

**ABSORPTION EVOLUTION FAMILIES AND
EXPONENTIAL STABILITY OF
NON-AUTONOMOUS DIFFUSION EQUATIONS***

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(Submitted by: J.A. Goldstein)

Abstract. For a positive, strongly continuous evolution family \mathcal{U} on $L^p(\Omega)$ and a positive, measurable, time dependent potential $V(\cdot)$ we construct a corresponding *absorption evolution family* \mathcal{U}_V by a procedure introduced by J. Voigt, [24, 25], in the autonomous case. We give sufficient conditions on \mathcal{U} and V such that \mathcal{U}_V is strongly continuous and satisfies variation of constants formulas. For an evolution family \mathcal{U} on $L^1(\mathbb{R}^N)$ satisfying upper and lower Gaussian estimates exponential stability of \mathcal{U}_V is characterized by a condition on the size of V extending recent results by W. Arendt and C.J.K. Batty, [2, 3, 6], in the autonomous case and by D. Daners, M. Hieber, P. Koch Medina, and S. Merino, [8, 12], in the time periodic case. An application to a second order parabolic equation with real coefficients and singular potential is given.

1. Introduction. In the present paper we prove the existence of mild solutions u for a non-autonomous diffusion equation with absorption on $X = L^p(\mathbb{R}^N)$, $1 \leq p < \infty$, given by

$$\begin{aligned} \frac{d}{dt} u(t) &= \sum_{k,l=1}^N D_k (a_{kl}(t) D_l u(t)) - V(t)u(t), \\ u(s) &= x, \quad t \geq s \geq 0. \end{aligned} \tag{1.1}$$

Here we assume $a_{kl} \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^N, \mathbb{R})$, uniform ellipticity of the differential operator and that the potential $0 \leq V \in L^1_{loc}(\mathbb{R}_+, L^p_{loc}(\mathbb{R}^N))$ is growing at

Received for publication October 1997.

*This paper is part of a research project supported by the Deutsche Forschungsgemeinschaft DFG.

AMS Subject Classifications: 47D06, 47A55, 47B65, 35K10, 35B35.

infinity less than $e^{\alpha|\xi|^2}$, $\alpha > 0$. Further, for $p = 1$ and a large class of potentials we characterize exponential stability of the solutions of (1.1) by a condition on the size of V . All this is part of Section 5 and follows from our investigations of the following more general situation.

Consider $X = L^p(\Omega)$ for a σ -finite measure space (Ω, μ) and $1 \leq p < \infty$. For $I \in \{[a, b], [a, \infty), \mathbb{R}\}$ set $D = D_I := \{(t, s) \in I^2 : t \geq s\}$. Let $\mathcal{U} = (U(t, s))_{(t, s) \in D} \subseteq \mathcal{L}(X)$ be a strongly continuous evolution family of positive operators on X (see Section 2 for this notion). At first, in Section 2, it is shown for potentials $V \in L^\infty(I \times \Omega)$ that there is a unique strongly continuous evolution family $\mathcal{U}_V = (U_V(t, s))_{(t, s) \in D}$ satisfying the *variation of constants formula*

$$U_V(t, s)x = U(t, s)x - \int_s^t U(t, \tau)V(\tau)U_V(\tau, s)x d\tau \quad (1.2)$$

for $(t, s) \in D$ and $x \in X$. Following a procedure introduced by J. Voigt, [24, 25], in the autonomous case, we then construct the so-called *absorption evolution family* \mathcal{U}_V for arbitrary measurable $0 \leq V : I \times \Omega \rightarrow \mathbb{R}$ in Section 3. The operators $U_V(t, s)$ are defined as the strong limit of the operators $U_n(t, s)$ determined by (1.2) where V is replaced by the truncated potential $V \wedge n\mathbb{1}$. We prove that \mathcal{U}_V is strongly continuous and that (1.2) holds if V satisfies the integrability condition mentioned above and \mathcal{U} satisfies an *upper Gaussian estimate*, that is, $0 \leq U(t, s) \leq Me^{\alpha(t-s)\Delta}$ for $(t, s) \in D$ and constants $M, \alpha > 0$. Here, Ω is an open subset of \mathbb{R}^N , $(e^{t\Delta})_{t \geq 0}$ is the semigroup generated by the Laplacian on $L^p(\mathbb{R}^N)$, and $L^p(\Omega)$ is embedded in $L^p(\mathbb{R}^N)$ by extending functions by 0. The *lower Gaussian estimate* is defined analogously.

In Section 4 we investigate for $\Omega = \mathbb{R}^N$ the asymptotic behavior of \mathcal{U}_V . If $U(t, s) = e^{(t-s)\Delta}$ and $0 \leq V(t) \equiv V_0 \in L^1(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$, then exponential stability of \mathcal{U}_V was characterized by W. Arendt and C.J.K. Batty, [3], by a condition on the size of V_0 . (See [2] and [6] for further results in the autonomous case.) For $p = 1$ this characterization was extended to the periodic case by D. Daners, M. Hieber, P. Koch Medina, and S. Merino, [8], [12]. We generalize these results to positive evolution families satisfying Gaussian estimates.

Finally, in the last section we apply our results to the diffusion equation (1.1). Here, \mathcal{U} is given by the weak fundamental solution of (1.1) with $V = 0$. Due to (1.2) we can interpret $t \mapsto U_V(t, s)x$ as a *mild solution* of (1.1). We

point out that for measurable coefficients and singular potentials we cannot expect to obtain a differentiable solution of (1.1).

2. Bounded perturbations. In this section we study bounded perturbations of evolution families on arbitrary Banach spaces X . Let $I \in \{[a, b], [a, \infty), \mathbb{R}\}$ and $D = D_I := \{(t, s) \in I^2 : t \geq s\}$. A family $\mathcal{U} = (U(t, s))_{(t, s) \in D}$ of bounded operators on X is called an *evolution family* if $U(s, s) = Id$ and $U(t, s) = U(t, r)U(r, s)$ for $t \geq r \geq s$. We say that \mathcal{U} is *strongly continuous* if the mapping $D \ni (t, s) \mapsto U(t, s)$ is strongly continuous. The *exponential growth bound* $\omega(\mathcal{U})$ of \mathcal{U} is defined by $\omega(\mathcal{U}) := \inf\{w \in \mathbb{R} : \text{there is } M_w \geq 1 \text{ with } \|U(t, s)\| \leq M_w e^{w(t-s)} \text{ for } (t, s) \in D\}$. The evolution family \mathcal{U} is called *exponentially bounded* if $\omega(\mathcal{U}) < \infty$, and \mathcal{U} is *exponentially stable* if $\omega(\mathcal{U}) < 0$.

In the sequel we make use of the following “truncation-extension” procedure. Let \mathcal{U} be a strongly continuous evolution family with index set D_I . Set $I_n = [a_n, b_n] := I \cap [-n, n]$ for $n \in \mathbb{N}$. Then we define

$$U_n(t, s) := \begin{cases} Id, & t \geq s \geq b_n \text{ or } a_n \geq t \geq s, \\ U(\min\{t, b_n\}, \max\{s, a_n\}), & \text{otherwise,} \end{cases} \quad (2.1)$$

cf. [16]. Clearly, \mathcal{U}_n is a strongly continuous evolution family with index set $D_{\mathbb{R}}$. Moreover, due to the principle of uniform boundedness we obtain

$$\sup_{(t, s) \in D_{\mathbb{R}}} \|U_n(t, s)\| < \infty.$$

For an exponentially bounded, strongly continuous evolution family \mathcal{U} with index set $D_{\mathbb{R}}$ we define an operator $T(t)$ on the Bochner-Lebesgue space $E = L^1(\mathbb{R}, X)$ by setting

$$T(t)f := U(\cdot, \cdot - t)f(\cdot - t), \quad f \in E, t \geq 0.$$

One can easily see that $\mathcal{T} = (T(t))_{t \geq 0}$ is a \mathcal{C}_0 -semigroup on E , cf. [22]. We call \mathcal{T} the *evolution semigroup* associated with \mathcal{U} . Its generator is denoted by $(G, D(G))$. For further information concerning this approach we refer to [14, 15, 16, 17, 18, 19, 21, 22, 23] and the references therein.

Denote by $L_{loc}^\infty(I, \mathcal{L}_s(X))$ the space of almost everywhere defined, strongly measurable, operator-valued functions $B(\cdot)$ (i.e., $B(\cdot)x$ is strongly measurable for all $x \in X$) such that $B(\cdot)$ is essentially bounded on compact intervals

in $I \subseteq \mathbb{R}$. Further, let $L^\infty(I, \mathcal{L}_s(X))$ be the subspace of a.e. defined, essentially bounded functions $B(\cdot)$. We stress that the operators $B(t)$ are defined for $t \in I \setminus N$, where the null set N only depends on $B(\cdot)$. Observe that $B(\cdot) \in L^\infty(I, \mathcal{L}_s(X))$ induces a bounded multiplication operator \mathcal{B} on E by setting $\mathcal{B}f := B(\cdot)f(\cdot)$, $f \in E$, and $\|\mathcal{B}\| \leq \|B(\cdot)\|_\infty$. With χ_A we denote the characteristic function of a set A .

We now obtain a result on bounded perturbations of evolution families by means of standard perturbation theory for \mathcal{C}_0 -semigroups applied to the evolution semigroup \mathcal{T} on E . Perturbations by strongly continuous $B(\cdot)$ were studied in [15] and [17] with the same approach. Other versions of Theorem 2.1 based on the Dyson-Phillips expansion are contained in [7, Ch. 2, 9].

Theorem 2.1. *Let $\mathcal{U} = (U(t, s))_{(t, s) \in D}$ be a strongly continuous evolution family on a Banach space X . Let $B(\cdot) \in L^\infty_{loc}(I, \mathcal{L}_s(X))$. Then there is a strongly continuous evolution family $\mathcal{U}_B = (U_B(t, s))_{(t, s) \in D}$ on X satisfying*

$$U_B(t, s)x = U(t, s)x + \int_s^t U(t, \tau)B(\tau)U_B(\tau, s)x \, d\tau, \quad (2.2)$$

$$U_B(t, s)x = U(t, s)x + \int_s^t U_B(t, \tau)B(\tau)U(\tau, s)x \, d\tau \quad (2.3)$$

for $x \in X$ and $(t, s) \in D$. Moreover, if $\mathcal{V} = (V(t, s))_{(t, s) \in D}$ is a family of bounded operators which is strongly continuous with respect to t and satisfies (2.2), then $\mathcal{V} = \mathcal{U}_B$.

Proof. Set $I_n := I \cap [-n, n]$ and let $B_n(\cdot) := \chi_{I_n} B(\cdot)$, $n \in \mathbb{N}$. We consider the bounded, strongly continuous evolution family \mathcal{U}_n with index set $D_{\mathbb{R}}$ as defined in (2.1). Let \mathcal{T}_n be the associated evolution semigroup on $E = L^1(\mathbb{R}, X)$ with generator $(G_n, D(G_n))$. As observed above, $B_n(\cdot)$ induces a bounded multiplication operator \mathcal{B}_n on E . Thus by [20], Thm. 3.1.1, Prop. 3.1.2, the perturbed operator $G_{n,B} := G_n + \mathcal{B}_n$ with domain $D(G_{n,B}) = D(G_n)$ generates a \mathcal{C}_0 -semigroup $\mathcal{T}_{n,B}$ on E satisfying

$$\mathcal{T}_{n,B}(t)f = \mathcal{T}_n(t)f + \int_0^t \mathcal{T}_n(t-\tau)\mathcal{B}_n\mathcal{T}_{n,B}(\tau)f \, d\tau, \quad (2.4)$$

$$\mathcal{T}_{n,B}(t)f = \mathcal{T}_n(t)f + \int_0^t \mathcal{T}_{n,B}(t-\tau)\mathcal{B}_n\mathcal{T}_n(\tau)f \, d\tau \quad (2.5)$$

for $f \in E$ and $t \geq 0$. From [21, Thm. 3.4] it follows that $\mathcal{T}_{n,B}$ is an evolution semigroup induced by a strongly continuous evolution family $\mathcal{U}_{n,B}$ on X . Using [18, Theorem 4.2] we derive from (2.4) and (2.5)

$$U_{n,B}(t, s)x = U(t, s)x + \int_s^t U(t, \tau)B(\tau)U_{n,B}(\tau, s)x \, d\tau, \quad (2.6)$$

$$U_{n,B}(t, s)x = U(t, s)x + \int_s^t U_{n,B}(t, \tau)B(\tau)U(\tau, s)x \, d\tau, \quad (2.7)$$

respectively, for $x \in X$ and $(t, s) \in D$ with $t, s \in [-n, n]$. A standard application of Gronwall's inequality shows that $U_B(t, s) := U_{n,B}(t, s)$ for $(t, s) \in D_I$ with $t, s \in [-n, n]$ is a (well-defined) strongly continuous evolution family \mathcal{U}_B which satisfies the assertions. \square

We now apply the above result to the following situation. Consider $X = L^p(\Omega)$ for a σ -finite measure space (Ω, μ) and $1 \leq p < \infty$. Assume that the evolution family \mathcal{U} on X is *positive*, that is, $U(t, s)x \geq 0$ for $(t, s) \in D$ and $0 \leq x \in X$. Let $V \in L^\infty(I \times \Omega)$. Then the function $V(t, \cdot) : \xi \mapsto V(t, \xi)$ is measurable and bounded for a.e. $t \in I$. So V induces bounded multiplication operators $V(t)$ on X with $\|V(t)\| \leq \|V\|_\infty$ for a.e. $t \in I$.

Lemma 2.2. *Let $V \in L^\infty(I \times \Omega)$ and $X = L^p(\Omega)$, $1 \leq p < \infty$. Then $V(\cdot) \in L^\infty(I, \mathcal{L}_s(X))$.*

Proof. By the preceding observations we only have to show that $t \mapsto V(t)x$ is strongly measurable for $x \in X$. There exist uniformly bounded, simple functions V_n converging to V pointwise a.e. as $n \rightarrow \infty$. Hence, for a.e. $t \in I$ we have $V_n(t, \cdot) \rightarrow V(t, \cdot)$ pointwise a.e.. By the dominated convergence theorem $V_n(t)x$ converges to $V(t)x$ in X as $n \rightarrow \infty$ for a.e. $t \in I$ and each $x \in X$. Moreover, it is easily seen that the mapping $t \mapsto V_n(t)x$ is strongly measurable. This yields the assertion.

Proposition 2.3. *Let $\mathcal{U} = (U(t, s))_{(t,s) \in D}$ be a strongly continuous evolution family on $X = L^p(\Omega)$, $1 \leq p < \infty$. Let $V \in L^\infty(I \times \Omega)$. Then the following assertions hold:*

- (a) *There is a strongly continuous evolution family $\mathcal{U}_V = (U_V(t, s))_{(t,s) \in D}$ on X satisfying*

$$U_V(t, s)x = U(t, s)x - \int_s^t U(t, \tau)V(\tau)U_V(\tau, s)x \, d\tau, \quad (2.8)$$

$$U_V(t, s)x = U(t, s)x - \int_s^t U_V(t, \tau)V(\tau)U(\tau, s)x \, d\tau \quad (2.9)$$

for $x \in X$ and $(t, s) \in D$. Moreover, \mathcal{U}_V is uniquely determined by (2.8).

- (b) If $V_n \in L^\infty(I \times \Omega)$ with $\|V_n\|_\infty \leq C$ and $V_n \rightarrow V$ pointwise a.e., then $U_{V_n}(t, s)$ converges strongly to $U_V(t, s)$ as $n \rightarrow \infty$ for each $(t, s) \in D$.
- (c) If \mathcal{U} is positive and $0 \leq \tilde{V} \leq V$ for $\tilde{V} \in L^\infty(I \times \Omega)$, then

$$0 \leq e^{-(t-s)\|V\|_\infty} U(t, s) \leq U_V(t, s) \leq U_{\tilde{V}}(t, s) \leq U(t, s) \quad (2.10)$$

for $(t, s) \in D$.

Proof. Assertion (a) follows from Lemma 2.2 and Theorem 2.1. Let V_n be as in (b). As in the proof of Lemma 2.2 one shows that for a.e. $t \in I$ the multiplication operators $V_n(t)$ converge strongly to $V(t)$ as $n \rightarrow \infty$. Fix $(t, s) \in D$ and $x \in X$. Then (2.8) yields

$$\begin{aligned} U_V(t, s)x - U_{V_n}(t, s)x &= \int_s^t U(t, \tau)(V_n(\tau) - V(\tau))U_V(\tau, s)x \, d\tau \\ &\quad - \int_s^t U(t, \tau)V_n(\tau)(U_V(\tau, s)x - U_{V_n}(\tau, s)x) \, d\tau. \end{aligned}$$

The first integral tends to 0 as $n \rightarrow \infty$ due to Lebesgue's theorem. Hence, for each $\varepsilon > 0$ there is an $N_\varepsilon \in \mathbb{N}$ such that

$$\|U_V(t, s)x - U_{V_n}(t, s)x\| \leq \varepsilon + CM \int_s^t \|U_V(\tau, s)x - U_{V_n}(\tau, s)x\| \, d\tau$$

for $n \geq N_\varepsilon$, where $M := \sup_{t \geq \tau \geq s} \|U(t, \tau)\|$. So Gronwall's inequality implies (b).

We consider the bounded, strongly continuous evolution family \mathcal{U}_n with index set $D_\mathbb{R}$ as defined in (2.1). Due to (a) there exists a perturbed evolution family $(\mathcal{U}_n)_V$. Let \mathcal{T}_n , resp. $(\mathcal{T}_n)_V$, be the associated evolution semigroup on $E = L^1(\mathbb{R}, X)$. Since V induces a bounded operator on E , the Trotter product formula, [20, Cor. 3.5.5], yields

$$(\mathcal{T}_n)_V(t)f = \lim_{m \rightarrow \infty} (\mathcal{T}_n(\frac{t}{m})e^{-\frac{t}{m}V})^m f$$

for $f \in E$. For $0 \leq \tilde{V} \leq V$ and positive \mathcal{U} we derive

$$0 \leq e^{-t\|V\|_\infty} \mathcal{T}_n(t) \leq (\mathcal{T}_n)_V(t) \leq (\mathcal{T}_n)_{\tilde{V}}(t) \leq \mathcal{T}_n(t). \quad (2.11)$$

Observe that $U_n(t, s) = U(t, s)$ and $(U_n)_V(t, s) = U_V(t, s)$ for $(t, s) \in D$ with $t, s \in [-n, n]$. Thus due to the strong continuity of $\mathcal{U}, \mathcal{U}_V, \mathcal{U}_{\tilde{V}}$, assertion (c) follows from (2.11).

3. Absorption evolution families. We now consider a positive, strongly continuous evolution family $\mathcal{U} = (U(t, s))_{(t,s) \in D}$ on $X = L^p(\Omega)$, $1 \leq p < \infty$, and a positive, measurable function $V : I \times \Omega \rightarrow \mathbb{R}$. Set $V^{(n)} := n\Pi \wedge V$, $n \in \mathbb{N}$, where Π denotes the constant function 1. Then $0 \leq V^{(n)} \in L^\infty(I \times \Omega)$ and $V^{(n)} \uparrow V$. Let \mathcal{U}_n be the evolution family corresponding to \mathcal{U} and $V^{(n)}$ as obtained in Proposition 2.3. By (2.10) we have $0 \leq U_n(t, s)x \leq U_m(t, s)x \leq U(t, s)x$ for $n \geq m$, $0 \leq x \in X$, and $(t, s) \in D$. Therefore we can define a bounded operator $U_V(t, s)$ on X by setting $U_V(t, s)x := \lim_{n \rightarrow \infty} U_n(t, s)x$, $(t, s) \in D$. We first collect several properties of the family $\mathcal{U}_V = (U_V(t, s))_{(t,s) \in D}$.

Lemma 3.1. *Let \mathcal{U} be a positive, strongly continuous evolution family on $X = L^p(\Omega)$, $1 \leq p < \infty$, and let $V, \tilde{V} : I \times \Omega \rightarrow \mathbb{R}$ be measurable such that $0 \leq \tilde{V} \leq V$. Define \mathcal{U}_V on X as above. Then the following assertions hold:*

- (a) $0 \leq U_V(t, s) \leq U_{\tilde{V}}(t, s) \leq U(t, s)$ for $(t, s) \in D$.
- (b) \mathcal{U}_V is a strongly measurable evolution family on X , i.e., the mapping $(t, s) \mapsto U_V(t, s)x$ is strongly measurable for each $x \in X$.
- (c) If $V_n : I \times \Omega \rightarrow \mathbb{R}$ is measurable such that $0 \leq V_n \leq V$ and $V_n \rightarrow V$ pointwise a.e. as $n \rightarrow \infty$, then $U_{V_n}(t, s)$ converges strongly to $U_V(t, s)$ for $(t, s) \in D$.

Proof. Assertion (a) is clear from the definition and (2.10). Since \mathcal{U}_n is a strongly continuous evolution family, assertion (b) follows from the strong convergence of $U_n(t, s)$ to $U_V(t, s)$ as $n \rightarrow \infty$. Consider V_n as in (c). Let $V^{(m)} = m\Pi \wedge V$ and $V_n^{(m)} = m\Pi \wedge V_n$ for $n, m \in \mathbb{N}$. We have $0 \leq V_n^{(m)} \leq V^{(m)} \in L^\infty(I \times \Omega)$ and $V_n^{(m)} \rightarrow V^{(m)}$ pointwise a.e. as $n \rightarrow \infty$. Hence, by Proposition 2.3(b) the operator $U_{V_n^{(m)}}(t, s)$ converges strongly to $U_m(t, s)$ as $n \rightarrow \infty$ for $(t, s) \in D$. On the other hand, $V_n^{(m)} \leq V_n$ yields $U_{V_n}(t, s) \leq U_{V_n^{(m)}}(t, s)$ for $n, m \in \mathbb{N}$ and $(t, s) \in D$. Thus

$$\begin{aligned} U_V(t, s)x &\leq \liminf_{n \rightarrow \infty} U_{V_n}(t, s)x \leq \limsup_{n \rightarrow \infty} U_{V_n}(t, s)x \\ &\leq \lim_{n \rightarrow \infty} U_{V_n^{(m)}}(t, s)x = U_m(t, s)x \end{aligned}$$

for $0 \leq x \in X$, $(t, s) \in D$, and $m \in \mathbb{N}$. Now (c) follows by letting $m \rightarrow \infty$. \square

As a consequence, the evolution family \mathcal{U}_V depends only on \mathcal{U} and V and not on a particular approximating sequence (V_n) (satisfying $0 \leq V_n \leq V$). The following definition generalizes Voigt's notion of an *absorption semigroup* (if $V \geq 0$), see [24], [25].

Definition 3.2. Let $0 \leq V : I \times \Omega \rightarrow \mathbb{R}$ be measurable and let $\mathcal{U} = (U(t, s))_{(t,s) \in D}$ be a positive, strongly continuous evolution family on $X = L^p(\Omega)$, $1 \leq p < \infty$. Then the evolution family \mathcal{U}_V on X as defined above is called the *absorption evolution family* corresponding to \mathcal{U} and V .

We stress that in general \mathcal{U}_V is not strongly continuous and there may exist $(t, s) \in D$ such that (2.8) fails. For instance, consider $X = \mathbb{C}$, $I = \mathbb{R}_+$, $U(t, s) \equiv 1$, and $V(t) = \frac{1}{t}$, yielding $U_V(t, s) = \exp(-\int_s^t V(\tau) d\tau)$, $t \geq s \geq 0$, see also [25, Ex. 4.1]. However, in a more specific situation we can show strong continuity and variation of constants formulas for \mathcal{U}_V .

For the next result we recall that a subspace Y of $X = L^p(\Omega)$ is called an *ideal* in X if $x \in X$, $0 \leq y \in Y$, and $|x| \leq y$ imply $x \in Y$. Further, we refer to [9, pp. 52] for the definition of the Pettis-integral. If $V : I \times \Omega \rightarrow \mathbb{C}$ is measurable we denote by $(V(t), D(V(t)))$, $t \in I$, the induced multiplication operators on X with maximal domain.

Theorem 3.3. Let $\mathcal{U} = (U(t, s))_{(t,s) \in D}$ be a positive, strongly continuous evolution family on $X = L^p(\Omega)$, $1 \leq p < \infty$, and let $V : I \times \Omega \rightarrow \mathbb{R}$ be positive and measurable. Consider a dense ideal Y in X .

- (a) Assume that for $s \in I$ and a.e. $t \in I$ with $t > s$ we have $U(t, s)Y \subseteq D(V(t))$, and

$$\begin{aligned} & \text{for } y \in Y \text{ and } [c, d] \subseteq D \text{ there is } \psi \in L^1[c, d] \text{ with} \\ & \|V(\cdot)U(\cdot, s)y\|_p \leq \psi \text{ on } [s, d] \text{ for all } s \in [c, d]. \end{aligned} \quad (3.1)$$

Then, $U_V(t, s)Y \subseteq D(V(t))$ for $s \in I$ and a.e. $I \ni t > s$ and \mathcal{U}_V satisfies (2.8) and (2.9) for $(t, s) \in D$ and $x \in Y$. In addition, suppose that the mapping $s \mapsto V(t)U(t, s)y$ is continuous for $y \in Y$, a.e. $t \in I$, and $s \in (-\infty, t) \cap I$. Then \mathcal{U}_V is strongly continuous.

- (b) Assume $U(t, s)X \subseteq D(V(t))$ for $s \in I$ and a.e. $t \in I$ with $t > s$ and that (3.1) holds. Then \mathcal{U}_V is strongly continuous. Moreover, $U_V(t, s)X \subseteq D(V(t))$ for $s \in I$ and a.e. $I \ni t > s$, and (2.8) and

(2.9) hold for $(t, s) \in D$ and $x \in X$, where the integrals are Pettis-integrals if $p > 1$.

Proof. (a): From $0 \leq U_V(t, s) \leq U(t, s)$ it follows that $U_V(t, s)Y \subseteq D(V(t))$ for $s \in I$ and a.e. $I \ni t > s$. Consider the absorption evolution family \mathcal{U}_n corresponding to $V^{(n)} = n\mathbb{I} \wedge V$, $n \in \mathbb{N}$. Using the dominated convergence theorem, we derive

$$\lim_{n \rightarrow \infty} V^{(n)}(t)U_n(t, s)y = V(t)U_V(t, s)y \quad (3.2)$$

for $y \in Y$, $s \in I$, and a.e. $I \ni t > s$. Further, by Proposition 2.3 we have

$$U_n(t, s)y = U(t, s)y - \int_s^t U(t, \tau)V^{(n)}(\tau)U_n(\tau, s)y \, d\tau, \quad (3.3)$$

$$U_n(t, s)y = U(t, s)y - \int_s^t U_n(t, \tau)V^{(n)}(\tau)U(\tau, s)y \, d\tau. \quad (3.4)$$

For $y \in Y$ and $(t, s) \in D$ we set

$$\begin{aligned} f_n(\tau) &:= U(t, \tau)V^{(n)}(\tau)U_n(\tau, s)y, & f(\tau) &:= U(t, \tau)V(\tau)U_V(\tau, s)y, \\ g_n(\tau) &:= U_n(t, \tau)V^{(n)}(\tau)U(\tau, s)y, & g(\tau) &:= U_V(t, \tau)V(\tau)U(\tau, s)y \end{aligned}$$

for a.e. $\tau \in (s, t]$ and $n \in \mathbb{N}$. Then (3.1) yields $\|f_n(\cdot)\|_p, \|g_n(\cdot)\|_p \leq \varphi$ for a function $\varphi \in L^1[s, t]$. Further, by (3.2) we obtain $f_n(\tau) \rightarrow f(\tau)$ and $g_n(\tau) \rightarrow g(\tau)$ in X for a.e. $\tau \in (s, t]$. Consequently, for $y \in Y$ equations (2.8) and (2.9) follow from (3.3) and (3.4), respectively, due to Lebesgue's theorem. To prove strong continuity of \mathcal{U}_V it suffices to show that $D \ni (t, s) \mapsto U_V(t, s)y \in X$ is continuous for $y \in Y$. By (3.1) and the strong continuity of \mathcal{U} we see that the mapping

$$t \mapsto \int_s^t U(t, \tau)V(\tau)U_V(\tau, s)y \, d\tau \in X, \quad t \geq s,$$

is continuous for each $s \in I$. Thus (2.8) implies that $t \mapsto U_V(t, s)$ is strongly continuous for $t \geq s$. Because of (2.9) it remains to show the continuity of the mapping

$$\Phi : (t, s) \mapsto \int_s^t U_V(t, \tau)V(\tau)U(\tau, s)y \, d\tau \in X.$$

By (3.1) this is clear at points (t_0, t_0) . So let $t > s$. By the assumption and the strong continuity of $U_V(\cdot, s)$, the mapping

$$D \ni (t, s) \mapsto U_V(t, \tau)V(\tau)U(\tau, s)y$$

is continuous for a.e. $\tau \in (s, t)$. Then we can derive the continuity of Φ by using (3.1).

(b) From the assumption we infer that $U_V(t, s)X \subseteq D(V(t))$ for $s \in I$ and a.e. $I \ni t > s$. Thus, $V(t)U(t, s)$ and $V(t)U_V(t, s)$ are everywhere defined and hence bounded for $s \in I$ and a.e. $I \ni t > s$. Since $V(t)U(t, s) = V(t)U(t, r)U(r, s)$ for $t > r > s$ this implies strong continuity of $s \mapsto V(t)U(t, s)$ and, due to (a), the first assertion in (b) follows. Further, for $0 \leq x \in X$ there is an increasing sequence $(y_n) \subseteq Y_+$ converging to x . By (a) we have

$$U(t, s)y_n - U_V(t, s)y_n = \int_s^t U(t, \tau)V(\tau)U_V(\tau, s)y_n d\tau =: \int_s^t h_n(\tau) d\tau, \quad (3.5)$$

$$U(t, s)y_n - U_V(t, s)y_n = \int_s^t U_V(t, \tau)V(\tau)U(\tau, s)y_n d\tau =: \int_s^t k_n(\tau) d\tau \quad (3.6)$$

for $(t, s) \in D$. Then $h_n(\tau)$ and $k_n(\tau)$ converge in X monotonically to $h(\tau) := U(t, \tau)V(\tau)U_V(\tau, s)x$ and $k(\tau) := U_V(t, \tau)V(\tau)U(\tau, s)x$, respectively, for a.e. $\tau \in (s, t]$. In particular, h and k are strongly measurable. Moreover, let $0 \leq x' \in X'$. The positive functions $\langle h_n(\cdot), x' \rangle$ and $\langle k_n(\cdot), x' \rangle$ converge monotonically to $\langle h(\cdot), x' \rangle$ and $\langle k(\cdot), x' \rangle$ as $n \rightarrow \infty$, respectively. On the other hand,

$$\int_s^t h_n(\tau) d\tau \rightarrow U(t, s)x - U_V(t, s)x \quad \text{and} \quad \int_s^t k_n(\tau) d\tau \rightarrow U(t, s)x - U_V(t, s)x$$

in X by (3.5) and (3.6), respectively. By the monotone convergence theorem we obtain that $\langle h(\cdot), x' \rangle$ and $\langle k(\cdot), x' \rangle$ are integrable and

$$\langle U(t, s)x - U_V(t, s)x, x' \rangle = \int_s^t \langle h(\tau), x' \rangle d\tau = \int_s^t \langle k(\tau), x' \rangle d\tau.$$

If $p = 1$ we can choose $x' = \mathbb{I}$. Since $\|z\|_1 = \langle z, \mathbb{I} \rangle$ for each $0 \leq z \in L^1(\Omega)$, the functions h and k are Bochner integrable, and (2.8) and (2.9) hold for each $x \in L^1(\Omega)$ with Bochner integrals. For $1 < p < \infty$ the functions h and k are Pettis integrable, and (2.8) and (2.9) hold for each $x \in L^p(\Omega)$, $1 < p < \infty$, where the integrals are Pettis integrals.

Corollary 3.4. *Let \mathcal{U} be a positive, strongly continuous evolution family on $X = L^p(\Omega)$, $1 \leq p < \infty$. Let $0 \leq V \in L^1_{loc}(I, L^\infty(\Omega))$. Then the conclusions of Theorem 3.3 hold for $Y = X$.*

In order to obtain further sufficient conditions for (3.1) we now introduce the concept of a Gaussian estimate. Let Ω be an open subset of \mathbb{R}^N . We identify $X = L^p(\Omega)$ with a subspace of $L^p(\mathbb{R}^N)$ by extending functions by 0. Consider the semigroup $(e^{t\Delta})_{t \geq 0}$ generated by the Laplacian on $L^p(\mathbb{R}^N)$, $1 \leq p \leq \infty$. It is known that $e^{t\Delta}x = K_t * x$, where $K_t(\xi) = (4\pi t)^{-N/2} \exp(-\frac{|\xi|^2}{4t})$, $t > 0$, $\xi \in \mathbb{R}^N$, is the Gaussian kernel. A positive evolution family $\mathcal{U} = (U(t, s))_{(t,s) \in D}$ on X is said to satisfy an *upper Gaussian estimate* if there are constants $M, a > 0$ such that $0 \leq U(t, s)x \leq M e^{a(t-s)\Delta}x$ for $0 \leq x \in L^p(\Omega)$. In Section 5 we consider examples for evolution families satisfying an upper Gaussian estimate. We now easily derive the following result.

Corollary 3.5. *Let $\Omega \subseteq \mathbb{R}^N$ be open and let $X = L^p(\Omega)$, $1 \leq p < \infty$. Let \mathcal{U} be a positive, strongly continuous evolution family on X satisfying an upper Gaussian estimate. Let $V = V_1 + V_2 + V_3$, where $0 \leq V_1 \in L^1_{loc}(I, X)$, $0 \leq V_2 \in L^1_{loc}(I, L^\infty(\Omega))$, $0 \leq V_3 \in L^\infty(I \times \Omega)$. Then the conclusions of Theorem 3.3 hold for $Y = X \cap L^\infty(\Omega)$.*

Proof. Observe that the Gaussian estimate and Young's inequality yield

$$|U(t, s)x| \leq M K_{a(t-s)} * |x| \in Y \subseteq D(V(t))$$

for $t > s$ and $x \in X$. In particular, $\|U(t, s)y\|_\infty \leq M\|y\|_\infty$ and

$$\|V(t)U(t, s)y\|_p \leq M(\|V_1(t)\|_p \|y\|_\infty + \|V_2(t)\|_\infty \|y\|_p + \|V_3\|_\infty \|y\|_p)$$

for $y \in Y = X \cap L^\infty(\Omega)$ and $(t, s) \in D$. \square

The assumptions of part (a) and (b) of Theorem 3.3 can be checked for a much larger class of potentials V . For simplicity we concentrate on the case $\Omega = \mathbb{R}^N$. By λ we denote the Lebesgue measure on \mathbb{R}^N . We set $\varepsilon_\alpha(\xi) := e^{-\alpha|\xi|^2}$.

Corollary 3.6. *Let $X = L^p(\mathbb{R}^N)$, $1 \leq p < \infty$. Let \mathcal{U} be a positive, strongly continuous evolution family on X satisfying an upper Gaussian estimate. Let $0 \leq V \in L^1_{loc}(I, L^p(\mathbb{R}^N, \varepsilon_\alpha \lambda))$ for all $\alpha > 0$. Then the conclusions of Theorem 3.3 (a) and (b) hold for $Y = \{y \in L^\infty(\mathbb{R}^N) : y \text{ has compact support}\}$.*

Proof. Fix $y \in Y$ and $d > 0$. Consider $t, s \in I$ with $d \geq t - s > 0$. Set $K := \text{supp } y$. Choose a ball B of radius r such that $\text{dist}(\xi, K) \geq 1$ for $\xi \notin B$. Further, observe that the Gaussian kernel K_τ satisfies

$$K_{a\tau}(\xi) \leq 2^{\frac{N}{2}} e^{-\frac{1}{8a\tau}} K_{2a\tau}(\xi) \quad (3.7)$$

for $a > 0$, $\tau > 0$, and $|\xi| \geq 1$. Then, by using the Gaussian estimate and (3.7) we compute

$$\begin{aligned} \|V(t)U(t, s)y\|_p^p &\leq M^p \int_{\mathbb{R}^N} V(t, \xi)^p \left(\int_K K_{a(t-s)}(\xi - \eta) |y(\eta)| d\eta \right)^p d\xi \\ &\leq M^p \|y\|_\infty^p \left(\int_B V(t, \xi)^p d\xi \right. \\ &\quad \left. + \int_{B^c} V(t, \xi)^p 2^{\frac{pN}{2}} e^{-\frac{p}{8a(t-s)}} \left(\int_K K_{2a(t-s)}(\xi - \eta) d\eta \right)^p d\xi \right) \\ &\leq M^p \|y\|_\infty^p \left(\|V(t)\|_{L^p(B)}^p + \lambda(K)^p \right. \\ &\quad \left. \times (4a\pi(t-s))^{-\frac{pN}{2}} e^{-\frac{p}{8a(t-s)}} \int_{B^c} V(t, \xi)^p e^{-\frac{p \text{dist}(\xi, K)^2}{8a(t-s)}} d\xi \right) \\ &\leq C_1 \left(\|V(t)\|_{L^p(B)}^p + \int_{B^c} V(t, \xi)^p e^{-\frac{p(|\xi|-r)^2}{8ad}} d\xi \right) \\ &\leq C_2 \left(\|V(t)\|_{L^p(B)}^p + \int_{B^c} V(t, \xi)^p e^{-\alpha|\xi|^2} d\xi \right) \leq C_3 \|V(t)\|_{L^p(\mathbb{R}^N, \varepsilon_\alpha \lambda)}^p \end{aligned}$$

for a.e. t and constants $C_k, \alpha > 0$ depending on y and d . This establishes (3.1).

Next, notice that for each sequence $(s_n) \subseteq I$ with $s_n < t$ and $s_n \rightarrow s$ there is a subsequence (s_{n_k}) such that $V(t)U(t, s_{n_k})y$ converges to $V(t)U(t, s)y$ pointwise a.e. Hence, by Lebesgue's theorem, for showing the continuity of $s \mapsto V(t)U(t, s)y$ it suffices to find $z \in X$ such that

$$z \in D(V(t)) \text{ for a.e. } t \in I \quad \text{and} \quad |U(t, r)y| \leq z \text{ for } 0 \leq t - r \leq 2d. \quad (3.8)$$

Clearly, by the Gaussian estimate we have $|U(t, r)y|(\xi) \leq M\|y\|_\infty$ for $\xi \in \mathbb{R}^N$. Moreover, for $\xi \notin B$ we infer from (3.7) that

$$\begin{aligned} |U(t, r)y|(\xi) &\leq M 2^{\frac{N}{2}} \|y\|_\infty e^{-\frac{1}{8a(t-r)}} \int_K K_{2a(t-r)}(\xi - \eta) d\eta \\ &\leq M 2^{\frac{N}{2}} \|y\|_\infty (8a\pi(t-r))^{-\frac{N}{2}} e^{-\frac{1}{8a(t-r)}} \int_K e^{-\frac{|\xi-\eta|^2}{16d}} d\eta \\ &\leq C \|y\|_\infty \lambda(K) e^{-\frac{(|\xi|-r)^2}{16d}} =: \varphi(\xi) \end{aligned}$$

for some constant $C > 0$. Setting $z(\xi) := M\|y\|_\infty$ for $\xi \in B$ and $z(\xi) := \varphi(\xi)$ for $\xi \notin B$ yields (3.8) for a.e. $t \in I$.

4. Exponential stability. In this section we assume that I is unbounded, $\Omega = \mathbb{R}^N$, and \mathcal{U} is positive. If \mathcal{U} satisfies an upper Gaussian estimate, then the absorption evolution family \mathcal{U}_V is uniformly bounded since $0 \leq U_V(t, s) \leq U(t, s)$. So the question arises under which conditions on V the evolution family \mathcal{U}_V is exponentially stable. For this we introduce the following notion. Let $B(\xi, r) := \{\eta \in \mathbb{R}^N : |\xi - \eta| \leq r\}$.

Definition 4.1. Let $0 \leq V \in L^1_{loc}(I \times \mathbb{R}^N)$ and $t > 0$. We say that V belongs to the class \mathcal{E}_t if there are constants $c, r > 0$ such that

$$\int_s^{s+t} \int_{B(\xi, r)} V(\tau, \eta) d\eta d\tau \geq c \quad \text{for all } s \in I, \xi \in \mathbb{R}^N. \quad (4.1)$$

This definition generalizes a concept introduced by W. Arendt and C.J.K. Batty for time independent potentials and by D. Daners and P. Koch-Medina for time periodic potentials; see [3, Prop. 1.4] and [8]. We begin with some preparations. We first give a sufficient condition for (4.1).

Lemma 4.2. Let $0 \leq V \in L^1_{loc}(I \times \mathbb{R}^N)$. Let $c, r, t > 0$ and $m \geq 0$ be constants such that

$$\int_s^{s+t} \int_{B(\xi, r)} V(\tau, \eta) d\eta d\tau \geq c$$

for $s \in I$ and $\xi \in \mathbb{R}^N \setminus A_s$, where $A_s \subseteq \mathbb{R}^N$ is measurable and $\lambda(A_s) \leq m$. Then $V \in \mathcal{E}_t$.

Proof. Without loss of generality let $m > 0$. Set

$$r_1 := r + (2m)^{\frac{1}{N}} \lambda(B(0, 1))^{-\frac{1}{N}}.$$

Fix $\xi \in \mathbb{R}^N$ and $s \in I$. Then there is a vector $\xi(s) \in \mathbb{R}^N \setminus A_s$ with $|\xi - \xi(s)| \leq r_1 - r$. (Otherwise, $B(\xi, r_1 - r) \subseteq A_s$, and hence $m \geq \lambda(A_s) \geq 2m$, which is a contradiction.) Thus

$$\int_s^{s+t} \int_{B(\xi, r_1)} V(\tau, \eta) d\eta d\tau \geq \int_s^{s+t} \int_{B(\xi(s), r)} V(\tau, \eta) d\eta d\tau \geq c.$$

So (4.1) is satisfied with constants $t, c, r_1 > 0$. \square

The following characterization of exponential stability of an evolution family can easily be shown by using that $\|T\| = \|T'\mathbb{I}\|_\infty$ for $0 \leq T \in \mathcal{L}(L^1(\Omega))$, where \mathbb{I} denotes the constant function 1.

Lemma 4.3. *Let \mathcal{U} be a positive, exponentially bounded evolution family on $L^1(\Omega)$. Then \mathcal{U} is exponentially stable if and only if there are constants $t_0 > 0$ and $\delta < 1$ such that $U(s + t_0, s)' \mathbb{I} \leq \delta \mathbb{I}$ for all $s \in I$. In this case, $\delta > 0$ can be chosen arbitrarily small.*

A positive evolution family \mathcal{U} on $L^p(\mathbb{R}^N)$ is said to satisfy a *lower Gaussian estimate* if there are constants $\tilde{M}, \tilde{a} > 0$ such that $U(t, s) \geq \tilde{M} e^{\tilde{a}(t-s)\Delta}$ for $(t, s) \in D$. For $p = 1$ this implies $U(t, s)' \mathbb{I} \geq \tilde{M} \mathbb{I}$. Moreover, \mathcal{U} is called *submarkovian* if $p = 1$ and $U(t, s)' \mathbb{I} \leq \mathbb{I}$ for $(t, s) \in D$. Further, we use the space $L^1_{loc,u}(I \times \mathbb{R}^N)$ of uniformly locally integrable functions f endowed with the norm

$$\|f\|_{loc,u} := \sup_{s \in I, \xi \in \mathbb{R}^N} \int_s^{s+1} \int_{B(\xi,1)} |f(\tau, \eta)| d\eta d\tau.$$

Notice that a different choice of constants in the integrals, say $t, r > 0$ instead of 1, yields an equivalent norm.

In Theorems 4.4 and 4.6 we present for $X = L^1(\mathbb{R}^N)$ necessary and sufficient conditions on \mathcal{U} and V such that the absorption evolution family \mathcal{U}_V is exponentially stable. The case $X = L^p(\mathbb{R}^N)$ is considered afterwards.

Theorem 4.4. *Let \mathcal{U} be a submarkovian, strongly continuous evolution family on $X = L^1(\mathbb{R}^N)$ satisfying a lower Gaussian estimate. Let*

$$0 \leq V \in \overline{L^\infty(I \times \mathbb{R}^N)}_{+}^{L^1_{loc,u}}.$$

If $V \in \mathcal{E}_t$ for some $t > 0$, then \mathcal{U}_V is exponentially stable.

Proof. (i) Let $V \in \mathcal{E}_t$ satisfy the above assumptions, where $t > 0$ is fixed. Then there are functions $0 \leq \tilde{V}_n \in L^\infty(I \times \mathbb{R}^N)$ converging to V in $L^1_{loc,u}(I \times \mathbb{R}^N)$. Set $V_n := \tilde{V}_n \wedge V$. Choose $c, r > 0$ as in (4.1). Then

$$\begin{aligned} \int_s^{s+t} \int_{B(\xi,r)} V_n(\tau, \eta) d\eta d\tau &\geq c - \int_s^{s+t} \int_{B(\xi,r)} (V(\tau, \eta) - V_n(\tau, \eta)) d\eta d\tau \\ &\geq c - C \|V - \tilde{V}_n\|_{loc,u}, \end{aligned}$$

where C only depends on t and r . Hence, $V_n \in \mathcal{E}_t$ for n large enough. We also have $0 \leq U_V(t, s) \leq U_{V_n}(t, s)$ by Lemma 3.1. As a consequence, it suffices to consider $0 \leq V \in L^\infty(I \times \mathbb{R}^N)$.

(ii) Now let $0 \leq V \in L^\infty(I \times \mathbb{R}^N)$. Choose $t_0 > t$. By assumption, $\mathbb{I} \geq U(t, s)' \mathbb{I} \geq \tilde{M} \mathbb{I}$ for $(t, s) \in D$. So we derive from (2.9), (2.10), and the lower Gaussian estimate that

$$\begin{aligned} & \langle x, U_V(s + t_0, s)' \mathbb{I} \rangle \\ &= \langle x, U(s + t_0, s)' \mathbb{I} \rangle - \int_s^{s+t_0} \langle x, U(\tau, s)' V(\tau) U_V(s + t_0, \tau)' \mathbb{I} \rangle d\tau \\ &\leq \langle x, \mathbb{I} \rangle - \tilde{M} e^{-t_0 \|V\|_\infty} \int_s^{s+t_0} \langle x, U(\tau, s)' V(\tau) \mathbb{I} \rangle d\tau \\ &\leq \langle x, \mathbb{I} \rangle - \tilde{M}^2 e^{-t_0 \|V\|_\infty} \int_s^{s+t_0} \langle x, e^{\tilde{a}(\tau-s)\Delta} V(\tau) \mathbb{I} \rangle d\tau \end{aligned}$$

for $s \in I$ and $0 \leq x \in X$. Also by Fubini's theorem,

$$\begin{aligned} & \int_s^{s+t_0} \langle x, e^{\tilde{a}(\tau-s)\Delta} V(\tau) \mathbb{I} \rangle d\tau \\ &= \int_s^{s+t_0} \int_{\mathbb{R}^N} \left(x(\eta) \int_{\mathbb{R}^N} K_{\tilde{a}(\tau-s)}(\xi - \eta) V(\tau, \xi) d\xi \right) d\eta d\tau \\ &= \left\langle x, \int_s^{s+t_0} \int_{\mathbb{R}^N} K_{\tilde{a}(\tau-s)}(\xi - \cdot) V(\tau, \xi) d\xi d\tau \right\rangle. \end{aligned}$$

Set $\varepsilon := \tilde{a}(t_0 - t) > 0$ and choose $r > 0$ as in (4.1). Then (4.1) yields

$$\begin{aligned} & \int_s^{s+t_0} \int_{\mathbb{R}^N} K_{\tilde{a}(\tau-s)}(\xi - \eta) V(\tau, \xi) d\xi d\tau \\ &\geq \int_{s+t_0-t}^{s+t_0} \int_{B(\eta, r)} K_{\tilde{a}(\tau-s)}(\xi - \eta) V(\tau, \xi) d\xi d\tau \geq \frac{c}{(4\pi\tilde{a}t_0)^{N/2}} e^{-\frac{r^2}{4\varepsilon}} =: ck_{\varepsilon, r} \end{aligned}$$

for $s \in I$ and a.e. $\eta \in \mathbb{R}^N$. Altogether we obtain

$$\langle x, U_V(s + t_0, s)' \mathbb{I} \rangle \leq (1 - ck_{\varepsilon, r} \tilde{M}^2 e^{-t_0 \|V\|_\infty}) \langle x, \mathbb{I} \rangle \quad (4.2)$$

for all $0 \leq x \in L^1(\mathbb{R}^N)$. Hence, $U_V(s + t_0, s)' \mathbb{I} \leq \delta \mathbb{I}$ for $\delta := 1 - ck_{\varepsilon, r} \tilde{M}^2 e^{-t_0 \|V\|_\infty} < 1$ and $s \in I$. Thus the assertion follows from Lemma 4.3.

Remark 4.5. (a) Theorem 4.4 is false if we only require $0 \leq V \in \mathcal{E}_t \cap L^1_{loc,u}(I \times \mathbb{R}^N)$. An example is given in [3, p. 1019], where $V(t) \equiv V_0$ and $U(t, s) = e^{(t-s)\Delta}$.

(b) Let $V = V_1 + V_2 + V_3 + V_4$, where $0 \leq V_1 \in L^1(I, L^1(\mathbb{R}^N))$, $0 \leq V_2 \in L^1(I, L^\infty(\mathbb{R}^N))$, $0 \leq V_3 \in L^\infty(I \times \mathbb{R}^N)$, and $0 \leq V_4 \in \{f \in L^\infty(I, L^1(\mathbb{R}^N)) : f(\tau, \cdot) = f(\tau + T, \cdot) \text{ for some } T > 0\}$. Consider $V_k^{(n)} = n\mathbb{I} \wedge V_k \in L^\infty(I \times \mathbb{R}^N)$ for $k = 1, 2, 3, 4$. Then $0 \leq V_n := \sum_{k=1}^4 V_k^{(n)}$ converges to V in $L^1_{loc,u}(I \times \mathbb{R}^N)$.

Proof of (b). It suffices to show $V_k^{(n)} \rightarrow V_k$ in $L^1_{loc,u}$ for $k = 1, 2, 3, 4$. This is clear for V_1 and V_3 . Further, for $s \in I$ and $\xi \in \mathbb{R}^N$

$$\begin{aligned} & \int_s^{s+1} \int_{B(\xi, 1)} (V_2(\tau, \eta) - V_2^{(n)}(\tau, \eta)) d\eta d\tau \\ & \leq \lambda(B(0, 1)) \int_I \|V_2(\tau, \cdot) - V_2^{(n)}(\tau, \cdot)\|_\infty d\tau. \end{aligned}$$

For a.e. τ the integrand on the right-hand side equals 0 for n large enough. So by the dominated convergence theorem the right-hand side converges to 0 as $n \rightarrow \infty$. Finally, for $s \in I$ and $\xi \in \mathbb{R}^N$

$$\int_s^{s+T} \int_{B(\xi, 1)} (V_4(\tau, \eta) - V_4^{(n)}(\tau, \eta)) d\eta d\tau \leq \int_0^{2T} \|V_4(\tau, \cdot) - V_4^{(n)}(\tau, \cdot)\|_1 d\tau.$$

Since for a.e. τ we have $V_4^{(n)}(\tau, \eta) \rightarrow V_4(\tau, \eta)$ for a.e. $\eta \in \mathbb{R}^N$, the integrand on the right-hand side tends to 0 as $n \rightarrow \infty$ for a.e. τ . By Lebesgue's theorem the claim follows.

Theorem 4.6. *Let \mathcal{U} be a positive, strongly continuous evolution family on $X = L^1(\mathbb{R}^N)$ satisfying an upper Gaussian estimate. Assume that there is a constant $d > 0$ such that $U(t, s)' \mathbb{I} \geq d\mathbb{I}$, $(t, s) \in D$. Let $V = V_1 + V_2 + V_3$, where $0 \leq V_1 \in L^1_{loc,u}(I, L^1(\mathbb{R}^N))$, $0 \leq V_2 \in L^1(I, L^\infty(\mathbb{R}^N))$, and $0 \leq V_3 \in L^\infty(I \times \mathbb{R}^N)$. Then exponential stability of \mathcal{U}_V implies $V \in \mathcal{E}_t$ for some $t > 0$.*

Proof. By Lemma 4.3 there are constants $t > 0$, $0 < \delta < d$ such that $U_V(s+t, s)' \mathbb{I} \leq \delta \mathbb{I}$ for $s \in I$. Fix $s \in I$ and $0 \leq x \in X \cap L^\infty(\mathbb{R}^N)$. By

Corollary 3.5 equation (2.9) holds. Then the assumptions yield

$$\begin{aligned}
 \delta \langle x, \mathbb{I} \rangle &\geq \langle x, U_V(s+t, s)' \mathbb{I} \rangle \\
 &= \langle x, U(s+t, s)' \mathbb{I} \rangle - \int_s^{s+t} \langle V(\tau) U(\tau, s) x, U_V(t+s, \tau)' \mathbb{I} \rangle d\tau \quad (4.3) \\
 &\geq d \langle x, \mathbb{I} \rangle - M^2 \int_s^{s+t} \langle V(\tau) e^{a(\tau-s)\Delta} x, \mathbb{I} \rangle d\tau.
 \end{aligned}$$

Since $0 \leq x \in L^1 \cap L^\infty$, Fubini's theorem implies

$$\begin{aligned}
 &\int_s^{s+t} \langle V(\tau) e^{a(\tau-s)\Delta} x, \mathbb{I} \rangle d\tau \\
 &= \int_s^{s+t} \int_{\mathbb{R}^N} V(\tau, \xi) \int_{\mathbb{R}^N} K_{a(\tau-s)}(\xi - \eta) x(\eta) d\eta d\xi d\tau \quad (4.4) \\
 &= \int_{\mathbb{R}^N} x(\eta) \left(\int_s^{s+t} \int_{\mathbb{R}^N} K_{a(\tau-s)}(\xi - \eta) V(\tau, \xi) d\xi d\tau \right) d\eta.
 \end{aligned}$$

From (4.3) and (4.4) we infer

$$\int_s^{s+t} \int_{\mathbb{R}^N} K_{a(\tau-s)}(\xi - \eta) V(\tau, \xi) d\xi d\tau \geq \frac{d - \delta}{M^2} \quad (4.5)$$

for $s \in I$ and a.e. $\eta \in \mathbb{R}^N$. On the other hand we have

$$\begin{aligned}
 \left\| \int_s^{s+t} \int_{\mathbb{R}^N} K_{a(\tau-s)}(\xi - \cdot) V_1(\tau, \xi) d\xi d\tau \right\|_1 &\leq \int_s^{s+t} \|V_1(\tau)\|_1 d\tau \\
 &\leq \tilde{C} \|V_1\|_{L^1_{loc,u}} =: C,
 \end{aligned}$$

where C does not depend on s . Thus the measure of the set

$$A_s := \left\{ \eta \in \mathbb{R}^N : \int_s^{s+t} \int_{\mathbb{R}^N} K_{a(\tau-s)}(\xi - \eta) V_1(\tau, \xi) d\xi d\tau \geq \frac{d - \delta}{2M^2} \right\}$$

is bounded by $m := \frac{2M^2 C}{d - \delta}$ for all $s \in I$. We set $V_\infty := V_2 + V_3$. Then (4.5) implies

$$\int_s^{s+t} \int_{\mathbb{R}^N} K_{a(\tau-s)}(\xi - \eta) V_\infty(\tau, \xi) d\xi d\tau \geq \frac{d - \delta}{2M^2}$$

for $s \in I$ and $\eta \in \mathbb{R}^N \setminus A_s$. Moreover,

$$\int_s^{s+\varepsilon} \int_{\mathbb{R}^N} K_{a(\tau-s)}(\xi - \eta) V_\infty(\tau, \xi) d\xi d\tau \leq \int_s^{s+\varepsilon} (\|V_2(\tau)\|_\infty + \|V_3\|_\infty) d\tau$$

for $\varepsilon > 0$. By assumption the integral on the right-hand side tends to 0 as $\varepsilon \rightarrow 0$ uniformly in $s \in I$. So we can fix $\varepsilon \in (0, t)$ such that

$$\int_{s+\varepsilon}^{s+t} \int_{\mathbb{R}^N} K_{a(\tau-s)}(\xi - \eta) V_\infty(\tau, \xi) d\xi d\tau \geq \frac{d - \delta}{4M^2}$$

for $s \in I$ and $\eta \in \mathbb{R}^N \setminus A_s$. We further compute

$$\begin{aligned} \frac{d - \delta}{4M^2} &\leq \int_{s+\varepsilon}^{s+t} \int_{B(\eta, r)} K_{a(\tau-s)}(\xi - \eta) V_\infty(\tau, \xi) d\xi d\tau \\ &\quad + \int_{s+\varepsilon}^{s+t} \int_{B(\eta, r)^c} K_{a(\tau-s)}(\xi - \eta) V_\infty(\tau, \xi) d\xi d\tau \\ &\leq \frac{1}{(4\pi a\varepsilon)^{N/2}} \left(\int_{s+\varepsilon}^{s+t} \int_{B(\eta, r)} V_\infty(\tau, \xi) d\xi d\tau \right. \\ &\quad \left. + \int_{s+\varepsilon}^{s+t} \int_{B(0, r)^c} e^{-\frac{1}{4a\varepsilon}|\xi|^2} V_\infty(\tau, \eta - \xi) d\xi d\tau \right) \\ &\leq \frac{1}{(4\pi a\varepsilon)^{N/2}} \left(\int_s^{s+t} \int_{B(\eta, r)} V_\infty(\tau, \xi) d\xi d\tau \right. \\ &\quad \left. + \int_s^{s+t} (\|V_2(\tau)\|_\infty + \|V_3\|_\infty) d\tau \int_{B(0, r)^c} e^{-\frac{1}{4a\varepsilon}|\xi|^2} d\xi \right) \end{aligned}$$

for $s \in I$, $r > 0$, and a.e. $\eta \in \mathbb{R}^N \setminus A_s$. Thus we can fix a small constant $c > 0$ and choose $r > 0$ such that

$$\int_s^{s+t} \int_{B(\xi, r)} V_\infty(\tau, \eta) d\eta d\tau \geq c$$

for each $s \in I$ and a.e. $\xi \in \mathbb{R}^N \setminus A_s$, where $\lambda(A_s) \leq m$. Therefore Lemma 4.2 yields $V_\infty \in \mathcal{E}_t$, and hence $V \in \mathcal{E}_t$. \square

Notice that potentials V as considered in Remark 4.5(b) satisfy the assumptions of Theorems 4.4 and 4.6. As a consequence, the next result follows immediately.

Corollary 4.7. *Let $V \geq 0$ be as in Remark 4.5(b). Let \mathcal{U} be a submarkovian, strongly continuous evolution family on $L^1(\mathbb{R}^N)$ satisfying an upper and a lower Gaussian estimate. Then \mathcal{U}_V is exponentially stable if and only if $V \in \mathcal{E}_t$ for some $t > 0$.*

The case $X = L^p(\mathbb{R}^N)$ is treated by means of the Riesz-Thorin interpolation theorem and the following lemma whose proof relies on an argument due to W. Arendt, cf. [1, p. 1160].

Lemma 4.8. *Let $\mathcal{U} = (U(t, s))_{(t,s) \in D}$ be a positive, strongly continuous evolution family on $L^p(\Omega)$, $1 \leq p < \infty$, where $\Omega \subseteq \mathbb{R}^N$ is open. Assume that \mathcal{U} satisfies an upper Gaussian estimate. Then for each $1 \leq r < \infty$ there exists a unique strongly continuous evolution family \mathcal{U}_r on $L^r(\Omega)$ which coincides with \mathcal{U} on $L^p(\Omega) \cap L^r(\Omega)$.*

Proof. Due to the upper Gaussian estimate the restriction of $U(t, s)$, $(t, s) \in D$, to $L^p(\Omega) \cap L^r(\Omega)$ has a unique bounded extension $U_r(t, s)$ to $L^r(\Omega)$, $1 \leq r < \infty$. Clearly, \mathcal{U}_r is a bounded evolution family. For proving strong continuity of \mathcal{U}_r it is enough to consider $x \in L^p(\Omega) \cap L^r(\Omega)$. So let $D \ni (t_n, s_n) \rightarrow (t, s) \in D$ and set $y_n := U(t_n, s_n)x$ and $z_n := M K_{a(t_n-s_n)} * |x|$. Observe that it suffices to show that every subsequence of (y_n) has a subsequence which converges in L^r to $y := U(t, s)x$. Since $y_n \rightarrow y$ in L^p we see by passing to a subsequence that $y_n \rightarrow y$ pointwise a.e. Taking another subsequence we obtain that $\|z_n - z_{n-1}\|_r \leq 2^{-n}$. Let

$$w := \sum_{n \geq 2} |z_n - z_{n-1}| + |z_1|.$$

Then $w \in L^r(\mathbb{R}^N)$. Further, $|y_n| \leq z_n \leq w$. Hence, Lebesgue's theorem implies $y_n \rightarrow y$ in L^r .

Corollary 4.9. *Assume*

$$0 \leq V \in \overline{L^\infty(I \times \mathbb{R}^N)_+}^{L^1_{loc,u}}.$$

Let $\mathcal{U} = (U(t, s))_{(t,s) \in D}$ be a strongly continuous evolution family on $L^p(\mathbb{R}^N)$, $1 < p < \infty$, satisfying an upper and a lower Gaussian estimate. Assume that the induced evolution family \mathcal{U}_1 on $L^1(\mathbb{R}^N)$ is submarkovian. If $V \in \mathcal{E}_t$ for some $t > 0$, then the absorption evolution family \mathcal{U}_V on $L^p(\mathbb{R}^N)$ is exponentially stable.

Proof. By Lemma 4.8 \mathcal{U} induces a strongly continuous evolution family \mathcal{U}_r on $L^r(\mathbb{R}^N)$, $1 \leq r < \infty$, satisfying an upper and a lower Gaussian estimate. By Lemma 3.1 there exists a bounded absorption evolution family $\mathcal{U}_{r,V}$ on $L^r(\mathbb{R}^N)$. Choose $r \in (p, \infty)$. By interpolation of $U_{1,V}(t, s)$ and $U_{r,V}(t, s)$ we obtain operators $W(t, s) : L^p(\mathbb{R}^N) \rightarrow L^p(\mathbb{R}^N)$, $(t, s) \in D$, which are exponentially stable due to Theorem 4.4. It remains to show that $W(t, s) = U_{p,V}(t, s)$ for $(t, s) \in D$. As in the proof of Theorem 4.4, it suffices to consider $V \in L^\infty(I \times \mathbb{R}^N)$. Then the evolution family $\mathcal{U}_{r,V}$, $1 \leq r < \infty$, is strongly continuous and satisfies (2.8) by Proposition 2.3. By interpolation, \mathcal{W} is strongly continuous and satisfies (2.8), too. Consequently, Proposition 2.3 implies $U_{p,V}(t, s) = W(t, s)$.

Remark 4.10. (a) The method of the proofs of Theorems 4.4 and 4.6 also works for $X = BUC(\mathbb{R}^N)$ and potentials V as considered in Theorems 4.4 and 4.6 provided that analogues of the results of Sections 2 and 3 can be established, see [23, §4.3].

(b) In the case of $U(t, s) = e^{(t-s)\Delta}$, $X = L^p(\mathbb{R}^N)$, and time independent potentials $V(t) \equiv V_0$ the above characterization was shown by W. Arendt and C.J.K. Batty for $0 \leq V_0 \in L^1(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ in [3, Thm. 1.2]. For arbitrary measurable $V_0 \geq 0$ similar results of probabilistic nature can be found in [2], [6]. The characterization was extended by D. Daners and P. Koch Medina, [8, Theorem 5.1], to the time periodic case for $X = BUC(\mathbb{R}^N)$ or $L^1(\mathbb{R}^N)$. In [12] a generalization to certain periodic evolution families satisfying Gaussian estimates was given. We have used several ideas from [8].

(c) If $U(t, s) = e^{(t-s)\Delta}$ and $V(t) \equiv V_0 \in L^1 + L^\infty$, then in Corollary 4.9 the converse also holds, see [2]. We do not know if this is true in the non-autonomous situation, cf. [8], Remark 5.2.

5. Applications.

In this section we study the initial value problem

$$\begin{aligned} D_t u(t, \xi) &= \sum_{k,l=1}^N D_k (a_{kl}(t, \xi) D_l u(t, \xi)) - V(t, \xi) u(t, \xi), \\ u(s, \xi) &= x(\xi), \quad t \geq s \geq 0, \quad \xi \in \mathbb{R}^N, \end{aligned} \tag{5.1}$$

where $x \in L^\infty(\mathbb{R}^N)$ and $D_t = \partial/\partial t$, $D_k = \partial/\partial \xi_k$ in the sense of distributions. We further assume $a_{kl} \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^N, \mathbb{R})$ for $k, l = 1, \dots, N$ and $\sum_{k,l=1}^N a_{kl}(t, \xi) \eta_k \eta_l \geq \mu |\eta|^2$ for $\eta, \xi \in \mathbb{R}^N$, $t \geq 0$, and a constant $\mu > 0$. For

$V \equiv 0$ it is known that there is a unique bounded weak solution u of (5.1) (see e.g. [10] for a definition) which is given by

$$u(t, \xi) = \int_{\mathbb{R}^N} k(t, s, \xi, \eta) x(\eta) d\eta, \quad \xi \in \Omega, \quad t > s, \quad (5.2)$$

where $k : Q := \{(t, s, \xi, \eta) \in \mathbb{R}_+^2 \times \mathbb{R}^{2N} : t > s\} \rightarrow \mathbb{R}$ is measurable. The kernel k is called *weak fundamental solution*. There are constants $M, a, \tilde{M}, \tilde{a} > 0$ such that

$$\tilde{M} K_{\tilde{a}(t-s)}(\xi - \eta) \leq k(t, s, \xi, \eta) \leq M K_{a(t-s)}(\xi - \eta) \quad (5.3)$$

for $t > s$ and a.e. $\xi, \eta \in \mathbb{R}^N$, where $K_t, t > 0$, is the Gaussian kernel. Moreover, k is Hölder continuous uniformly on compact subsets of Q , and we have

$$k(t, s, \xi, \eta) = \int_{\Omega} k(t, r, \xi, \zeta) k(r, s, \zeta, \eta) d\zeta \quad \text{and} \quad (5.4)$$

$$\int_{\mathbb{R}^N} k(t, s, \xi, \eta) d\eta = 1 \quad (5.5)$$

on Q . These properties were established by S.D. Eidel'man and F.O. Porper in [10, Ch. 5]. (For further results see the paper [5] by D.G. Aronson, the monographs [11], [13], and the references therein.) We now define

$$U(t, s)x := \int_{\Omega} k(t, s, \cdot, \eta) x(\eta) d\eta$$

for $t > s \geq 0$ and $x \in X = L^p(\mathbb{R}^N)$, $1 \leq p < \infty$. Further, we set $U(s, s) := Id$ for $s \geq 0$. Using the above mentioned properties of k we can check the assumptions on the evolution family \mathcal{U} we have made in the preceding sections. For $X = BUC(\mathbb{R}^N)$ a similar result can be found in [12].

Lemma 5.1. *The above defined family $\mathcal{U} = (U(t, s))_{t \geq s \geq 0}$ is a positive, strongly continuous evolution family on $X = L^p(\mathbb{R}^N)$, $1 \leq p < \infty$. Moreover, \mathcal{U} satisfies an upper and a lower Gaussian estimate and is submarkovian on $L^1(\mathbb{R}^N)$.*

Proof. Since k is measurable and satisfies (5.3), $U(t, s)$ is a bounded operator on X . By (5.4) and (5.3) \mathcal{U} is an evolution family satisfying an upper

and a lower Gaussian estimate. Observe that the weak fundamental solution of the adjoint problem, cf. [5, p. 621], is given by $\tilde{k}(t, s, \xi, \eta) = k(s, t, \eta, \xi)$ for $0 \leq t < s$ and $\xi, \eta \in \mathbb{R}^N$, [5, Theorem 10]. Moreover, \tilde{k} satisfies (5.5). Hence, \mathcal{U} is (sub-)markovian on $L^1(\mathbb{R}^N)$.

It remains to prove strong continuity of \mathcal{U} . Since \mathcal{U} is uniformly bounded, it suffices to consider $x \in C_c(\mathbb{R}^N)$. So fix $x \in C_c(\mathbb{R}^N)$ and let $(t', s') \rightarrow (t, s)$ in $D_{\mathbb{R}_+}$. We may assume that $0 \leq t - s, t' - s' \leq c$ for a constant c . As in the proof of Corollary 3.6, due to the upper Gaussian estimate there exists $y \in X$ such that $|U(t', s')x| \leq y$ for $0 \leq t' - s' \leq c$. Consequently, by the dominated convergence theorem, we have to show that

$$U(t', s')x(\xi) \rightarrow U(t, s)x(\xi) \quad \text{for } \xi \in \mathbb{R}^N. \quad (5.6)$$

So let $\xi \in \mathbb{R}^N$. At first, consider $t > s$. We can fix constants $d, \tilde{c} > 0$ such that $\tilde{c} \leq t - s, t' - s' \leq c$ and $|\xi| < d$. Since k is Hölder continuous uniformly in $Q_1 := \{(t', s', \zeta, \eta) \in Q : \tilde{c} \leq t - s, t' - s' \leq c, |\zeta| \leq d, \eta \in \text{supp } x\}$, there are constants $C \geq 0$ and $0 < \alpha \leq 1$ depending on Q_1 such that

$$\begin{aligned} |U(t, s)x(\xi) - U(t', s')x(\xi)| &\leq \|x\|_\infty \int_{\text{supp } x} |k(t, s, \xi, \eta) - k(t', s', \xi, \eta)| d\eta \\ &\leq C \|x\|_\infty \lambda(\text{supp } x) (|t - t'|^\alpha + |s - s'|^\alpha). \end{aligned} \quad (5.7)$$

In the case $(t', s') \rightarrow (s, s)$ we may assume $t' > s'$. For a given $\varepsilon > 0$, choose $\delta > 0$ such that $|\xi - \eta| \leq \delta$ implies $|x(\xi) - x(\eta)| \leq \varepsilon$. Observe that the Gaussian kernel K_t satisfies

$$K_{a(t'-s')}(\xi - \eta) \leq 2^{\frac{N}{2}} e^{-\frac{\delta^2}{8a(t'-s')}} K_{2a(t'-s')}(\xi - \eta)$$

for $t' > s'$ and $|\xi - \eta| \geq \delta > 0$. Then by (5.5) and (5.3) we obtain that

$$\begin{aligned} |U(t', s')x(\xi) - x(\xi)| &= \left| \int_{\mathbb{R}^N} k(t', s', \xi, \eta)(x(\eta) - x(\xi)) d\eta \right| \\ &\leq M \int_{\mathbb{R}^N} K_{a(t'-s')}(\xi - \eta) |x(\eta) - x(\xi)| d\eta \\ &\leq \varepsilon M + 2M \|x\|_\infty \int_{|\xi - \eta| \geq \delta} K_{a(t'-s')}(\xi - \eta) d\eta \\ &\leq \varepsilon M + 2M \|x\|_\infty 2^{\frac{N}{2}} e^{-\frac{\delta^2}{8a(t'-s')}} \int_{\mathbb{R}^N} K_{2a(t'-s')}(\xi - \eta) d\eta \\ &= \varepsilon M + 2M \|x\|_\infty 2^{\frac{N}{2}} e^{-\frac{\delta^2}{8a(t'-s')}}. \end{aligned} \quad (5.8)$$

Now (5.7) and (5.8) yield (5.6). \square

A measurable function $u : [s, \infty) \times \mathbb{R}^N \rightarrow \mathbb{C}$ is called a *mild solution* of (5.1) if

$$u(t, \xi) = \int_{\mathbb{R}^N} k(t, \tau, \xi, \eta) x(\eta) d\eta - \int_s^t \int_{\mathbb{R}^N} k(t, \tau, \xi, \eta) V(\tau, \eta) u(\tau, \eta) d\eta d\tau \quad (5.9)$$

holds for a.e. (t, ξ) . Thanks to Lemma 5.1 we can derive from Corollary 3.6 the existence of a mild solution of (5.1). Moreover, by Corollaries 4.7 and 4.9 we find conditions for the exponential stability of such a solution. Recall that $\varepsilon_\alpha(\xi) = e^{-\alpha|\xi|^2}$.

Proposition 5.2. *Let \mathcal{U} be the evolution family on $X = L^p(\mathbb{R}^N)$, $1 \leq p < \infty$, as considered in Lemma 5.1. Further, consider a potential $0 \leq V \in \bigcap_{\alpha > 0} L^1_{loc}(\mathbb{R}_+, L^p(\mathbb{R}^N, \varepsilon_\alpha \lambda))$. Then there is a strongly continuous absorption evolution family \mathcal{U}_V satisfying (2.8) and (2.9) for $x \in L^\infty(\mathbb{R}^N)$ with compact support. For such x , there exists a mild solution u of (5.1) with $u(t, \cdot) = U_V(t, s)x$ for a.e. t .*

If, in addition, $0 \leq V \in \overline{L^\infty(\mathbb{R}_+ \times \mathbb{R}^N)}^{L^1_{loc, u}} \cap \mathcal{E}_t$ for some $t > 0$, then \mathcal{U}_V (and hence u) is exponentially stable on X . If $p = 1$ and $V \geq 0$ is given as in Remark 4.5(b), then \mathcal{U}_V (and hence u) is exponentially stable on $L^1(\mathbb{R}^N)$ if and only if $V \in \mathcal{E}_t$ for some $t > 0$.

Proof. Due to [4, Lemma 2.2] there is a measurable function $u : [s, \infty) \times \mathbb{R}^N \rightarrow \mathbb{C}$ such that $u(t, \cdot) = U_V(t, s)x$. By using [4, Lemma 2.2] we see that (2.8) implies (5.9).

We note that in [5, Theorems 3, 10] the existence of a unique bounded weak solution of (5.1) on a bounded interval I was shown under a stronger integrability assumption on V . Moreover, this weak solution coincides a.e. with the mild solution obtained above, see [5, Theorem 10].

Finally, we point out that it is possible to derive similar results for uniformly elliptic operators of second order with smooth real coefficients on bounded domains Ω with smooth boundary and Dirichlet boundary conditions. In fact, if one defines \mathcal{U} on $L^p(\Omega)$ as in (5.2) (by replacing the fundamental solution by the Green's function, cf. e.g. [11], [13]) then it can be shown as in Lemma 5.1 that \mathcal{U} is strongly continuous and satisfies an upper Gaussian estimate, see [23, Lemma 3.4].

Acknowledgment. We like to thank Daniel Daners for comments on his paper [8].

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