smart material), which is mathematically modeled by the derivative of a Dirac measure concentrated at a point of the wall, through which the scalar control influences the system's dynamics.

More precisely, let Ω be a two-dimensional domain ("the chamber"). We consider explicitly two cases:

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(i) either Ω is a 2-dimensional rectangle with three consecutive hard walls comprising the boundary Γ_1, and one vibrating wall comprising the boundary Γ_0 fixed at its extremes, where $\Gamma_0 \cup \Gamma_1 = \partial\Omega$ (Figure 1);

(ii) or else Ω is a general two-dimensional bounded domain, whose smooth boundary Γ is divided into two parts Γ_0 and Γ_1, $\Gamma = \Gamma_0 \cup \Gamma_1$, with Γ_1 a portion of the boundary acting as the hard wall, and Γ_0 a flat portion of the boundary acting as the moving wall fixed at its extremes (Figure 2).

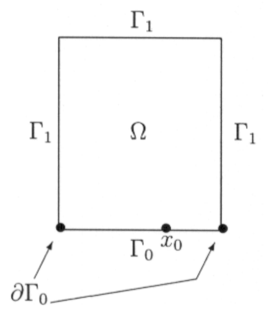

Figure 1. A rectangular Ω; $\alpha = \frac{3}{4}$.

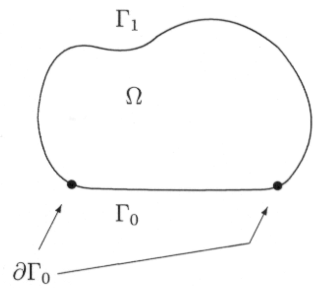

Figure 2. A general Ω; $\alpha = \frac{3}{5} - \epsilon$.

If $z(t,x)$ denotes the acoustic wave (unwanted noise) in the chamber, and $v(t,x)$, $x \in \Gamma_0$, denotes the displacement of Γ_0, then the relevant system of partial differential equations describing the given problem is

$$\begin{cases} z_{tt} = \Delta z & \text{on } (0,T] \times \Omega \equiv Q; & (1.1a) \\ \dfrac{\partial z}{\partial \nu}\Big|_{\Sigma_1} \equiv 0 & \text{on } (0,T] \times \Gamma_1 \equiv \Sigma_1; & (1.1b) \\ \dfrac{\partial z}{\partial \nu}\Big|_{\Sigma_0} \equiv -v_t & \text{on } (0,T] \times \Gamma_0 \equiv \Sigma_0; & (1.1c) \end{cases}$$

$$\begin{cases} v_{tt} - \gamma\Delta v_{tt} + \Delta^2 v - z_t = \delta'(x_0)u(t) & \text{on } \Sigma_0; & (1.1d) \\ \text{either clamped B.C. on } \partial\Gamma_0 & & \\ v \equiv \dfrac{\partial v}{\partial \nu} \equiv 0, & \text{on } (0,T] \times \partial\Gamma_0; & (1.1e_c) \\ \text{or else hinged B.C. on } \partial\Gamma_0, & & \\ v = \Delta v \equiv 0 & \text{on } (0,T] \times \partial\Gamma_0 & (1.1e_h) \end{cases}$$

$$\begin{cases} z(0,\ \cdot\ ) = z_0; \quad z_t(0,\ \cdot\ ) = z_1 & \text{in } \Omega; \\ v(0,\ \cdot\ ) = v_0; \quad v_t(0,\ \cdot\ ) = v_1 & \text{in } \Gamma_0. \end{cases} \qquad (1.1\text{f})$$

where $\nu(x)$ = unit outward normal vector at $x \in \Gamma$ and $\gamma > 0$ is a constant. In (1.1d), x_0 is a chosen point on Γ_0, $u(t)$ the scalar control function, and

$\delta'(x_0)$ denotes the derivative of the Dirac measure at x_0, which models the action of the bending moment caused by the piezo-ceramic patch. This model differs in a critical way from other models recently studied in noise reduction (see [1], [2] and references therein) in that the elastic dynamics of the wall is more realistically modeled by a hyperbolic Kirchoff equation, rather than by a structurally damped Euler-Bernoulli equation with so-called Kelvin-Voight damping, as in past models, such as:

$$v_{tt} + \Delta^2 v + \Delta^2 v_t - z_t = \delta'(x_0)u(t) \quad \text{on} \quad \Sigma_0,$$

which is instead parabolic. Thus, our model couples *two hyperbolic equations*, rather than a hyperbolic and a parabolic equation as in past models.

The basic structure of acoustic flow models has been known for a long time, see e.g., [17; example at p. 263]; some related mathematics questions regarding spectral properties or strong stabilization of this model in [17] are studied e.g., in [3] and [10]. Perhaps the key contribution of smart materials to the modeling, as supplied e.g., by NASA Langley acoustic groups, is the presence of (finitely many) δ' on the moving wall. Mathematically, it suffices to incorporate only one such δ' in (1.1d).

The main goal of the present paper is two-fold, as expressed by the following two main results.

Theorem 1.1. *Let $y(t) = [z(t), z_t(t), v(t), v_t(t)]$. Then the coupled P.D.E. system* (1.1) *can be rewritten abstractly as the equation*

$$\dot{y} = Ay + Bu \in [\mathcal{D}(A^*)]', \quad y(0) = y_0 = [z_0, z_1, v_0, v_1] \in Y, \tag{1.2}$$

where A and B are suitable operators, given by (2.16) *and* (2.19) *below, and where Y is the Hilbert space*

$$Y = H^1(\Omega) \times L_2(\Omega) \times H^2(\Gamma_0) \times H^1(\Gamma_0) \tag{1.3}$$

(*indeed, the last two components of* (1.3) *are refined as $H_0^2(\Gamma_0) \times H_0^1(\Gamma_0)$ for clamped B.C.; $[H^2(\Gamma_0) \cap H_0^1(\Gamma_0)] \times H_0^1(\Gamma_0)$ for hinged B.C.; see* (2.10), (2.11) *below*). *When Y is topologized by an equivalent norm as in* (2.10) *below, then the operator A is skew-adjoint, $A^* = -A$, and thus generates a s.c. unitary group e^{At} on Y, $t \geq 0$* (*conservative homogeneous problem*), *see* (2.21), (2.22).

As a consequence of Theorem 1.1, the solution of the abstract version (1.2) of the coupled P.D.E. problem (1.1) is then

$$y(t) = e^{At}y_0 + (Lu)(t); \qquad (Lu)(t) = \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau. \tag{1.4}$$

Since the operator B given by (2.19) is *not* bounded from $U = \mathbf{R}$ to Y, but instead satisfies the regularity $A^{-1}B \in \mathcal{L}(U;Y)$ [14, Chapter 10, Section 10.2] or $B : U \to [\mathcal{D}(A^*)]'$ where $[\quad]'$ denotes duality of $\mathcal{D}(A^*)$ with respect to the pivot space Y, it is necessary to specify the regularity of the operator L in (1.4). This is given next. The main result of the paper is the following regularity Theorem for (1.2), or (1.4).

Theorem 1.2. *With reference to the abstract version* (1.2), *or* (1.4), *of the coupled P.D.E. system* (1.1), *and to the space* Y *in* (1.3), *we have that*

(i) *For each* $0 < T < \infty$, *the operator* L *in* (1.4) *satisfies the property*

$$L: \text{ continuous } L_2(0,T) \to C([0,T];Y). \tag{1.5}$$

In P.D.E. terms, the meaning of (1.5) *is as follows. Set* $z_0 = z_1 = v_0 = v_1$ *in* (1.1f), *then the corresponding solution of* (1.1) *satisfies*

$$u \in L_2(0,T) \to [z(t), z_t(t), v(t), v_t(t)] \in C([0,T];Y). \tag{1.6}$$

(ii) *Equivalently, by duality* [14], *then the following 'abstract trace regularity' property holds true*: *for each* $0 < T < \infty$, *there exists a constant* $C_T > 0$ *such that*

$$\int_0^T \left| B^* e^{A^* t} y \right|^2 \leq C_T \|y\|_Y^2, \quad \forall\, y \text{ first in } \mathcal{D}(A^*), \text{ next extended to all of } Y. \tag{1.7}$$

In P.D.E. terms the meaning of (1.7) is as follows. Let $u \equiv 0$ in (1.1d), then the corresponding *homogeneous* problem (1.1) satisfies the estimate

$$\int_0^T |v_{tx}(t, x_0; y_0)|^2 dt \leq C_T \|y_0\|_Y^2, \quad y_0 \in Y, \tag{1.8}$$

$y_0 = [z_0, z_1, v_0, v_1]$, where $v_{tx}(t, x_0; y_0)$ is the second partial derivative of the solution v in t and x, evaluated at the point $x = x_0 \in \Gamma_0$ (point observation), and due to the Initial Condition y_0 and to $u \equiv 0$.

Remark 1.1. Theorem 1.2 is sharp. The regularity

$$[v, v_t] \in C([0,T]; H^2(\Gamma_0) \times H^1(\Gamma_0))$$

of the Kirchoff component (the one subject directly to the control action u) of the coupled problem (1.1) is exactly the same as the regularity described by

Theorem 3.1 for the *uncoupled* Kirchoff problem (3.1). As noted in Remark 3.1, such regularity is '$\frac{1}{2}+\epsilon$' *higher* in Sobolev space units, in the space variable, over the regularity that is obtained by the variation of parameter formula of the corresponding semigroup, based simply on the membership property that $\delta'(x_0) \in [H^{\frac{3}{2}+\epsilon}(\Gamma_0)]' \subset [\mathcal{D}(\mathcal{A}^{\frac{3}{8}+\frac{\epsilon}{4}})]'$, or $\mathcal{A}^{-\frac{3}{8}-\frac{\epsilon}{4}}\delta'(x_0) \in L_2(\Gamma_0)$, where $\dim \Gamma_0 = 1$, and where $\mathcal{A}$ is the bi-harmonic operator defined by (2.1) below (in the hinged case). A similar loss of '$\frac{1}{2}+\epsilon$' would occur, if one used directly and analogously the abstract model (1.4b) with e^{At} the s.c. semigroup of Theorem 1.1 and with B given by (2.19). Similarly, (1.8) does *not* follow from energy regularity. In conclusion, Theorem 3.1 cannot be obtained by 'standard' methods: it requires a combination of sharp regularity results for the uncoupled Kirchoff part (Theorem 3.1 from [18]) and for the uncoupled wave part (Theorem 3.2 from [11], [12]).

The present note gives a direct proof of the regularity statement (1.5). A different proof of the equivalent dual trace regularity statement (1.8) was recently given in [21]. The present proof is preferable, as it is simpler and more streamlined than that of [21] for the dual version in (1.8). Both proofs, however, rely critically on the sharp regularity results of the two basic dynamical components of the noise reduction model: [18] for the Kirchoff equation with point control and [11], or related results, for the wave equation with Neumann control.

Implications of Theorem 1.1 on control problems. Sharp (optimal) regularity results for mixed P.D.E. problems, besides representing the first fundamental step in the analysis of these dynamics, have also key implications in the study of associated optimal control problems. In the case in question, Theorem 1.1 and Theorem 1.2 permit us to invoke a large body of abstract results on, say, quadratic optimal control theory, min-max game theory, etc., see [14], [15], [16], [19], and apply them to the coupled P.D.E. problem (1.1). A brief sample follows.

Case $T<\infty$. No further requirement is needed in optimal control problems on a finite interval $[0,T]$, $T<\infty$, over Theorem 1.1 and Theorem 1.2.

Case $T=\infty$. In this case, in order to invoke the abstract theory as in [14], [15], [16], [19], which is allowed by Theorem 1.1, additional control theoretic hypotheses are needed, such as the Finite Cost condition and the Detectability Condition. The Finite Cost Condition amounts to the property of exact controllability of the underlying dynamics (1.1) on the state space Y (of regularity) in (1.3) with $L_2(0,T)$-controls [or of uniform stabilization with $L_2(0,\infty;U)$-feedback control]. However, such exact controllability property on the space of regularity Y given by (1.3) fails for problem (1.1). This is a general *pathology* of hyperbolic or Petrowski-type P.D.E. dynamics with point control, acting through δ or δ', etc. See e.g. [14] and the introduction of [7], [8]. To remedy the situation, we modify the original *conservative* dynamics (1.1) (recall $A^* = -A$) by adding damping terms as to (i) make it *uniformly stable* on Y, while (ii) preserving the *same regularity* in $C([0,T];Y)$. This can be accomplished by replacing the conservative problem (1.1) with the following damped version say, in the hinged case:

$$\begin{cases} z_{tt} = \Delta z - d_0 z_t & \text{on } Q; & (1.9a) \\ \left.\frac{\partial z}{\partial \nu}\right|_{\Sigma_1} \equiv -d_1 z_t & \text{on } \Sigma_1; & (1.9b) \\ \left.\frac{\partial z}{\partial \nu}\right|_{\Sigma_0} \equiv -d_1 z_t - v_t & \text{on } \Sigma_0; & (1.9c) \end{cases}$$

$$\begin{cases} v_{tt} - \gamma \Delta v_{tt} + \Delta^2 v - z_t = \delta'(x_0)u(t) - k_0 \Delta v_t & \text{on } \Sigma_0; & (1.9d) \\ v = 0;\ \Delta v = -k_1 \frac{\partial v_t}{\partial \nu} & \text{on } (0,T] \times \partial\Gamma_0, & (1.9e_h) \end{cases}$$

plus initial conditions as in (1.1f), where the damping constants d_i and k_i satisfy the conditions

$$d_i \geq 0,\ k_i \geq 0; \quad d_0 + d_1 > 0; \quad k_0 + k_1 > 0. \tag{1.10}$$

The regularity of the corresponding mixed problem (1.9a–e) *is still the same as that given by Theorem* 1.2 (see Remark 3.4).

The needed uniform stabilization result on the space Y in (1.3) is given in [4].

2. Abstract model of the original problem (1.1) in $\{z, v\}$ (Hinged B.C.): Theorem 1.1. In this section we introduce the relevant functional analytic setting for problem (1.1), which culminates with the proof of Theorem 1.1. For simplicity of notation, we restrict to the case of hinged B.C. The case of clamped B.C. can be handled similarly by using [18, Section 3]. Some relevant operators, in this latter case, are given explicitly at the end of this section.

Hinged B.C.

(i) Let $\mathcal{A}:\ L_2(\Gamma_0) \supset \mathcal{D}(\mathcal{A}) \to L_2(\Gamma_0)$ be the positive self-adjoint operator

$$\mathcal{A}f = \Delta^2 f,\ \mathcal{D}(\mathcal{A}) = \{f \in H^4(\Gamma_0):\ f|_{\partial\Gamma_0} = \Delta f|_{\partial\Gamma_0} = 0\}. \tag{2.1}$$

(ii) Next, introduce the operator $\mathbf{A}$ [13], [14],

$$\mathbf{A} = \left(I + \gamma\mathcal{A}^{\frac{1}{2}}\right)^{-1}\mathcal{A}: \; L_2(\Gamma_0) \supset \mathcal{D}(\mathbf{A}) = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \to L_2(\Gamma_0). \tag{2.2}$$

The operator $\mathbf{A}$ is positive self-adjoint on the space $\mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{4}})$ topologized by the inner product

$$(x, y)_{\mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{4}})} = \left(\left(I + \gamma\mathcal{A}^{\frac{1}{2}}\right)x, y\right)_{L_2(\Gamma_0)}, \quad x, y \in \mathcal{D}(\mathcal{A}^{\frac{1}{4}}), \tag{2.3}$$

[13], [14]. We have [5],

$$\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^2(\Gamma_0) \cap H_0^1(\Gamma_0); \quad \mathcal{D}(\mathcal{A}^{\frac{1}{4}}) = H_0^1(\Gamma_0). \tag{2.4}$$

(iii) Define the Neumann map N by [14],

$$h = Ng \iff \begin{cases} \Delta h = 0 \;\text{ in } \Omega; & (2.5a) \\ \dfrac{\partial h}{\partial \nu} = 0 \text{ on } \Gamma_1; \quad \dfrac{\partial h}{\partial \nu} = g \text{ on } \Gamma_0. & (2.5b) \end{cases}$$

(iv) Let $\mathcal{A}_N : L_2^0(\Omega) \equiv L_2(\Omega)/\mathcal{N}(\mathcal{A}_N) \to L_2^0(\Omega)$ be the positive self-adjoint operator

$$\mathcal{A}_N f = -\Delta f; \quad \mathcal{D}(\mathcal{A}_N) = \left\{ f \in H^2(\Omega) : \left.\frac{\partial f}{\partial \nu}\right|_\Gamma = 0 \right\}, \tag{2.6}$$

where $\mathcal{N}(\mathcal{A}_N)$ is the one-dimensional nullspace of $\mathcal{A}_N$ of constant functions. We have [14],

$$N^*\mathcal{A}_N f = \begin{cases} 0 & \text{on } \Gamma_1 \\ -f|_{\Gamma_0} & \text{on } \Gamma_0 \end{cases} \quad f \in \mathcal{D}(\mathcal{A}_N). \tag{2.7}$$

Elliptic theory and [5] yield, see [14], for any $\epsilon > 0$,

$$N: \text{ continuous } L_2(\Gamma) \to H^{\frac{3}{2}}(\Omega) \supset H^{\frac{3}{2}-2\epsilon}(\Omega) \equiv \mathcal{D}\left(\mathcal{A}_N^{\frac{3}{4}-\epsilon}\right); \tag{2.8}$$

$$\mathcal{A}_N^{\frac{3}{4}-\epsilon} N: \text{ continuous } L_2(\Gamma) \to L_2^0(\Omega). \tag{2.9}$$

(v) Finally, we introduce the space (norm equivalent to (1.3))

$$Y \equiv \mathcal{D}(\mathcal{A}_N^{\frac{1}{2}}) \times L_2(\Omega) \times \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times \mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{4}}) \equiv Y_W \times Y_K; \tag{2.10}$$

$$Y_W = H^1(\Omega) \times L_2(\Omega); \quad Y_K = [H^2(\Gamma_0) \cap H_0^1(\Gamma_0)] \times H_0^1(\Gamma_0) \tag{2.11}$$

(norm equivalence). Thus, by (2.1), (2.6), (2.7), the coupled problem (1.1) in $\{z, v\}$ can be rewritten as [13]

$$\begin{cases} z_{tt} = -\mathcal{A}_N z + \mathcal{A}_N N\,(v_t|_{\Gamma_0}) & \text{on } [\mathcal{D}(\mathcal{A}_N)]'; \quad (2.12) \\ (I + \gamma \mathcal{A}^{\frac{1}{2}}) v_{tt} + \mathcal{A} v = -N^* \mathcal{A}_N z_t + \delta'(x_0) u(t) & \text{on } L_2(\Gamma_0), \quad (2.13) \end{cases}$$

and the second equation (2.13) becomes via (2.2)

$$v_{tt} = -\mathbf{A} v - (I + \gamma \mathcal{A}^{\frac{1}{2}})^{-1} N^* \mathcal{A}_N z_t + (I + \gamma \mathcal{A}^{\frac{1}{2}})^{-1} \delta'(x_0) u(t). \tag{2.14}$$

The corresponding first-order system of (2.12), (2.14) with

$$y(t) = [z(t), z_t(t), v(t), v_t(t)]$$

is

$$\dot{y} = Ay + Bu, \qquad y(0) = y_0 = [z_0, z_1, v_0, v_1] \in Y; \tag{2.15}$$

$$A = \begin{bmatrix} 0 & I & 0 & 0 \\ -\mathcal{A}_N & 0 & 0 & \mathcal{A}_N N(\,\cdot\,|_{\Gamma_0}) \\ 0 & 0 & 0 & I \\ 0 & -(I + \gamma \mathcal{A}^{\frac{1}{2}})^{-1} N^* \mathcal{A}_N & -\mathbf{A} & 0 \end{bmatrix}; \tag{2.16}$$

$$\begin{aligned} Y \supset \mathcal{D}(A) = \big\{ & y \in Y : y_2 \in \mathcal{D}(\mathcal{A}_N^{\frac{1}{2}}), y_3 \in \mathcal{D}(\mathcal{A}^{\frac{3}{4}}), \\ & y_4 \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}), \mathcal{A}_N^{\frac{1}{2}}[y_1 - N(y_4|_{\Gamma_0})] \in \mathcal{D}(\mathcal{A}_N^{\frac{1}{2}}) \big\} \to Y. \end{aligned} \tag{2.17}$$

The Y-adjoint of A is

$$A^* = \begin{bmatrix} 0 & I & 0 & 0 \\ \mathcal{A}_N & 0 & 0 & -\mathcal{A}_N N(\,\cdot\,|_{\Gamma_0}) \\ 0 & 0 & 0 & -I \\ 0 & (I + \gamma \mathcal{A}^{\frac{1}{2}})^{-1} N^* \mathcal{A}_N & \mathbf{A} & 0 \end{bmatrix} = -A; \tag{2.18}$$

$\mathcal{D}(A^*) = \mathcal{D}(A)$. The operator $B : U \to [\mathcal{D}(A^*)]'$, $U = \mathbf{R}$, and its L_2-adjoint B^* are [here $[\quad]'$ is the dual of $\mathcal{D}(A^*)$ w.r.t. the pivot space Y]:

$$Bu = \begin{bmatrix} 0 \\ 0 \\ 0 \\ (I + \gamma \mathcal{A}^{\frac{1}{2}})^{-1} \delta'(x_0) u \end{bmatrix}; \quad B^* \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = -\frac{d}{dx}\, y_4|_{x=x_0}, \; y \in \mathcal{D}(A^*). \tag{2.19}$$

Indeed, to find B^* by (2.10), (2.3), we compute

$$(Bu, y)_Y = \left(\left(I + \gamma \mathcal{A}^{\frac{1}{2}} \right)^{-1} \delta'(x_0)u, y_4 \right)_{\mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{4}})} = (\delta'(x_0)u, y_4)_{L_2(\Gamma_0)}$$
$$= u \left(-\frac{d}{dx}\, y_4|_{x=x_0} \right) = (u, B^* y)_{U=\mathbf{R}}, \tag{2.20}$$

and (2.19) follows for B^*.

From (2.16)–(2.18) we have that A generates a s.c. unitary group e^{At} on Y:

$$e^{A^*t} = e^{-At}; \quad \left\| e^{A^*t} \right\|_{\mathcal{L}(Y)} \equiv \left\| e^{-At} \right\|_{\mathcal{L}(Y)} \equiv 1, \tag{2.21}$$

$$\mathrm{Re}(Ax, x)_Y = \mathrm{Re}(A^*x, x)_Y \equiv 0, \quad x \in \mathcal{D}(A) = \mathcal{D}(A^*), \tag{2.22}$$

Y as in (2.10). With A and B given by (2.16), (2.19), the solution of problem (1.1) is given by (1.4).

Theorem 1.1 is proved (in the hinged case).

Clamped B.C. Here we introduce the following positive, self-adjoint operators [18, Section 3]:

$$\mathcal{A}f = \Delta^2 f, \quad \mathcal{D}(\mathcal{A}) = \left\{ f \in H^4(\Omega) : \ f|_{\partial\Gamma_0} = \left.\frac{\partial f}{\partial \nu}\right|_{\partial\Gamma_0} = 0 \right\}; \tag{2.23}$$

$$\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H_0^2(\Gamma_0); \quad \mathcal{D}(\mathcal{A}^{\frac{1}{4}}) = H_0^1(\Gamma_0) \tag{2.24}$$

(counterparts of (2.2), (2.4)). Moreover, let

$$\mathcal{A}_D f = -\Delta f, \quad \mathcal{D}(\mathcal{A}_D) = H^2(\Gamma_0) \cap H_0^1(\Gamma_0), \quad \mathbf{A} = (I + \gamma \mathcal{A}_D)^{-1} \mathcal{A}, \tag{2.25}$$

counterpart of (2.2).

3. Proof of Theorem 1.2: Regularity of L in (1.5). In this section we prove the regularity of L in (1.5).

Step 1 (Uncoupled non-homogeneous Kirchoff equation on Γ_0). Henceforth, $\phi(t, x)$ will denote the solution of the following mixed problem for the Kirchoff equation (which is problem (1.1d-e) this time uncoupled)

$$\begin{cases} \phi_{tt} - \gamma \Delta \phi_{tt} + \Delta^2 \phi = \delta'(x_0) u(t) & \text{in } (0, T] \times \Gamma_0; & (3.1a) \\ \phi(0, \cdot\,) = \phi_0, \quad \phi_t(0, \cdot\,) = \phi_1 & \text{in } \Gamma_0, & (3.1b) \\ \text{either clamped B.C. on } \partial\Gamma_0 & & \\ \phi \equiv \frac{\partial \phi}{\partial \nu} \equiv 0 & \text{on } (0, T] \times \partial\Gamma_0; & (3.1c_c) \\ \text{or else hinged B.C. on } \partial\Gamma_0, & & \\ \phi \equiv \Delta \phi \equiv 0 & \text{in } (0, T] \times \partial\Gamma_0 & (3.1c_h) \end{cases}$$

An optimal regularity result for problem (3.1) will be given next.

Theorem 3.1 (ϕ-problem). *Recall that* $\dim \Gamma_0 = 1$. *Consider the* ϕ*-problem in* (3.1) *with, say, hinged B.C., and thus with*

$$\{\phi_0, \phi_1\} \in [H^2(\Gamma_0) \cap H_0^1(\Gamma_0)] \times H_0^1(\Gamma_0); \quad u \in L_2(0,T). \tag{3.2}$$

Then, continuously,

$$\{\phi, \phi_t, \phi_{tt}\} \in C([0,T]; [H^2(\Gamma_0) \cap H_0^1(\Gamma_0)] \times H_0^1(\Gamma_0)) \times L_2(0,T; L_2(\Gamma_0)). \tag{3.3}$$

For clamped B.C. the first component space is $H_0^2(\Gamma_0)$.

Proof. It suffices to take zero Initial Condition by Theorem 1.1. Then the proof follows directly from [18, Theorem 2.1(c), Eqn. (2.9a-c)] in the case of hinged B.C., and from [18, Theorem 3.1(c), Eqn. (3.15a-c)] in the case of clamped B.C. As a matter of fact, paper [18] treats explicitly the case with point control through the Dirac measure $\delta(\,\cdot\,)$ concentrated at interior point of a domain of dimension 1, 2, 3, rather than through $\delta'(\,\cdot\,)$. One readily shows, however, that the space regularity results with $\delta'(\,\cdot\,)$ are *one Sobolev unit less* than the space regularity results with $\delta(\,\cdot\,)$ given in [18].

Remark 3.1. Theorem 3.1 is sharp. Its regularity results are '$\frac{1}{2} + \epsilon$' *higher* in Sobolev space units, in the space variable, over those that can be obtained via the variation of parameter formula and semigroup methods based simply on the membership property that $\delta'(x_0) \in [H^{\frac{3}{2}+\epsilon}(\Gamma_0)]' \subset [\mathcal{D}(\mathcal{A}^{\frac{3}{8}+\frac{\epsilon}{4}})]'$, or $\mathcal{A}^{-\frac{3}{8}-\frac{\epsilon}{4}}\delta'(x_0) \in L_2(\Gamma_0)$, $\dim \Gamma_0 = 1$. See [18, Remark 2.1, p. 400].

Step 2 (Uncoupled non-homogeneous wave equation on Ω). Henceforth, $\psi(t,x)$ will denote the solution of the following mixed problem for the wave equation (which is problem (1.1a-b-c), this time uncoupled):

$$\begin{cases} \psi_{tt} = \Delta\psi & (0,T] \times \Omega \equiv Q; & (3.4a) \\ \psi(0,\,\cdot\,) = 0,\ \psi_t(0,\,\cdot\,) = 0 & \text{in } \Omega; & (3.4b) \\ \left.\frac{\partial\psi}{\partial\nu}\right|_{\Sigma} \equiv g & (0,T] \times \partial\Omega \equiv \Sigma. & (3.4c) \end{cases}$$

A sharp regularity theory for problem (2.1) is given in [11], [12], from which we shall quote below.

Henceforth, α is a constant taking up the following values (where $\epsilon > 0$ is arbitrary):

$$\begin{cases} \alpha = \frac{3}{5} - \epsilon & \text{for a general smooth domain } \Omega \text{ of } R^n,\ n \geq 2; & (3.5a) \\ \alpha = \frac{3}{4} & \text{for a parallelepiped } \Omega \text{ of } R^n,\ n \geq 2; & (3.5b) \\ \alpha = \frac{2}{3} & \text{for a sphere } \Omega \text{ of } R^n,\ n \geq 2. & (3.5c) \end{cases}$$

Theorem 3.2 ([11], [12]). *With reference to problem* (3.4) (*where, actually,* $\dim \Omega = 2$), *we have*

(i) (*interior regularity*) *Let*

$$g \in H^1(0,T;L_2(\Gamma)) \cap C\left([0,T];H^{\alpha-\frac{1}{2}}(\Gamma)\right), \ g|_{t=0} = 0. \tag{3.6}$$

Then, continuously,

$$\{\psi, \psi_t, \psi_{tt}\} \in C\left([0,T];H^{\alpha+1}(\Omega) \times H^{\alpha}(\Omega) \times H^{\alpha-1}(\Omega)\right); \tag{3.7}$$

(ii) (*boundary regularity*) *Let*

$$g \in H^1(\Sigma) \equiv L_2(0,T;H^1(\Gamma)) \cap H^1(0,T;L_2(\Gamma)), \ g|_{t=0} = 0. \tag{3.8}$$

Then, continuously,

$$\psi|_{\Sigma} \in H^{2\alpha}(\Sigma) = L_2(0,T;H^{2\alpha}(\Gamma)) \cap H^{2\alpha}(0,T;L_2(\Gamma)). \tag{3.9}$$

Proof. Part (i) is contained in [11, Theorem 3.1(ii), Eqns. (3.5), (3.6); and Theorem 3.1(iii), Eqns. (3.7)–(3.10)], where α is as in (3.5). Part (ii) is contained in [11, Theorem 4.1]. □

The case of interest for the present acoustic model is captured next. With ϕ_t provided by Theorem 3.1 for the Kirchoff problem (3.1), consider the specialization of the wave equation problem (3.4) given by

$$\begin{cases} w_{tt} = \Delta w & \text{in } Q; & (3.10a)\\ w(0,\ \cdot\ ) = 0; \quad w_t(0,\ \cdot\ ) = 0 & \text{in } \Omega; & (3.10b)\\ \left.\frac{\partial w}{\partial \nu}\right|_{\Sigma_1} \equiv 0 & \text{in } \Sigma_1; & (3.10c)\\ \left.\frac{\partial w}{\partial \nu}\right|_{\Sigma_0} = -\phi_t & \text{in } \Sigma_0, & (3.10d) \end{cases}$$

where, by Theorem 3.1, Eqn. (3.3), we have

$$\begin{cases} -\phi_t & \in C\left([0,T];H_0^1(\Gamma_0)\right) \cap H^1(0,T;L_2(\Gamma_0)) & (3.11a)\\ & \subset H^1(\Sigma_0). & (3.11b) \end{cases}$$

Moreover, by (3.1b), we verify that

$$\text{on } \Gamma_0: \ -\phi_t|_{t=0} = -\phi_1 = 0, \tag{3.12}$$

so that the Compatibility Relation $g|_{t=0} = 0$ required by Theorem 3.2 is satisfied with $g = 0$ on Σ_1, and $g = -\phi_t$ on Σ_0. As a corollary of Theorem 3.2, we thus obtain the result of our interest.

Corollary 3.3. *With reference to the wave problem* (3.10) *we have*

(i) (*interior regularity*)

$$\{w, w_t, w_{tt}\} \in C\left([0,T]; H^{\alpha+1}(\Omega) \times H^{\alpha}(\Omega) \times H^{\alpha-1}(\Omega)\right); \tag{3.13}$$

(ii) (*boundary regularity*)

$$w_t \in H^{2\alpha-1}(0,T; L_2(\Gamma_0)). \tag{3.14}$$

Proof. (i) Since $H^1(\Gamma_0) \subset H^{\alpha-\frac{1}{2}}(\Gamma_0)$ by (3.5), we see by (3.11) and (3.12) that we can invoke Theorem 3.2 for problem (3.10), and obtain (3.13) from (3.7), and (3.14) from (3.9), i.e., from $w \in H^{2\alpha}(0,T; L_2(\Gamma))$. □

Step 3. With w_t given by (3.14), we next consider the following coupled system of two P.D.E.'s, in the variables $\zeta(t,x)$ and $h(t,x)$:

$$\begin{cases} \zeta_{tt} = \Delta\zeta & \text{on } (0,T] \times \Omega = Q; & (3.15a) \\ \left.\dfrac{\partial\zeta}{\partial\nu}\right|_{\Sigma_1} \equiv 0 & \text{on } (0,T] \times \Gamma_1 \equiv \Sigma_1; & (3.15b) \\ \left.\dfrac{\partial\zeta}{\partial\nu}\right|_{\Sigma_0} \equiv -h_t & \text{on } (0,T] \times \Gamma_0 \equiv \Sigma_0; & (3.15c) \end{cases}$$

$$\begin{cases} h_{tt} - \gamma\Delta h_{tt} + \Delta^2 h - \zeta_t = w_t & \text{on } \Sigma_0; & (3.15d) \\ \text{either clamped B.C. on } \partial\Gamma_0 & & \\ h \equiv \dfrac{\partial h}{\partial\nu} \equiv 0; & \text{on } (0,T] \times \partial\Gamma_0; & (3.15e_c) \\ \text{or else hinged B.C. on } \partial\Gamma_0, & & \\ h \equiv \Delta h \equiv 0 & \text{on } (0,T] \times \partial\Gamma_0 & (3.15e_h) \end{cases}$$

$$\zeta(0,\,\cdot\,) = 0;\ \zeta_t(0,\,\cdot\,) = 0 \text{ in } \Omega;\ h(0,\,\cdot\,) = 0;\ h_t(0,\,\cdot\,) = 0 \text{ in } \Gamma_0. \tag{3.15f}$$

Remark 3.2. As a motivation, we have that problem (3.15) is obtained by setting

$$\zeta(t,x) \equiv z(t,x) - w(t,x); \quad h(t,x) \equiv v(t,x) - \phi(t,x), \tag{3.16}$$

differentiating formally and using problems (1.1), (3.1), and (3.10); this procedure is justified in Step 5 below.

Remark 3.3. The coupled problem (3.15) in the new variables $\{\zeta, h\}$ is the same as the original coupled problem (1.1) in the variables $\{z, v\}$, except that: on Σ_0, the point control term $\delta'(x_0)u(t)$ in (1.1d) is replaced with the non-homogeneous term w_t in (3.15d), for which the *a-priori* regularity (3.14) is available. Thus, problem $\{\zeta, h\}$ is easier to analyze than problem $\{z, v\}$. In fact, we shall obtain a regularity result for problem $\{\zeta, h\}$ by semigroup methods, using the unitary group obtained in Theorem 1.1. By contrast, semigroup methods applied to even a single hyperbolic equation—such as the wave equation on the Kirchoff equation—acted upon by interior point control, are *definitely non-optimal*, as they lose '$\frac{1}{2}+\epsilon$' regularity in Sobolev space units, in the space variable; see [21, Remark 2.1] for the wave equation; and [18, Remark 2.1] for the Kirchoff equation. Refer to Remark 3.1.

According to (2.1), (2.6), (2.7), problem (3.15) can be rewritten abstractly as

$$\zeta_{tt} = -\mathcal{A}_N\zeta + \mathcal{A}_N N\left(h_t|_{\Gamma_0}\right) \quad \text{on } [\mathcal{D}(\mathcal{A}_N)]', \tag{3.17a}$$

$$\left(I+\gamma\mathcal{A}^{\frac{1}{2}}\right)h_{tt} + \mathcal{A}h = -N^*\mathcal{A}_N\zeta_t + w_t \quad \text{on } L_2(\Gamma_0), \tag{3.17b}$$

at least in the hinged case: Compare with Eqns. (2.12), (2.13). [In the clamped case, we use the operators in (2.23)–(2.25).] The corresponding first-order system of (3.17), (3.18) is

$$\frac{d}{dt}\begin{bmatrix}\zeta\\ \zeta_t\\ h\\ h_t\end{bmatrix} = A\begin{bmatrix}\zeta\\ \zeta_t\\ h\\ h_t\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 0\\ \left(I+\gamma\mathcal{A}^{\frac{1}{2}}\right)^{-1} w_t\end{bmatrix}, \tag{3.18}$$

where the operator A is given by (2.16), (2.17). Thus, according to Theorem 1.1, the solution of (3.18) with zero initial condition (by (3.15f)) is given by

$$\eta(t) = \begin{bmatrix}\zeta(t)\\ \zeta_t(t)\\ h(t)\\ h_t(t)\end{bmatrix} = \int_0^t e^{A(t-\tau)}\begin{bmatrix}0\\ 0\\ 0\\ \left(I+\gamma\mathcal{A}^{\frac{1}{2}}\right)^{-1} w_t(\tau)\end{bmatrix} d\tau, \tag{3.19}$$

where e^{At} is the unitary group on Y, see (2.10), of Theorem 1.1. We next establish the regularity of $\eta(t)$ in (3.19).

Step 4. Proposition 3.4. *With reference to* (3.19), *and* Y *in* (2.10), *we have that*

$$\eta(t) \equiv [\zeta(t), \zeta_t(t), h(t), h_t(t)] \in C([0,T];Y). \tag{3.20}$$

Proof. We return to the *a-priori* regularity (3.14) for w_t and obtain

$$\left(I+\gamma\mathcal{A}^{\frac{1}{2}}\right)^{-1} w_t \in H^{2\alpha-1}\left(0,T;\mathcal{D}(\mathcal{A}^{\frac{1}{2}})\right) \tag{3.21}$$

$$\subset L_2\left(0,T;\mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{4}})\right), \tag{3.22}$$

since $2\alpha-1>0$ by (3.5). It will suffice to proceed with the *loss of space regularity* of one Sobolev unit of (3.22) over (3.21). Using (3.22) in (3.19) and recalling that $\mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{4}})$ is the last component space of Y in (2.10), we readily obtain (3.20). □

Step 5. We now return from the $\{\zeta,h\}$ problem to the $\{z,v\}$ original problem. It is readily checked (recall Remark 3.2) that, having $\{\zeta,h\}$ from problem (3.15) [Proposition 3.4], and having ϕ and w from problem (3.1) [Theorem 3.1], and problem (3.10) [Corollary 3.3], respectively, the functions $\{\zeta(t,x)+w(t,x),h(t,x)+\phi(t,x)\}$ satisfy problem (1.1), so that we, in fact, have

$$z(t,x)\equiv\zeta(t,x)+w(t,x);\quad v(t,x)\equiv h(t,x)+\phi(t,x). \tag{3.23}$$

Thus, via (3.23), we can obtain the regularity of the original variables $\{z(t,x),v(t,x)\}$ from the regularity of the new variables $\{\zeta(t,x),h(t,x)\}$ given by Proposition 3.4, Eqn. (3.20), and from the regularity of $w(t,x)$ and $\phi(t,x)$ given by Corollary 3.3, Eqn. (3.13), and Theorem 3.1, Eqn. (3.3), respectively. We obtain, recalling (3.20) with Y as in (2.10), in the *hinged case*:

$$\begin{bmatrix}\zeta\\ \zeta_t\\ h\\ h_t\end{bmatrix}\in C\left([0,T];\begin{bmatrix}\mathcal{D}(\mathcal{A}_N^{\frac{1}{2}})=H^1(\Omega)\\ L_2(\Omega)\\ \mathcal{D}(\mathcal{A}^{\frac{1}{2}})=H^2(\Gamma_0)\cap H_0^1(\Gamma_0)\\ \mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{4}})=H_0^1(\Gamma_0)\end{bmatrix}\right), \tag{3.24}$$

and recalling Eqns. (3.13) and (3.3),

$$\begin{aligned}&\begin{bmatrix}w\\ w_t\end{bmatrix}\in C\left([0,T];\begin{bmatrix}H^{\alpha+1}(\Omega)\\ H^{\alpha}(\Omega)\end{bmatrix}\right);\\ &\begin{bmatrix}\phi\\ \phi_t\end{bmatrix}\in C\left([0,T];\begin{bmatrix}H^2(\Gamma_0)\cap H_0^1(\Gamma_0)\\ H_0^1(\Gamma_0)\end{bmatrix}\right).\end{aligned} \tag{3.25}$$

Thus, via (3.23), using (3.24), (3.25), we finally obtain

$$\begin{bmatrix} z \\ z_t \\ v \\ v_t \end{bmatrix} \in C\left([0,T]; \begin{bmatrix} \mathcal{D}(\mathcal{A}_N^{\frac{1}{2}}) = H^1(\Omega) \\ L_2(\Omega) \\ H^2(\Gamma_0) \cap H_0^1(\Gamma_0) \\ H_0^1(\Gamma_0) \end{bmatrix}\right) = C([0,T];Y), \tag{3.26}$$

and Theorem 1.2 is proved, at least in the hinged case.

For the *clamped case*, we use instead the operators in (2.23), (2.24). The proof of Theorem 1.2 is complete. □

Remark 3.4. This remark points out the main changes which are needed in the above proof, when the conservative problem (1.1) is replaced by the damped problem (1.9), to obtain *the same regularity as in* Theorem 1.2. It will suffice to focus on the boundary-damped case:

$$d_0 = k_0 = 0; \quad d_1 > 0, \ k_1 > 0, \tag{3.27}$$

for no changes really occur if $d_0 > 0$, $k_0 > 0$, while $d_1 = k_1 = 0$.

(i) First, problem (3.1) now replaces (3.1c$_h$) with

$$\phi \equiv 0; \quad \Delta\phi = -k_1 \frac{\partial \phi_t}{\partial \nu}. \tag{3.28}$$

The proof of [18] readily adapts to this case and Theorem 3.1 continues to hold true.

(ii) Second, problem (3.4) now replaces (3.4c) with

$$\left.\frac{\partial \psi}{\partial \nu}\right|_{\Sigma} = -d_1\psi_t + g. \tag{3.29}$$

This new damped problem (3.4a–b), (3.29) provides the regularity

$$g \in L_2(0,T;L_2(\Gamma)) \Rightarrow \{\psi,\psi_t\} \in C([0,T];H^1(\Omega) \times L_2(\Omega)) \tag{3.30}$$

(rather than

$$g \in L_2(0,T;L_2(\Gamma)) \Rightarrow \{\psi,\psi_t\} \in C([0,T];H^\alpha(\Omega) \times H^{\alpha-1}(\Omega)), \tag{3.31}$$

$\alpha < 1$ as in (3.5), for problem (3.4) [11], [12].) Thus, not only the damped B.C. (3.29) is responsible now for the *higher* regularity result (3.30), which is false in the case of problem (3.4) with $\dim\Omega \geq 2$; but, moreover, the proof of

(3.30) is *far easier* than that of (3.31), as it only requires a direct (operator) energy method [9, Section 4] rather than pseudo-differential analysis [11], [12].

4. The Case Where $\dim\Omega = 3$**.** Let Ω be a three-dimensional chamber with a two-dimensional smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$; or else a three-dimensional parallelopiped, with Γ_0 being a face of it. As in Section 1, here Γ_1 denotes the hard wall, and Γ_0 the moving wall fixed at its extreme curve. Let β be a curve contained in Γ_0, and let $\frac{\partial\delta_\beta}{\partial\nu}$ denote the derivative of the Dirac mass concentrated in β with respect to the normal to β. On the triple $\{\Omega, \Gamma_0, \Gamma_1\}$ we consider again problem (1.1), except that Eqn. (1.1d) on $\Gamma_0 \times (0, T] \equiv \Sigma_0$ is now replaced by

$$v_{tt} - \gamma\Delta v_{tt} + \Delta^2 v - z_t = \frac{\partial\delta_\beta}{\partial\nu}\, u(t) \quad \text{on } \Sigma_0,\ \gamma > 0. \tag{4.1}$$

This model was considered in [6]. The curve β satisfies some geometric assumptions [6]: namely, β in Γ_0 is the union of a finite number of curves β_i, $i = 1, \dots, m$, each of which, in turn, obeys one of the following conditions: (H.1) there exist $n_i \geq 1$ such that β_i is of class C^{n_i+3} and the curvature of β vanishes at most at one end of β_i where the tangent line has contact of order n_i; (H.2) β_i is a segment.

Paper [6] estimates the Fourier transform of the distribution $\frac{\partial\beta_\beta}{\partial\nu}$, as to fall into the Laplace-Fourier setting of [18], in particular, in order to invoke the basic estimate in [18, Lemma 1.1]. This way [6] proves that the Kirchoff problem on Γ_0, $\dim\Gamma_0 = 2$:

$$\begin{cases} \phi_{tt} - \gamma\Delta\phi_{tt} + \Delta^2\phi = \frac{\partial\delta_\beta}{\partial\nu} u(t) & \text{in } (0,T] \times \Gamma_0; & (4.2a) \\ \phi(0, \cdot) = 0, \quad \phi_t(0, \cdot) = 0 & \text{in } \Gamma_0, & (4.2b) \\ \text{either clamped B.C. on } \partial\Gamma_0 & & \\ \phi = \frac{\partial\phi}{\partial\nu} \equiv 0 & \text{in } (0,T] \times \partial\Gamma_0; & (4.2c_c) \\ \text{or else hinged B.C. on } \partial\Gamma_0, & & \\ \phi \equiv \Delta\phi \equiv 0 & \text{in } (0,T] \times \partial\Gamma_0 & (4.2c_h) \end{cases}$$

has exactly the *same regularity* as problem (3.1) on Γ_0, $\dim\Gamma_0 = 1$; i.e.,

$$u \in L_2(0,T) \Rightarrow \{\phi, \phi_t, \phi_{tt}\} \in C([0,T]; H^2(\Gamma_0) \times \Gamma^1(\Gamma_0)) \times L_2(0,T; L_2(\Gamma_0)) \tag{4.3}$$

[6, Theorem 2.1]. Instead, the regularity of the wave problem (3.4) given in Theorem 3.2, and then specialized in Corollary 3.3, holds true for $\dim\Omega \geq 2$. We conclude that *Theorem* 1.1 *and Theorem* 1.2 *hold true also for problem* (1.1) *on* $\{\Omega, \Gamma_0, \Gamma_1\}$, $\dim\Omega = 3$, $\dim\Gamma_0 = 2$, *where Eqn.* (1.1*d*) *is replaced by Eqn.* (4.1).

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