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GALERKIN APPROXIMATION, STRONG CONTINUITY OF THE RELATIVE REARRANGEMENT MAP AND APPLICATION TO PLASMA PHYSICS EQUATIONS

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Abstract. We prove a strong continuity result for the relative rearrangement map. This new result and its corollary are used for the resolution of equations of the form $-\Delta u = F(u)$ via a Brouwer fixed point theorem. The nonlocal nonlinearity F might depend on the monotone rearrangement u_*, its derivative u'_* and the relative rearrangement of u with respect to the data.

I. Introduction. In this paper, we propose the Galerkin method to solve some nonlocal equations of the form $-\Delta u = F(u)$, where F can depend on the monotone rearrangement of u, u_* the generalized inverse of the distribution function $m(t) = \text{meas}\,\{x : u(x) > t\}$, its first derivative and the relative rearrangement of u with respect to the data. To be more precise, let us consider Ω an open bounded connected set of $\mathbb{R}^N, N \geq 2$ and let $b\ : \Omega \to]0, +\infty[$ be a bounded function. Then the decreasing rearrangement with respect to the weight b of a measurable function u defined on Ω is given by

$$u_*^b(s) = \inf\{t \in \mathbb{R} : \int_{\{u>t\}} b(x)\,dx \leq s\},$$

for any $s \in (0, |\Omega|_b)$, where $|\Omega|_b = \int_\Omega b(x)\,dx$ and more generally, $|E|_b = \int_E b(x)\,dx$ for a Lebesgue measurable set E. If $b(x) = 1$, we simply denote by $u_* = u_*^b$ and $|E| = |E|_b$. For a measurable function u defined on Ω, there exists at most a countable set D of points t_i such that meas $\{x : u(x) = t_i\} \doteq |\{u > t_i\}| > 0$. We set $P(u) = \bigcup_{t_i \in D}\{u = t_i\}$ and $\{u = t_i\}$ is called a plateau of u at t_i or a flat region.

The directional derivative of the map : $u \longrightarrow u_*^b$ was studied in [17] (see [9] for the case $b = 1$) and we have :

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Lemma 1. *Let* $u \in L^1(\Omega, b), v \in L^p(\Omega, b), 1 \leq p \leq +\infty$ *(which are Lebesgue spaces associated with the measure* $b(x)dx$*) and define on* $\Omega_*^b = (0, |\Omega|_b)$ *:*

$$w(s) = \int_{u>u_*^b(s)} v(x)b(x)\,dx + \int_0^{s-|u>u_*^b(s)|_b} (v|_{P(s)})_*^b(\sigma)\,d\sigma,$$

where $P(s) = \{x \in \Omega : u(x) = u_*^b(s)\}$ *and* $v|_{P(s)}$ *is the restriction of* v *to* $P(s)$*. Then*

i) $w \in W^{1,p}(\Omega_*^b)$.

ii) $\frac{(u+\lambda v)_*^b - u_*^b}{\lambda}$ *converges to* $\frac{dw}{ds}$ *as* λ *goes to zero, weakly in* $L^p(\Omega, b)$ *if* $1 \leq p < +\infty$ *and in the weak-star topology if* $p = +\infty$.

iii) *The map* $v \in L^p(\Omega, b) \longrightarrow \frac{dw}{ds} \in L^p(\Omega_*^b)$ *is a contraction.*

Definition 1. Under the same assumptions as in Lemma 1, the function $\frac{dw}{ds}$ is called the b-relative rearrangement of u with respect to v and is denoted by $v_{*u}^b = \frac{dw}{ds}$.

For more details on the notion of relative rearrangement, one can consult [7], [9], [8], [16], [11], [12], [14], [15], [17].

Motivated by the use of Galerkin approximation for some plasma physics problems (see [13]), we were lead to study the continuity of the maps :

i) $u \longrightarrow b_{*u} = \lim_{\lambda \to 0^+} \frac{(u+\lambda b)_* - u_*}{\lambda}$

ii) $u \longrightarrow b_{*u}(|u > u(x)|) = (\varphi \circ g)(x)$, where $g(x) = |u > u(x)|$ and $\varphi(s) = b_{*u}(s)$.

More precisely, the main purpose of this paper is to show the following new theorem:

Theorem 1. *Let* $v \in W^{1,\infty}(\Omega)$ *and let* $(v_j)_{j\geq 0}$ *be a sequence from* $W^{1,\infty}(\Omega)$ *such that* $|\{x \in \Omega : \nabla v(x) = 0\}| = |\{x \in \Omega : \nabla v_j(x) = 0\}| = 0$*. If* v_j *converges to* v *in* $W^{1,q}(\Omega)$ *for some* $q > N$*, then* b_{*v_j} *tends to* b_{*v} *strongly in* $L^p(0, |\Omega|)$ *provided that* $b \in L^p(\Omega)$, $1 \leq p < +\infty$*. Furthermore,* $b_{*v_j}(|v_j > v_j(.)|)$ *converges to* $b_{*v}(|v > v(.)|)$ *strongly in* $L^p(\Omega)$*.*

As a consequence of the above theorem, which we will use later, we have:

Theorem 2. *Let* $(\lambda_k, \varphi_k)_{k\geq 1}$ *be a sequence of eigenvalues and eigenfunctions of -*Δ *with Dirichlet boundary conditions and let* V_m *be the vector space spanned by* $\{\varphi_1, \ldots, \varphi_m\}$*. Then, for* $1 \leq p < +\infty$*, the maps*

i) $u \in V_m \backslash \{0\} \longrightarrow b_{*u} \in L^p(0, |\Omega|)$*-strong,*

ii) $u \in V_m \backslash \{0\} \longrightarrow b_{*u}(|u > u(.)|) \in L^p(\Omega)$*-strong,*

are continuous provided that $b \in L^p(\Omega)$*.*

A slight improvement of Theorem 1 will be given by A. Ferone, M. Jalal and R. Volpicelli. Theorem 1 and Theorem 2 have been announced in [13].

The proof of Theorem 1 relies on some lemmas already proved in previous papers.

Lemma 2. *We assume that* $ess\inf_\Omega b > 0$ *and we set for* $N < p \le +\infty$, $q_c = \frac{1}{1+\frac{1}{p}-\frac{1}{N}}$. *If* $u \in W^{1,p}(\Omega)$, *then* $u_*^b(\Omega_*^b)$ *is in* $W^{1,q}(\Omega_*^b)$ *for all* $q \in [1, q_c)$. *Furthermore, there exists a constant* c_q *(independent of* u*) such that*

$$|(u_*^b)'|_{L^q(\Omega_*^b)} \le c_q |\nabla u|_{L^p(\Omega)}.$$

The following lemmas were introduced in [4, 5] (see Lemma 16 in [5], replacing H_1 by H_{N-1}, ρ by 1 and $b\rho$ by b in the proof) and are true in any dimension.

Lemma 3. *We assume that* $ess\inf_\Omega b > 0$ *and we set* $C_\infty = \{v \in W^{1,\infty}(\Omega) : |\{x : \nabla v(x) = 0\}| = 0\}$. *Then, for any* $u \in C_\infty$, *one has*

$$b_{*u}(|u > u(x)|) = \frac{u_*'(|u > u(x)|)}{(u_*^b)'(|u > u(x)|_b)}.$$

Lemma 4. *Under the same assumptions as in Lemma* 3, *let* $u \in C_\infty$ *and let* $(u_n)_{n\ge 0}$ *be a sequence from* C_∞ *such that* u_n *converges to* u *in* $W^{1,p}(\Omega)$ *for some* $p > N$. *Then,* $\frac{du_{n*}^b}{ds}(|u_n > u_n(.)|_b)$ *converges weakly to* $\frac{du_*^b}{ds}(|u > u(.)|_b)$ *in* $L^q(\Omega)$ *for* q *in* $]1, q_c[$.

We also need some properties of the relative rearrangement (see [8,10,5]).

Lemma 5. *Let* $b > 0$, $b \in L^\infty(\Omega)$. *For* $(u,\varphi) \in L^1(\Omega, b)^2$ *:*

i) $(\varphi + c)_{*u}^b = \varphi_{*u}^b + c$ *for all constant* c.

ii) *Let us denote by* χ_A *the characteristic function of a set* A. *Then*

$$(\varphi\chi_{\Omega\setminus P(u)})_*^b = \varphi_{*u}^b \chi_{\Omega_*^b \setminus P(u_*^b)},$$
$$\big(P(u_*^b) = \bigcup_{t_i \in D} \{u_*^b = t_i\} = \bigcup_{t_i\in D} \{u = t_i\}\big).$$

iii) *If* $|P(u)| = 0$, *then* $(\varphi_1 + \varphi_2)_*^b = \varphi_{1*u}^b + \varphi_{2*u}^b$ *for all* $(\varphi_1, \varphi_2) \in L^1(\Omega, b)^2$.

To end this section, we recall the continuity result for the maps $u \longrightarrow (u_*^b)'$ (see [1], [10], [5] for details). Let $b > 0,\ b \in L^\infty(\Omega)$ and $u \in W^{1,1}_{loc}(\Omega)$, and set for $t \in \mathbb{R}$

$$m_0^b(t) = |\{x \in \Omega : u(x) > t, \nabla u(x) = 0\}|_b \doteq m_{u,0}^b(t),$$
$$m_1^b(t) = |\{x \in \Omega : u(x) > t\}|_b - m_0^b(t).$$

Definition 2. (see [1]) We say that u is co-area regular if the Radon measure $(m_{u,0}^b)'$ is singular with respect to the Lebesgue measure on $\mathbb{R}$.

The following theorem is proved in [10], [5].

Theorem 3. *Let Ω be an open subset of $\mathbb{R}^N$, $u \in W^{N,p}_{loc}(\Omega)$, and $p > 1$. Then u is co-area regular.*

A more simple criterion for deciding co-area regularity is the following.

Lemma 6. *If $u \in W^{1,1}_{loc}(\Omega)$ is such that $|\{x \in \Omega : \nabla u(x) = 0\}| = 0$, then u is co-area regular.*

The following lemma is derived from Almgrem and Lieb's results, concerns the strong convergence of a sequence $(u_{n*}^b)'$ to $(u_*^b)'$. More precisely, one has:

Lemma 7. *Let $b \in L^\infty(\Omega)$ be such that $\operatorname{ess\,inf}_\Omega b > 0$. If $p > N$ and $(u_n)_{n\geq 0}$ is a bounded sequence from $W^{1,p}(\Omega)$ converging to u in $W^{1,1}(\Omega)$, where u is a co-area regular function, then $(u_{n*}^b)'$ converges to $(u_*^b)'$ for all $q \in [1, q_c[$.*

For more details on the strong convergence of the first derivative of monotone rearrangement, see for instance [1], [5], [10].

Before proving the continuity of $v \longrightarrow \big(b_{*v}, b_{*v}(|v > v(.)|)\big)$ we recall the following weak-continuity result proved in [12], [10].

Lemma 8. *Let $b \in L^r(\Omega)$, $1 < r \leq +\infty$, and let $v \in L^1(\Omega)$ be such that $|P(v)| = 0$. If $(v_j)_{j\geq 0}$ is a sequence converging to v in $L^1(\Omega)$, then b_{*v_j} converges to b_{*v} weakly in $L^r(0, |\Omega|)$ if $1 < r < +\infty$ and for the weak-star topology if $r = +\infty$.*

The application of Theorem 1 and Theorem 2 on plasma physics that we will present here is *not new* but the proof is a more simple presentation of the results obtained in [5], where we used weighted Sobolev spaces. Here, since we use a Laplacian operator, we can provide some new and explicit estimates on the L^∞-norm of the solution (see also the L^2-norm).

II. A strong continuity result. The proof of Theorem 1 will need the following intermediate result.

Lemma 9. *Let $b \in L^\infty(\Omega)$ be such that ess $\inf_\Omega b > 0$, let $u \in C_\infty$ and let $(u_n)_{n\geq 0}$ be a sequence from C_∞ converging to u in $W^{1,p}(\Omega)$, for some $p > N$. Then, $(u^b_{n*})'(|u_n > u_n(.)|_b)$ converges strongly to $(u^b_*)'(|u > u(.)|_b)$ in $L^q(\Omega)$ for all $q \in [1, q_c[$.*

Proof of Lemma 9. From Lemma 7, we know that $(u^b_{n*})'$ converges to $(u^b_*)'$ in $L^q(\Omega^b_*)$ for all $q \in [1, q_c[$. Using a standard property of equimeasurability for rearrangements (see [2], [17], [7]), we know that:

$$\int_\Omega |(u^b_{n*})'(|u_n > u_n(x)|_b)|^q b(x)dx = \int_{\Omega^b_*} |(u^b_{n*})'(\sigma)|^q d\sigma.$$

We then deduce that

$$\lim_{n\to+\infty} \int_\Omega |(u^b_{n*})'(|u_n > u_n(x)|_b)|^q b(x)dx = \int_\Omega |(u^b_*)'(|u > u(x)|_b)|^q b(x)dx.$$

But Lemma 4 and the hypothesis on b ensure that $(u^b_{n*})'(|u_n > u_n(.)|_b)$ converges weakly to $(u^b_*)'(|u > u(.)|_b)$ in $L^q(\Omega)$ $q > 1$. The uniform convexity of $L^q(\Omega, b)$ $(q > 1)$, shows that $(u^b_{n*})'(|u_n > u_n(.)|_b)$ converges strongly to $(u^b_*)'(|u > u(.)|_b)$ in $L^q(\Omega, b)$, at first for $q > 1$ then for $q = 1$.

Now, we come to the proof of Theorem 1, which will be divided into 4 steps.

Proof of Theorem 1.

First case: $b \in L^\infty(\Omega)$ and ess $\inf_\Omega b > 0$. Since v_j is in C_∞, we then have for almost every $x \in \Omega$ (see Lemma 3)

$$b_{*v_j}(|v_j > v_j(x)|) = \frac{v'_{j*}(|v_j > v_j(x)|)}{(v^b_{j*})'(|v_j > v_j(x)|_b)}.$$

Furthermore, by Lemma 9 the sequence

$$v'_{j*}(|v_j > v_j(.)|)(\text{resp. } (v^b_{j*})'(|v_j > v_j(.)|_b)),$$

converges to $v'_*(|v > v(.)|)$ (resp. $(v^b_*)'(|v > v(.)|_b)$) in $L^1(\Omega)$-strong. Since $v \in C_\infty$, we deduce that (at least for a subsequence which we still denote v_j), $b_{*v_j}(|v_j > v_j(x)|)$ converges to $b_{*v}(|v > v(x)|) = \frac{v'_*(|v>v(x)|)}{(v^b_*)'(|v>v(x)|_b)}$ almost everywhere. But $|b_{*v_j}(|v_j > v_j(x)|)| \leq |b|_\infty$ a.e., thus (for the whole sequence)

$b_{*v_j}(|v_j > v_j(.)|)$ converges to $b_{*v}(|v > v(x)|)$ in $L^r(\Omega)$-strong for all finite r. It remains to show that b_{*v_j} converges to b_{*v} in $L^r(\Omega_*)$-strong. Lemma 8 implies already that b_{*v_j} converges to b_{*v} weakly if $1 < r < +\infty$. But by equimeasurability, we have:

$$\int_{\Omega*} |b_{*v_j}(\sigma)|^r d\sigma = \int_{\Omega} |b_{*v_j}(|v_j > v_j(x)|)|^r dx.$$

Thus, the preceding strong convergence and the equimeasurability lead to:

$$\lim_{j\to+\infty} \int_{\Omega*} |b_{*v_j}(\sigma)|^r d\sigma = \int_{\Omega*} |b_{*v}(\sigma)|^r d\sigma.$$

Since $L^r(\Omega_*)$ is uniformly convex for $1 < r < +\infty$, we deduce that b_{*v_j} converges to b_{*v} in $L^r(\Omega_*)$-strong for $1 < r < +\infty$, hence for $r = 1$.

Second case: $b \geq 0,\ b \in L^\infty(\Omega)$. We introduce $\varepsilon > 0$ and $b_\varepsilon = b + \varepsilon$. Then, we have

$$|b_{*v_j} - b_{*v}|_r \leq |b_{*v_j} - b_{\varepsilon*v_j}|_r + |b_{\varepsilon*v_j} - b_{\varepsilon*v}|_r + |b_{\varepsilon*v} - b_{*v}|_r, \quad 1 \leq r < +\infty. \tag{1}$$

Here, $|.|_r$ is the norm in $L^r(\Omega_*)$. Furthermore

$$(b+\varepsilon)_{*v_j} = b_{*v_j} + \varepsilon, \quad (b+\varepsilon)_{*v} = b_{*v} + \varepsilon, \tag{2}$$

and $\lim_j |b_{\varepsilon*v_j} - b_{\varepsilon*v}|_r = 0$ (first case).

Using relations (1) and (2), we then deduce that

$$\lim_{j\to+\infty} \sup |b_{*v_j} - b_{*v}|_r \leq \varepsilon|\Omega|^{1/r} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

By the same argument, we also have

$$\lim_{j\to+\infty} \sup |b_{*v_j}(|v_j > v_j(.)|) - b_{*v}(|v > v(.)|)|_r \leq \varepsilon|\Omega|^{1/r}.$$

Third case: b is only in $L^\infty(\Omega)$. Then, we write $b = b_+ - b_-$, since meas $(P(v))$ = meas $(P(v_j)) = 0$. Thus, we have $b_{*v} = b_{+*v} - b_{-*v}$, $b_{*v_j} = b_{+*v_j} - b_{-*v_j}$. Applying the above result with $b_{+*v_j},\ b_{+*v_j}(|v_j > v_j(.)|), b_{-*v_j},\ b_{-*v_j}(|v_j > v_j(.)|)$, we obtain the strong convergence results : $b_{*v_j} \underset{j\to+\infty}{\longrightarrow} b_{*v}$ in $L^r(\Omega_*)$ $\forall r < +\infty$, $b_{*v_j}(|v_j > v_j(.)|) \underset{j\to+\infty}{\longrightarrow} b_{*v}(|v > v(.)|)$ in $L^r(\Omega)$.

Fourth case: If $b \in L^p(\Omega),\ 1 \leq p < +\infty$, then we define the truncation

$$T_k(\sigma) = \begin{cases} \sigma & \text{if } |\sigma| \leq k \\ k\,\text{sign}(\sigma) & \text{otherwise,} \end{cases}$$

and we set $b^k = T_k(b)$. It is clear that for a fixed k, we have the strong convergences. But from equimeasurability and the contraction property of the relative rearrangement, one has:

$$|b_{*v}(|v > v(.)|) - b^k_{*v}(|v > v(.)|)|_p = |b^k_{*v} - b_{*v}|_p \leq |b - b^k|_p,$$

and the same relation holds with v replaced by v_j. Moreover by a simple decomposition, one has:

$$\begin{aligned} &|b_{*v_j} - b_{*v}|_p \leq 2|b - b^k|_p + |b^k_{*v_j} - b^k_{*v}|_p, \\ &|b_{*v_j}(|v_j > v_j(.)|) - b_{*v}(|v > v(.)|)|_p \leq 2|b - b^k|_p + o(1)_{j \to +\infty}. \end{aligned}$$

Letting first j go to infinity and then k go to infinity we find the result.

Theorem 2 is a direct consequence of Theorem 1 but is useful. Its proof is:

Proof of Theorem 2. Elements of $V_m \backslash \{0\}$ are analytic in Ω. Thus, $V_m \backslash \{0\} \subset C_\infty$.

The following strong convergences are also true on V_m.

Lemma 11. *Let $\Phi \in W^{1,\infty}(\Omega)$, such that Φ is analytic in Ω. Then the map* $v \in V_m \longrightarrow (v + \Phi)'_* \in L^q(\Omega_*)$ $q \in [1, \frac{N}{N-1}[$.

Proof. Under the assumptions of Lemma 11, for all $v \in V_m,\ v + \Phi$ is a co-area regular function and there exists a constant C_q independent of v and Φ such that:

$$|(v + \Phi)'_*|_q \leq C_q |\nabla(v + \Phi)|_\infty \quad \text{(see Lemma 2).}$$

The continuity can be obtained using these facts and the finiteness of the dimension of V_m (in which all norms are equivalent).

III. Application. As an application of the above result, we want to solve the following problem, which appears in plasma physics (see [4, 5]).

Let Ω be a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$, and let $\gamma \in]-\infty, 0[$. We want to find u solution of $(\mathcal{P})$ $-\Delta u = F(u)$, $u - \gamma \in H_0^1(\Omega) \cap W^{2,p}(\Omega)$ for all finite p. Here F is given by

$$F(u)(x) = aG(u)(x) + J(u)(x),\ a \in L^\infty(\Omega),$$
$$G(u)(x) = [F_v^2 - 2\int_{|u>0|}^{|u>u_+(x)|} [p(u_*)]'(\sigma) b_{*u}(\sigma) d\sigma]_+^{1/2},$$
$$J(u)(x) = p'(u(x))[b(x) - b_{*u}(|u > u(x)|)],$$

where $b \in L^\infty(\Omega)$, $p(t) = \lambda \frac{t_+^2}{2}$, $t_+ = \max(t, 0)$, $\lambda > 0$ and F_v is a constant > 0.

As before, we denote by $(\lambda_k, \varphi_k)_{k\geq 1}$ the eigenvalues and eigenfunctions associated to $-\Delta$ with Dirichlet boundary condition and we define on $H_0^1(\Omega) \cap W^{1,\infty}(\Omega)$ the maps $w \longrightarrow J(w + \gamma)$ and $w \longrightarrow G(w + \gamma)$. We then have:

Theorem 5. *Assume that $\lambda.osc_\Omega\ b < \lambda_1$. Then, there exists $w_m \in V_m$ satisfying, $\forall \varphi \in V_m$,*

$$\int_\Omega \nabla w_m . \nabla \varphi dx = \int_\Omega a(x) G(w_m + \gamma)(x) \varphi(x) dx + \int_\Omega J(w_m + \gamma)(x) \varphi(x) dx.$$

Proof of Theorem 5. We introduce the map T: $V_m \longrightarrow V_m$ by setting

$$Tv = \sum_{k=1}^m [\int_\Omega \nabla v . \nabla \varphi_k dx - \int_\Omega a.G(v + \gamma)\varphi_k dx - \int_\Omega J(v + \gamma)\varphi_k dx] \varphi_k.$$

This map is continuous. Indeed, since the maps $v \in V_m \backslash \{0\} \longrightarrow b_{*v} \in L^r(\Omega_*)$-strong and $v \in V_m \longrightarrow [p(v_* + \gamma)]' \in L^q(\Omega_*)$-strong are continuous for any finite r for all $q \in [1, \frac{N}{N-1}[$, and since $p'(\gamma) = 0$, we deduce that $v \in V_m \longrightarrow [p(v_* + \gamma)]'.b_{*v} \in L^q(\Omega_*)$-strong is also continuous. Let $(v_j)_{j\geq 0}$ be a sequence of V_m converging to v. We denote by $I(v, x)$ the interval : $I(v, x) = [|v + \gamma > (v + \gamma)_+(x)|,\ |v + \gamma > 0|]$ and $I(v_j, x) = [|v_j + \gamma > (v_j + \gamma)_+(x)|,\ |v_j + \gamma > 0|]$, $x \in \bar{\Omega}$. If $v \neq 0$, the characteristic function $\chi_{I(v_j,x)}$ converges to $\chi_{I(v,x)}$ in $L^r(\Omega_*)$-strong for every r finite and every $x \in \bar{\Omega}$. Again, since $p'(\gamma) = 0$ $(\gamma \leq 0)$ for every $x \in \bar{\Omega}$:

$$\lim_{j\to+\infty} \int_{\Omega*} \chi_{I(v_j,x)}(\sigma)[p(v_{j*} + \gamma)]'(\sigma) b_{*v_j}(\sigma) d\sigma$$
$$= \int_{\Omega*} \chi_{I(v,x)}(\sigma)[p(v_* + \gamma)]'(\sigma) b_{*v}(\sigma) d\sigma.$$

Noting that $(v+\gamma)_* = v_* + \gamma$, $b_{*(v+\gamma)} = b_{*v}$ (and the same equalities are true where we replace v by v_j), we find:

$$G(v_j+\gamma)(x) \underset{j\to+\infty}{\longrightarrow} G(v+\gamma)(x) \text{ for all } x\in\Omega.$$

Now $0 \le G(v_j+\gamma)(x) \le F_v$. Thus the Lebesgue dominate convergence gives:

$$\lim_{j\to+\infty}\int_\Omega a(x)G(v_j+\gamma)(x)\varphi(x)dx = \int_\Omega a(x)G(v+\gamma)(x)\varphi(x)dx \text{ for all } \varphi\in V_m. \tag{3}$$

Again noting that $b_{*(v+\gamma)}(|v+\gamma > (v+\gamma)(.)|) = b_{*v}(|v > v(.)|)$, using the continuity of $v\in V_m\backslash\{0\} \longrightarrow b_{*v}(|v > v(.)|)$ in $L^r(\Omega)$-strong and the fact that $p'\in C(\mathbb{R})$ with $p'(\gamma)=0$, we easily have:

$$\lim_{j\to+\infty}\int_\Omega J(v_j+\gamma).\varphi = \int_\Omega J(v+\gamma).\varphi \quad \forall\varphi\in V_m. \tag{4}$$

From (3) and (4) we deduce the continuity of $T : Tv_j \underset{j\to+\infty}{\longrightarrow} Tv$ in V_m.

The second step is to show that the map T is coercive in the following sense:

Lemma 12. *If $\lambda.osc_\Omega\, b < \lambda_1$, then there exists $M > 0$, such that $[Tv, v] > 0$ provided that $|v|_m = M$. Here, $[,]$ denotes the scalar product on V_m, i.e., if $v = \sum_{k=1}^m a_k\varphi_k$ and $w = \sum_{k=1}^m b_k\varphi_k$ then $[v,w] = \sum_{k=1}^m a_k.b_k$ and $|v|_m = [v,v]^{1/2}$.*

Proof of Lemma 12. Let $v\in V_m$. Then

$$[Tv,v] = \int_\Omega |\nabla v|^2dx - \int_\Omega aG(v+\gamma)v - \int_\Omega J(v+\gamma).v.$$

But, we have the following inequalities:

$$|a.G(v+\gamma)|(x) \le |a|_\infty F_v, \tag{5}$$

$$|J(v+\gamma)|(x) \le \lambda|v(x)|.\ \mathrm{osc}_\Omega b \text{ (since } \gamma\le 0). \tag{6}$$

From (5) and (6), we then have (via Poincaré 's inequality)

$$[Tv,v] \ge (\lambda_1 - \lambda\ \mathrm{osc}_\Omega\ b)\int_\Omega v(x)^2dx - |a|_\infty F_v\int_\Omega |v(x)|dx. \tag{7}$$

Clearly, relation (7) gives the result. Applying the Brouwer fixed point theorem (see [20]), we deduce the existence of $w_m\in V_m$ such that $Tw_m = 0$, which finishes the proof of Theorem 5.

It remains to pass to the limit for w_m. We begin with an apriori estimate.

Lemma 13. *If $\lambda.osc_\Omega\ b < \lambda_1$, then the sequence $(w_m)_{m\geq 1}$ remains in a bounded subset of $H_0^1(\Omega)\cap W^{2,2}(\Omega)$. Furthermore,*

$$|w_m|_{L^2(\Omega)} \leq \frac{|a|_\infty.F_v.|\Omega|^{1/2}}{\lambda_1 - \lambda.\ osc_\Omega b}, \tag{8}$$

$$\int_\Omega |\nabla w_m|^2 dx \leq |a|_\infty F_v |\Omega|^{1/2}.|w_m|_2 + (\lambda\ osc_\Omega b).|w_m|_2^2. \tag{9}$$

Proof of Lemma 13. We have:

$$\int_\Omega |\nabla w_m|^2 dx = \int_\Omega a.G(w_m+\gamma)w_m + \int_\Omega J(w_m+\gamma).w_m. \tag{10}$$

Using relations (5), (6) and (10), we have

$$\int_\Omega |\nabla w_m|^2 dx \leq |a|_\infty F_v \int_\Omega |w_m(x)|dx + \lambda\ \mathrm{osc}_\Omega b \int_\Omega w_m(x)^2 dx. \tag{11}$$

Using Poincaré's inequality and Cauchy-Schwartz's inequality, we obtain

$$(\lambda_1 - \lambda\ \mathrm{osc}_\Omega b)\Big(\int_\Omega |w_m|^2 dx\Big)^{1/2} \leq |a|_\infty.F_v.|\Omega|^{1/2},$$

which gives relation (8). Relation (9) comes from (11).

In order to show that w_m remains in a bounded set of $H_0^1(\Omega)\cap W^{2,2}(\Omega)$, we introduce the orthogonal projection of $L^2(\Omega)$ onto V_m, $P_m : L^2(\Omega) \longrightarrow V_m$. The equation satisfied by w_m is equivalent to:

$$(\mathcal{P}_m) \qquad \begin{cases} -\Delta w_m = P_m(aG(w_m+\gamma) + J(w_m+\gamma)), \\ w_m \in V_m. \end{cases}$$

Relations (5) and (6) ensure that $aG(w_m+\gamma)+J(w_m+\gamma)$ remains in a bounded set of $L^2(\Omega)$. Using the equation in $(\mathcal{P}_m)$, we infer that Δw_m remains in a bounded set of $L^2(\Omega)$ thus w_m remains in a bounded set of $W^{2,2}(\Omega)$.

We may assume that there exists $w \in H_0^1(\Omega)\cap W^{2,2}(\Omega)$, G_∞, $\tilde{b}_w \in L^\infty(\Omega)$, $\hat{b}_w \in L^\infty(\Omega_*)$ and a subsequence, which we still denote by w_m, satisfying

i) w_m converges to w weakly in $W^{2,2}(\Omega)$,

ii) w_m converges to w in $W^{1,1}(\Omega)$-strong,

iii) $b_{*w_m} \underset{m\to+\infty}{\rightharpoonup} \hat{b}_w$ in $L^\infty(\Omega_*)$-weak-star,

iv) $b_{*w_m}(|w_m > w_m(.)|) \underset{m\to+\infty}{\rightharpoonup} \tilde{b}_w$ in $L^\infty(\Omega)$-weak-star (since $|b_{*w_m}|_\infty \leq |b|_\infty$, $|b_{*w_m}(|w_m > w_m(x)|)| \leq |b|_\infty$),

v) $G(w_m+\gamma) \underset{m\to+\infty}{\rightharpoonup} G_\infty$ in $L^\infty(\Omega)$-weak-star.

Thus, w is a solution of

$$(\mathcal{P}_l) \qquad \begin{cases} -\Delta w = aG_\infty + p'(w+\gamma)[b-\tilde{b}_w] \\ w \in H_0^1(\Omega)\cap W^{2,p}(\Omega)\ p\in[1,+\infty[. \end{cases}$$

Another novelty of this study with respect to our result in [4, 5] is the following L^∞-estimate.

Lemma 14. *Assume that $N=2$ (for simplicity), then*

$$|w|_\infty \leq \frac{|a|_\infty F_v|\Omega|}{4\pi}\big[1+\frac{2\lambda_1}{\lambda_1-\lambda\ osc_\Omega b}\big] = M_\infty$$

Proof of Lemma 14. We use the standard technique of Talenti (see [19], [7]). Multiplying the equation of $(\mathcal{P}_l)$ by $v=(|w|-t)_+ sign(w)$, $t>0$ and differentiating with respect to t, we obtain

$$-\frac{d}{dt}\int_{\{|w|>t\}}|\nabla w|^2dx = \int_{\{|w|>t\}} aG_\infty.\text{sign}(w) + \lambda\int_{\{|w|>t\}}(w+\gamma)_+[b-\tilde{b}_w]\ sign(w).$$

Setting $\mu(t)=\text{meas}\{x : |w|(x)>t\}$, relations (5) and (6) imply for a.e. $t>0$

$$-\frac{d}{dt}\int_{\{|w|>t\}}|\nabla w|^2dx \leq |a|_\infty F_v\mu(t)+\lambda\ (\text{osc}_\Omega b).|w|_{L^2(\Omega)}.\mu(t)^{\frac{1}{2}}.$$

Using the De-Giorgi-Talenti inequality as in [19], [7], we find (following standard arguments):

$$-\frac{d}{ds}|w|_*(s) \leq \frac{|a|_\infty.F_v}{4\pi}+\frac{\lambda(\text{osc}_\Omega b)|w|_2}{4\pi}\ s^{-\frac{1}{2}}\ \text{a.e}\ s\in\Omega_*.$$

Integrating this last relation and using the L^2-estimate, one has Lemma 14.

To be clearer on the passage to the limit, we introduce an intermediate result:

Proposition 1. *We assume that :* **(C1)** *meas* $\{x : \nabla w(x) = 0\} = 0$ *and* $N = 2$. *Then,* $u = w + \gamma$ *is a solution of the problem* $(\mathcal{P})$ *stated above.*

Proof of Proposition 1. Assumption **(C1)** and the regularity of w imply that $w \in C_\infty$. Applying Theorem 1, we deduce that

$$\begin{aligned}
&b_{*w_m} \underset{m\to+\infty}{\longrightarrow} b_{*w} \text{ in } L^r(\Omega_*)\text{-strong } \forall r \in [1, +\infty[,\\
&b_{*w_m}(|w_m > w_m(.)|) \underset{m\to+\infty}{\longrightarrow} b_{*w}(|w > w(.)|) \text{ in } L^r(\Omega) - \text{strong},\\
&w'_{m*} \underset{m\to+\infty}{\longrightarrow} w'_* \text{ in } L^q(\Omega_*)\ \forall q \in [1, 2[.
\end{aligned}$$

(Thus $[p(w_{m*})]' \underset{m\to+\infty}{\longrightarrow} [p(w_*)]'$ in $L^q(\Omega_*)$). Since w has no flat region, we also have

$$\chi_{I(w_m,x)} \longrightarrow \chi_{I(w,x)} \text{ in } L^r(\Omega_*) \text{ for } r < +\infty,$$

and for every $x \in \bar{\Omega}$. Here as before, $I(w, x)$ is the interval $[|w + \gamma > (w + \gamma)_+(x)|, |w + \gamma > 0|]$ (the same definition holds for $I(w_m, x)$). Those strong convergences ensure that

$$\begin{aligned}
&\lim_{m\to+\infty} \int_\Omega aG(w_m + \gamma)\psi = \int_\Omega aG(w + \gamma)\psi \quad \forall \varphi \in L^\infty(\Omega),\\
&\lim_{m\to+\infty} \int_\Omega J(w_m + \gamma)\psi = \int_\Omega J(w + \gamma)\psi \quad \forall \psi \in H^1_0(\Omega),
\end{aligned}$$

and thus

$$\int_\Omega \nabla w.\nabla\psi = \int_\Omega aG(w + \gamma)\psi + \int_\Omega J(w + \gamma)\psi.$$

The next step is to find a sufficient condition on the data for condition **(C1)** to be fulfilled. We follow the same argument as in [5].

Lemma 15. *As before, we set* $|w|_\infty = M_\infty$ *and* $\nu = \frac{\lambda|b|_\infty.M^2_\infty}{F^2_v}$. *If* $\nu < 1$, *then*

$$G_\infty(x) \geq (1 - \nu)^{1/2}.F_v.$$

Proof. Since $|b_{*w_m}|_\infty \leq |b|_\infty$, we have

$$\begin{aligned}
&2\int_{|w_m+\gamma>0|}^{|w_m+\gamma>(w_m+\gamma)_+(x)|} [p(w_{m*} + \gamma)]'.b_{*w_m}(\sigma)d\sigma \leq \lambda|b|_\infty(w_m + \gamma)^2_+(x)\\
&\leq \lambda|b|_\infty.M^2_\infty + o(1) \ \textit{as} \ \ m \to +\infty.
\end{aligned}$$

(Here, we have used Lemma 14.) Then we deduce

$$o(1) + G(w_m + \gamma)(x) \geq (1-\nu)^{1/2} F_v,$$

which implies that

$$G_\infty(x) \geq (1-\nu)^{1/2} F_v.$$

Lemma 16. *Assume that* $d \doteq \ ess\inf_\Omega \ a^2 > 0$, $\frac{\lambda|b|_\infty . M_\infty^2}{F_v^2} = \nu < 1$ *and* $\lambda|b|_\infty < \frac{d(1-\nu)}{4\nu}$. *Then condition* **(C1)** *is fulfilled.*

Proof. If meas $\{x : \nabla w(x) = 0\} \neq 0$, the regularity of w and equation $(\mathcal{P}_l)$ implies that, a.e. on $\{x : \nabla w(x) = 0\}$, we have

$$\lambda(w+\gamma)_+(x)[b - \tilde{b}_w](x) = a(x) G_\infty(x).$$

Raising both sides of this last relation to the power 2 and using Lemma 15, we find

$$\lambda^2 (w+\gamma)_+^2 [b - \tilde{b}_w]^2(x) \geq d.(1-\nu) F_v^2.$$

From the L^∞-estimates and the fact that $|b - \tilde{b}_w| \leq 2|b|_\infty$, we obtain

$$4[\lambda|b|_\infty].\lambda|b|_\infty M_\infty^2 \geq d(1-\nu) F_v^2$$

that is $\lambda|b|_\infty \geq \frac{d(1-\nu)}{4\nu}$, which contradicts the choice of $\lambda|b|_\infty$.

Theorem 6. *Under the same assumptions as in Lemma* 16 *and the assumptions that* $N = 2$, *there exists a solution of* $-\Delta u = aG(u) + J(u)$ *such that* $u - \gamma \in W^{2,p}(\Omega) \cap H_0^1(\Omega)$ *for all finite* p. *Furthermore,*

$$|u-\gamma|_\infty \leq \frac{|a|_\infty F_v |\Omega|}{4\pi} \big[1 + \frac{2\lambda_1}{\lambda_1 - \lambda \ osc_\Omega b}\big].$$

Other applications of Theorem 1 and Theorem 2 related to plasma physics in the Tokomak configuration will be made by A. Ferone, M. Jalal and R. Volpicelli extending the work of R. Temam [21] and [5].

Recently, J.I. Diaz, G. Galiano and J.F. Padial obtained a partial result on the uniqueness of the solution of $(\mathcal{P})$ (see [4]).

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