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A LOCAL PARTIAL REGULARITY THEOREM FOR WEAK SOLUTIONS OF DEGENERATE ELLIPTIC EQUATIONS AND ITS APPLICATION TO THE THERMISTOR PROBLEM

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Abstract. A partial regularity theorem is established for weak solutions of elliptic equations of the form $\operatorname{div}(A(y)\nabla\psi) = 0$. Here we allow the possibility that the eigenvalues of $A(y)$ are not bounded away from 0 below. This result is then used to prove an everywhere regularity theorem for weak solutions of the initial- boundary-value problem for the system $\frac{\partial u}{\partial t} - \Delta u = \sigma(u)|\nabla\varphi|^2$, $\operatorname{div}(\sigma(u)\nabla\varphi) = 0$ in the case where σ may decay exponentially.

1. Introduction. Let Ω be a bounded domain in $\mathbf{R}^N, N \geq 2$, with Lipschitz boundary $\partial\Omega$ and let $A(y) = (a_{ij}(y))$ be a measurable, symmetric, nonnegative matrix defined on Ω, with smallest eigenvalue $\lambda(y)$, and biggest eigenvalue $\Lambda(y) \leq c\lambda(y)$, where $c \geq 1$ and $\lambda(y)$ is a measurable function on Ω. Consider the boundary value problem

$$-\operatorname{div}(A(y)\nabla\psi) = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$\psi = \psi_0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where ψ_0 is a given function in $W^{1,2}(\Omega)$. A function ψ is said to be a weak solution of (1.1)–(1.2) if

(i) $\psi \in W^{1,p}(\Omega)$ for some $p \geq 1$;
(ii) $\psi - \psi_0 \in W_0^{1,1}(\Omega), \lambda(y)|\nabla\psi|^2 \in L^1(\Omega)$; and
(iii) (1.1) is satisfied in the sense of distributions.

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Theorem A. *Let $A(y), \lambda(y)$ be given as above. Assume*

(H1) $\lambda(y) \in L^\infty(\Omega)$, $\frac{1}{\lambda(y)} \in L^k(\Omega)$ *for some* $k > N/2$.

Then there exists a weak solution ψ to (1.1)–(1.2) *with the following regularity properties*

(P1) $\psi \in L^\infty_{\text{loc}}(\Omega)$;

(P2) *If $x \in \Omega$ is such that*

$$\limsup_{R\to 0^+} \fint_{B(x,R)} \big(\frac{1}{\lambda(y)}\big)^k \, dy < \infty, \tag{1.3}$$

then

$$\lim_{R\to 0^+} osc_{B(x,R)}\psi = 0,$$

where $osc_{B(x,R)} = ess\ sup_{B(x,R)}\psi - ess\ inf_{B(x,R)}\psi$.

That is, ψ is continuous at x. Since (1.3) *holds for almost every x in Ω, one has that ψ is continuous almost everywhere on Ω.*

The weak solution in Theorem A is regular off a singular set whose Lebesgue measure is zero. Thus this is a partial regularity result. The problem of studying extensions of everywhere regularity results to degenerate elliptic equations was first considered in the late 60's and early 70's (see [9, 10, 16, 17]). The conditions these authors found for the weight λ, in order to have local Hölder continuity, is that $(\fint_B(\lambda(y))^t dy)^{\frac{1}{t}}(\fint_B(\lambda(y))^{-s}dy)^{\frac{1}{s}} \le c$ for all balls B in Ω, where $c > 0, \frac{1}{t} + \frac{1}{s} < \frac{2}{N}$. In [5] this condition was weakened to the following

$$(\fint_B \lambda(y)d)(\fint_B (\lambda(y))^{-1}dy) \le c$$

for all balls B in Ω, i.e., λ is an A_2-weight in Ω. We do not believe that (H1) implies that λ is an A_2-weight, even though we have not been able to construct a counter example.

Our interest in Theorem A arose from the study of the following initial boundary value problem

$$\frac{\partial u}{\partial t} - \Delta u = \sigma(u)|\nabla\varphi|^2 \quad \text{in } \Omega_T \equiv \Omega \times (0,T), \tag{1.4}$$

$$\operatorname{div}(\sigma(u)\nabla\varphi) = 0 \quad \text{in } \Omega_T, \tag{1.5}$$

$$u(x,t) = u_0(x,t) \quad \text{on } \partial_p\Omega_T, \tag{1.6}$$

$$\varphi(x,t) = \varphi_0(x,t) \quad \text{on } S_T \equiv \partial\Omega \times (0,T). \tag{1.7}$$

Here, $T > 0$, $\partial_p \Omega_T$ is the parabolic boundary of Ω_T, and $u_0(x,t), \varphi_0(x,t)$, $\sigma(u)$ are known functions of their arguments, satisfying the following conditions:

(H2) $\sigma \in C(\mathbf{R})$ is such that $0 < \sigma(s) \le M$ on $\mathbf{R}$ for some $M > 0$.

(H3) $u_0 \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; W^{1,2}(\Omega))$, $\frac{\partial u_0}{\partial t} \in L^2(0,T; W^{-1,2}(\Omega))$, and $\varphi_0 \in L^2(0,T; W^{1,2}(\Omega)) \cap L^\infty(\Omega_T)$.

This problem is often called the thermistor problem and it can be proposed as a model for the electrical heating of a conductor (see [15, 1]). Then u is the temperature of the conductor, and φ the electrical potential. The first equation describes the diffusion of heat, while the second equation represents the conservation of electrical charges. The function $\sigma(u)$ is the electrical conductivity. Its precise form is determined by the particular physical application one has in mind. See, e.g., [3] for various forms suggested for σ in industrial applications.

Concerning the weak formulation of the initial boundary value problem, we remark that several weak notions of solutions have been developed to study the system (1.4)–(1.5) (see [19, 20, 23]). Observe that under (H2)–(H3) the following definition is meaningful.

Definition. The pair (u, φ) is a classical weak solution of the initial boundary value problem on Ω_T if the following conditions are satisfied:

1. (Integrability hypotheses) $u \in C([0,T]; L^2(\Omega)) \cap L^2\big(0,T; W^{1,2}(\Omega)\big)$, $\varphi \in L^2(0,T; W^{1,2}(\Omega)) \cap L^\infty(\Omega_T)$.
2. (Equations) u, φ satisfy

$$\frac{\partial u}{\partial t} - \Delta u = \operatorname{div}(\varphi \sigma(u) \nabla \varphi) \quad \text{in } \Omega_T, \tag{1.8}$$

$$\operatorname{div}(\sigma(u) \nabla \varphi) = 0 \quad \text{in } \Omega_T \tag{1.9}$$

   in the sense of distributions.
3. (Initial boundary conditions) $u - u_0 \in L^2\big(0,T; W_0^{1,2}(\Omega)\big)$, $u(x,0) = u_0(x,0)$ in $L^2(\Omega)$, and $\varphi - \varphi_0 \in L^2\big(0,T; W_0^{1,2}(\Omega)\big)$.

Note that our definition shows that

$$u - u_0 \in W_2(0,T) \equiv \{v \in L^2\big(0,T; W_0^{1,2}(\Omega)\big) : \frac{\partial v}{\partial t} \in L^2\big(0,T; W^{-1,2}(\Omega)\big)\}.$$

Of course, $W_2(0,T) \subset C([0,T]; L^2(\Omega))$, and thus the initial condition in (3) makes sense. Also note that

$$\operatorname{div}(\sigma(u) \varphi \nabla \varphi) = \sigma(u) |\nabla \varphi|^2 \quad \text{in } \Omega_T \tag{1.10}$$

in the weak sense.

For simplicity, we shall further assume that

$$u_0 \geq 0 \quad \text{on } \Omega_T. \tag{1.11}$$

Consequently, $u \geq 0$ on Ω_T. We are ready to state our second result.

Theorem B. *Let* (H2), (H3) *and* (1.11) *be satisfied. Assume:*

(H4) $\lim_{s\to\infty} \sigma(s) = 0$.

(H5) $\lim_{\tau\to 0} \frac{\sigma(s+\tau)}{\sigma(s)} = 1$ *uniformly on* $\mathbf{R}$.

Then problem (1.4)–(1.7) *has a classical weak solution* (u, φ) *in* Ω_T. *In addition,*

(P3) *u is locally Hölder continuous in Ω_T.*

In many electrical devices, too high a temperature can cause very undesirable side effects. Thus it is interesting to know under what conditions on σ we can obtain a bounded temperature. Once one knows that u is bounded one can derive high regularity for the solution via a bootstrap argument [23]. The difficulty in establishing the boundedness of u lies in the fact that the system is quadratically nonlinear and degenerate. The first regularity result is obtained in [1, 25] where it is assumed that σ is at least Hölder continuous and satisfies $m_1 \leq \sigma \leq M_1$ on $\mathbf{R}$ for some $0 < m_1 \leq M_1$. That is, the second equation in the system is uniformly elliptic in the spatial variables. However, in many physical applications [18, 3], one has that $\sigma(s) \to 0$ as $s \to \infty$. This complicates the mathematical analysis of the problem a great deal. Recently, we [21] obtained the Hölder continuity of u under the assumption

$$c_1 e^{-\beta|s|} \leq \sigma(s) \leq c_2 e^{-\beta|s|} \quad \text{on } \mathbf{R} \tag{1.12}$$

for some $c_1, c_2, \beta > 0$. The key observation in this case is that $\sigma(u)$ is an A_2-weight, and hence the results of [5, 8] are applicable. Clearly, (H4) and (H5) are more general than (1.12). In [23, 24], the author began to study the partial regularity theory of system (1.4)–(1.5), giving a description of the set of possible singularities. Let us call a point $z = (x, t)$ singular if u is not L^∞_{loc} in any neighborhood of z; the remaining points, where u is locally essentially bounded, will be called regular points. The main result of [24] is that if the assumptions of Theorem B are satisfied with (H4) and (H5) being replaced by

$$M \geq \sigma(s) \geq \frac{1}{c} e^{-\beta|s|} \quad \text{on } (-\infty, \infty)$$

for some $M, c > 0$, and $\frac{2}{N\|\sigma\varphi_0^2\|_{\infty,\Omega_T}} > \beta > 0$, then there exists a suitable weak solution (u, φ) to (1.4)–(1.7) such that the parabolic Hausdorff dimension of the set where u may blow up is at most N. A different partial regularity result is presented in [23]. We refer the reader to [2, 22] for results on the corresponding stationary problem.

Clearly, Theorem B improves the main result in [23]. The strategy here is to first show that the partial regularity theorem in [24] still holds under (H4)–(H5). Then we further establish that the singular set in that theorem is actually empty. This is where Theorem A plays a crucial role.

Finally, we make some remarks about the notation. The letter c is used to denote the generic constant. Furthermore, if $r > 0, z = (x, t) \in \mathbf{R}^N \times (0, \infty)$, and u, φ are locally integrable, then

$$Q(z, r) = \{(y, \tau) : |y - x| < r, t - r^2 < \tau < t\}, \quad u_{z,r} = \fint_{Q(z,r)} u\, dy\, d\tau,$$

$$\varphi_{x,r}(\tau) = \fint_{B(x,r)} \varphi(y, \tau)\, dy, \quad B(x, r) = \{y : |y - x| < r\}.$$

When the notation we use is standard, no explanation is given.

2. Proof of Theorem A.

The proof of Theorem A relies on the following two lemmas whose proofs are modifications of some of the arguments in [7].

Lemma 2.1. *Let the assumptions of Theorem A be satisfied. Assume that* $\psi \in W^{1,2}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ *is a subsolution of* (1.1) *in* Ω, *i.e.,*

$$-div(A(y)\nabla\psi) \le 0 \quad \text{in } \Omega \tag{2.1}$$

in the sense of distributions. Then for any $x \in \Omega, R > 0$ *with* $B(x, R) \subset \Omega$ *and* $p > 1$, *we have*

$$\sup_{B(x,R/2)} \psi \le c\Big(\fint_{B(x,R)} \Big(\frac{1}{\lambda(y)}\Big)^k dy\Big)^{\frac{N}{(2k-N)p}} \Big(\fint_{B(x,R)} (\psi^+)^p dy\Big)^{\frac{1}{p}}, \tag{2.2}$$

where $c = c(N, p, k, \|\lambda(y)\|_{\infty,\Omega})$.

Proof. Fix an x in Ω. For any $0 < R_1 < R_2 \le \operatorname{dist}(x, \partial\Omega) \equiv$ the distance between x and $\partial\Omega$, we can select a function $\xi \in C_0^\infty(B(x, R_2))$ with the following properties:

$$\xi = 1 \quad \text{on } B(x, R_1),$$
$$|\nabla\xi| = c/(R_2 - R_1) \quad \text{on } B(x, R_2),$$
$$\xi \ge 0 \quad \text{on } B(x, R_2).$$

Define, for $n > 0, \epsilon > 0$, the test function $(\psi^+ + \epsilon)^n \xi^2$. Use it in (2.1) to obtain

$$\begin{aligned} & n \int_{B(x,R_2)} \lambda(y)(\psi^+ + \epsilon)^{n-1} |\nabla \psi^+|^2 \xi^2 \, dy \\ \le\; & n \int_{B(x,R_2)} A(y) \nabla \psi (\psi^+ + \epsilon)^{n-1} \nabla \psi^+ \xi^2 dy \\ =\; & - \int_{B(x,R_2)} A(y) \nabla \psi (\psi^+ + \epsilon)^n 2\xi \nabla \xi \, dy. \end{aligned} \tag{2.3}$$

Here, we used the fact that $A(y)\nabla\psi \cdot \nabla\psi^+ = A(y)\nabla\psi^+ \cdot \nabla\psi^+$. Now send ϵ to 0 to deduce

$$\begin{aligned} n \int_{B(x,R_2)} \lambda(y)(\psi^+)^{n-1} |\nabla \psi^+|^2 \xi^2 \, dy &\le - \int_{B(x,R_2)} A(y) \nabla \psi (\psi^+)^n 2\xi \nabla \xi \, dy \\ &= - \int_{B(x,R_2)} A(y) \nabla \psi^+ (\psi^+)^n 2\xi \nabla \xi \, dy. \end{aligned}$$

An application of Hölder's inequality yields

$$\int_{B(x,R_2)} \lambda(y)(\psi^+)^{n-1} |\nabla \psi^+|^2 \xi^2 dy \le \frac{c}{n^2} \int_{B(x,R_2)} (\psi^+)^{n+1} |\nabla \xi|^2 dy. \tag{2.4}$$

Set $q = \frac{2k}{1+k}$. Since $k > N/2$, we have $\frac{2N}{N+2} < q < 2$. Consequently, q^*, the Sobolev conjugate of q, $= \frac{Nq}{N-q} = \frac{2Nk}{Nk+N-2k} > 2$. By the Sobolev inequality,

$$\begin{aligned} \Big(\int_{B(x,R_2)} |(\psi^+)^{\frac{n+1}{2}} \xi|^{q^*} dy \Big)^{\frac{1}{q^*}} &\le c \Big(\int_{B(x,R_2)} |\nabla \big((\psi^+)^{\frac{n+1}{2}} \xi \big)|^q dy \Big)^{\frac{1}{q}} \\ &\le c \Big(\int_{B(x,R_2)} \Big(\frac{1}{\lambda(y)} \Big)^{q/2} \lambda(y)^{q/2} |\nabla \big((\psi^+)^{\frac{n+1}{2}} \xi \big)|^q dy \Big)^{\frac{1}{q}} \\ &\le c \Big(\int_{B(x,R_2)} \Big(\frac{1}{\lambda(y)} \Big)^{\frac{q}{2-q}} dy \Big)^{\frac{2-q}{2q}} \cdot \Big(\int_{B(x,R_2)} \lambda(y) |\nabla \big((\psi^+)^{\frac{n+1}{2}} \xi \big)|^2 dy \Big)^{\frac{1}{2}}. \end{aligned} \tag{2.5}$$

Set

$$D(R_2) = \Big(\int_{B(x,R_2)} \Big(\frac{1}{\lambda(y)} \Big)^k dy \Big)^{\frac{1}{k}}.$$

Recall (2.4) to obtain

$$\Big(\int_{B(x,R_1)}(\psi^+)^{\frac{(n+1)q^*}{2}}dy\Big)^{\frac{2}{q^*(n+1)}} \tag{2.6}$$
$$\leq \Big(\frac{cD(R_2)}{R_2-R_1}\big[\big(\tfrac{n+1}{n}\big)^2+1\big]\Big)^{\frac{1}{n+1}}\Big(\int_{B(x,R_2)}(\psi^+)^{n+1}dy\Big)^{\frac{1}{n+1}}.$$

Now set $\chi = q^*/2 > 1$. Taking $p > 1, R \in (0, \mathrm{dist}(x, \partial\Omega))$, we let $n+1 = \chi^m p, R_2 = r_m \equiv (1+2^{-m})\frac{R}{2}, R_1 = r_{m+1}, m = 0, 1, \cdots$, so that, by inequality (2.6),

$$\|\psi^+\|_{\chi^{m+1}p,B(x,r_{m+1})} \leq \Big(\frac{cD(r_m)}{R^2}\Big)^{\frac{1}{\chi^m p}}4^{\frac{m}{\chi^m p}}\|\psi^+\|_{\chi^m p,B(x,r_m)}$$
$$\leq \Big(\frac{cD(R)}{R^2}\Big)^{\frac{1}{p}(1+\frac{1}{\chi}+\cdots+\frac{1}{\chi^m})}4^{\frac{1}{p}(\frac{1}{\chi}+\cdots+\frac{m}{\chi^m})}\|\psi^+\|_{p,B(x,R)}. \tag{2.7}$$

Consequently, taking $m \to \infty$ yields

$$\|\psi^+\|_{\infty,B(x,R/2)} \leq c\Big(\frac{D(R)}{R^2}\Big)^{\frac{\chi}{p(\chi-1)}}\|\psi^+\|_{p,B(x,R)}. \tag{2.8}$$

We complete the proof by recalling the definitions of $D(R), \chi$.

Lemma 2.2. *Let the assumptions of Theorem A be satisfied and $\psi \in W^{1,2}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ be a weak solution of (1.1) on Ω. Then for any $x \in \Omega, R > 0$ such that $B(x, 5R) \subset \Omega$, we have*

$$\Big(1 - \frac{1}{2e^{c\big(\fint_{B(x,4R)}\big(\frac{1}{\lambda(y)}\big)^k dy\big)^{\frac{2+N}{2(2k-N)}}}}\Big)(M(5R) - m(5R)) \geq M(2R) - m(2R), \tag{2.9}$$

where $c = c(N, k, \|\lambda(y)\|_{\infty,\Omega}), M(r) = ess\ sup_{B(x,r)}\psi, m(r) = ess\ inf_{B(x,r)}\psi$ for $r > 0$.

Proof. Let $x \in \Omega, 0 < R < \frac{1}{5}\mathrm{dist}(x, \partial\Omega)$ be given. Consider

$$w_1 = \ln\frac{M(5R) - m(5R)}{2(M(5R) - \psi) + \epsilon}, \qquad w_2 = \ln\frac{M(5R) - m(5R)}{2(\psi - m(5R)) + \epsilon}$$

on $B(x, 5R)$, where $\epsilon > 0$. Clearly, $w_1, w_2 \in W^{1,2}_{\mathrm{loc}}(B(x, 5R)) \cap L^\infty_{\mathrm{loc}}(B(x, 5R))$. It is easy to verify that

$$-\mathrm{div}(A(y)\nabla w_1) \leq 0 \quad \text{on } B(x, 5R), \tag{2.10}$$
$$-\mathrm{div}(A(y)\nabla w_2) \leq 0 \quad \text{on } B(x, 5R). \tag{2.11}$$

Assume that

$$|\{y\in B(x,4R):\psi(y)\le (M(5R)+m(5R))/2\}|\ge \tfrac{1}{2}|B(x,4R)|.$$

(Otherwise we have that $|\{y\in B(x,4R):\psi(y)\ge (M(5R)+m(5R))/2\}|\ge \frac{1}{2}|B(x,4R)|$. In this case all subsequent arguments will be carried out with the function w_2.) Set $S=\{y\in B(x,4R):{w_1}^+=0\}$. Then we have

$$\begin{aligned}|S| &= |\{y\in B(x,4R):w_1\le 0\}|\\ &= |\{y\in B(x,4R):\psi(y)\le (M(5R)+m(5R))/2+\epsilon/2\}| \qquad (2.12)\\ &\ge |\{y\in B(x,4R):\psi(y)\le (M(5R)+m(5R))/2\}|\ge \tfrac{1}{2}|B(x,4R)|.\end{aligned}$$

In light of Poincaré's inequality in [7, p.164], we can derive

$$\Big(\int_{B(x,4R)}|{w_1}^+|^{\frac{2k}{1+k}}dy\Big)^{\frac{1+k}{2k}}\le cR\Big(\int_{B(x,4R)}|\nabla {w_1}^+|^{\frac{2k}{1+k}}dy\Big)^{\frac{1+k}{2k}}, \qquad (2.13)$$

where $c=c(N)$. Now let $\Phi=2(M(5R)-\psi)+\epsilon$. Obviously,

$$\mathrm{div}\big(A(y)\nabla\Phi\big)=0 \quad \text{in } B(x,5R). \qquad (2.14)$$

Pick a function $\xi\in C_0^\infty(B(x,5R))$ so that

$$\begin{aligned}\xi&=1 && \text{on } B(x,4R),\\ |\nabla\xi|&\le c/R && \text{on } B(x,5R),\\ \xi&\ge 0 && \text{on } B(x,5R).\end{aligned}$$

Note that $w_1=\ln(M(5R)-m(5R))-\ln\Phi$. We can use $\frac{1}{\Phi}\xi^2$ as a test function in (2.14) to get

$$\int_{B(x,4R)}\lambda(y)|\nabla w_1|^2dy\le cR^{N-2}. \qquad (2.15)$$

We estimate, with the aid of (2.15), that

$$\begin{aligned}&\Big(⨍_{B(x,4R)}|\nabla {w_1}^+|^{\frac{2k}{1+k}}dy\Big)^{\frac{1+k}{2k}}\\ &\le \Big(⨍_{B(x,4R)}\Big(\frac{1}{\lambda(y)}\Big)^{\frac{k}{1+k}}\lambda(y)^{\frac{k}{1+k}}|\nabla {w_1}^+|^{\frac{2k}{1+k}}dy\Big)^{\frac{1+k}{2k}} \qquad (2.16)\\ &\le \Big(⨍_{B(x,4R)}\Big(\frac{1}{\lambda(y)}\Big)^{k}dy\Big)^{\frac{1}{2k}}\Big(⨍_{B(x,4R)}\lambda(y)|\nabla {w_1}^+|^2dy\Big)^{\frac{1}{2}}\\ &\le \frac{c}{R}\Big(⨍_{B(x,4R)}\Big(\frac{1}{\lambda(y)}\Big)^{k}dy\Big)^{\frac{1}{2k}}.\end{aligned}$$

Appealing to Lemma 2.1 and keeping in mind (2.13) and (2.16), we derive

$$\begin{aligned}\text{ess sup}_{B(x,2R)} w_1 &= \text{ess sup}_{B(x,2R)} \ln \frac{M(5R)-m(5R)}{2(M(5R)-\psi)+\epsilon} \\ &\le c\Big(\fint_{B(x,4R)} \Big(\frac{1}{\lambda(y)}\Big)^k dy\Big)^{\frac{N(1+k)}{(2k-N)2k}} \Big(\fint_{B(x,4R)} |w_1{}^+|^{\frac{2k}{1+k}} dy\Big)^{\frac{1+k}{2k}} \\ &\le c\Big(\fint_{B(x,4R)} \Big(\frac{1}{\lambda(y)}\Big)^k dy\Big)^{\frac{N+2}{2(2k-N)}}.\end{aligned} \tag{2.17}$$

Send ϵ to 0 to get

$$\psi(y) \le M(5R) - \frac{1}{2e^{c\left(\fint_{B(x,4R)}\left(\frac{1}{\lambda(y)}\right)^k dy\right)^{\frac{2+N}{2(2k-N)}}}}(M(5R)-m(5R)) \text{ on } B(x,2R). \tag{2.18}$$

Subtracting $m(2R)$ from both sides of the inequality yields the desired result. The proof is complete.

Proof of Theorem A. Denote by I the $N \times N$ identity matrix. For each $m \in \{1, 2, \cdots\}$, consider

$$-\text{div}(A_m(y)\nabla\psi_m) = 0 \quad \text{in } \Omega, \tag{2.19}$$

$$\psi_m = \psi_0 \quad \text{on } \partial\Omega, \tag{2.20}$$

where $A_m(y) = A(y) + \frac{1}{m}I$. It is well-known that for each m the problem (2.19)–(2.20) has a unique solution in the space $W^{1,2}(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$. Use $\psi_m - \psi_0$ as a test function in (2.19) to get

$$\int_\Omega (\lambda(y) + 1/m)|\nabla\psi_m|^2 dy \le c\int_\Omega (\lambda(y)+1/m)|\nabla\psi_0|^2 dy \le c. \tag{2.21}$$

We are ready to estimate

$$\begin{aligned}\int_\Omega |\nabla\psi_m|^{\frac{2k}{1+k}} dy &= \int_\Omega \Big(\frac{1}{\lambda(y)}\Big)^{\frac{k}{1+k}} \lambda(y)^{\frac{k}{1+k}} |\nabla\psi_m|^{\frac{2k}{1+k}} dy \\ &\le \Big(\int_\Omega \Big(\frac{1}{\lambda(y)}\Big)^k dy\Big)^{\frac{1}{k+1}} \Big(\int_\Omega \lambda(y)|\nabla\psi_m|^2 dy\Big)^{\frac{k}{k+1}} \le c.\end{aligned} \tag{2.22}$$

Thus $\{\psi_m\}$ is bounded in $W^{1,\frac{2k}{1+k}}(\Omega)$. We can extract a subsequence of $\{\psi_m\}$, still denoted by $\{\psi_m\}$, so that

$$\psi_m \to \ \psi \text{ weakly in } W^{1,\frac{2k}{1+k}}(\Omega), \tag{2.23}$$

$$\psi_m \to \psi \text{ strongly in } L^{\frac{2k}{1+k}}(\Omega) \text{ and a.e. on } \Omega. \tag{2.24}$$

We can infer from Lemma 2.1 that

$$\begin{aligned} &\|\psi_m\|_{\infty,B(x,R/2)} \\ &\leq c\Big(\fint_{B(x,R)}\big(\frac{1}{\lambda(y)+1/m}\big)^k dy\Big)^{\frac{N(1+k)}{2k(2k-N)}}\Big(\fint_{B(x,R)}|\psi_m|^{\frac{2k}{1+k}}dy\Big)^{\frac{1+k}{2k}} \end{aligned} \tag{2.25}$$

for any $x \in \Omega, R > 0$ with $B(x,R) \subset \Omega$. This immediately implies that $\psi \in L^\infty_{\text{loc}}(\Omega)$. Now let $x \in \Omega$ be such that

$$\limsup_{R\to 0^+}\fint_{B(x,R)}\big(\frac{1}{\lambda(y)}\big)^k dy < \infty. \tag{2.26}$$

Then

$$G(x) \equiv \Big(\sup_{0<R\leq \text{dist}(x,\partial\Omega)}\fint_{B(x,R)}\big(\frac{1}{\lambda(y)}\big)^k dy\Big)^{\frac{2+N}{2(2k-N)}}$$

is finite. By virtue of Lemma 2.2 , we obtain

$$\text{osc}_{B(x,2R)}\psi_m \leq \Big(1 - \frac{1}{2e^{cG(x)}}\Big)\text{osc}_{B(x,5R)}\psi_m \tag{2.27}$$

for $R > 0$ such that $B(x,5R) \subset \Omega$. Invoking Lemma 8.23 in [7, p.201], we arrive at

$$\text{osc}_{B(x,R)}\psi_m \leq c\big(\frac{R}{R_0}\big)^\alpha \tag{2.28}$$

for all $0 < R \leq R_0 \equiv \text{dist}(x,\partial\Omega)$, where c, α are two positive numbers depending only on the term $cG(x)$ in (2.27). Passing to the limit in (2.28) gives

$$\text{osc}_{B(x,R)}\psi \leq c\big(\frac{R}{R_0}\big)^\alpha. \tag{2.29}$$

Hence, ψ is continuous at x. It is an elementary exercise to show that ψ is a weak solution of (1.1)–(1.2). This concludes the proof of Theorem A.

3. Proof of Theorem B. Before we prove Theorem B, we collect a few preliminary results.

The following simple lemma is useful in our later development.

Lemma 3.1. *For $x, y \in \mathbf{R}^N$, define $P(x,y) = a|x|^2 + bx \cdot y + c|y|^2$, where a, b, c are real numbers with $a \geq 0, c \geq 0$. Then $P(x,y) \geq 0$ for all $x, y \in \mathbf{R}^N$ if and only if $b^2 - 4ac \leq 0$.*

The proof of this lemma is elementary, and we shall omit it here.

Next we cite a lemma from [23].

Lemma 3.2. *Let* (H2) *and* (H3) *be satisfied and (u, φ) be a classical weak solution of* (1.4)-(1.7). *Then there exists a positive constant $c = c(M, \|\varphi\|_{\infty,\Omega_T})$ such that*

$$\int_{Q(z,r)} \sigma(u)|\nabla\varphi|^2 dy \leq cr^N \tag{3.1}$$

for all $z = (x,t) \in \Omega_T, 0 < r \leq (1/2)dist_p(z, \partial_p\Omega_T)$. Here $dist_p(z, \partial_p\Omega_T)$ denotes the parabolic distance between z and $\partial_p\Omega_T$.

The following theorem is essential to the proof of Theorem B.

Theorem 3.3. *Let* (H2) *and* (H3) *be satisfied and (u, φ) be a classical weak solution of* (1.4)–(1.7) *with $u \in L^\infty_{\text{loc}}(\Omega_T)$. Assume that*

$$\liminf_{s\to\infty} \frac{1}{\sigma(s)\|\varphi^2\|_{\infty,\Omega_T}} \equiv \beta > 0. \tag{3.2}$$

Then for all $z \in \Omega_T, 0 < r \leq (1/2)dist_p(z, \partial_p\Omega_T)$, $0 < \alpha < \beta$, we have

$$\sup_{t-(1/4)r^2 \leq \tau \leq t} \int_{B(x,(1/2)r)} e^{\alpha|u|} dy \leq cr^N e^{c \fint_{Q(z,r)} udyds},$$

where $c = c(N, \beta, M, \|\varphi\|_{\infty,\Omega_T})$.

This theorem can be viewed as a local version of a result in [3].

Proof. Let $z \in \Omega_T, 0 < r \leq (1/2)\text{dist}_p(z, \partial_p\Omega_T)$ be given. Consider the problem

$$\frac{\partial v}{\partial \tau} - \Delta v = \text{div}\big(\sigma(u)\nabla\varphi\varphi\big) \quad \text{in } Q(z,r), \tag{3.3}$$

$$v = 0 \quad \text{on } \partial_p Q(z,r). \tag{3.4}$$

Clearly, this problem has a unique classical weak solution v in the space

$$W_2(t-r^2, t) \equiv \{w \in L^2(t-r^2, t; W_0^{1,2}(B(x,r)) : \\ w_\tau \in L^2(t-r^2, t; W^{-1,2}(B(x,r))\}.$$

Remember that $u \geq 0$ a.e. on Ω_T and that the right hand side of (3.3) is nonnegative in the sense of distributions. By the weak comparison principle, we have

$$0 \leq v \leq u \qquad \text{a.e. on } Q(z,r). \tag{3.5}$$

Choose $f \in C^1(\mathbf{R})$ so that

$$f > 0, f' > 0 \qquad \text{on } \mathbf{R}. \tag{3.6}$$

For each $K > 0$, $(f(v) - f(K))^+$ is a legitimate test function, and upon utilizing it in (3.3), we obtain, with the aid of the chain rule, that

$$\begin{aligned} &\int_{B(x,r)\times\{\tau\}} \Big(\int_0^v (f(s) - f(K))^+ ds\Big) dy \\ &+ \int_{B(x,r)\times(t-r^2,\tau)\cap\{v\geq K\}} (f'(v)|\nabla v|^2 + \sigma(u)\varphi\nabla\varphi f'(v)\nabla v) dy ds = 0 \end{aligned} \tag{3.7}$$

for $t - r^2 \leq \tau \leq t$. On the other hand, use $(f(v) - f(K))^+\varphi$ as a test function in (1.5) to get

$$\begin{aligned} &\int_{B(x,r)\times(t-r^2,\tau)\cap\{v\geq K\}} (f(v)\sigma(u)|\nabla\varphi|^2 + \sigma(u)\varphi\nabla\varphi f'(v)\nabla v) dy\, ds \\ &= f(K)\int_{B(x,r)\times(t-r^2,\tau)\cap\{v\geq K\}} \sigma(u)|\nabla\varphi|^2 dy\, ds \leq cf(K)r^N. \end{aligned} \tag{3.8}$$

The last step is due to (3.1). Adding (3.8) to (3.7) yields

$$\begin{aligned} &\int_{B(x,t)\times\{\tau\}} \int_0^v (f(s) - f(K))^+ ds\, dy + \int_{B(x,r)\times(t-r^2,\tau)\cap\{v\geq K\}} (f'(v)|\nabla(v)|^2 \\ &\quad + 2\sigma(u)f'(v)\varphi\nabla\varphi\nabla v + \sigma(u)f(v)|\nabla\varphi|^2)\, dy\, ds \leq cf(K)r^N. \end{aligned} \tag{3.9}$$

According to Lemma 3.1, the integrand in the second integral in (3.9) is nonnegative if f is so chosen that

$$4(\sigma(u)f'(v)\varphi)^2 - 4\sigma(u)f(v)f'(v) \leq 0 \text{ on } B(x,r) \times (t - r^2, \tau) \cap \{v \geq K\}. \tag{3.10}$$

Set $L = \|\varphi^2\|_{\infty,\Omega_T}$. Then (3.10) is an easy consequence of the inequality

$$\frac{f'(v)}{f(v)} \leq \frac{1}{\sigma(u)L} \qquad \text{on } B(x,r) \times (t - r^2, \tau) \cap \{v \geq K\}. \tag{3.11}$$

We take $f(s) = e^{\alpha s}$, where $\alpha \in (0, \beta)$. Clearly, $f(s)$ satisfies (3.6). Since

$$\liminf_{s\to\infty} \frac{1}{\sigma(s)L} > \alpha, \tag{3.12}$$

we can find $K > 0$ so that

$$\frac{1}{\sigma(u)L} > \alpha \tag{3.13}$$

on $B(x,r) \times (t - r^2, \tau) \cap \{u \geq K\} \supset B(x,r) \times (t - r^2, \tau) \cap \{v \geq K\}$. The last step is due to (3.5). Fix such a K. We conclude from (3.9) that

$$\sup_{t-r^2\leq\tau\leq t} \int_{B(x,r)} \int_0^v (e^{\alpha s} - e^{\alpha K})^+ ds dy \leq c e^{\alpha K} r^N. \tag{3.14}$$

Consequently,

$$\sup_{t-r^2\leq\tau\leq t} \int_{B(x,r)} e^{\alpha v} dy \leq c r^N + e^{\alpha K} \sup_{t-r^2\leq\tau\leq t} \int_{B(x,r)} v dy. \tag{3.15}$$

For $\epsilon > 0$, define

$$\theta_\epsilon(s) = \begin{cases} 1 & \text{if } s > \epsilon \\ s/\epsilon & \text{if } |s| \leq \epsilon \\ -1 & \text{if } s < -\epsilon. \end{cases}$$

Using $\theta_\epsilon(v)$ as a test function in (3.3), we derive that

$$\begin{aligned} &\sup_{t-r^2\leq\tau\leq t} \int_{B(x,r)} \int_0^v \theta_\epsilon(s) ds\, dy + \int_{Q(z,r)} \theta_\epsilon'(v) |\nabla v|^2 dy\, ds \\ &\leq 2 \int_{Q(z,r)} \sigma(u) |\nabla\varphi|^2 \theta_\epsilon(v) dy ds \leq 2 \int_{Q(z,r)} \sigma(u) |\nabla\varphi|^2 dy\, ds. \end{aligned} \tag{3.16}$$

Dropping the second integral in (3.16) and then sending ϵ to 0, we get

$$\sup_{t-r^2\leq\tau\leq t} \int_{B(x,r)} |v| dy \leq 2 \int_{Q(z,r)} \sigma(u) |\nabla\varphi|^2 dy\, ds. \tag{3.17}$$

It follows from (3.15), (3.17) and (3.1) that

$$\sup_{t-r^2\leq\tau\leq t} \int_{B(x,r)} e^{\alpha v} dy \leq c r^N. \tag{3.18}$$

We easily see that $w \equiv u - v$ satisfies

$$\frac{\partial w}{\partial \tau} - \Delta w = 0 \quad \text{in } Q(z,r), \tag{3.19}$$

$$w = u \quad \text{on } \partial_p Q(z,r). \tag{3.20}$$

Thus, by Lemma 1 in [13],

$$\sup_{Q(z,(1/2)r)}|w| \le c \fint_{Q(z,r)} |w| dy\, ds \le c \fint_{Q(z,r)} u\, dy\, ds. \tag{3.21}$$

We estimate

$$\begin{aligned} \sup_{t-(1/4)r^2 \le \tau \le t} \int_{B(x,(1/2)r)} e^{\alpha|u|}\, dy &\le e^{\alpha \sup_{Q(z,1/2r)}|w|} \sup_{t-r^2\le\tau\le t} \int_{B(x,r)} e^{\alpha v}\, dy \\ &\le c r^N e^{c \fint_{Q(z,r)} u\, dy\, ds}. \end{aligned} \tag{3.22}$$

This completes the proof.

Lemma 3.4. *Assume that σ satisfies* (H5)*. Then:*

(a) *For each $K > 0$ there exist $0 < m_K \le M_K$ with $m_K \le \frac{\sigma(s+\tau)}{\sigma(s)} \le M_K$ for all $s \in (-\infty, \infty)$, $\tau \in [-K, K]$.*
(b) *$\sigma(s) \ge \sigma(0) m_1 b^s$ for all $s \ge 0$, where $b = m_1 < 1$.*

Proof. (a) is proved in [20]. To see (b), for each $s \ge 0$ there is a nonnegative integer m such that $m \le s < m+1$. Thus, we may write $s = m + s_0$, where $s_0 = s - m \in [0,1)$. We conclude from (a) that

$$\sigma(s) = \sigma(m+s_0) \ge m_1 \sigma(m) \ge m_1^2 \sigma(m-1) \ge \cdots \ge m_1^{m+1}\sigma(0).$$

Note that $m_1 \le 1, m \le s$. We derive $\sigma(s) \ge \sigma(0) m_1 m_1^s$. This completes the proof.

Once we have established Theorem A and Theorem 3.3, we are ready to prove Theorem B.

Proof of Theorem B. We consider a sequence of approximate problems.

$$\frac{\partial u_n}{\partial t} - \Delta u_n = \sigma_n(u_n)\left|\nabla \varphi_n\right|^2 \quad \text{in } \Omega_T, \tag{3.23}$$

$$\operatorname{div}(\sigma_n(u_n)\nabla\varphi_n) = 0 \quad \text{in } \Omega_T, \tag{3.24}$$

$$u_n = u_0 \quad \text{on } \partial_p \Omega_T, \tag{3.25}$$

$$\varphi_n = \varphi_0 \quad \text{on } S_T\ (n = 1, 2, \dots), \tag{3.26}$$

where $\sigma_n(s) = \sigma(s) + \frac{1}{n}$. Then by virtue of results in [20, 23] for each fixed n, there exists a classical weak solution (u_n, φ_n) to (3.23)–(3.26) with $u_n \in L^\infty_{\text{loc}}(\Omega_T)$. Furthermore, we have

$$\|\varphi_n\|_{\infty,\Omega_T} \le c, \qquad \int_{\Omega_T} |\nabla\varphi_n|^2 dx\,dt \le c,$$
$$\sup_{[0,T]} \int_\Omega u_n^2(x,t)dx + \int_{\Omega_T} |\nabla u_n|^2 dx\,dt \le c$$

uniformly in n. Again, by [20] we can conclude that there is a a subsequence of $\{u_n, \varphi_n\}$, still denoted by $\{u_n, \varphi_n\}$, so that

$$u_n \to u \text{ weakly in } L^2(0,T;W^{1,2}(\Omega)) \text{ and strongly in } L^2(\Omega_T), \tag{3.27}$$
$$\varphi_n \to \varphi \text{ weak}^* \text{ in } L^\infty(\Omega_T), \text{ weakly in } L^2(0,T;W^{1,2}(\Omega)) \text{ and a.e. on } \Omega_T \tag{3.28}$$

and that (u, φ) is a classical weak solution of (1.4)–(1.7). The main result in [23] asserts that if $z = (x,t) \in \Omega_T$ is such that $\limsup_{r\to 0^+} \fint_{Q(z,r)} udyds$ is sufficiently large then u is Hölder continuous in a neighborhood of z. Thus it is enough to establish the Hölder continuity of u in neighborhoods of those z's for which

$$\limsup_{r\to 0^+} \fint_{Q(z,r)} u\,dy\,ds < \infty. \tag{3.29}$$

To this end, observe that $\lim_{s\to\infty} \sigma(s) = 0$ and $\sigma_n(s) = \sigma(s) + 1/n$. We can infer from the proof of Theorem 3.3 that for all $z = (x,t) \in \Omega_T, 0 < r \le (1/2)\text{dist}_p(z, \partial_p\Omega_T)$, $0 < \alpha$, we have

$$\sup_{t-(1/4)r^2 \le \tau \le t} \int_{B(x,(1/2)r)} e^{\alpha|u|} dy \le cr^N e^{c\fint_{Q(z,r)} udy\,ds}, \tag{3.30}$$

where $c = c(N, \alpha, M, \|\varphi\|_{\infty,\Omega_T})$. It follows from Lemma 3.4 that

$$\sigma(s) \ge ce^{-\gamma s} \text{ on } [0,\infty) \tag{3.31}$$

for some $c, \gamma > 0$. This combined with (3.30) shows that for any $k > N/2$ there exists a positive number c such that

$$\sup_{t-(1/4)r^2 \le \tau \le t} \int_{B(x,(1/2)r)} \Big(\frac{1}{\sigma(u)}\Big)^k dy \le cr^N e^{c\fint_{Q(z,r)} udy\,ds} \tag{3.32}$$

all $z \in \Omega_T, 0 < r \leq (1/2)\text{dist}_p(z, \partial_p\Omega_T)$. This enables us to utilize the proof in [24] to conclude that u is Hölder continuous in a neighborhood of z if both

$$\limsup_{r\to 0} \sup_{t-r^2\leq\tau\leq t} \fint_{B(x,r)} u^2(y,\tau)dy < \infty \tag{3.33}$$

and

$$\liminf_{r\to 0} \sup_{t-r^2\leq\tau\leq t} \fint_{B(x,r)} (u - u_{x,r}(\tau))^2 dy = 0. \tag{3.34}$$

Now fix $z = (x,t) \in \Omega_T$ so that (3.29) is satisfied. Then by virtue of (3.32),

$$P(z) \equiv \text{ess sup}_{0<r<\text{dist}_p(z,\partial_p\Omega_T)}\Big(\sup_{t-(1/4)r^2\leq\tau\leq t} \int_{B(x,(1/2)r)} \big(\frac{1}{\sigma(u)}\big)^k dy\Big)^{\frac{N+2}{2(2k-N)}} \tag{3.35}$$

is finite. For $0 < r < \text{dist}_p(z, \partial_p\Omega_T)$ define

$$\omega(z,r) = \text{ess sup}_{t-r^2\leq\tau\leq t}\big(\text{ess sup}_{B(x,r)}\varphi(y,\tau) - \text{ess inf}_{B(x,r)}\varphi(y,\tau)\big).$$

Note that $\varphi(\cdot,\tau) \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ for a.e. $\tau \in (0,T)$. We also have

$$\text{div}(\sigma(u)\nabla\varphi) = 0 \quad \text{ on } \Omega$$

for a.e. $\tau \in (0,T)$. Thus we are in a position to apply Lemma 2.2, thereby obtaining

$$\omega(z,2r) \leq (1 - \frac{1}{2e^{cP(z)}})\omega(z,5r) \tag{3.36}$$

for all $r > 0$ such that $Q(z,5r) \subset \Omega_T$. Therefore,

$$\omega(z,r) \leq cr^\gamma \tag{3.37}$$

for some $c, \gamma > 0$ depending only on the $cP(z)$ in (3.36) and $\text{dist}_p(z, \partial_p\Omega_T)$. Once this is established, we can employ the proof of Theorem 7 in [23] to obtain

$$\fint_{Q(z,r)} (u - u_{z,r})^2 dy\, ds \leq cr^\varepsilon \tag{3.38}$$

for some $c, \varepsilon > 0$ independent of r. In light of Theorem 2.1 in [24], there exists a positive constant $c = c(M, \|\varphi\|_{\infty,\Omega_T})$ such that

$$\begin{aligned} &\sup_{t-r^2\le\tau\le t}\int_{B(x,r)}\big(u-u_{z,2r}\big)^2dy+\int_{Q(z,r)}\big|\nabla u\big|^2dyd\tau \\ &\quad\le\frac{c}{r^2}\Big(\int_{Q(z,2r)}\big(u-u_{z,2r}\big)^2dyd\tau+\int_{Q(z,2r)}\big(\varphi-\varphi_{x,2r}(\tau)\big)^2dyd\tau\Big) \end{aligned} \tag{3.39}$$

for all $z = (x,t) \in \Omega_T, 0 < r \le (1/2)\text{dist}_p(z, \partial_p\Omega_T)$. This, along with (3.38) and (3.37), implies (3.34), while (3.33) follows from (3.29) and (3.30). The proof is complete.

REFERENCES

[1] S.N. Antontsev and M. Chipot, *The thermistor problem: existence, smoothness, uniqueness, and blow up,* SIAM J. Math. Anal., 25 (1994), 1128–1156.

[2] X. Chen, *Existence and regularity of solutions of a nonlinear degenerate elliptic system arising from a thermistor problem,* J. Partial Differential Equations, 7 (1994), 19–34.

[3] X. Chen and H.M. Yin, *On a strongly coupled elliptic-parabolic system,* preprint.

[4] E. Dibenedetto, "Degenerate Parabolic Equations," Springer-Verlag, New York, 1993.

[5] E.B. Fabes, C.E. Kenig, and R.P. Serapioni, *The local regularity of solutions of degenerate elliptic equations,* Comm. Partial Differential Equations, 7 (1982), 77–116.

[6] M. Gianquinta, "Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems," Princeton Univ. Press, Princeton, NJ, 1983.

[7] D. Gilbarg and N.S. Trudinger, "Elliptic Partial Differential Equations of Second Order," Springer-Verlag, 2nd Edition, New York, 1983.

[8] P.J. Holden, *Extension domains for A_p weights,* Michigan Math. J., 39 (1992), 353–368.

[9] S.N. Kruzkov, *Certain properties of solutions to elliptic equations,* Soviet Math., 4 (1963), 686–695.

[10] D. Kinderlehrer and G. Stapmpacchia, "An Introduction to Variational Inequalities and their Applications," Academic Press, New York, 1980.

[11] J.L. Lewis, *On very weak solutions of certain elliptic systems,* Comm. Partial Differential Equations, 18 (1993), 1515–1537.

[12] O.A. Ladyzenskaja, V.A. Solonnikov, and N.N. Uralceva, "Linear and Quasilinear Equations of Parabolic Type," AMS, Providence, RI, 1968.

[13] J. Moser, *On a pointwise estimate for parabolic differential equations,* Comm. Pure Appl. Math., XXIV (1971), 727–740.

[14] M.K.V. Murthy and G. Stampacchia, *Boundary value problems for some degenerate elliptic operators,* Ann. Mat. Pura Appl., 80 (1968), 1–122.

[15] P. Shi, M. Shillor, and X. Xu, *Existence of a solution to the Stefan problem with Joule's heating,* J. Differential Equations, 105 (1993), 239–263.

[16] N. Trudinger, *On the regularity of generalized solutions of linear, non-uniformly elliptic equations,* Arch. Rat. Mech. Anal., 42 (1971), 51–62.

[17] N. Trudinger, *Linear elliptic operators with measurable coefficients,* Ann. Sc. Norm. Sup. Pisa, 27 (1973), 265–308.

[18] H. Xie and W. Allegretto, $C^\alpha(\overline{\Omega})$*-solution of a class of nonlinear degenerate systems arising in the thermister problem,* SIAM J. Math. Anal., 22 (1991), 1491–1499.

[19] X. Xu, *A strongly degenerate system involving an equation of parabolic type and an equation of elliptic type,* Comm. Partial Differential Equations, 18 (1993), 199–213.

[20] X. Xu, *A p-Laplacian problem in* L^1 *with nonlinear boundary conditions,* Comm. Partial Differential Equations, 19 (1994), 143–176.

[21] X. Xu, *Local and global existence of continuous temperature in the electrical heating of conductors,* Houston J. Math., 22 (1996), 435–455.

[22] X. Xu, *Existence and regularity theorems for the thermistor problem with degeneracy,* preprint, 1996.

[23] X. Xu, *Partial regularity of solutions to a class of degenerate systems,* Trans. Amer. Math. Soc., 349 (1997), 1973–1992.

[24] X. Xu, *Local partial regularity theorems for suitable weak solutions of a class of degenerate systems,* Appl. Math. & Optim., 34 (1996), 299–324.

[25] G. Yuan, *Regularity of solutions of the thermistor problem,* Applicable Anal., 53 (1994), 149–164.

[26] G. Yuan and Z. Liu, *Existence and uniqueness of the* C^α*-solution for the thermistor problem with mixed boundary values,* SIAM J. Math. Anal., 25 (1994), 1157–1166.