A LOCAL PARTIAL REGULARITY THEOREM FOR WEAK SOLUTIONS OF DEGENERATE ELLIPTIC EQUATIONS AND ITS APPLICATION TO THE THERMISTOR PROBLEM

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Abstract. A partial regularity theorem is established for weak solutions of elliptic equations of the form \( \text{div}(A(y)\nabla \psi) = 0 \). Here we allow the possibility that the eigenvalues of \( A(y) \) are not bounded away from 0 below. This result is then used to prove an everywhere regularity theorem for weak solutions of the initial-boundary-value problem for the system \( \frac{\partial u}{\partial t} - \Delta u = \sigma(u)|\nabla \varphi|^2 \), \( \text{div}(\sigma(u)\nabla \varphi) = 0 \) in the case where \( \sigma \) may decay exponentially.

1. Introduction. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \), \( N \geq 2 \), with Lipschitz boundary \( \partial \Omega \) and let \( A(y) = (a_{ij}(y)) \) be a measurable, symmetric, nonnegative matrix defined on \( \Omega \), with smallest eigenvalue \( \lambda(y) \), and biggest eigenvalue \( \Lambda(y) \leq c\lambda(y) \), where \( c \geq 1 \) and \( \lambda(y) \) is a measurable function on \( \Omega \). Consider the boundary value problem

\[
-\text{div}(A(y)\nabla \psi) = 0 \quad \text{in } \Omega, \\
\psi = \psi_0 \quad \text{on } \partial \Omega,
\]

where \( \psi_0 \) is a given function in \( W^{1,2}(\Omega) \). A function \( \psi \) is said to be a weak solution of (1.1)–(1.2) if

(i) \( \psi \in W^{1,p}(\Omega) \) for some \( p \geq 1 \);
(ii) \( \psi - \psi_0 \in W^{1,1}_0(\Omega) \), \( \lambda(y)|\nabla \psi|^2 \in L^1(\Omega) \); and
(iii) (1.1) is satisfied in the sense of distributions.

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**Theorem A.** Let \( A(y), \lambda(y) \) be given as above. Assume

(H1) \( \lambda(y) \in L^\infty(\Omega), \frac{1}{\lambda(y)} \in L^k(\Omega) \) for some \( k > N/2 \).

Then there exists a weak solution \( \psi \) to (1.1)–(1.2) with the following regularity properties

(P1) \( \psi \in L^\infty_{loc}(\Omega) \);

(P2) If \( x \in \Omega \) is such that

\[
\limsup_{R \to 0^+} \int_{B(x,R)} \left( \frac{1}{\lambda(y)} \right)^k \, dy < \infty, 
\]

then

\[
\lim_{R \to 0^+} \sup_{B(x,R)} \psi = 0,
\]

where \( \sup_{B(x,R)} \psi - \inf_{B(x,R)} \psi \).

That is, \( \psi \) is continuous at \( x \). Since (1.3) holds for almost every \( x \) in \( \Omega \), one has that \( \psi \) is continuous almost everywhere on \( \Omega \).

The weak solution in Theorem A is regular off a singular set whose Lebesgue measure is zero. Thus this is a partial regularity result. The problem of studying extensions of everywhere regularity results to degenerate elliptic equations was first considered in the late 60’s and early 70’s (see [9, 10, 16, 17]). The conditions these authors found for the weight \( \lambda \), in order to have local Hölder continuity, is that \( \left( \int_B (\lambda(y))^{\frac{1}{k}} \, dy \right)^{\frac{1}{k}} \left( \int_B (\lambda(y))^{-s} \, dy \right)^{\frac{1}{s}} \leq c \) for all balls \( B \) in \( \Omega \), where \( c > 0, \frac{1}{k} + \frac{s}{k} < \frac{2}{N} \). In [5] this condition was weakened to the following

\[
\left( \int_B \lambda(y) \, dy \right) \left( \int_B (\lambda(y))^{-1} \, dy \right) \right) \leq c
\]

for all balls \( B \) in \( \Omega \), i.e., \( \lambda \) is an \( A_2 \)-weight in \( \Omega \). We do not believe that (H1) implies that \( \lambda \) is an \( A_2 \)-weight, even though we have not been able to construct a counter example.

Our interest in Theorem A arose from the study of the following initial boundary value problem

\[
\frac{\partial u}{\partial t} - \Delta u = \sigma(u)|\nabla \varphi|^2 \quad \text{in } \Omega_T \equiv \Omega \times (0,T), 
\]

\[
\text{div}(\sigma(u)\nabla \varphi) = 0 \quad \text{in } \Omega_T, 
\]

\[
u(x,t) = u_0(x,t) \quad \text{on } \partial \Omega_T, 
\]

\[
\varphi(x,t) = \varphi_0(x,t) \quad \text{on } S_T \equiv \partial \Omega \times (0,T).
\]
Here, $T > 0$, $\partial_p \Omega_T$ is the parabolic boundary of $\Omega_T$, and $u_0(x,t), \varphi_0(x,t)$, $\sigma(u)$ are known functions of their arguments, satisfying the following conditions:

\begin{enumerate}[(H2)]
    
    \item $\sigma \in C(\mathbb{R})$ is such that $0 < \sigma(s) \leq M$ on $\mathbb{R}$ for some $M > 0$.
    \item $u_0 \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; W^{1,2}(\Omega)), \frac{\partial u_0}{\partial t} \in L^2(0, T; W^{-1,2}(\Omega))$, and $\varphi_0 \in L^2(0, T; W^{1,2}(\Omega)) \cap L^\infty(\Omega_T)$.
\end{enumerate}

This problem is often called the thermistor problem and it can be proposed as a model for the electrical heating of a conductor (see [15, 1]). Then $u$ is the temperature of the conductor, and $\varphi$ the electrical potential. The first equation describes the diffusion of heat, while the second equation represents the conservation of electrical charges. The function $\sigma(u)$ is the electrical conductivity. Its precise form is determined by the particular physical application one has in mind. See, e.g., [3] for various forms suggested for $\sigma$ in industrial applications.

Concerning the weak formulation of the initial boundary value problem, we remark that several weak notions of solutions have been developed to study the system (1.4)–(1.5) (see [19, 20, 23]). Observe that under (H2)–(H3) the following definition is meaningful.

**Definition.** The pair $(u, \varphi)$ is a classical weak solution of the initial boundary value problem on $\Omega_T$ if the following conditions are satisfied:

1. (Integrability hypotheses) $u \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; W^{1,2}(\Omega)), \varphi \in L^2(0, T; W^{1,2}(\Omega)) \cap L^\infty(\Omega_T)$.
2. (Equations) $u, \varphi$ satisfy
   \begin{align}
   \frac{\partial u}{\partial t} - \Delta u &= \text{div}(\varphi \sigma(u) \nabla \varphi) \quad \text{in } \Omega_T, \tag{1.8} \\
   \text{div}(\sigma(u) \nabla \varphi) &= 0 \quad \text{in } \Omega_T \tag{1.9}
   \end{align}
   in the sense of distributions.
3. (Initial boundary conditions) $u - u_0 \in L^2(0, T; W^{1,2}_0(\Omega))$, $u(x, 0) = u_0(x, 0)$ in $L^2(\Omega)$, and $\varphi - \varphi_0 \in L^2(0, T; W^{1,2}_0(\Omega))$.

Note that our definition shows that

$$u - u_0 \in W_2(0, T) \equiv \{v \in L^2(0, T; W^{1,2}_0(\Omega)) : \frac{\partial v}{\partial t} \in L^2(0, T; W^{-1,2}(\Omega))\}.$$

Of course, $W_2(0, T) \subset C([0, T]; L^2(\Omega))$, and thus the initial condition in (3) makes sense. Also note that

$$\text{div}(\sigma(u) \varphi \nabla \varphi) = \sigma(u) |\nabla \varphi|^2 \quad \text{in } \Omega_T \tag{1.10}$$
in the weak sense.

For simplicity, we shall further assume that

\[ u_0 \geq 0 \quad \text{on } \Omega_T. \]  

(1.11)

Consequently, \( u \geq 0 \) on \( \Omega_T \). We are ready to state our second result.

**Theorem B.** Let (H2), (H3) and (1.11) be satisfied. Assume:

(H4) \( \lim_{s \to -\infty} \sigma(s) = 0. \)

(H5) \( \lim_{r \to 0} \frac{\sigma(s r)}{\sigma(s)} = 1 \) uniformly on \( \mathbb{R} \).

Then problem (1.4)–(1.7) has a classical weak solution \((u, \varphi)\) in \( \Omega_T \). In addition,

(P3) \( u \) is locally Hölder continuous in \( \Omega_T \).

In many electrical devices, too high a temperature can cause very undesirable side effects. Thus it is interesting to know under what conditions on \( \sigma \) we can obtain a bounded temperature. Once one knows that \( u \) is bounded, one can derive high regularity for the solution via a bootstrap argument [23]. The difficulty in establishing the boundedness of \( u \) lies in the fact that the system is quadratically nonlinear and degenerate. The first regularity result is obtained in [1, 25] where it is assumed that \( \sigma \) is at least Hölder continuous and satisfies \( m_1 \leq \sigma \leq M_1 \) on \( \mathbb{R} \) for some \( 0 < m_1 \leq M_1 \). That is, the second equation in the system is uniformly elliptic in the spatial variables. However, in many physical applications [18, 3], one has that \( \sigma(s) \to 0 \) as \( s \to \infty \). This complicates the mathematical analysis of the problem a great deal. Recently, we [21] obtained the Hölder continuity of \( u \) under the assumption

\[ c_1 e^{-\beta|s|} \leq \sigma(s) \leq c_2 e^{-\beta|s|} \quad \text{on } \mathbb{R} \]  

(1.12)

for some \( c_1, c_2, \beta > 0 \). The key observation in this case is that \( \sigma(u) \) is an \( A_2 \)-weight, and hence the results of [5, 8] are applicable. Clearly, (H4) and (H5) are more general than (1.12). In [23, 24], the author began to study the partial regularity theory of system (1.4)–(1.5), giving a description of the set of possible singularities. Let us call a point \( z = (x, t) \) singular if \( u \) is not \( L^\infty_{\text{loc}} \) in any neighborhood of \( z \); the remaining points, where \( u \) is locally essentially bounded, will be called regular points. The main result of [24] is that if the assumptions of Theorem B are satisfied with (H4) and (H5) being replaced by

\[ M \geq \sigma(s) \geq \frac{1}{c} e^{-\beta|s|} \quad \text{on } (-\infty, \infty) \]
for some $M, c > 0$, and $\frac{2}{N\|\sigma \varphi_0\|_{\infty, \Omega}} > \beta > 0$, then there exists a suitable weak solution $(u, \varphi)$ to (1.4)–(1.7) such that the parabolic Hausdorff dimension of the set where $u$ may blow up is at most $N$. A different partial regularity result is presented in [23]. We refer the reader to [2, 22] for results on the corresponding stationary problem.

Clearly, Theorem B improves the main result in [23]. The strategy here is to first show that the partial regularity theorem in [24] still holds under (H4)–(H5). Then we further establish that the singular set in that theorem is actually empty. This is where Theorem A plays a crucial role.

Finally, we make some remarks about the notation. The letter $c$ is used to denote the generic constant. Furthermore, if $r > 0, z = (x, t) \in \mathbb{R}^N \times (0, \infty)$, and $u, \varphi$ are locally integrable, then

$$Q(z, r) = \{(y, \tau) : |y - x| < r, t - r^2 < \tau < t\}, \quad u_{z, r} = \int_{Q(z, r)} u \, dy \, d\tau,$$

$$\varphi_{x, r} = \int_{B(x, r)} \varphi(y, \tau) \, dy, \quad B(x, r) = \{y : |y - x| < r\}.$$

When the notation we use is standard, no explanation is given.

2. Proof of Theorem A. The proof of Theorem A relies on the following two lemmas whose proofs are modifications of some of the arguments in [7].

Lemma 2.1. Let the assumptions of Theorem A be satisfied. Assume that $\psi \in W^{1,2}_{\text{loc}}(\Omega) \cap L^\infty(\Omega)$ is a subsolution of (1.1) in $\Omega$, i.e.,

$$-\operatorname{div}(A(y)\nabla \psi) \leq 0 \quad \text{in } \Omega \quad (2.1)$$

in the sense of distributions. Then for any $x \in \Omega, R > 0$ with $B(x, R) \subset \Omega$ and $p > 1$, we have

$$\sup_{B(x, R/2)} \psi \leq c\left(\int_{B(x, R)} \left(\frac{1}{\lambda(y)}\right)^k \, dy\right)^{\frac{N}{(2k-N)p}} \left(\int_{B(x, R)} (\psi^+)^p \, dy\right)^{\frac{1}{p}}, \quad (2.2)$$

where $c = c(N, p, k, \|\lambda(y)\|_{\infty, \Omega})$.

Proof. Fix an $x$ in $\Omega$. For any $0 < R_1 < R_2 \leq \text{dist}(x, \partial \Omega) \equiv$ the distance between $x$ and $\partial \Omega$, we can select a function $\xi \in C^\infty_0(B(x, R_2))$ with the following properties:

$$\xi = 1 \quad \text{on } B(x, R_1),$$

$$|\nabla \xi| = c/(R_2 - R_1) \quad \text{on } B(x, R_2),$$

$$\xi \geq 0 \quad \text{on } B(x, R_2).$$
Define, for $n > 0, \epsilon > 0$, the test function $(\psi^+ + \epsilon)^n \xi^2$. Use it in (2.1) to obtain

\begin{align}
    n \int_{B(x,R_2)} \lambda(y)(\psi^+ + \epsilon)^{n-1}|\nabla \psi^+|^2 \xi^2 \, dy \\
    \leq n \int_{B(x,R_2)} A(y)\nabla\psi(\psi^+ + \epsilon)^{n-1} \nabla \psi^+ \xi^2 dy \\
    = -\int_{B(x,R_2)} A(y)\nabla \psi(\psi^+ + \epsilon)^n 2\xi \nabla \xi \, dy.
\end{align}

(2.3)

Here, we used the fact that $A(y)\nabla \psi \cdot \nabla \psi^+ = A(y)\nabla \psi^+ \cdot \nabla \psi^+$. Now send $\epsilon$ to 0 to deduce

\begin{align}
    n \int_{B(x,R_2)} \lambda(y)(\psi^+)^{n-1}|\nabla \psi^+|^2 \xi^2 \, dy &\leq -\int_{B(x,R_2)} A(y)\nabla \psi(\psi^+)^n 2\xi \nabla \xi \, dy \\
    &= -\int_{B(x,R_2)} A(y)\nabla \psi(\psi^+)^n 2\xi \nabla \xi \, dy.
\end{align}

An application of Hölder’s inequality yields

\begin{align}
    \int_{B(x,R_2)} \lambda(y)(\psi^+)^{n-1}|\nabla \psi^+|^2 \xi^2 \, dy &\leq \frac{c}{n^2} \int_{B(x,R_2)} (\psi^+)^{n+1} |\nabla \xi|^2 \, dy. \tag{2.4}
\end{align}

Set $q = \frac{2k}{1 + k}$. Since $k > N/2$, we have $\frac{2N}{N+2} < q < 2$. Consequently, $q^*$, the Sobolev conjugate of $q$, $= \frac{Nq}{N-q} = \frac{2Nk}{Nk+N-2k} > 2$. By the Sobolev inequality,

\begin{align}
    (\int_{B(x,R_2)} |(\psi^+)^{n+1} \xi|^{\frac{q}{2}} \, dy)^{\frac{2}{q}} &\leq c(\int_{B(x,R_2)} |\nabla ((\psi^+)^{n+1} \xi)|^q \, dy)^{\frac{1}{q}} \\
    &\leq c(\int_{B(x,R_2)} \left(\frac{1}{\lambda(y)}\right)^{q/2} \lambda(y)^{q/2} |\nabla ((\psi^+)^{n+1} \xi)|^q \, dy)^{\frac{1}{q}} \tag{2.5} \\
    &\leq c(\int_{B(x,R_2)} \left(\frac{1}{\lambda(y)}\right)^{\frac{q}{2}} \lambda(y)^{\frac{q}{2}} \cdot (\int_{B(x,R_2)} \lambda(y) |\nabla ((\psi^+)^{n+1} \xi)|^2 \, dy)^{\frac{1}{2}}. \\
    &\leq c(\int_{B(x,R_2)} \left(\frac{1}{\lambda(y)}\right)^k \, dy)^{\frac{1}{k}}.
\end{align}

Set

$$D(R_2) = \left(\int_{B(x,R_2)} \left(\frac{1}{\lambda(y)}\right)^k \, dy\right)^{\frac{1}{k}}.$$
Recall (2.4) to obtain
\[
\left( \int_{B(x, R_1)} (\psi^+)^{(n+1)\sigma^*} dy \right)^{\frac{2}{\sigma(n+1)}} \leq \left( \frac{cD(R_2)}{R_2 - R_1} \left[ (\frac{n+1}{n})^2 + 1 \right] \right)^{\frac{1}{n+1}} \left( \int_{B(x, R_2)} (\psi^+)^{n+1} dy \right)^{\frac{1}{n+1}}.
\]

(2.6)

Now set \( \chi = q^*/2 > 1 \). Taking \( p > 1, R \in (0, \text{dist}(x, \partial \Omega)) \), we let \( n + 1 = \chi^mp, R_2 = r_m \equiv (1 + 2^{-m})\frac{R}{2}, R_1 = r_{m+1}, m = 0, 1, \ldots \), so that, by inequality (2.6),
\[
\|\psi^+\|_{\chi^m+1, n, B(x, r_m+1)} \leq \left( \frac{cD(r_m)}{R^2} \right)^{\frac{1}{\chi^m+1}} 4^{\frac{m}{\chi^m+1}} \|\psi^+\|_{\chi^m, n, B(x, r_m)}
\]
\[
\leq \left( \frac{cD(R)}{R^2} \right)^{\frac{1}{\chi^m+1}} 4^{\frac{m}{\chi^m+1}} \|\psi^+\|_{\chi^m+1, n, B(x, R)}.
\]

(2.7)

Consequently, taking \( m \to \infty \) yields
\[
\|\psi^+\|_{\infty, B(x, R/2)} \leq c\left( \frac{D(R)}{R^2} \right)^{\frac{1}{\chi+1}} \|\psi^+\|_{\chi^m, B(x, R)}.
\]

(2.8)

We complete the proof by recalling the definitions of \( D(R), \chi \).

**Lemma 2.2.** Let the assumptions of Theorem A be satisfied and \( \psi \in W^{1,2}_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega) \) be a weak solution of (1.1) on \( \Omega \). Then for any \( x \in \Omega, R > 0 \) such that \( B(x, 5R) \subset \Omega \), we have
\[
(1 - \frac{1}{2e^c(\frac{1}{n})^k dy})^2(M(5R) - m(5R)) \geq M(2R) - m(2R),
\]

(2.9)

where \( c = c(N, k, \|\lambda(y)\|_{\infty, \Omega}), M(r) = \text{ess sup}_{B(x, r)} \psi, m(r) = \text{ess inf}_{B(x, r)} \psi \) for \( r > 0 \).

**Proof.** Let \( x \in \Omega, 0 < R < \frac{1}{2} \text{dist}(x, \partial \Omega) \) be given. Consider
\[
w_1 = \ln \frac{M(5R) - m(5R)}{2(M(5R) - \psi)} + \epsilon, \quad w_2 = \ln \frac{M(5R) - m(5R)}{2(\psi - m(5R))} + \epsilon
\]
on \( B(x, 5R) \), where \( \epsilon > 0 \). Clearly, \( w_1, w_2 \in W^{1,2}_{\text{loc}}(B(x, 5R)) \cap L^\infty_{\text{loc}}(B(x, 5R)) \).

It is easy to verify that
\[
- \text{div}(A(y) \nabla w_1) \leq 0 \quad \text{on} \ B(x, 5R), \quad (2.10)
\]
\[
- \text{div}(A(y) \nabla w_2) \leq 0 \quad \text{on} \ B(x, 5R). \quad (2.11)
\]
Assume that
\[ |\{y \in B(x, 4R) : \psi(y) \leq (M(5R) + m(5R))/2\}| \geq \frac{1}{2}|B(x, 4R)|. \]
(Otherwise we have that \(|\{y \in B(x, 4R) : \psi(y) \geq (M(5R) + m(5R))/2\}| \geq \frac{1}{2}|B(x, 4R)|\). In this case all subsequent arguments will be carried out with the function \(w_2\). Set \(S = \{y \in B(x, 4R) : w_1^+ = 0\}\). Then we have
\[
|S| = |\{y \in B(x, 4R) : w_1 \leq 0\}|
\geq |\{y \in B(x, 4R) : \psi(y) \leq (M(5R) + m(5R))/2\}| \geq \frac{1}{2}|B(x, 4R)|.
\]
In light of Poincaré’s inequality in [7, p.164], we can derive
\[
\left( \int_{B(x,4R)} |w_1^+|^{\frac{2k}{1+k}} dy \right)^{\frac{1+k}{2k}} \leq cR \left( \int_{B(x,4R)} |\nabla w_1^+|^{\frac{2k}{1+k}} dy \right)^{\frac{1+k}{2k}},
\]
where \(c = c(N)\). Now let \(\Phi = 2(M(5R) - \psi) + \epsilon\). Obviously,
\[
div(A(y)\nabla \Phi) = 0 \quad \text{in} \quad B(x, 5R).
\]
Pick a function \(\xi \in C_0^\infty(B(x, 5R))\) so that
\[
\xi = 1 \quad \text{on} \quad B(x, 4R),
\]
\[|\nabla \xi| \leq c/R \quad \text{on} \quad B(x, 5R),
\]
\[\xi \geq 0 \quad \text{on} \quad B(x, 5R).
\]
Note that \(w_1 = \ln(M(5R) - m(5R)) - \ln \Phi\). We can use \(\frac{1}{\Phi} \xi^2\) as a test function in (2.14) to get
\[
\int_{B(x,4R)} \lambda(y)|\nabla w_1|^2 dy \leq cR^{N-2}.
\]
We estimate, with the aid of (2.15), that
\[
\left( \int_{B(x,4R)} |\nabla w_1^+|^{\frac{2k}{1+k}} dy \right)^{\frac{1+k}{2k}}
\leq \left( \int_{B(x,4R)} \left( \frac{1}{\lambda(y)} \right)^{\frac{k}{1+k}} \lambda(y)^{\frac{k}{1+k}} |\nabla w_1^+|^{\frac{2k}{1+k}} dy \right)^{\frac{1+k}{2k}}
\leq \left( \int_{B(x,4R)} \left( \frac{1}{\lambda(y)} \right)^k dy \right)^{\frac{2}{k}} \left( \int_{B(x,4R)} \lambda(y)|\nabla w_1^+|^2 dy \right)^{\frac{1}{2}}
\leq \frac{c}{R} \left( \int_{B(x,4R)} \left( \frac{1}{\lambda(y)} \right)^k dy \right)^{\frac{2}{k}}.
\]
Appealing to Lemma 2.1 and keeping in mind (2.13) and (2.16), we derive

\[
\text{ess sup}_{B(x,2R)} w_1 = \text{ess sup}_{B(x,2R)} \frac{M(5R) - m(5R)}{2(M(5R) - \psi)} + \epsilon \\
\leq c \left( \int_{B(x,4R)} \left( \frac{1}{\lambda(y)} \right)^k dy \right)^{\frac{N(1+k)}{2k(N-k)}} \left( \int_{B(x,4R)} |w_1|^\frac{2k}{\tau+k} dy \right)^{\frac{k}{2k}} \\
\leq c \left( \int_{B(x,4R)} \left( \frac{1}{\lambda(y)} \right)^k dy \right)^{\frac{N+2}{2k(N-k)}} 
\]

Send \( \epsilon \) to 0 to get

\[
\psi(y) \leq M(5R) - \frac{1}{2c} \left( \int_{B(x,4R)} \left( \frac{1}{\lambda(y)} \right)^k dy \right)^{\frac{2kN}{2k(N-k)}} (M(5R) - m(5R)) \text{ on } B(x,2R).
\]

Subtracting \( m(2R) \) from both sides of the inequality yields the desired result. The proof is complete.

**Proof of Theorem A.** Denote by \( I \) the \( N \times N \) identity matrix. For each \( m \in \{1, 2, \cdots\} \), consider

\[
\begin{align*}
-\text{div}(A_m(y) \nabla \psi_m) &= 0 \quad \text{in } \Omega, \\
\psi_m &= \psi_0 \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( A_m(y) = A(y) + \frac{1}{m} I \). It is well-known that for each \( m \) the problem (2.19)–(2.20) has a unique solution in the space \( W^{1,2}(\Omega) \cap L^\infty_{\text{loc}}(\Omega) \). Use \( \psi_m - \psi_0 \) as a test function in (2.19) to get

\[
\int_{\Omega} (\lambda(y) + 1/m) |\nabla \psi_m|^2 dy \leq c \int_{\Omega} (\lambda(y) + 1/m) |\nabla \psi_0|^2 dy \leq c.
\]

We are ready to estimate

\[
\int_{\Omega} |\nabla \psi_m|^\frac{2k}{\tau+k} dy = \int_{\Omega} \left( \frac{1}{\lambda(y)} \right)^{\frac{k}{\tau+k}} \lambda(y) |\nabla \psi_m|^\frac{2k}{\tau+k} dy \\
\leq \left( \int_{\Omega} \left( \frac{1}{\lambda(y)} \right)^k dy \right)^{\frac{1}{k+1}} \left( \int_{\Omega} \lambda(y) |\nabla \psi_m|^2 dy \right)^{\frac{k}{k+1}} \leq c.
\]
Thus \( \{\psi_m\} \) is bounded in \( W^{1,\frac{2k}{2+k}}(\Omega) \). We can extract a subsequence of \( \{\psi_m\} \), still denoted by \( \{\psi_m\} \), so that

\[
\psi_m \rightharpoonup \psi \text{ weakly in } W^{1,\frac{2k}{2+k}}(\Omega),
\]

\[
\psi_m \rightarrow \psi \text{ strongly in } L^{\frac{2k}{2+k}}(\Omega) \text{ and a.e. on } \Omega.
\]

We can infer from Lemma 2.1 that

\[
\|\psi_m\|_{\infty, B(x,R/2)} \leq c \left( \int_{B(x,R)} \left( \frac{1}{\lambda(y)} + 1/m \right)^{k} dy \right)^{\frac{N(1+k)}{2k(N-\xi)}} \left( \int_{B(x,R)} |\psi_m|^{\frac{2k}{1+k}} dy \right)^{\frac{1+k}{2k}}
\]

for any \( x \in \Omega, R > 0 \) with \( B(x,R) \subset \Omega \). This immediately implies that \( \psi \in L^{\infty}_{\text{loc}}(\Omega) \). Now let \( x \in \Omega \) be such that

\[
\limsup_{R \rightarrow 0^+} \int_{B(x,R)} \left( \frac{1}{\lambda(y)} \right)^k dy < \infty.
\]

Then

\[
G(x) \equiv (\sup_{0 < R \leq \text{dist}(x,\partial\Omega)} \int_{B(x,R)} \left( \frac{1}{\lambda(y)} \right)^k dy)^{\frac{2+N}{N(2k-N)}}
\]

is finite. By virtue of Lemma 2.2, we obtain

\[
\text{osc}_{B(x,2R)} \psi_m \leq \left( 1 - \frac{1}{2cG(x)} \right) \text{osc}_{B(x,5R)} \psi_m
\]

for \( R > 0 \) such that \( B(x,5R) \subset \Omega \). Invoking Lemma 8.23 in [7, p.201], we arrive at

\[
\text{osc}_{B(x,R)} \psi_m \leq c \left( \frac{R}{R_0} \right)^{\alpha}
\]

for all \( 0 < R \leq R_0 \equiv \text{dist}(x,\partial\Omega) \), where \( c, \alpha \) are two positive numbers depending only on the term \( cG(x) \) in (2.27). Passing to the limit in (2.28) gives

\[
\text{osc}_{B(x,R)} \psi \leq c \left( \frac{R}{R_0} \right)^{\alpha}.
\]

Hence, \( \psi \) is continuous at \( x \). It is an elementary exercise to show that \( \psi \) is a weak solution of (1.1)–(1.2). This concludes the proof of Theorem A.

3. Proof of Theorem B. Before we prove Theorem B, we collect a few preliminary results.

The following simple lemma is useful in our later development.
Lemma 3.1. For \( x, y \in \mathbb{R}^N \), define \( P(x, y) = a|x|^2 + bx \cdot y + c|y|^2 \), where \( a, b, c \) are real numbers with \( a \geq 0, c \geq 0 \). Then \( P(x, y) \geq 0 \) for all \( x, y \in \mathbb{R}^N \) if and only if \( b^2 - 4ac \leq 0 \).

The proof of this lemma is elementary, and we shall omit it here.

Next we cite a lemma from [23].

Lemma 3.2. Let (H2) and (H3) be satisfied and \((u, \varphi)\) be a classical weak solution of (1.4)–(1.7). Then there exists a positive constant \( c = c(M, \|\varphi\|_{\infty, \Omega_T}) \) such that

\[
\int_{Q(z,r)} \sigma(u)|\nabla \varphi|^2 \, dy \leq cr^N \tag{3.1}
\]

for all \( z = (x, t) \in \Omega_T, 0 < r \leq (1/2)\text{dist}_p(z, \partial_p \Omega_T) \). Here \( \text{dist}_p(z, \partial_p \Omega_T) \) denotes the parabolic distance between \( z \) and \( \partial_p \Omega_T \).

The following theorem is essential to the proof of Theorem B.

Theorem 3.3. Let (H2) and (H3) be satisfied and \((u, \varphi)\) be a classical weak solution of (1.4)–(1.7) with \( u \in L^\infty_{\text{loc}}(\Omega_T) \). Assume that

\[
\liminf_{s \to \infty} \frac{1}{\sigma(s)\|\varphi\|^2_{\infty, \Omega_T}} \equiv \beta > 0. \tag{3.2}
\]

Then for all \( z \in \Omega_T, 0 < r \leq (1/2)\text{dist}_p(z, \partial_p \Omega_T), 0 < \alpha < \beta \), we have

\[
\sup_{t-(1/4)r^2 \leq \tau \leq t} \int_{B(x,(1/2)r)} e^{\alpha|u|} \, dy \leq cr^N e^{c \int_{Q(z,r)} u \, dy \, ds},
\]

where \( c = c(N, \beta, M, \|\varphi\|_{\infty, \Omega_T}) \).

This theorem can be viewed as a local version of a result in [3].

Proof. Let \( z \in \Omega_T, 0 < r \leq (1/2)\text{dist}_p(z, \partial_p \Omega_T) \) be given. Consider the problem

\[
\frac{\partial v}{\partial r} - \Delta v = \text{div}(\sigma(u)\nabla \varphi) \quad \text{in} \ Q(z,r), \tag{3.3}
\]

\[
v = 0 \quad \text{on} \ \partial_p Q(z,r). \tag{3.4}
\]

Clearly, this problem has a unique classical weak solution \( v \) in the space

\[
W_2(t-r^2, t) \equiv \{ w \in L^2(t-r^2, t; W^{1,2}_0(B(x,r)) : w_r \in L^2(t-r^2, t; W^{-1,2}(B(x,r))) \}. 
\]
Remember that \( u \geq 0 \) a.e. on \( \Omega_T \) and that the right hand side of (3.3) is nonnegative in the sense of distributions. By the weak comparison principle, we have

\[
0 \leq v \leq u \quad \text{a.e. on } Q(z, r). \tag{3.5}
\]

Choose \( f \in C^1(\mathbb{R}) \) so that

\[
f > 0, f' > 0 \quad \text{on } \mathbb{R}. \tag{3.6}
\]

For each \( K > 0 \), \((f(v) - f(K))^+ \) is a legitimate test function, and upon utilizing it in (3.3), we obtain, with the aid of the chain rule, that

\[
\begin{align*}
&\int_{B(x,r) \times \{r-t^2, r\} \cap \{v \geq K\}} (f(s) - f(K))^+ dsdy \\
&+ \int_{B(x,r) \times \{r-t^2, r\} \cap \{v \geq K\}} (f'(v)|\nabla v|^2 + \sigma(u)\varphi\nabla \varphi f'(v)\nabla v)dyds = 0
\end{align*}
\tag{3.7}
\]

for \( t - r^2 \leq \tau \leq t \). On the other hand, use \((f(v) - f(K))^+ \varphi \) as a test function in (1.5) to get

\[
\begin{align*}
&\int_{B(x,r) \times \{r-t^2, r\} \cap \{v \geq K\}} (f(v)\sigma(u)|\nabla \varphi|^2 + \sigma(u)\varphi\nabla \varphi f'(v)\nabla v)dyds \\
&= f(K)\int_{B(x,r) \times \{r-t^2, r\} \cap \{v \geq K\}} \sigma(u)|\nabla \varphi|^2 dyds \leq cf(K)r^N.
\end{align*}
\tag{3.8}
\]

The last step is due to (3.1). Adding (3.8) to (3.7) yields

\[
\begin{align*}
&\int_{B(x,t) \times \{r\}} 0 (f(s) - f(K))^+ dsdy + \int_{B(x,r) \times \{r-t^2, r\} \cap \{v \geq K\}} (f'(v)|\nabla v|^2 \\
&+ 2\sigma(u)f'(v)\varphi\nabla \varphi \nabla v + \sigma(u)f(v)|\nabla \varphi|^2 dyds \leq cf(K)r^N.
\end{align*}
\tag{3.9}
\]

According to Lemma 3.1, the integrand in the second integral in (3.9) is nonnegative if \( f \) is so chosen that

\[
4(\sigma(u)f'(v)\varphi)^2 - 4\sigma(u)f(v)f'(v) \leq 0 \quad \text{on } B(x, r) \times (t - r^2, \tau) \cap \{v \geq K\}. \tag{3.10}
\]

Set \( L = \|\varphi^2\|_{\infty, \Omega_T} \). Then (3.10) is an easy consequence of the inequality

\[
\frac{f'(v)}{f(v)} \leq \frac{1}{\sigma(u)L} \quad \text{on } B(x, r) \times (t - r^2, \tau) \cap \{v \geq K\}. \tag{3.11}
\]
We take \( f(s) = e^{\alpha s} \), where \( \alpha \in (0, \beta) \). Clearly, \( f(s) \) satisfies (3.6). Since
\[
\liminf_{s \to \infty} \frac{1}{\sigma(s)L} > \alpha, \tag{3.12}
\]
we can find \( K > 0 \) so that
\[
\frac{1}{\sigma(u)L} > \alpha \tag{3.13}
\]
on \( B(x, r) \times (t - r^2, \tau) \cap \{ u \geq K \} \supset B(x, r) \times (t - r^2, \tau) \cap \{ v \geq K \} \). The last step is due to (3.5). Fix such a \( K \). We conclude from (3.9) that
\[
\sup_{t - r^2 \leq \tau \leq t} \int_{B(x, r)} \int_0^v (e^{\alpha s} - e^{\alpha K})^+ ds dy \leq ce^{\alpha K} r^N. \tag{3.14}
\]
Consequently,
\[
\sup_{t - r^2 \leq \tau \leq t} \int_{B(x, r)} e^{\alpha v} dy \leq cr^N + e^{\alpha K} \sup_{t - r^2 \leq \tau \leq t} \int_{B(x, r)} v dy. \tag{3.15}
\]
For \( \epsilon > 0 \), define
\[
\theta_\epsilon(s) = \begin{cases} 
1 & \text{if } s > \epsilon \\
\frac{s}{\epsilon} & \text{if } |s| \leq \epsilon \\
-1 & \text{if } s < -\epsilon.
\end{cases}
\]
Using \( \theta_\epsilon(v) \) as a test function in (3.3), we derive that
\[
\sup_{t - r^2 \leq \tau \leq t} \int_{B(x, r)} \int_0^v \theta_\epsilon(s) ds dy + \int_{Q(z, r)} \theta_\epsilon'(v) |\nabla v|^2 dy ds \\
\leq 2 \int_{Q(z, r)} \sigma(u) |\nabla \varphi|^2 \theta_\epsilon(v) dy ds \leq 2 \int_{Q(z, r)} \sigma(u) |\nabla \varphi|^2 dy ds. \tag{3.16}
\]
Dropping the second integral in (3.16) and then sending \( \epsilon \) to 0, we get
\[
\sup_{t - r^2 \leq \tau \leq t} \int_{B(x, r)} |v| dy \leq 2 \int_{Q(z, r)} \sigma(u) |\nabla \varphi|^2 dy ds. \tag{3.17}
\]
It follows from (3.15), (3.17) and (3.1) that
\[
\sup_{t - r^2 \leq \tau \leq t} \int_{B(x, r)} e^{\alpha v} dy \leq cr^N. \tag{3.18}
\]
We easily see that \( w \equiv u - v \) satisfies
\[
\frac{\partial w}{\partial t} - \Delta w = 0 \quad \text{in } Q(z, r),
\]
\[
w = u \quad \text{on } \partial_p Q(z, r).
\]
(3.19) (3.20)

Thus, by Lemma 1 in [13],
\[
\sup_{Q(z,(1/2)r)} |w| \leq c \int_{Q(z,r)} |w| dy \, ds \leq c \int_{Q(z,r)} u \, dy \, ds.
\]
(3.21)

We estimate
\[
\sup_{t-(1/4)r^2 \leq \tau \leq t} \int_{B(x,(1/2)r)} e^{\alpha |u|} \, dy \leq e^{\alpha \sup_{Q(z,1/2r)} |w|} \sup_{t-r^2 \leq \tau \leq t} \int_{B(x,r)} e^{\alpha u} \, dy \\
\leq cr^N e^{\int_{Q(x,r)} u \, dy \, ds}.
\]
(3.22)

This completes the proof.

**Lemma 3.4.** Assume that \( \sigma \) satisfies (H5). Then:

(a) For each \( K > 0 \) there exist \( 0 < m_K \leq M_K \) with \( m_K \leq \frac{\sigma(s+\tau)}{\sigma(s)} \leq M_K \) for all \( s \in (-\infty, \infty) \), \( \tau \in [-K, K] \).

(b) \( \sigma(s) \geq \sigma(0) m_1 b^s \) for all \( s \geq 0 \), where \( b = m_1 < 1 \).

**Proof.** (a) is proved in [20]. To see (b), for each \( s \geq 0 \) there is a nonnegative integer \( m \) such that \( m \leq s < m + 1 \). Thus, we may write \( s = m + s_0 \), where \( s_0 = s - m \in [0, 1) \). We conclude from (a) that
\[
\sigma(s) = \sigma(m + s_0) \geq m_1 \sigma(m) \geq m_1^2 \sigma(m - 1) \geq \cdots \geq m_1^m \sigma(0).
\]

Note that \( m_1 \leq 1, m \leq s \). We derive \( \sigma(s) \geq \sigma(0) m_1 m_1^s \). This completes the proof.

Once we have established Theorem A and Theorem 3.3, we are ready to prove Theorem B.

**Proof of Theorem B.** We consider a sequence of approximate problems.

\[
\frac{\partial u_n}{\partial t} - \Delta u_n = \sigma_n(u_n)|\nabla \varphi_n|^2 \quad \text{in } \Omega_T,
\]
(3.23)
\[
\text{div}(\sigma_n(u_n) \nabla \varphi_n) = 0 \quad \text{in } \Omega_T,
\]
(3.24)
\[
u_n = u_0 \quad \text{on } \partial_p \Omega_T,
\]
(3.25)
\[
\varphi_n = \varphi_0 \quad \text{on } S_T \ (n = 1, 2, \ldots),
\]
(3.26)
where $\sigma_n(s) = \sigma(s) + \frac{1}{n}$. Then by virtue of results in [20, 23] for each fixed $n$, there exists a classical weak solution $(u_n, \varphi_n)$ to (3.23)–(3.26) with $u_n \in L^\infty_{\text{loc}}(\Omega_T)$. Furthermore, we have
\[
\|\varphi_n\|_{\infty, \Omega_T} \leq c, \quad \int_{\Omega_T} |\nabla \varphi_n|^2 \, dx \, dt \leq c,
\]
\[
\sup_{[0,T]} \int_{\Omega} u_n^2(x,t) \, dx + \int_{\Omega_T} |\nabla u_n|^2 \, dx \, dt \leq c
\]
uniformly in $n$. Again, by [20] we can conclude that there is a a subsequence of $\{u_n, \varphi_n\}$, still denoted by $\{u_n, \varphi_n\}$, so that
\[
u_n \to u \text{ weakly in } L^2(0,T; W^{1,2}(\Omega)) \text{ and strongly in } L^2(\Omega_T),
\]
\[
\varphi_n \to \varphi \text{ weak}^* \text{ in } L^\infty(\Omega_T), \text{ weakly in } L^2(0,T; W^{1,2}(\Omega))
\]
and a.e. on $\Omega_T$
(3.28)
and that $(u, \varphi)$ is a classical weak solution of (1.4)–(1.7). The main result in [23] asserts that if $z = (x,t) \in \Omega_T$ is such that $\limsup_{r \to 0^+} \int_{Q(z,r)} u \, dy \, ds$ is sufficiently large then $u$ is Hölder continuous in a neighborhood of $z$. Thus it is enough to establish the Hölder continuity of $u$ in neighborhoods of those $z$'s for which
\[
\limsup_{r \to 0^+} \int_{Q(z,r)} u \, dy \, ds < \infty.
\]
(3.29)
To this end, observe that $\lim_{s \to \infty} \sigma(s) = 0$ and $\sigma_n(s) = \sigma(s) + 1/n$. We can infer from the proof of Theorem 3.3 that for all $z = (x,t) \in \Omega_T, 0 < r \leq \left(1/2\right)\text{dist}_P(z, \partial_p \Omega_T), 0 < \alpha$, we have
\[
\sup_{t-(1/4)r^2 \leq \tau \leq t} \int_{B(x,(1/2)r)} e^{\alpha|u|} \, dy \leq c r^N e^{c \|\varphi\|_{\infty, \Omega_T}} e^{ \int_{Q(z,r)} u \, dy \, ds},
\]
(3.30)
where $c = c(N, \alpha, M, \|\varphi\|_{\infty, \Omega_T})$. It follows from Lemma 3.4 that
\[
\sigma(s) \geq ce^{-\gamma s} \text{ on } [0, \infty)
\]
(3.31)
for some $c, \gamma > 0$. This combined with (3.30) shows that for any $k > N/2$ there exists a positive number $c$ such that
\[
\sup_{t-(1/4)r^2 \leq \tau \leq t} \int_{B(x,(1/2)r)} \left(\frac{1}{\sigma(u)}\right)^k \, dy \leq c r^N e^{c \|\varphi\|_{\infty, \Omega_T}} e^{ \int_{Q(z,r)} u \, dy \, ds}
\]
(3.32)
all \( z \in \Omega_T, 0 < r \leq (1/2) \text{dist}_p(z, \partial \Omega_T) \). This enables us to utilize the proof in [24] to conclude that \( u \) is Hölder continuous in a neighborhood of \( z \) if both

\[
\limsup_{r \to 0} \sup_{t-r^2 \leq \tau \leq t} \int_{B(x,r)} u^2(y, \tau) \, dy < \infty \tag{3.33}
\]

and

\[
\liminf_{r \to 0} \sup_{t-r^2 \leq \tau \leq t} \int_{B(x,r)} (u - u_{x,r}(\tau))^2 \, dy = 0. \tag{3.34}
\]

Now fix \( z = (x, t) \in \Omega_T \) so that (3.29) is satisfied. Then by virtue of (3.32),

\[
P(z) \equiv \text{ess sup}_{0 < r < \text{dist}_p(z, \partial \Omega_T)} \left( \sup_{t-(1/4)r^2 \leq \tau \leq t} \int_{B(x,(1/2)r)} \left( \frac{1}{\sigma(u)} \right)^{k \frac{N+2}{2(N-k)}} \, dy \right)
\]

is finite. For \( 0 < r < \text{dist}_p(z, \partial \Omega_T) \) define

\[
\omega(z, \tau) = \text{ess sup}_{t-r^2 \leq \tau \leq t} \left( \text{ess sup}_{B(x,r)} \varphi(y, \tau) - \text{ess inf}_{B(x,r)} \varphi(y, \tau) \right).
\]

Note that \( \varphi(\cdot, \tau) \in W^{1,2}(\Omega) \cap L^\infty(\Omega) \) for a.e. \( \tau \in (0, T) \). We also have

\[
\text{div}(\sigma(u) \nabla \varphi) = 0 \quad \text{on} \quad \Omega
\]

for a.e. \( \tau \in (0, T) \). Thus we are in a position to apply Lemma 2.2, thereby obtaining

\[
\omega(z, 2r) \leq (1 - \frac{1}{2cP(z)}) \omega(z, 5r) \tag{3.36}
\]

for all \( r > 0 \) such that \( Q(z, 5r) \subset \Omega_T \). Therefore,

\[
\omega(z, r) \leq cr^\gamma \tag{3.37}
\]

for some \( c, \gamma > 0 \) depending only on the \( cP(z) \) in (3.36) and \( \text{dist}_p(z, \partial \Omega_T) \). Once this is established, we can employ the proof of Theorem 7 in [23] to obtain

\[
\int_{Q(z,r)} (u - u_{z,r})^2 \, dy \, ds \leq cr^\varepsilon \tag{3.38}
\]
for some $c, \varepsilon > 0$ independent of $r$. In light of Theorem 2.1 in [24], there exists a positive constant $c = c(M, \|\varphi\|_{\infty, \Omega_T})$ such that

$$
\sup_{t-r^2 \leq \tau \leq t} \int_{B(x,r)} (u - u_{z,2r})^2 \, dy + \int_{Q(z,r)} |\nabla u|^2 \, dyd\tau
\leq \frac{c}{r^2} \left( \int_{Q(z,2r)} (u - u_{z,2r})^2 \, dyd\tau + \int_{Q(z,2r)} (\varphi - \varphi_{x,2r}(\tau))^2 \, dyd\tau \right)
$$

(3.39)

for all $z = (x,t) \in \Omega_T, 0 \leq r \leq (1/2)\text{dist}_p(z, \partial \Omega_T)$. This, along with (3.38) and (3.37), implies (3.34), while (3.33) follows from (3.29) and (3.30). The proof is complete.

REFERENCES


