Differential and Integral Equations Volume 12, Number 1, January 1999, pp. 1–22

EXPONENTIAL ATTRACTORS FOR NONAUTONOMOUS PARTIALLY DISSIPATIVE EQUATIONS

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(Submitted by: Roger Temam)

Abstract. Our aim in this article is to construct exponential attractors for nonautonomous partially dissipative systems by adapting to the nonautonomous case the method presented in [1] and based on a decomposition of the difference of two trajectories. We follow the ideas developed in [2] which consist in studying an equation on an extended space. As an example, we prove the existence of exponential attractors for the slightly compressible Navier-Stokes equations with quasiperiodic in time volume forces.

0. Introduction. In [8], by adapting the construction presented in [4] for autonomous systems, we studied the existence of exponential attractors for nonautonomous first-order dissipative equations (see also [9] where a construction specific to semilinear wave equations in space dimension one is given). The notion of exponential attractor was introduced in [4]; and, when it exists, an exponential attractor is a compact set with finite fractal dimension which attracts the trajectories at an exponential rate, in contrast with the global attractor which may attract the orbits at an arbitrarily slow rate.

Received for publication March 1998.
AMS Subject Classifications: 35B40, 35B45, 35Q30.

To do so, following the construction of attractors for nonautonomous evolution equations given in [2], in order to construct attractors for an equation of the form

$$\frac{du}{dt} = \mathcal{G}(\sigma_0(t), u), \tag{0.1}$$

where $u \in E$, E being a Banach space, and $\sigma_0(t)$ contains all the time dependent terms of the equation and is called the time symbol, we consider in fact the family of equations

$$\frac{du}{dt} = \mathcal{G}(\sigma(t), u), \tag{0.2}$$

where $\sigma(t)$ describes a closed functional space Σ which contains $\sigma_0(t)$ and is invariant by translation, called the symbol space. We then construct a semigroup $S(t)$ acting on $E \times \Sigma$ which possesses an exponential attractor $\mathcal{M}$. It is then natural to study the properties of the set $\mathcal{M}_\Sigma = \Pi_1\mathcal{M}$, where Π_1 denotes the orthogonal projector on E. In [8], for quasiperiodic symbols, we proved that $\mathcal{M}_\Sigma$ is a compact, uniformly (with respect to σ) exponentially attracting set with finite fractal dimension, and we called this set a uniform (with respect to σ) exponential attractor for the family of equations (0.2).

Our aim in this article is to study the existence of uniform (with respect to σ) exponential attractors for partially dissipative equations with quasiperiodic time symbols. For such systems, the problems come from the lack of dissipativity in some of the variables. In order to overcome this difficulty, we prove the squeezing property (see [4]) using a method developed in [1] for the study of reaction-diffusion equations in an unbounded domain (see also [6] and [7] for further applications of this method). This method is based on a decomposition of the difference of two trajectories.

We then apply this result to a singular perturbation problem, namely the slightly compressible Navier-Stokes equations introduced in [14] in order to overcome the computational difficulties related to the incompressibility. These equations read

$$\frac{\partial u^\epsilon}{\partial t} - \nu\Delta u^\epsilon - \nu'\nabla(\operatorname{div} u^\epsilon) + (u^\epsilon.\nabla)u^\epsilon + \frac{1}{2}(\operatorname{div} u^\epsilon)u^\epsilon + \nabla p^\epsilon = f(x,t), \tag{0.3}$$

$$\epsilon\frac{\partial p^\epsilon}{\partial t} + \operatorname{div} u^\epsilon = 0, \tag{0.4}$$

where $\epsilon > 0$. When $\epsilon = 0$, these equations become the incompressible Navier-Stokes equations. We prove the existence of a local uniform (with respect to

σ) exponential attractor for (0.3)–(0.4). We also prove that the exponential attractor we construct is, in a sense that will be precised below, uniform in ϵ. Indeed, by adapting to the nonautonomous case the techniques developed in [6], we prove that if we work with the variables u^ϵ and $q^\epsilon = \sqrt{\epsilon}p^\epsilon$, the fractal dimension of the exponential attractor (and thus of the global attractor), and also the rate of attraction of the trajectories (this will then insure the stability of the exponential attractor), are bounded independently of ϵ for $\epsilon \leq \epsilon_0$, $\epsilon_0 > 0$ (we note here that the bounds on the dimension of the global attractor obtained in [10] explode as $\epsilon \to 0^+$). It is thus natural to study the behavior of the exponential attractor as $\epsilon \to 0^+$. We can obtain here a lower-continuity type property of the uniform (with respect to σ) exponential attractor associated to (0.3)–(0.4) to a uniform (with respect to σ) exponential attractor associated to the incompressible Navier-Stokes equations. The difficulty here, due to the singular perturbation aspect of the problem, is that the attractor associated to the slightly compressible Navier-Stokes equations is described by the variables u^ϵ and q^ϵ, whereas the attractor associated to the incompressible Navier-Stokes equations is only described by the variable u. We thus first truncate the pressure component in order to study the lower-semicontinuity of the attractors (see Definition 2.1 below). Similar results were derived in [6] in the autonomous case.

Throughout this article, the same letter c (and sometimes c' or c'') shall denote constants which may change from line to line.

1. An abstract result. Let H be a Hilbert space. We consider the following equation in H

$$\frac{du}{dt} = \mathcal{G}(\alpha t, u), \tag{1.1}$$

$$u(\tau) = u_\tau, \tag{1.2}$$

$t \geq \tau$, $\tau \in R$; where $u_\tau \in H$ and $\alpha = (\alpha_1, ..., \alpha_k)$; the α_i being rationally independent. Furthermore, we assume that $\mathcal{G}(\omega_1, ..., \omega_k, .)$ is 2π-periodic in each argument ω_i, $i = 1, ..., k$.

Following the construction presented in [2] for the study of nonautonomous systems, we associate to (1.1)–(1.2) the family of equations depending on the parameter $\sigma \in T^k$, T^k being the k-dimensional torus

$$\frac{du}{dt} = \mathcal{G}(\alpha t + \sigma, u), \tag{1.3}$$

$$u(\tau) = u_\tau, \tag{1.4}$$

$t \geq \tau,\ \tau \in R.$

We assume that (1.3)–(1.4) is well posed and that we can define the family of operators

$$U_\sigma(t,\tau) : H \to H$$
$$u_\tau \mapsto u(t), \tag{1.5}$$

where u is the solution of (1.3)–(1.4), $t \geq \tau$ and $\sigma \in T^k$. This family of operators forms a family of processes, i.e.,

$$U_\sigma(t,t) = Id, \quad \forall t \in R, \quad \forall \sigma \in T^k, \tag{1.6}$$

$$U_\sigma(t,s) \circ U_\sigma(s,\tau) = U_\sigma(t,\tau),\ \forall t \geq s \geq \tau,\ \forall \sigma \in T^k. \tag{1.7}$$

We now give the following definitions (see [2], [4] and [8]):

Definition 1.1.

(i) A closed set $\mathcal{A}_{T^k} \subset H$ is called the uniform (with respect to σ) attractor for the family of processes (1.5) if
 (a) $\mathcal{A}_{T^k}$ is a compact set of H;
 (b) $\forall B \subset H$ bounded,

$$\mathrm{Lim}_{t\to+\infty} Sup_{\sigma\in T^k} \mathrm{dist}_H(U_\sigma(t,\tau)B, \mathcal{A}_{T^k}) = 0;$$

 (c) (minimality property) $\forall \mathcal{A}' \subset H$ closed and satisfying (b), $\mathcal{A}_{T^k} \subset \mathcal{A}'$.

(ii) Let V be a second Hilbert space such that the injection $V \subset H$ is compact. A closed set $\mathcal{A}_{T^k} \subset H$ is called the uniform (with respect to σ) $H - V$ attractor for the family of processes (1.5) if (a) and (c) hold and (b) is replaced by
 (b1) $\forall B \subset V$ bounded,

$$\mathrm{Lim}_{t\to+\infty} \mathrm{Sup}_{\sigma\in T^k} dist_H(U_\sigma(t,\tau)B, \mathcal{A}_{T^k}) = 0.$$

Definition 1.2.

(i) Let $Y \subset H$ be a closed set. A set $\mathcal{M}_{T^k} \subset H$ is a uniform (with respect to σ) exponential attractor for the family of processes (1.5) on Y if
 (a) $\mathcal{A}_{T^k} \subset \mathcal{M}_{T^k} \subset Y$, where $\mathcal{A}_{T^k}$ is the uniform (with respect to σ) attractor;

(b) $\mathcal{M}_{T^k}$ is compact;
(c) $\mathcal{M}_{T^k}$ has finite fractal dimension;
(d) $\forall B \subset Y$ bounded, there exist two constants c_1 and c_2 that depend only on B such that

$$\mathrm{Sup}_{\sigma\in T^k}\mathrm{dist}_H(U_\sigma(t,\tau)B,\mathcal{M}_{T^k}) \le c_1 e^{-c_2(t-\tau)},$$

$\forall t \ge \tau,\ \tau \in R.$

(ii) Let V be a second Hilbert space such that $Y \subset V \subset H$, the injection $V \subset H$ being compact. A closed set $\mathcal{M}_{T^k} \subset H$ is a uniform (with respect to σ) $H-V$ exponential attractor for the family of processes (1.5) and for a set of initial data in Y if (a), (b) and (c) hold and (d) is replaced by

(d1) $\forall B \subset Y$ bounded for the topology of V, there exist two constants c_1 and c_2 that depend only on B such that

$$\mathrm{Sup}_{\sigma\in T^k}\mathrm{dist}_H(U_\sigma(t,\tau)B,\mathcal{M}_{T^k}) \le c_1 e^{-c_2(t-\tau)},$$

$\forall t \ge \tau,\ \tau \in R.$

Remark 1.1. (i) A first consequence of Definition 1.2 is that once the existence of an exponential attractor $\mathcal{M}_{T^k}$ has been obtained, then the uniform (with respect to σ) attractor (or the global attractor for autonomous systems) has finite fractal dimension. This can be interesting when one cannot use the Lyapunov exponents' method to prove the finite dimensionality of the attractor. Indeed, to use this method we need some differentiability property, whereas only a Lipschitz property will be necessary to obtain the existence of an exponential attractor (this is discussed for instance in [3] or [13]). This shows another interest for the study of exponential attractors.

(ii) When $Y \neq H$, we say that $\mathcal{M}_{T^k}$ is a local uniform (with respect to σ) exponential attractor.

In [8], we proved the existence of uniform (with respect to σ) exponential attractors for nonautonomous dissipative first-order evolution equations. To do so, we proved by adapting the construction given in [4] that the semigroup

$$\begin{aligned} S(t) : H \times T^k &\to H \times T^k \\ (u,\sigma) &\mapsto (U_\sigma(t,0)u, \alpha t + \sigma\ (mod\ T^k)), \end{aligned} \tag{1.8}$$

possesses an exponential attractor $\mathcal{M}$. Then, we proved that $\mathcal{M}_{T^k} = \Pi_1\mathcal{M}$, where Π_1 denotes the orthogonal projector on H, is a uniform (with respect to σ) exponential attractor on a proper compact set $Y \subset H$.

For dissipative equations, the existence of the global attractor $\mathcal{A}$ for the semigroup $S(t)$ is usually based on some compactness property satisfied by $S(t)$ (see [15] for a review on this subject for autonomous systems, and [2] for nonautonomous systems). Such compactness properties were also essential for the construction of the exponential attractor $\mathcal{M}$ (and thus $\mathcal{M}_{T^k}$) given in [4] and [8]. Indeed, the proof is based on the existence of a compact absorbing set.

For partially dissipative systems, where typically there is a lack of dissipativity on some components of u, the construction given in [4] is no longer valid. For such systems, in order to prove the existence of the global attractor, one solution, which has been successfully applied to partially dissipative reaction-diffusion systems in the autonomous case (see [12]), consists in proving that the semigroup $S(t)$ can be decomposed as

$$S(t) = S_1(t) + S_2(t), \tag{1.9}$$

where $S_1(t)$ enjoys some compactness property (say is uniformly compact for t large), and where $S_2(t)$ is continuous from $H \times T^k$ onto itself, and satisfies

$$\mathrm{Sup}_{\varphi \in C} |S_2(t)\varphi|_{H \times T^k} \to 0 \ as\ t \to +\infty, \tag{1.10}$$

for every bounded set $C \subset H \times T^k$.

In [1], the authors introduced a decomposition similar to (1.9) but on the difference of two solutions to construct exponential attractors. Before discussing this point, we give the following definition (see [4]).

Definition 1.3. We say that the semigroup $S(t)$ enjoys the squeezing property on a positively invariant closed set $X \subset H \times T^k$ if $\forall \delta \in]0, \frac{1}{4}[$, there exist an orthogonal projector $\mathcal{P}$ with finite rank $N_0(\delta)$ and a time $t^\star > 0$ such that $\forall (\varphi, \psi) \in X^2$, either

$$|S^\star \varphi - S^\star \psi|_{H \times T^k} \leq \delta |\varphi - \psi|_{H \times T^k},$$

or

$$|(I - \mathcal{P})(S^\star \varphi - S^\star \psi)|_{H \times T^k} \leq |\mathcal{P}(S^\star \varphi - S^\star \psi)|_{H \times T^k},$$

where $S^\star = S(t^\star)$.

It is proved in [4] (see also [1] when X is not compact, and [5] for another approach) that if the semigroup $S(t)$ enjoys the squeezing property and is Lipschitz on X (and if X can be covered by a finite number of balls of radius one when it is not compact), then it possesses an exponential attractor on X. We now have the following result, which is an extension to the nonautonomous case of the result obtained in [1].

Proposition 1.1.. *Let V be a Hilbert space such that the inclusion $V \subset H$ is compact. Let us furthermore assume that there exists an orthogonal projector $P_n : H \to H,\ n \in N^\star$, with finite rank, such that*

$$|Q_n y|_H \le c(n)|y|_V,$$

$\forall y \in V$, where $Q_n = I - P_n$ and $c(n) \to 0$ as $n \to +\infty$. If

$$S(t)\varphi - S(t)\psi = \varphi^1(t) + \varphi^2(t),$$

$$|\varphi^1(t)|^2_{H\times T^k} \le d(t)|\varphi - \psi|^2_{H\times T^k}, \qquad |\varphi^2(t)|^2_{V\times T^k} \le h(t)|\varphi - \psi|^2_{H\times T^k},$$

$\forall \varphi, \psi \in X$, where $d(t)$ is continuous and satisfies $d(t) \to 0$ as $t \to +\infty$ and $h(t)$ is continuous, then $S(t)$ enjoys the squeezing property on X.

Such a decomposition was first introduced in [1] for the study of reaction-diffusion equations in an unbounded domain, and has then been successfully applied in [6] to the slightly compressible Navier-Stokes equations and in [7] to partially dissipative reaction systems in the autonomous case. Actually, this method is more general than the one described in [4], and can also be applied to the dissipative case. However, this method necessitates more regularity on the initial data, especially on the non fully dissipative unknowns. Consequently, in the references [6] and [7], the authors only obtained the existence of $H - E$ exponential attractors, for a proper Hilbert space E. Furthermore, the estimates of the fractal dimension of the attractors obtained by this method are less sharp than the ones obtained by the method of [4].

Proof of Proposition 1.1 We set $\varphi = (u, \sigma),\ \psi = (v, \overline{\sigma}),\ \mathcal{P}_n(u, \sigma) = (P_n u, \sigma)$ and $\mathcal{Q}_n(u, \sigma) = (Q_n u, 0)$. Thus, $\mathcal{P}_n$ is an orthogonal projector with finite rank on $H \times T^k$.

Let $\eta < \frac{1}{8}$ and $t^\star > 0$ be such that $\sqrt{d(t^\star)} \le \frac{\eta}{8}$. We set $R = \sqrt{h(t^\star)}$ and we choose n such that $c(n)^{-1} \ge \frac{16R^2}{\eta^2}$. We finally set $y(t) = S(t)\varphi - S(t)\psi$. We then introduce the sets

$$B = \{z \in V \times T^k,\ |z|_{V\times T^k} \le R|y(0)|_{H\times T^k}\},$$

$$B_1 = \{z \in B,\ |\mathcal{P}_n z|^2_{H\times T^k} \ge \frac{9\eta^2}{16}|y(0)|^2_{H\times T^k}\},$$

$B_2 = B\backslash B_1$. Writing $\varphi^2(t) = (w^2(t), \theta^2(t))$, we have

$$\begin{aligned}|\mathcal{Q}_n\varphi^2(t^\star)|^2_{H\times T^k} &= |Q_n w^2(t^\star)|^2_H \le c(n)|w^2(t^\star)|^2_V \le c(n)|\varphi^2(t^\star)|^2_{V\times T^k}\\ &\le R^2 c(n)|y(0)|^2_{H\times T^k},\end{aligned}$$

which gives

$$|\mathcal{Q}_n\varphi^2(t^\star)|_{H\times T^k} \le \frac{\eta}{4}|y(0)|_{H\times T^k}. \tag{1.11}$$

By construction, $\varphi^2(t^\star)$ belongs to B. If $\varphi^2(t^\star) \in B_1$, then with our choice of $t^\star$

$$|\mathcal{P}_n\varphi^1(t^\star)|_{H\times T^k} \le \frac{\eta}{8}|y(0)|_{H\times T^k}, \tag{1.12}$$

and since

$$|\mathcal{P}_n y(t^\star)|_{H\times T^k} \ge |\mathcal{P}_n\varphi^2(t^\star)|_{H\times T^k} - |\mathcal{P}_n\varphi^1(t^\star)|_{H\times T^k},$$

we deduce thanks to (1.11) and (1.12) that

$$|\mathcal{P}_n y(t^\star)|_{H\times T^k} \ge \frac{5}{8}\eta|y(0)|_{H\times T^k}. \tag{1.13}$$

Furthermore

$$\begin{aligned}|\mathcal{Q}_n y(t^\star)|_{H\times T^k} &\le |\mathcal{Q}_n\varphi^1(t^\star)|_{H\times T^k} + |\mathcal{Q}_n\varphi^2(t^\star)|_{H\times T^k}\\ &\le |\varphi^1(t^\star)|_{H\times T^k} + |\mathcal{Q}_n\varphi^2(t^\star)|_{H\times T^k}\\ &\le \sqrt{d(t^\star)}|y(0)|_{H\times T^k} + \frac{\eta}{4}|y(0)|_{H\times T^k},\end{aligned}$$

which yields, since $\sqrt{d(t^\star)} \le \frac{\eta}{8}$

$$|\mathcal{Q}_n y(t^\star)|_{H\times T^k} \le \frac{5}{8}\eta|y(0)|_{H\times T^k},$$

and thanks to (1.13), we then find

$$|\mathcal{Q}_n y(t^\star)|_{H\times T^k} \le |\mathcal{P}_n y(t^\star)|_{H\times T^k}. \tag{1.14}$$

If now $\varphi^2(t^\star) \in B_2$, then

$$\begin{aligned}|\varphi^2(t^\star)|^2_{H\times T^k} &= |\mathcal{Q}_n\varphi^2(t^\star)|^2_{H\times T^k} + |\mathcal{P}_n\varphi^2(t^\star)|^2_{H\times T^k}\\ &\le (\frac{\eta^2}{16} + \frac{9\eta^2}{16})|y(0)|^2_{H\times T^k} \le \frac{5}{8}\eta^2|y(0)|^2_{H\times T^k}.\end{aligned}$$

Therefore, since

$$|\mathcal{Q}_n y(t^\star)|_{H\times T^k} \le |\mathcal{Q}_n \varphi^1(t^\star)|_{H\times T^k} + |\varphi^2(t^\star)|_{H\times T^k},$$

we have

$$|\mathcal{Q}_n y(t^\star)|_{H\times T^k} \le (\frac{1}{8} + \sqrt{\frac{5}{8}})\eta |y(0)|_{H\times T^k},$$

and thus

$$|\mathcal{Q}_n y(t^\star)|_{H\times T^k} \le \eta |y(0)|_{H\times T^k}. \tag{1.15}$$

Finally, (1.14) and (1.15) (with $\eta = \frac{\delta}{2}$) give the squeezing property. This finishes the proof of Proposition 1.1.

If $S(t)$ satisfies the squeezing property on X and is Lipschitz (and if X can be covered by a finite number of balls of radius one when it is not compact (see [1])), then it possesses an exponential attractor $\mathcal{M}$ on X (resp. a $H-V$ exponential attractor for a set of initial data in X), and exactly as in [8], Sec. 1, we have the following result:

Proposition 1.2.. *We assume that X has the form*

$$X = \overline{\cup_{t\ge t_1} S(t)(B\times T^k)},$$

where B is bounded in H, then $\mathcal{M}_{T^k} = \Pi_1\mathcal{M}$ is a uniform (with respect to σ) exponential attractor for the family of processes (1.5) *on*

$$Y = \overline{\cup_{\sigma\in T^k}\cup_{t\ge t_1} U_\sigma(t,0)B},$$

(*resp. a $H-V$ uniform* (*with respect to σ*) *exponential attractor for a set of initial data in Y*).

Remark 1.2. When $X = Y\times T^k$, we easily show that $\mathcal{M}_{T^k} = \Pi_1\mathcal{M}$ is a uniform (with respect to σ) exponential attractor for the family of processes (1.5) on Y.

2. Uniform exponential attractors for the slightly compressible 2D Navier-Stokes equations. In this section, we apply the abstract result derived in Section 1 to a singular perturbation problem depending on a small parameter $\epsilon > 0$. The exponential attractors we shall construct will be uniform with respect to σ, but also with respect to ϵ as far as the fractal dimension and the rate of attraction of the trajectories are concerned

(Subsection 2.2). This will allow us to obtain a lower-semicontinuity type property as $\epsilon \to 0^+$ (Subsection 2.3).

2.1. Mathematical setting of the problem. Let $\Omega \subset R^2$ be a bounded regular domain. We consider the following equations on Ω:

$$\frac{\partial u^\epsilon}{\partial t} - \nu \Delta u^\epsilon - \nu' \nabla(divu^\epsilon) + (u^\epsilon.\nabla)u^\epsilon + \frac{1}{2}(divu^\epsilon)u^\epsilon + \nabla p^\epsilon = f(x,t), \quad (2.1)$$

$$\epsilon \frac{\partial p^\epsilon}{\partial t} + divu^\epsilon = 0, \quad (2.2)$$

where ϵ, ν and ν' are strictly positive real numbers; together with the boundary conditions:

$$u^\epsilon = 0 \ on \ \partial\Omega. \quad (2.3)$$

The hypotheses satisfied by f will be precised below.

These equations were proposed in [14] as a means to overcome the computational difficulties due to the incompressibility condition (i.e., $\epsilon = 0$ in (2.2)). The long time behavior (i.e. the existence of the global attractor) of these equations are studied in [10], and in [6] for the existence of exponential attractors.

For the mathematical setting of the problem, we introduce the following spaces

$$H = \{u \in L^2(\Omega)^2, \ u.n = 0 \ on \ \partial\Omega\},$$

$$V = \{u \in H^1(\Omega)^2 \cap H, \ u = 0 \ on \ \partial\Omega\},$$

where n denotes the unit outer normal vector, which we endow with their usual scalar products and norms, which we denote $(.,.)$ and $|.|$ for H, and $((.,.))$ and $\|.\|$ for V. Furthermore, we consider the operator

$$A = -\Delta - \mu \nabla div, \quad (2.4)$$

where $\mu = \frac{\nu}{\nu'}$, with domain $D(A) = H^2(\Omega)^2 \cap H_0^1(\Omega)^2$, and we set

$$(B(u,v),w) = b(u,v,w) = \frac{1}{2}\int_\Omega (divu)v.wdx + \int_\Omega (u.\nabla)v.wdx. \quad (2.5)$$

We refer the reader to [14] for the properties satisfied by B and b.

We now rewrite (2.1)–(2.3) in the form

$$\frac{du^\epsilon}{dt} + \nu A u^\epsilon + B(u^\epsilon, u^\epsilon) + \nabla p^\epsilon = f(t) \ in \ H, \quad (2.6)$$

$$\epsilon \frac{dp^\epsilon}{dt} + divu^\epsilon = 0. \tag{2.7}$$

We assume that f is bounded and continuous from R into V (actually, it suffices to have $f(t) \in H^1(\Omega)^2$ and $f.n = 0$ on $\partial\Omega$, see [10]) and that f is quasiperiodic with respect to t, and more precisely that

$$f(.,t) = \tilde{f}(.,\alpha_1 t, ..., \alpha_k t), \tag{2.8}$$

where $\tilde{f}(.,\omega_1, ..., \omega_k)$ is 2π-periodic in each argument ω_i; the α_i being rationally independent. For dimensional reasons, we endow T^k with the norm $|\sigma| = U^2(\sum_{i=1}^k |\sigma_i|^2)^{\frac{1}{2}}$, $\sigma = (\sigma_1, ..., \sigma_k)$, where U is a characteristic velocity.

As described in Section 1, we can write (2.6)–(2.7) in the form

$$\frac{d\Theta^\epsilon}{dt} = \mathcal{G}(\omega_0(t), \Theta^\epsilon), \tag{2.9}$$

where

$$\Theta^\epsilon = \begin{pmatrix} u^\epsilon \\ p^\epsilon \end{pmatrix}, \tag{2.10}$$

$$\omega_0(t) = \alpha t, \tag{2.11}$$

$$\alpha = (\alpha_1, ..., \alpha_k), \tag{2.12}$$

and if $\Theta = \begin{pmatrix} u \\ p \end{pmatrix}$

$$\mathcal{G}(\omega_0(t), \Theta) = \begin{pmatrix} \tilde{f}(x, \omega_0(t)) - \nu Au - B(u,u) - \nabla p \\ -\frac{1}{\epsilon} divu \end{pmatrix}; \tag{2.13}$$

and we study in fact the family of equations depending on the parameter $\sigma \in T^k$

$$\frac{d\Theta^\epsilon}{dt} = \mathcal{G}(\omega_0(t) + \sigma, \Theta^\epsilon), \tag{2.14}$$

$$\Theta^\epsilon(\tau) = \Theta_\tau, \ \tau \in R \tag{2.15}$$

i.e., we study the family of equations

$$\frac{du^\epsilon}{dt} + \nu Au^\epsilon + B(u^\epsilon, u^\epsilon) + \nabla p^\epsilon = \tilde{f}(\omega_0(t) + \sigma), \tag{2.16}$$

$$\epsilon \frac{dp^\epsilon}{dt} + divu^\epsilon = 0, \tag{2.17}$$

$$u^\epsilon(\tau) = u_\tau,\ p^\epsilon(\tau) = p_\tau, \tag{2.18}$$

$\sigma \in T^k$; to which we associate the autonomous system

$$\frac{du^\epsilon}{dt} + \nu A u^\epsilon + B(u^\epsilon, u^\epsilon) + \nabla p^\epsilon = \tilde{f}(\omega), \tag{2.19}$$

$$\epsilon \frac{dp^\epsilon}{dt} + div u^\epsilon = 0, \tag{2.20}$$

$$\frac{d\omega}{dt} = \alpha, \tag{2.21}$$

$$u^\epsilon(0) = u_0,\ p^\epsilon(0) = p_0,\ \omega(0) = \sigma. \tag{2.22}$$

We obtain existence and uniqueness results exactly as in the autonomous case (see [10]) by making Galerkin approximations. These results allow us to define the family of processes $U^\epsilon_\sigma(t, \tau)$ associated to (2.16)–(2.18) and the semigroup $S^\epsilon(t)$ associated to (2.19)–(2.22).

Taking the scalar product of (2.16) by u^ϵ in H, we prove that if $|u_\tau|^2 + \epsilon_0 |p_\tau|^2 \leq \rho_0^2,\ \epsilon_0 > 0$, then there exist $R_0 = R_0(|f|_{L^\infty(R;L^2)})$ and a time $\tau_0 = \tau_0(R_0, \rho_0) > 0$ such that if $t - \tau \geq \tau_0$ and $0 < \epsilon \leq \epsilon_0$

$$|u^\epsilon(t)|^2 + \epsilon |p^\epsilon(t)|^2 \leq R_0^2. \tag{2.23}$$

To do so, we proceed exactly as in [6] for the autonomous case, noting that the estimates on f are uniform in time (for instance, we replace $|f|$ by $|f|_{L^\infty(R;L^2(\Omega)^2)}$).

Similarly, by taking the scalar product of (2.16) by Au^ϵ, we prove that if $\|u_\tau\|^2 + \epsilon_0(1+\mu)|\nabla p_\tau|^2 \leq \rho_1^2$, there exist R_1 and a time $\tau_1 = \tau_1(R_0, R_1, \rho_0, \rho_1) > 0$ such that if $t - \tau \geq \tau_1$ and $0 < \epsilon \leq \epsilon_0$, then

$$\|u^\epsilon(t)\|^2 + \epsilon(1+\mu)|\nabla p^\epsilon(t)|^2 \leq R_1^2. \tag{2.24}$$

We now note that if $p_\tau \in \dot{H}^2_n(\Omega) = \{p \in H^2(\Omega),\ \frac{\partial p}{\partial n} = 0$ on $\partial\Omega$ and $\int_\Omega p dx = 0\}$, then, integrating (2.20) over Ω, we easily prove that $p^\epsilon(t) \in \dot{H}^2_n(\Omega),\ \forall t \geq \tau$ (the condition $\frac{\partial p^\epsilon}{\partial n} = 0$ on $\partial\Omega$ is obtained by using (2.16) and noting that $\tilde{f}.n = 0$ on $\partial\Omega$). Taking then the scalar product of (2.16) by $A^2 u^\epsilon$, we prove that if $|Au_\tau|^2 + \epsilon_0(1+\mu)^2|\Delta p_\tau|^2 \leq \rho_2^2$, there exist R_2 and a time $\tau_2 = \tau_2(R_0, R_1, R_2, \rho_0, \rho_1, \rho_2) > 0$ such that if $t - \tau \geq \tau_2$ and $0 < \epsilon \leq \epsilon_0$, then

$$|Au^\epsilon(t)|^2 + \epsilon(1+\mu)^2|\Delta p^\epsilon(t)|^2 \leq R_2^2. \tag{2.25}$$

Again, these formal estimates can be justified by making Galerkin approximations.

Remark 2.1. We note here that in the above estimates, the $R_i s$ and the $\tau_i s$ do not depend on ϵ.

Remark 2.2. We note that there is no regularizing effect on the variable p^ϵ.

2.2. Construction of uniform (with respect to σ) exponential attractors. Thanks to the estimates derived in the previous subsection, we prove as in [10] for the autonomous case the existence of the global attractor $\mathcal{A}^\epsilon$ for the weak topology of $H^1(\Omega)^3 \times T^k$ for $S^\epsilon(t)$, i.e.,

(i) $\mathcal{A}^\epsilon$ is bounded in $H^1(\Omega)^3 \times T^k$ and is weakly closed;
(ii) $\mathcal{A}^\epsilon$ is invariant for $S^\epsilon(t)$;
(iii) $\mathcal{A}^\epsilon$ attracts the bounded sets of $V \times H^1(\Omega) \times T^k$ for the weak topology.

In particular, $\mathcal{A}^\epsilon$ is a $L^2(\Omega)^3 \times T^k - H^1(\Omega)^3 \times T^k$ global attractor for $S^\epsilon(t)$. Furthermore, the set $\mathcal{A}^\epsilon_{T^k} = \Pi_1 \mathcal{A}^\epsilon$ is bounded and weakly closed in $V \times H^1(\Omega)$ and attracts uniformly (with respect to σ) the bounded sets of $V \times H^1(\Omega)$ for the weak topology (we shall say that $\mathcal{A}^\epsilon_{T^k}$ is a weak uniform (with respect to σ) attractor). In particular, $\mathcal{A}^\epsilon_{T^k}$ is a $L^2(\Omega)^3 - H^1(\Omega)^3$ uniform (with respect to σ) attractor for the family of processes $U^\epsilon_\sigma(t,\tau)$.

In order to prove the existence of uniform (with respect to σ) exponential attractors, we shall prove that all the assumptions of Proposition 1.1 are satisfied.

We set $X = D(A) \times \dot{H}^2_n(\Omega) \times T^k$ and we consider the operator $\overline{A}$ defined by

$$\overline{A} = \begin{pmatrix} A & 0 \\ 0 & -\Delta \end{pmatrix}, \tag{2.26}$$

with domain $D(\overline{A}) = D(A) \times \dot{H}^2_n(\Omega)$. Let (Λ_n), $n \in N^\star$, be the eigenvalues of $\overline{A}$, $0 < \Lambda_1 < \cdots < \Lambda_n < \cdots$, $\Lambda_n \to +\infty$ as $n \to +\infty$, and let (w_n), $n \in N^\star$, be the associated eigenvectors. We then introduce the orthogonal projector P_n on $\mathrm{Span}(w_1, ..., w_n)$. If $\Theta = \begin{pmatrix} u \\ p \end{pmatrix}$, then we immediately have

$$|\Theta|_{V \times H^1(\Omega)} \geq \sqrt{\Lambda_{n+1}} |Q_n \Theta|_{H \times L^2(\Omega)}, \tag{2.27}$$

so that the first assumption of Proposition 1.1 is satisfied.

Let now $\varphi = \begin{pmatrix} u \\ p \\ \omega \end{pmatrix}$ and $\psi = \begin{pmatrix} v \\ q \\ \overline{\omega} \end{pmatrix}$ be two solutions of (2.19)–(2.22) with initial values $\varphi_0 = \begin{pmatrix} u_0 \\ p_0 \\ \sigma \end{pmatrix}$ and $\psi_0 = \begin{pmatrix} v_0 \\ q_0 \\ \overline{\sigma} \end{pmatrix}$ respectively (for the sake of simplicity we shall omit temporarily the superscript ϵ). Thus, $\varphi(t) - \psi(t)$ satisfies the equations

$$\frac{dw}{dt} + \nu Aw + B(w,u) + B(v,w) + \nabla r = \tilde{f} - \overline{f}, \tag{2.28}$$

$$\epsilon \frac{dr}{dt} + \operatorname{div} w = 0, \tag{2.29}$$

$$\frac{d}{dt}(\omega - \overline{\omega}) = 0, \tag{2.30}$$

$$w(0) = w_0 = u_0 - v_0, \quad r(0) = r_0 = p_0 - q_0, \quad (\omega - \overline{\omega})(0) = \sigma - \overline{\sigma}, \tag{2.31}$$

where $w = u - v$, $r = p - q$ and $\overline{f} = \tilde{f}(x, \overline{\omega}(t))$.

We then introduce the decomposition

$$\varphi(t) - \psi(t) = \varphi^1(t) + \varphi^2(t), \tag{2.32}$$

where $\varphi^1(t) = (u^1, p^1, 0)$ and $\varphi^2(t) = (u^2, p^2, \omega - \overline{\omega})$ satisfy the equations

$$\frac{du^1}{dt} + \nu Au^1 + \nabla p^1 = 0, \tag{2.33}$$

$$\epsilon \frac{dp^1}{dt} + \operatorname{div} u^1 = 0, \tag{2.34}$$

$$u^1(0) = w_0, \ p^1(0) = r_0 \tag{2.35}$$

and

$$\frac{du^2}{dt} + \nu Au^2 + B(w,u) + B(v,w) + \nabla p^2 = \tilde{f} - \overline{f}, \tag{2.36}$$

$$\epsilon \frac{dp^2}{dt} + \operatorname{div} u^2 = 0, \tag{2.37}$$

$$u^2(0) = 0, \quad p^2(0) = 0. \tag{2.38}$$

Taking the scalar product in H of (2.33) with u^1, we have

$$\frac{1}{2}\frac{d}{dt}|u^1|^2 + \nu\|u^1\|^2 - (p^1, \operatorname{div} u^1) = 0, \tag{2.39}$$

which yields, thanks to (2.34)

$$\frac{1}{2}\frac{d}{dt}(|u^1|^2 + \epsilon|p^1|^2) + \nu\|u^1\|^2 = 0. \tag{2.40}$$

Integrating (2.40) between 0 and t, we also obtain

$$\int_0^t \|u^1(s)\|^2 ds \leq \frac{1}{2\nu}(|w_0|^2 + \epsilon|r_0|^2). \tag{2.41}$$

We now consider the function

$$E_0(u,p) = |u|^2 + 2\delta\epsilon(u, \nabla p) + \epsilon(1+\nu\epsilon\delta)|p|^2, \tag{2.42}$$

$\delta \geq 0$ to be fixed later and $0 < \epsilon \leq \epsilon_0$, $\epsilon_0 > 0$, which was introduced in [10]. We note that in (2.42) and also in what follows, the constants will be independent of ϵ for $\epsilon \leq \epsilon_0$. As in [6], Sec. 6.2, we find, thanks to (2.39)

$$\frac{d}{dt}E_0(u^1, p^1) + c\delta(|u^1|^2 + 2\epsilon|p^1|^2) \leq 0, \tag{2.43}$$

for $\delta \leq \delta_0$; which yields, using Gronwall's lemma

$$|u^1(t)|^2 + \epsilon|p^1(t)|^2 \leq ce^{-\delta c' t}(|w_0|^2 + \epsilon|r_0|^2). \tag{2.44}$$

Taking then the scalar product of (2.36) and (2.37) with Au^2, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u^2\|^2 + \nu|Au^2|^2 + b(u^1+u^2, u, Au^2) + b(v, u^1+u^2, Au^2)\\ + (\nabla p^2, Au^2) = (\tilde{f} - \overline{f}, Au^2),\end{aligned} \tag{2.45}$$

$$\frac{\epsilon}{2}(1+\mu)\frac{d}{dt}|\nabla p^2|^2 = (\nabla p^2, Au^2). \tag{2.46}$$

Therefore, thanks to the properties satisfied by b (see [6])

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u^2(t)\|^2 + \frac{\epsilon}{2}(1+\mu)\frac{d}{dt}|\nabla p^2(t)|^2 + \frac{\nu}{2}|Au^2(t)|^2\\ \leq g_1^4(t)\|u^2(t)\|^2 + g_2^2(t)\|u^2(t)\|^2 + k_1^2\|u^1(t)\|^2\\ + k_2^2(|u^1(t)|^2 + \|u^1(t)\|^2) + c|\sigma - \overline{\sigma}|^2,\end{aligned} \tag{2.48}$$

where

$$g_1(t) = c_1|u(t)|^{\frac{1}{2}}\|u(t)\|^{\frac{1}{2}} + c_1'|v(t)|^{\frac{1}{2}}\|v(t)\|^{\frac{1}{2}}, \tag{2.49}$$

$$g_2(t) = c_2|Au(t)|^{\frac{1}{2}}\|u(t)\|^{\frac{1}{2}} + c_2'|Av(t)|^{\frac{1}{2}}\|v(t)\|^{\frac{1}{2}}, \tag{2.50}$$

$$k_1 = Sup_{t\geq 0}(c_3|Au(t)|^{\frac{1}{2}}\|u(t)\|^{\frac{1}{2}} + c_3'|Av(t)|^{\frac{1}{2}}|v(t)|^{\frac{1}{2}}), \tag{2.51}$$

$$k_2 = Sup_{t\geq 0}(c_4|Au(t)|^{\frac{1}{2}}\|u(t)\|^{\frac{1}{2}} + c_4'|Av(t)|^{\frac{1}{2}}\|v(t)\|^{\frac{1}{2}}). \tag{2.52}$$

We prove, exactly as in [6], that the functions g_1 and g_2 are continuous and that k_1 and k_2 are finite. We set

$$g_3 = g_1^4 + g_2^2, \tag{2.53}$$

and integrating (2.48) between 0 and t, we obtain

$$\begin{aligned} \|u^2(t)\|^2 \leq \int_0^t g_3(s)\|u^2(s)\|^2 ds + k_1^2\int_0^t \|u^1(s)\|^2 ds \\ + k_2^2\int_0^t (\|u^1(s)\|^2 + |u^1(s)|^2)ds + ct|\sigma - \overline{\sigma}|^2, \end{aligned} \tag{2.54}$$

and thus, thanks to Gronwall's lemma

$$\begin{aligned} \|u^2(t)\|^2 \leq (k_1^2 + k_2^2)\int_0^t e^{\int_s^t g_3(\tau)d\tau}\|u^1(s)\|^2 ds \\ + k_2^2\int_0^t e^{\int_s^t g_3(\tau)d\tau}|u^1(s)|^2 ds + ce^{\int_0^t g_3(\tau)d\tau}t|\sigma - \overline{\sigma}|^2, \end{aligned}$$

which yields, using (2.41) and (2.44)

$$\begin{aligned} \|u^2(t)\|^2 \leq (k_1^2 + k_2^2)e^{\int_0^t g_3(\tau)d\tau}(|w_0|^2 + \epsilon|r_0|^2) \\ + ck_2^2 e^{\int_0^t g_3(\tau)d\tau}(|w_0|^2 + \epsilon|r_0|^2) + c'te^{\int_0^t g_3(\tau)d\tau}|\sigma - \overline{\sigma}|^2. \end{aligned} \tag{2.55}$$

We also derive an estimate on the pressure by integrating (2.46) between 0 and t, and we find, thanks to (2.54)

$$\begin{aligned} \|u^2(t)\|^2 + \epsilon(1+\mu)|\nabla p^2(t)|^2 \leq \int_0^t g_3(s)\|u^2(s)\|^2 ds + k_1^2\int_0^t \|u^1(s)\|^2 ds \\ + k_2^2\int_0^t (\|u^1(s)\|^2 + |u^1(s)|^2)ds + ct|\sigma - \overline{\sigma}|^2, \end{aligned} \tag{2.56}$$

which gives, using again (2.41) and (2.44)

$$\|u^2(t)\|^2 + \epsilon(1+\mu)|\nabla p^2(t)|^2 \le \int_0^t g_3(s)\|u^2(s)\|^2 ds + c(k_1^2+k_2^2)(|w_0|^2 + \epsilon|r_0|^2) + c't|\sigma-\overline{\sigma}|^2,$$

and thanks to (2.55), we find

$$\begin{aligned}\|u^2(t)\|^2 &+ \epsilon(1+\mu)|\nabla p^2(t)|^2 \\ &\le c(k_1^2+k_2^2)(1+\int_0^t g_2(\tau)d\tau e^{\int_0^t g_3(\tau)d\tau})(|w_0|^2+\epsilon|r_0|^2) \\ &+ c't(1+\int_0^t g_3(\tau)d\tau e^{\int_0^t g_3(\tau)d\tau})|\sigma-\overline{\sigma}|^2,\end{aligned} \tag{2.57}$$

and we finally deduce that

$$\|u^2(t)\|^2+\epsilon(1+\mu)|\nabla p^2(t)|^2+|\sigma-\overline{\sigma}|^2 \le h(t)(|w_0|^2+\epsilon|r_0|^2+|\sigma-\overline{\sigma}|^2), \tag{2.58}$$

where

$$h(t) = c(k_1^2+k_2^2)(1+\int_0^t g_3(\tau)d\tau e^{\int_0^t g_3(\tau)d\tau})+c't(1+\int_0^t g_3(\tau)d\tau e^{\int_0^t g_3(\tau)d\tau})+1. \tag{2.59}$$

We can then obtain estimates on $h(t)$ (i.e. on the $g_i s$ and the $k_i s$) by using the a priori estimates derived in Subsection 1.1, and we easily check that $t \mapsto h(t)$ is continuous (see [6] for more details).

Thus, thanks to (2.44) and (2.58), we see that all the assumptions of Proposition 1.1 are satisfied. Therefore, $S^\epsilon(t)$ enjoys the squeezing property on $X = D(A) \times \dot{H}_n^2(\Omega) \times T^k$, and consequently, $S^\epsilon(t)$ possesses a $L^2(\Omega)^3 \times T^k - X$ exponential attractor $\mathcal{M}^\epsilon$ for a set of initial data in X (the Lipschitz property of the semigroup is easy to check, and we can choose the initial data in a bounded absorbing set which can be covered by a finite number of balls of radius one (see Subsection 2.3 below)). Furthermore, thanks to Remark 1.2 and since all the estimates we have derived are uniform in ϵ for $\epsilon \le \epsilon_0$, we have the

Proposition 2.1. *We set* $Y = D(A) \times \dot{H}_n^2(\Omega)$ *and* $\mathcal{M}_{T^k}^\epsilon = \Pi_1 \mathcal{M}^\epsilon$. *Then,* $\mathcal{M}_{T^k}^\epsilon$ *is a local* $L^2(\Omega)^3 - Y$ *uniform (with respect to* σ*) exponential attractor* *for the family of processes associated to* (2.16)–(2.18) *and for a set of initial data in* Y. *Furthermore, if we consider the variables* u^ϵ *and* $q^\epsilon = \epsilon^{\frac{1}{2}} p^\epsilon$, *then the fractal dimension of the corresponding uniform* (*with respect to* σ) *exponential attractor and the rate of attraction of the trajectories are bounded independently of* ϵ *for* $\epsilon \leq \epsilon_0$, $\epsilon_0 > 0$.

2.3. Lower-semicontinuity of the exponential attractor. We proved in the previous subsection the existence of a local exponential attractor $\mathcal{M}^\epsilon$ for the semigroup $S^\epsilon(t)$ (when $\epsilon \leq \epsilon_0$, $\epsilon_0 > 0$) with an exponential rate of attraction which is uniform with respect to ϵ and for the variables $(u^\epsilon, \epsilon^{\frac{1}{2}} p^\epsilon)$. We are here interested in studying the continuity of $\mathcal{M}^\epsilon$ as $\epsilon \to 0^+$. The equations (2.16)–(2.18) become (formally) when ϵ tends to 0 :

$$\frac{du}{dt} + \nu A u + B(u,u) + \nabla p = \tilde{f}(\omega_0(t) + \sigma), \tag{2.60}$$

$$\operatorname{div} u = 0, \tag{2.61}$$

$$u(\tau) = u_\tau, \; p(\tau) = p_\tau, \tag{2.62}$$

$\sigma \in T^k$, where we have omit the superscript ($\epsilon =$)0.

Let $\overline{\Pi}$ denote the projector on the velocity component (i.e., $\overline{\Pi}(u,p) = u$). We give the following definition:

Definition 2.1. We shall say that the family of uniform exponential attractors $\mathcal{M}^\epsilon_{T^k}$, $0 < \epsilon \leq \epsilon_0$, is lower-semicontinuous to $\mathcal{M}_{T^k}$ if $\forall s \in R$, $\forall \delta > 0$, $\exists \tau = \tau(\delta, s) > s$, $\exists 0 < \epsilon = \epsilon(\delta, s) \leq \epsilon_0$ such that

$$Sup_{\sigma \in T^k} dist(U_\sigma(\tau, s)\mathcal{M}_{T^k}, \overline{\Pi}\mathcal{M}^\epsilon_{T^k}) \leq \delta, \tag{2.63}$$

where $U_\sigma(t,s)$ is the family of processes associated to (2.60)–(2.62).

We note that the exponential attractor $\mathcal{M}^\epsilon$ obtained in the previous subsection is an exponential attractor obtained for initial data in $Y = D(A) \times \dot{H}^2_n(\Omega)$, and we can actually see that the initial data can be chosen in

$$Y^\epsilon = \{(u, \epsilon^{\frac{1}{2}} p) \in D(A) \times \dot{H}^2_n(\Omega), \; |u|_{D(A)} \leq c, \; |\epsilon^{\frac{1}{2}} p|_{\dot{H}^2_n(\Omega)} \leq c'\},$$

when c and c' are large enough (i.e. when Y^ϵ is absorbing for (2.16)–(2.18)). Furthermore, it is proved in [8] that the semigroup $S(t)$ constructed with (2.60)–(2.62) possesses an exponential attractor $\mathcal{M}$ for initial data in

$$Z = \overline{\cup_{\sigma \in T^k} \cup_{t \geq \tau_1} S(t)(B \times T^k)},$$

where B is of the form

$$B=\{u\in D(A),\ divu=0,\ |u|_{D(A)}\leq c\},$$

and where τ_1 and c are properly chosen (we note that thanks to the regularizing effects of (2.60)–(2.62), it is not a restriction to take the initial data in Z, and this exponential attractor is actually global). We thus consider here very regular solutions of (2.60)–(2.62) in order to study the continuity of the trajectories as $\epsilon\to 0^+$. However, thanks to the regularizing effects of the equations, this is again not a restriction.

Let $(u,\sigma)\in\mathcal{M}\cap S(t)Z$. Then, there exists $(u_0,\sigma_0)\in Z$ such that

$$(u,\sigma)=S(t)(u_0,\sigma_0).$$

Since $divf=0$, we define p_0 by

$$-\Delta p_0=divB(u_0,u_0), \tag{2.64}$$

so that $p\in\dot{H}^2_n(\Omega)$ and $(u_0,\epsilon^{\frac{1}{2}}p_0)\in Y^\epsilon$ for ϵ small enough. We set

$$(v(t),q(t))=(u(t),p(t))-(u^\epsilon(t),p^\epsilon(t)), \tag{2.65}$$

where $(u,p,\sigma)=S(t)(u_0,p_0,\sigma_0)$ and $(u^\epsilon,p^\epsilon,\sigma)=S^\epsilon(t)(u_0,p_0,\sigma_0)$; and (v,q) is the solution of

$$\frac{dv}{dt}+\nu Av+B(u,v)+B(v,u^\epsilon)+\nabla q=0, \tag{2.66}$$

$$\epsilon\frac{dq}{dt}+divv=\epsilon\frac{dp}{dt}, \tag{2.67}$$

$$v(0)=0,\ q(0)=0. \tag{2.68}$$

We easily derive the following energy estimate :

$$\frac{1}{2}\frac{d}{dt}|v|^2+\frac{1}{2}\epsilon\frac{d}{dt}|q|^2+\nu\|v\|^2\leq\epsilon|\frac{dp}{dt}||v|+|b(v,u^\epsilon,v)|, \tag{2.69}$$

and the properties satisfied by b (see [6] and [10]) yield

$$|b(v,u^\epsilon,v)|\leq\nu\|v\|^2+\frac{c}{\nu}\|u^\epsilon\|^2|v|^2+\frac{c'}{\nu^3}|u^\epsilon|^2\|u^\epsilon\|^2|v|^2, \tag{2.70}$$

and thus

$$\frac{d}{dt}|v|^2 + \epsilon\frac{d}{dt}|q|^2 \le \epsilon|\frac{dp}{dt}|^2 + \alpha(t)|v|^2, \tag{2.71}$$

where

$$\alpha(t) = 1 + \frac{c}{\nu}\|u^\epsilon(t)\|^2 + \frac{c'}{\nu^3}|u^\epsilon(t)|^2\|u^\epsilon(t)\|^2. \tag{2.72}$$

Therefore, using Gronwall's lemma

$$|v(t)|^2 + \epsilon|q(t)|^2 \le \epsilon\beta(t), \tag{2.73}$$

where

$$\beta(t) = \int_0^t |\frac{dp}{dt}(s)|^2 e^{\int_s^t \alpha(\tau)d\tau} ds, \tag{2.74}$$

and we easily prove that β is well defined and continuous (see [6] for the details).

Now, since $\mathcal{M}^\epsilon$ is an exponential attractor for $S^\epsilon(t)$, for every $\delta > 0$, there exists a time $\tau(\delta) > 0$ independent of ϵ such that

$$\forall t \ge \tau(\delta),\ dist(S^\epsilon(t)Y^\epsilon, \mathcal{M}^\epsilon) \le \frac{\delta}{2}. \tag{2.75}$$

We then choose $\epsilon_1 = \epsilon_1(\delta) \le \epsilon_0$ such that

$$\beta(\tau(\delta))\epsilon_1 \le \frac{\delta}{2}. \tag{2.76}$$

Thus, thanks to (2.75), for every $\epsilon \le \epsilon_1$, there exists $(\overline{u}^\epsilon, \epsilon^{\frac{1}{2}}\overline{p}^\epsilon, \overline{\sigma}^\epsilon) \in \mathcal{M}^\epsilon$ such that

$$|(u^\epsilon(\tau), \epsilon^{\frac{1}{2}}p^\epsilon(\tau), \sigma) - (\overline{u}^\epsilon, \epsilon^{\frac{1}{2}}\overline{p}^\epsilon, \overline{\sigma}^\epsilon)|_{L^2(\Omega)^3 \times T^k} \le \frac{\delta}{2}. \tag{2.77}$$

Furthermore, thanks to (2.76)

$$|(u^\epsilon(\tau), \epsilon^{\frac{1}{2}}p^\epsilon(\tau), \sigma) - (u(\tau), \epsilon^{\frac{1}{2}}p(\tau), \sigma)|_{L^2(\Omega)^3 \times T^k} \le \frac{\delta}{2}. \tag{2.78}$$

Therefore, combining (2.77) and (2.78), we obtain

$$\begin{aligned} &|(\overline{u}^\epsilon, \epsilon^{\frac{1}{2}}\overline{p}^\epsilon, \overline{\sigma}^\epsilon) - (u(\tau), \epsilon^{\frac{1}{2}}p(\tau), \sigma)|^3_{L^2(\Omega)} \times T^k \\ &\quad \le |(u^\epsilon(\tau), \epsilon^{\frac{1}{2}}p^\epsilon(\tau), \sigma) - (\overline{u}^\epsilon, \epsilon^{\frac{1}{2}}\overline{p}^\epsilon, \overline{\sigma}^\epsilon)|_{L^2(\Omega)^3 \times T^k} \\ &\quad + |(u^\epsilon(\tau), \epsilon^{\frac{1}{2}}p^\epsilon(\tau), \sigma) - (u(\tau), \epsilon^{\frac{1}{2}}p(\tau), \sigma)|_{L^2(\Omega)^3 \times T^k} \\ &\quad \le \delta. \end{aligned} \tag{2.79}$$

In particular, we have

$$|\overline{u}^\epsilon - u| \le \delta, \tag{2.80}$$

and since this result is uniform in u and σ, we find

$$Sup_{(u,\sigma)\in\mathcal{M}\cap S(\tau)Z} Inf_{(\overline{u}^\epsilon,\overline{\sigma}^\epsilon)\in\Pi\mathcal{M}^\epsilon} |(\overline{u}^\epsilon,\overline{\sigma}^\epsilon)-(u,\sigma)|_{L^2(\Omega)^2\times T^k} \le \delta, \tag{2.81}$$

where Π denotes the projector on the velocity and time symbol components.

We have thus proved the

Proposition 2.2.. *The family of exponential attractors $\mathcal{M}^\epsilon$ for the semigroup $S^\epsilon(t)$ associated to (2.19)–(2.22) is lower-semicontinuous to $\mathcal{M}$, where $\mathcal{M}$ is an exponential attractor for the semigroup $S(t)$ associated to the incompressible Navier-Stokes equations.*

We now prove the lower-semicontinuity of the uniform (with respect to σ) exponential attractors. Since $\mathcal{M}_{T^k} = \Pi_1\mathcal{M}$ and $\mathcal{M}^\epsilon_{T^k} = \Pi_1\mathcal{M}^\epsilon$, a consequence of (2.80) is that

$$Sup_{\sigma\in T^k} Sup_{u\in\mathcal{M}_{T^k}\cap U_\sigma(t,0)\Pi_1 Z} Inf_{\overline{u}^\epsilon\in\overline{\Pi}\mathcal{M}^\epsilon_{T^k}} |\overline{u}^\epsilon - u| \le \delta, \tag{2.82}$$

hence, (2.63) when $s=0$. Actually, in (2.66)-(2.68), we could have taken the initial data for $t=s$, instead of $t=0$, and set $(u,p,\sigma) = S(t-s)(u_0,p_0,\sigma_0)$, $(u^\epsilon,p^\epsilon,\sigma) = S^\epsilon(t-s)(u_0,p_0,\sigma_0)$ if $s \ge 0$; and $(u,p,\sigma) = S(t-s)(u_0,p_0,T(s)\sigma_0)$, $(u^\epsilon,p^\epsilon,\sigma) = S^\epsilon(t-s)(u_0,p_0,T(s)\sigma_0)$, where $T(s)\sigma = \alpha s+\sigma \ (mod\, T^k)$, if $s<0$ (it is proved in [2] that $U_{T(s)\sigma}(t,\tau) = U_\sigma(t+s,\tau+s)$, $\forall s\ge 0,\ t\ge\tau,\ \tau\in R$, and since this relation is generally only valid for $s\ge 0$, we need to consider two cases). Repeating the same proof, we then obtain (2.63) for any given $s\in R$. We thus have the

Proposition 2.3. *The family of uniform (with respect to σ) exponential attractors $\mathcal{M}^\epsilon_{T^k}$ associated to the slightly compressible Navier-Stokes equations is lower-semicontinuous to $\mathcal{M}_{T^k}$, where $\mathcal{M}_{T^k}$ is a uniform (with respect to σ) exponential attractor associated to the incompressible Navier-Stokes equations.*

Remark 2.3. We can obtain similar results if instead of Dirichlet boundary conditions, we consider free surface boundary conditions:

$$u^\epsilon.n = curl u^\epsilon = 0 \ on\ \partial\Omega,$$

where $curl v = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}$; or periodic boundary conditions :

$$u^\epsilon \ and \ p^\epsilon \ are \ \Omega - periodic, \quad \int_\Omega u^\epsilon dx = 0.$$

In the case of periodic boundary conditions, we also assume that f is Ω-periodic and that $\int_\Omega f dx = 0$.

Acknowledgments. The authors wish to thank Professor P. Fabrie for many stimulating discussions. They also wish to thank the referees, whose remarks and comments helped improve the presentation of this article.

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