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INTEGRABILITY AND BOUNDEDNESS OF LOCAL SOLUTIONS TO SINGULAR AND DEGENERATE QUASILINEAR PARABOLIC EQUATIONS

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Abstract. Integral and supremum estimates are proven for local solutions of degenerate and singular quasilinear parabolic equations. This is done with the aid of a local energy inequality and estimates in L_q^{weak} spaces.

1. Introduction and results. In this paper we shall obtain $L_{q,loc}(\Omega_T)$ and $L_{\infty,loc}(\Omega_T)$ estimates for local solutions to a class of quasilinear degenerate or singular parabolic equations modeled after

$$u_t - \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) = f(x,t) + \operatorname{div}\mathbf{g}; \qquad (p>1). \tag{1}$$

If $p>2$ the problem is degenerate, while if $p<2$ the problem is singular.

Indeed, let $\Omega \subseteq \mathbf{R}^N$ be a domain, let $T>0$, and let $\Omega_T = \Omega\times(0,T)$. Consider a general quasilinear equation of the form

$$u_t - \operatorname{div} a(x,t,u,\nabla u) = b(x,t,u,\nabla u) \tag{2}$$

where for almost every $(x,t,u,\mathbf{v}) \in \Omega\times[0,T]\times\mathbf{R}\times\mathbf{R}^N$ the following structure conditions hold:

(H1) $1 < p \le \delta < p\big(\frac{N+2}{N}\big)$, $c_i \ge 0$ for $0 \le i \le 5$, $c_0 > 0$, and $\phi_j \ge 0$ for $0 \le j \le 2$,

(H2) $a(x,t,u,\mathbf{v})\cdot\mathbf{v} \ge c_o|\mathbf{v}|^p - c_3|u|^\delta - \phi_o(x,t)$,

(H3) $|a(x,t,u,\mathbf{v})| \le c_1|\mathbf{v}|^{p-1} + c_4|u|^{\delta(1-\frac{1}{p})} + \phi_1(x,t)$,

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(H4) $|b(x,t,u,\mathbf{v})| \le c_2|\mathbf{v}|^{p(1-\frac{1}{\delta})} + c_5|u|^{\delta-1} + \phi_2(x,t)$,

(H5) $\phi_1 \in L_{\frac{p}{p-1},loc}(\Omega_T)$,

(H6) $\phi_o \in L_{\mu,loc}(\Omega_T)$ with $\mu > 1$, and $\phi_1, \phi_2 \in L_{s,loc}(\Omega_T)$ with $s > \frac{(N+2)p}{(N+2)p-N}$,

(H7) $u \in L_{r,loc}(\Omega_T)$ for some $r \ge 1$ for which $N(p-2)+rp > 0$. In particular if $p > 2N/(N+2)$ we may set $r = 2$.

We shall then prove

Theorem 1. *Let* $u \in L_{\infty,loc}(0,T;L_{2,loc}(\Omega)) \cap L_{p,loc}(0,T;W^1_{p,loc}(\Omega))$ *satisfy* (2) *in the sense of distributions, and suppose that the structure conditions* (H1)-(H7) *are satisfied.*

If both $s > (N+p)/p$ *and* $\mu > (N+p)/p$*, then* $u \in L_{\infty,loc}(\Omega_T)$*;*

if both $s = (N+p)/p$ *and* $\mu = (N+p)/p$*, then* $u \in L_{q,loc}(\Omega_T)$ *for all* $q < \infty$*;*

if both $s < (N+p)/p$ *and* $\mu < (N+p)/p$*, then* $u \in L_{q,loc}(\Omega_T)$ *for all* $q < q^*$*, where*

$$q^* = \min\Big\{\frac{p+\frac{p}{N}-1}{1-(1-\frac{1}{s})(1+\frac{p}{N})}, \frac{p+\frac{2p}{N}}{1-(1-\frac{1}{\mu})(1+\frac{p}{N})}\Big\}.$$

Regularity properties of solutions of these types of equations have been studied extensively over the past 15 years, first in an attempt to prove Hölder continuity of solutions, and later to study conditions which guarantee the boundedness of solutions. Indeed in the degenerate case, Hölder continuity was proven by DiBenedetto and Friedman [6, 7], and in the singular case by Y.Z. Chen and DiBenedetto, [3, 4]. Local boundedness under suitable structure conditions was proven by Porzio [11]; such results have been extended to equations with more general structures by Andreucci [1] and Lieberman [10]. An excellent discussion of known results is the book of DiBenedetto [5].

The results contained in this paper have the following new features. First, to the best of this author's knowledge, this is the only result which yields information about the degree of local integrability of solutions which are not necessarily bounded. Secondly, this result extends the class of equations for which the local boundedness of solutions is guaranteed. Indeed, for the case

$p > \frac{2N}{N+2}$, in [5, Chp. 5, Thm 3.1] boundedness of solutions was proven only if

$$\phi_1^{\frac{p}{p-1}}, \phi_2^{\frac{\delta}{\delta-1}} \in L_{s,loc}(\Omega_T) \qquad \text{for } s > \frac{N+p}{p}. \tag{3}$$

In the case $p \leq \frac{2N}{N+2}$, local boundedness was proven in [5, Chp. 5, Thm. 5.1] only if the problem had homogeneous structure, meaning (H2), (H3) and (H4) are replaced by the requirements $a(x,t,u,\mathbf{v})\cdot\mathbf{v} \geq c_o|\mathbf{v}|^p$, $|a(x,t,u,\mathbf{v})| \leq c_1|\mathbf{v}|^{p-1}$ and $b(x,t,u,\mathbf{v}) = 0$; moreover further global information was required, to the effect that the solution could be approximated weakly in $L_{r,loc}(\Omega_T)$ by bounded solutions. Only under these additional circumstances, now no longer necessary, was boundedness proven.

The results of Theorem 1 are almost optimal in the sense that they almost agree with the results in the linear case. Indeed, consider the linear problem (so $p = 2$)

$$u_t - \frac{\partial}{\partial x_i}\left\{a_{ij}(x,t)u_{x_j} + a_i(x,t)u\right\} + b_i(x,t)u_{x_i} + a(x,t)u = \phi(x,t) + \frac{\partial \phi_i}{\partial x_i} \tag{4}$$

for $\phi \in L_{s,loc}(\Omega_T)$, $\phi_i \in L_{\mu,loc}(\Omega_T)$. Ladyzhenskaya, Solonnikov and Uraltseva showed in [8, Chap. 3, Secs. 8,9] that local solutions of this problem are locally bounded if $s, \mu > \frac{N+2}{2}$, that they are in $L_{q,loc}(\Omega_T)$ for all $q < \infty$ if $s = \mu = \frac{N+2}{2}$, and that they are in $L_{q^*,loc}(\Omega_T)$ if $s, \mu < \frac{N+2}{2}$, where q^* is the number in Theorem 1 with $p = 2$.

A few comments about our assumptions are now in order. At first, the requirement (H5) that $\phi_1 \in L_{\frac{p}{p-1},loc}(\Omega_T)$ may seem out of place; its purpose is to guarantee that terms of the form $a(x,t,u,\nabla u)\cdot\nabla u$ are locally integrable. It is natural in some sense as this same condition was imposed by Ladyzhenskaya, Solonnikov and Uraltseva to obtain boundedness of solutions to quasilinear equations in the case $p = 2$; see [8, Chap. 5, Thm. 2.1, Eqn. 2.2].

The restrictions on s and μ in (H6) are exactly the conditions that guarantee $q^* > \frac{N+2}{N}p$; recall that Sobolev embedding will imply that any function which satisfies the integrability hypotheses of Theorem 1 will be in $L_{\frac{N+2}{N}p,loc}(\Omega_T)$.

2. Sketch of proof. The proof of Theorem 1 is based upon the following result.

Proposition 2. (Local Energy Estimate) *Suppose that u is a solution of* (2) *in the sense of distributions, and that the assumptions of Theorem* 1 *hold. Then for any $Q_R(x_o, t_o) = B_R(x_o) \times (t_o - R^p, t_o) \subset\subset \Omega_T$, for any $0 < \sigma < 1$, and for any $k > 0$, we have*

$$\begin{aligned} \Big\{ \iint_{Q_{\sigma R}} (u \mp k)_\pm^{\left(\frac{N+2}{N}\right)p} \, dx\, dt \Big\}^{\frac{1}{1+p/N}} &\leq \frac{\gamma}{(1-\sigma)^p R^p} \iint_{Q_R} (u \mp k)_\pm^2 \, dx\, dt \\ + \frac{\gamma}{(1-\sigma)^p R^p} \iint_{Q_R} (u \mp k)_\pm^p \, dx\, dt &+ \gamma \iint_{Q_R} |u|^\delta \chi[(u \mp k)_\pm > 0] \, dx\, dt \\ + \gamma \Big(\frac{\|\phi_1\|_{L_s(Q_R)}}{(1-\sigma)R} &+ \|\phi_2\|_{L_s(Q_R)} \Big) \Big(\iint_{Q_R} (u \mp k)_\pm^{\frac{s}{s-1}} \, dx\, dt \Big)^{1-\frac{1}{s}} \\ &+ \gamma \|\phi_o\|_{L_\mu(Q_R)} \big(\text{meas}[(u \mp k)_\pm > 0] \big)^{1-\frac{1}{\mu}}. \end{aligned} \tag{5}$$

where γ is a constant that depends only on c_i, N, p, δ, s, and μ but is independent of k and $\|\phi_1\|_{L_{\frac{p}{p-1}, loc}(\Omega_T)}$.

This is a standard result following from the use of a smooth cutoff approximation of $(u \mp k)_\pm$ as a testing function; see for example [5, Chap. 5, Prop. 6.1]. For convenience we have included a sketch of the proof in §.

Once the local energy estimate is known, the proof proceeds in two steps. First observe that if $u \in L_{\beta, loc}(\Omega_T)$ for some β, then the local energy inequality can be used to obtain an estimate of the form

$$\text{meas}\big\{(x,t) \in Q_{\sigma R} : |u(x,t)| > k\big\} \leq \gamma\big(\|u\|_{L_\beta(Q_R)}\big)\big(\frac{1}{k}\big)^{\alpha(\beta)} \tag{6}$$

for some exponent $\alpha(\beta)$. This is sufficient to give us an estimate for $|u|_{L^{\text{weak}}_{\alpha(\beta)}(Q_{\sigma R})}$. Indeed, recall the definitions of the spaces $L_q^{\text{weak}}(\mathcal{U})$; a measurable function f is an element of $L_q^{\text{weak}}(\mathcal{U})$ if and only if

$$|f|_{L_q^{\text{weak}}(\mathcal{U})} \equiv \sup_{k>0} k \big(\text{meas}\{x \in \mathcal{U} : |f(x)| > k\}\big)^{\frac{1}{q}} < \infty. \tag{7}$$

The quantity $|f|_{L_q^{\text{weak}}(\mathcal{U})}$ is not a norm, but it is a quasinorm. The inequality

$$|f|_{L_q^{\text{weak}}(\mathcal{U})} \leq \|f\|_{L_q(\mathcal{U})} \tag{8}$$

follows immediately from

$$k^q \operatorname{meas}[|f| > k] \le \int_{\mathcal{U}} |f|^q \chi[|f| > k]\, dx \le \int_{\mathcal{U}} |f|^q\, dx \tag{9}$$

so that $L_q(\mathcal{U}) \subset L_q^{\mathrm{weak}}(\mathcal{U})$. However $L_q^{\mathrm{weak}}(\mathcal{U}) \ne L_q(\mathcal{U})$, as the function $f(x) = 1/x$ satisfies $f \in L_1^{\mathrm{weak}}(0,1)$, but $f \notin L_1(0,1)$. Finally, if $q' < q$ and $\mathcal{U}$ is bounded, then $L_q^{\mathrm{weak}}(\mathcal{U}) \subset L_{q'}(\mathcal{U})$; indeed, [9, Theorem 1.13] implies

$$\begin{aligned} \|f\|^{q'}_{L_{q'}(\mathcal{U})} &= q' \int_0^\infty k^{q'-1} \operatorname{meas}[|f| > k]\, dk \\ &\le q' \operatorname{meas}\mathcal{U} + q' |f|^q_{L_q^{\mathrm{weak}}(\mathcal{U})} \int_1^\infty k^{q'-q-1}\, dk < \infty. \end{aligned} \tag{10}$$

For further details about the spaces $L_q^{\mathrm{weak}}(\mathcal{U})$ see [2, Chp. 1] or [12, IX.4].

The energy inequality in the form (6) then tells us that if $u \in L_{\beta,loc}(\Omega_T)$, then $u \in L_{q,loc}(\Omega_T)$ for all $q < \alpha(\beta)$. Carefully calculating $\alpha(\beta)$ and iterating this process, we shall prove

Proposition 3. *Under the hypotheses of Theorem* 1, *the following is true. If $s, \mu \ge (N+p)/p$, then $u \in L_{q,loc}(\Omega_T)$ for all $q < \infty$ while if $s, \mu < (N+p)/p$, then $u \in L_{q,loc}(\Omega_T)$ for all $q < q^*$.*

To prove the boundedness of solutions, we shall use the integrability guaranteed by Proposition 3, the energy estimates, and the usual DeGiorgi iteration techniques, coupled with an interpolation in the case $\frac{N+2}{N} p \le 2$ and proceed in what are now standard ways; cf. [5, Chap. 5].

3. The $L_{q,loc}(\Omega_T)$ estimates for $q < \infty$. The left side of (5) in Proposition 2 with the + choice is estimated as

$$\iint_{Q_{\sigma R}} (u-k)_+^{\left(\frac{N+2}{N}\right)p}\, dx\, dt \ge k^{\left(\frac{N+2}{N}\right)p} \operatorname{meas}_{Q_{\sigma R}}[u > 2k], \tag{11}$$

thus

$$\begin{aligned} \Big\{k^{\left(\frac{N+2}{N}\right)p} \operatorname{meas}_{Q_{\sigma R}}[u > 2k]\Big\}^{\frac{1}{1+p/N}} &\le C \iint_{Q_R} (u-k)_+^2\, dx\, dt \\ &+ C \iint_{Q_R} (u-k)_+^p\, dx\, dt + C \iint_{Q_R} |u|^\delta \chi[u > k]\, dx\, dt \\ &+ C\Big(\iint_{Q_R} (u-k)_+^{\frac{s}{s-1}}\, dx\, dt\Big)^{1-\frac{1}{s}} + C\big(\operatorname{meas}[u > k]\big)^{1-\frac{1}{\mu}} \end{aligned} \tag{12}$$

where $C = C(\sigma, R, c_i, N, p, \delta, s, \mu, \|\phi_o\|_{L_{\mu,loc}(\Omega_T)}, \|\phi_1, \phi_2\|_{L_{s,loc}(\Omega_T)})$. We shall estimate $|u|_{L_q^{\text{weak}}(Q_{\sigma R})}$ by analyzing the right side of this equation.

Suppose that $u \in L_{\beta,loc}(\Omega_T)$ for some β sufficiently large. Then, for any exponent $\theta < \beta$, note that

$$\begin{aligned}\iint_{Q_R} (u-k)_+^\theta \, dx \, dt &\leq \Big(\iint_{Q_R} (u-k)_+^\beta \, dx \, dt \Big)^{\frac{\theta}{\beta}} \big(\text{meas}[u > k]\big)^{1-\frac{\theta}{\beta}} \\ &\leq \|u\|^\theta_{L_\beta(Q_R)} \Big(\frac{1}{k^\beta} |u|^\beta_{L_\beta^{\text{weak}}(Q_R)} \Big)^{1-\frac{\theta}{\beta}} \leq \|u\|^\beta_{L_\beta(Q_R)} \Big(\frac{1}{k} \Big)^{\beta-\theta}.\end{aligned} \tag{13}$$

Thus we can estimate

$$\begin{aligned}&\Big\{ k^{\left(\frac{N+2}{N}\right)p} \text{meas}_{Q_{\sigma R}}[u > 2k] \Big\}^{\frac{1}{1+p/N}} \\ &\qquad \leq C\|u\|^\beta_{L_\beta(Q_R)} \Big\{ \big(\frac{1}{k}\big)^{\beta-2} + \big(\frac{1}{k}\big)^{\beta-p} + \big(\frac{1}{k}\big)^{\beta-\delta} \Big\} \\ &\qquad + C\|u\|^{\beta\left(1-\frac{1}{s}\right)}_{L_\beta(Q_R)} \big(\frac{1}{k}\big)^{\beta\left(1-\frac{1}{s}\right)-1} + C\|u\|^{\beta\left(1-\frac{1}{\mu}\right)}_{L_\beta(Q_R)} \big(\frac{1}{k}\big)^{\beta\left(1-\frac{1}{\mu}\right)}.\end{aligned} \tag{14}$$

Repeat the same process with $(u+k)_-$ replacing $(u-k)_+$; we shall obtain a constant

$$\tilde{C} = \tilde{C}(\|u\|_{L_\beta(Q_R)}, \sigma, R, c_i, N, p, \delta, s, \mu, \|\phi_o\|_{L_{\mu,loc}(\Omega_T)}, \|\phi_1, \phi_2\|_{L_{s,loc}(\Omega_T)})$$

but independent of k so that

$$\begin{aligned}\text{meas}_{Q_{\sigma R}}[|u| > k] &\leq \tilde{C} \Big\{ \big(\frac{1}{k}\big)^{\left(1+\frac{p}{N}\right)\beta + (p-2)} + \big(\frac{1}{k}\big)^{\left(1+\frac{p}{N}\right)\beta + \frac{p}{N}(2-p)} \\ &\quad + \big(\frac{1}{k}\big)^{\left(1+\frac{p}{N}\right)\beta + \frac{p}{N}(2-\delta) + (p-\delta)} + \big(\frac{1}{k}\big)^{\left(1+\frac{p}{N}\right)\left(1-\frac{1}{s}\right)\beta + p + \frac{p}{N} - 1} \\ &\quad + \big(\frac{1}{k}\big)^{\left(1+\frac{p}{N}\right)\left(1-\frac{1}{\mu}\right)\beta + p + \frac{2p}{N}} \Big\}.\end{aligned} \tag{15}$$

Set

$$\begin{aligned}\alpha(\beta) = \min \Big\{ &\big(1 + \frac{p}{N}\big)\beta + (p-2), \big(1 + \frac{p}{N}\big)\beta + \frac{p}{N}(2-p), \\ &\big(1 + \frac{p}{N}\big)\beta + \frac{p}{N}(2-\delta) + (p-\delta), \big(1 + \frac{p}{N}\big)\big(1 - \frac{1}{s}\big)\beta + p + \frac{p}{N} - 1, \\ &\big(1 + \frac{p}{N}\big)\big(1 - \frac{1}{\mu}\big)\beta + p + \frac{2p}{N} \Big\};\end{aligned} \tag{16}$$

we then have proven the following

Lemma 4. *If $u \in L_{\beta,loc}(\Omega_T)$, then $u \in L^{\text{weak}}_{\alpha(\beta),loc}(\Omega_T)$ and $u \in L_{q,loc}(\Omega_T)$ for all $q < \alpha(\beta)$.*

With this in mind, we shall analyze the iterations $\beta_o, \alpha(\beta_o), \alpha(\alpha(\beta_0)), \ldots$ where

$$\beta_o = \max\bigl\{2, \frac{N+2}{N}p, r\bigr\} \tag{17}$$

is chosen so that our hypotheses guarantee that $u \in L_{\beta_o,loc}(\Omega_T)$. Lemma 4 will then guarantee that $u \in L_{q,loc}(\Omega_T)$ for any $q < (\alpha\circ\alpha\circ\cdots\circ\alpha)(\beta_o)$, regardless of the number of compositions. To analyze the sequence of iterations, set

$$\alpha_1(\beta) = \bigl(1+\frac{p}{N}\bigr)\beta + (p-2), \tag{18}$$

$$\alpha_2(\beta) = \bigl(1+\frac{p}{N}\bigr)\beta + \frac{p}{N}(2-p), \tag{19}$$

$$\alpha_3(\beta) = \bigl(1+\frac{p}{N}\bigr)\beta + \frac{p}{N}(2-\delta) + (p-\delta), \tag{20}$$

$$\alpha_4(\beta) = \bigl(1+\frac{p}{N}\bigr)\bigl(1-\frac{1}{s}\bigr)\beta + p + \frac{p}{N} - 1, \tag{21}$$

$$\alpha_5(\beta) = \bigl(1+\frac{p}{N}\bigr)\bigl(1-\frac{1}{\mu}\bigr) + p + \frac{2p}{N}; \tag{22}$$

we shall analyze each in turn.

Case 1: α_1. Note that $\alpha_1(\beta) > \beta$ if and only if

$$\beta > \frac{N}{p}(2-p) \tag{23}$$

so that the sequence $\beta_o, \alpha_1(\beta_o), \alpha_1(\alpha_1(\beta_o)), \ldots$ will tend to infinity if $\beta_o > \frac{N}{p}(2-p)$. Indeed, the above shows that this sequence is monotone increasing; if it tended to a finite limit, that limit would be a fixed point of α_1 that was greater than β_o; since there are no such fixed points the sequence tends to infinity.

To see that $\beta_o > \frac{N}{p}(2-p)$, note that $\beta_o \geq r$ and $N(p-2) + pr > 0$ by hypothesis (H7).

Case 2: α_2. Now $\alpha_2(\beta) > \beta$ if and only if

$$\beta > (p-2). \tag{24}$$

Clearly $\beta_o > p-2$, so the sequence $\beta_o, \alpha_2(\beta_o), \alpha_2(\alpha_2(\beta_o)), \ldots$ tends to infinity for the same reasons as before.

Case 3: α_3. Now $\alpha_3(\beta) > \beta$ if and only if

$$\beta > (\delta - 2) + \frac{N}{p}(\delta - p). \tag{25}$$

Note that

$$\begin{aligned}(\delta - 2) + \frac{N}{p}(\delta - p) &< \delta - 2 + \frac{N}{p}\Big[\big(\frac{N+2}{N}\big)p - p\Big] \\ &< \delta - 2 + N + 2 - N < \delta \le \beta_o\end{aligned} \tag{26}$$

and thus the sequence $\beta_o, \alpha_3(\beta_o), \alpha_3(\alpha_3(\beta_o)), \ldots$ tends to infinity.

Case 4: α_4. Here the situation is somewhat different; since $p + \frac{p}{N} - 1 > \frac{p}{N} > 0$, if $\big(1+\frac{p}{N}\big)\big(1-\frac{1}{s}\big) \ge 1$ we can immediately conclude that the sequence $\beta_o, \alpha_4(\beta_o), \alpha_4(\alpha_4(\beta_o)), \ldots$ tends to infinity. The condition $\big(1+\frac{p}{N}\big)\big(1-\frac{1}{s}\big) \ge 1$ is equivalent to the condition

$$s \ge \frac{N+p}{p}. \tag{27}$$

Now suppose $s < \frac{N+p}{p}$. Then $\alpha_4(\beta) > \beta$ if and only if

$$\beta < \frac{p + \frac{p}{N} - 1}{1 - \big(1 - \frac{1}{s}\big)\big(1 + \frac{p}{N}\big)}, \tag{28}$$

while $\alpha_4(\beta) < \beta$ if and only if

$$\beta > \frac{p + \frac{p}{N} - 1}{1 - \big(1 - \frac{1}{s}\big)\big(1 + \frac{p}{N}\big)}. \tag{29}$$

Thus, if $s < \frac{N+p}{p}$, the sequence $\beta_o, \alpha_4(\beta_o), \alpha_4(\alpha_4(\beta_o)), \ldots$ tends to

$$q_s^* = \frac{p + \frac{p}{N} - 1}{1 - \big(1 - \frac{1}{s}\big)\big(1 + \frac{p}{N}\big)}. \tag{30}$$

Case 5: As in case 4, if $\mu \ge (N+p)/p$, we know immediately that the sequence $\beta_o, \alpha_5(\beta_o), \alpha_5(\alpha_5(\beta_o)), \ldots$ tends to infinity. If $\mu < (N+p)/p$, then $\alpha_5(\beta) > \beta$ if and only if

$$\beta < \frac{p + \frac{2p}{N}}{1 - \big(1 - \frac{1}{\mu}\big)\big(1 + \frac{p}{N}\big)} \tag{31}$$

and $\alpha_5(\beta) < \beta$ if and only if the above inequality is reversed. Thus, if $\mu < (N+p)/p$, we know that the sequence $\beta_o, \alpha_5(\beta_o), \alpha_5(\alpha_5(\beta_o)), \dots$ tends to

$$q^*_\mu = \frac{p + \frac{2p}{N}}{1 - \big(1 - \frac{1}{\mu}\big)\big(1 + \frac{p}{N}\big)}. \tag{32}$$

Summarizing, we have the following.

Lemma 5. *The sequence $\beta_o, \alpha(\beta_o), \alpha(\alpha(\beta_o)), \alpha(\alpha(\alpha(\beta_o))), \dots$ tends to infinity if $s, \mu \ge \frac{N+p}{p}$ and it tends to q^* if $s, \mu < \frac{N+p}{p}$.*

Proposition 3 then follows immediately.

4. The $L_{\infty,loc}(\Omega_T)$ estimates. In this section, we shall show the boundedness of solutions if $s > \frac{N+p}{p}$ and $\mu > \frac{N+p}{p}$. Proposition 2 implies that, for any $0 < \tilde\sigma < 1$, for any $\tilde k$, and for any $Q_R \subset\subset \Omega_T$,

$$\begin{aligned}\Big(\iint_{Q_{\tilde\sigma R}} (u-\tilde k)_+^{\left(\frac{N+2}{N}\right)p}\,dx\,dt\Big)^{\frac{1}{1+p/N}} &\le \frac{\gamma}{(1-\tilde\sigma)^p R^p}\iint_{Q_R}(u-\tilde k)_+^2\,dx\,dt\\ &+ \frac{\gamma}{(1-\tilde\sigma)^p R^p}\iint_{Q_R}(u-\tilde k)_+^p\,dx\,dt + \gamma\iint_{Q_R}|u|^\delta\chi[u>\tilde k]\,dx\,dt\\ &+ \gamma\Big(1+\frac{1}{(1-\tilde\sigma)R}\Big)\Big(\iint_{Q_R}(u-\tilde k)_+^{\frac{s}{s-1}}\,dx\,dt\Big)^{1-\frac{1}{s}}\\ &+ \gamma\big(\mathrm{meas}_{Q_R}[u>\tilde k]\big)^{1-\frac{1}{\mu}}. \end{aligned} \tag{33}$$

Fix $\rho > 0$, $\sigma > 0$, and let $Q_\rho \subset\subset \Omega_T$. For each integer n, set

$$\rho_n = \sigma\rho + \frac{(1-\sigma)}{2^n}\rho, \tag{34}$$

and let $Q^n = Q_{\rho_n}$ so that $Q^0 = Q_\rho$ and $Q^\infty = Q_{\sigma\rho}$. Let $k > 0$ be chosen later, and set

$$k_n = k\big(1 - \frac{1}{2^{n+1}}\big). \tag{35}$$

We wish to apply (33) above with $\tilde k = k_{n+1}$, $Q_R = Q^n$, and $Q_{\tilde\sigma R} = Q^{n+1}$. To do so, note that in this instance

$$(1-\tilde\sigma)R = \big(\frac{1-\sigma}{2^{n+1}}\big)\rho. \tag{36}$$

Substituting this above yields

$$\begin{aligned}
&\Big(\iint_{Q^{n+1}} (u-k_{n+1})_+^m \,dx\,dt\Big)^{\frac{1}{1+p/N}} \le \frac{\gamma 2^{np}}{(1-\sigma)^p\rho^p}\iint_{Q^n}(u-k_{n+1})_+^2\,dx\,dt\\
&+\frac{\gamma 2^{np}}{(1-\sigma)^p\rho^p}\iint_{Q^n}(u-k_{n+1})_+^p\,dx\,dt+\gamma\iint_{Q^n}|u|^\delta\chi[u>k_{n+1}]\,dx\,dt\\
&+\frac{\gamma 2^n}{(1-\sigma)\rho}\Big(\iint_{Q^n}(u-k_{n+1})_+^{\frac{s}{s-1}}\,dx\,dt\Big)^{1-\frac{1}{s}}+\gamma\big(\operatorname{meas} A_{n+1}\big)^{1-\frac{1}{\mu}}
\end{aligned} \tag{37}$$

where $m = \frac{N+2}{N}p$ and

$$A_{n+1} = \{(x,t)\in Q^n : u(x,t) > k_{n+1}\}\,. \tag{38}$$

Note that for every $\theta > 1$,

$$\begin{aligned}
\iint_{Q^n}(u-k_n)_+^\theta\,dx\,dt &\ge \iint_{Q_n}(u-k_n)_+^\theta\chi[u>k_{n+1}]\,dx\,dt\\
&\ge (k_{n+1}-k_n)^\theta \operatorname{meas} A_{n+1} \ge \Big(\frac{k}{2^{n+2}}\Big)^\theta \operatorname{meas} A_{n+1}
\end{aligned} \tag{39}$$

so that

$$\operatorname{meas} A_{n+1} \le \Big(\frac{2^{n+2}}{k}\Big)^\theta \iint_{Q^n}(u-k_n)_+^\theta\,dx\,dt. \tag{40}$$

To obtain our boundedness results, we shall use (37) and (40) to develop an iterative inequality, however the choice of what to estimate shall depend upon the parameters of the problem.

4.1. Case 1: $\frac{N+2}{N}p > 2$. We shall obtain an iterative inequality for

$$Y_n = ⨌_{Q^n}(u-k_n)_+^m\,dx\,dt = \frac{1}{\operatorname{meas} Q^n}\iint_{Q^n}(u-k_n)_+^m\,dx\,dt. \tag{41}$$

We begin by estimating the first term on the right side of (37) with the aid of (40) as

$$\begin{aligned}
&\frac{\gamma 2^{np}}{(1-\sigma)^p\rho^p}\iint_{Q^n}(u-k_{n+1})_+^2\,dx\,dt\\
&\qquad\le \frac{\gamma 2^{np}}{(1-\sigma)^p\rho^p}\Big(\iint_{Q^n}(u-k_{n+1})_+^m\,dx\,dt\Big)^{\frac{2}{m}}\{\operatorname{meas}A_{n+1}\}^{1-\frac{2}{m}}\\
&\qquad\le \frac{\gamma 2^{np}}{(1-\sigma)^p\rho^p}\Big(\frac{2^{n+2}}{k}\Big)^{m-2}\iint_{Q^n}(u-k_n)_+^m\,dx\,dt\\
&\qquad\le \frac{\gamma}{(1-\sigma)^p k^{m-2}}\rho^N 2^{(p+m-2)n}Y_n
\end{aligned}\tag{42}$$

where we have used the fact that $m=\frac{N+2}{N}p>2$. Similarly

$$\frac{\gamma 2^{np}}{(1-\sigma)^p\rho^p}\iint_{Q^n}(u-k_{n+1})_+^p\,dx\,dt\le\frac{\gamma}{(1-\sigma)^p k^{m-p}}\rho^N 2^{mn}Y_n.\tag{43}$$

To estimate the third term on the right of (37), first note that if $C>B$, then

$$\sup_{y>C}\frac{y}{y-B}=\frac{C}{C-B}\tag{44}$$

so that for each $(x,t)\in[u>k_{n+1}]$ we have

$$\frac{u}{u-k_n}\le\frac{k_{n+1}}{k_{n+1}-k_n}.\tag{45}$$

Thus, for each θ,

$$|u|^\theta\chi[u>k_{n+1}]\le 2^{(n+2)\theta}(u-k_n)_+^\theta.\tag{46}$$

Consequently, since $\delta<m$

$$\gamma\iint_{Q^n}|u|^\delta\chi[u>k_{n+1}]\,dx\,dt\le\frac{\gamma}{k^{m-\delta}}\rho^{N+p}2^{mn}Y_n.\tag{47}$$

To estimate the next term in a similar fashion, we need to know $\frac{s}{s-1}\le m$; however this is a simple consequence of the fact that $s\ge\frac{N+p}{p}$ and the definition of m. As this requirement is satisfied, we proceed as follows:

$$\begin{aligned}\frac{\gamma 2^n}{(1-\sigma)\rho}\Big(\iint_{Q^n}(u-k_{n+1})_+^{\frac{s}{s-1}}\Big)^{1-\frac{1}{s}}&\\ &\le \frac{\gamma 2^n}{(1-\sigma)\rho}\Big(\iint_{Q^n}(u-k_{n+1})_+^m\,dx\,dt\Big)^{\frac{1}{m}}\{\text{meas}\,A_{n+1}\}^{1-\frac{1}{s}-\frac{1}{m}}\\ &\le \frac{\gamma 2^n}{(1-\sigma)\rho}\Big(\frac{2^{n+2}}{k}\Big)^{m\left(1-\frac{1}{s}\right)-1}\Big(\iint_{Q^n}(u-k_n)_+^m\Big)^{1-\frac{1}{s}}\\ &\le \frac{\gamma}{(1-\sigma)k^{m\left(1-\frac{1}{s}\right)-1}}\rho^{(N+p)\left(1-\frac{1}{s}\right)-1}2^{\left[m\left(1-\frac{1}{s}\right)-1\right]n}Y_n^{1-\frac{1}{s}}.\end{aligned}\tag{48}$$

Lastly,

$$\begin{aligned}\gamma\big(\text{meas}\,A_{n+1}\big)^{1-\frac{1}{\mu}}&\le \gamma\Big(\frac{2^n m}{k^m}\rho^{N+p}Y_n\Big)^{1-\frac{1}{\mu}}\\ &\le \frac{\gamma}{k^{m\left(1-\frac{1}{\mu}\right)}}\rho^{(N+p)\left(1-\frac{1}{\mu}\right)}2^{m\left(1-\frac{1}{\mu}\right)n}Y_n^{1-\frac{1}{\mu}}\end{aligned}\tag{49}$$

Thus

$$\begin{aligned}\rho^{N+p}Y_{n+1}\le&\frac{\gamma\rho^{N+p}2^{(p+m-2)\left(1+\frac{p}{N}\right)n}}{(1-\sigma)^{p+N}k^{(m-2)\left(1+\frac{p}{N}\right)}}Y_n^{1+\frac{p}{N}}+\frac{\gamma\rho^{N+p}2^{(p+m)\left(1+\frac{p}{N}\right)n}}{(1-\sigma)^{p+N}k^{(m-p)\left(1+\frac{p}{N}\right)}}Y_n^{1+\frac{p}{N}}\\ &+\frac{\gamma\rho^{(N+p)\left(1+\frac{p}{N}\right)}2^{m\left(1+\frac{p}{N}\right)n}}{k^{(m-\delta)\left(1+\frac{p}{N}\right)}}Y_n^{1+\frac{p}{N}}\\ &+\frac{\gamma\rho^{\left[(N+p)\left(1-\frac{1}{s}\right)-1\right]\left(1+\frac{p}{N}\right)}2^{\left[m\left(1-\frac{1}{s}\right)-1\right]\left(1+\frac{p}{N}\right)n}}{(1-\sigma)^{1+\frac{p}{N}}k^{\left[m\left(1-\frac{1}{s}\right)-1\right]\left(1+\frac{p}{N}\right)}}Y_n^{\left(1-\frac{1}{s}\right)\left(1+\frac{p}{N}\right)}\\ &+\frac{\gamma\rho^{\left(N+p\right)\left(1-\frac{1}{\mu}\right)\left(1+\frac{p}{N}\right)}2^{m\left(1-\frac{1}{\mu}\right)\left(1+\frac{p}{N}\right)n}}{k^{m\left(1-\frac{1}{\mu}\right)\left(1+\frac{p}{N}\right)}}Y_n^{\left(1-\frac{1}{\mu}\right)\left(1+\frac{p}{N}\right)}.\end{aligned}\tag{50}$$

Now note that

$$m\Big(1-\frac{1}{s}\Big)-1>0\tag{51}$$

so that if we require $k \geq 1$, $0 < \sigma < \sigma_o < 1$, and fix ρ, we obtain constants $\tilde{\gamma} = \tilde{\gamma}(\sigma_o, \rho, c_i, N, p, \delta, s, \mu. \|\phi_o\|_{L_{\mu,loc}(\Omega_T)}, \|\phi_1, \phi_2\|_{L_{s,loc}(\Omega_T)})$ and $B = B(N, p, \delta, s, \mu)$ so that

$$Y_{n+1} \leq \tilde{\gamma} B^n Y_n^{1+\frac{p}{N}} + \tilde{\gamma} B^n Y_n^{(1-\frac{1}{s})(1+\frac{p}{N})} + \tilde{\gamma} B^n Y_n^{(1-\frac{1}{\mu})(1+\frac{p}{N})}. \tag{52}$$

Now if $s > \frac{N+p}{p}$, then

$$(1 - \frac{1}{s})(1 + \frac{N}{p}) > (\frac{N}{N+p})(\frac{N+p}{N}) = 1, \tag{53}$$

while similarly the assumption $\mu > \frac{N+p}{p}$ implies

$$\big(1 - \frac{1}{\mu}\big)\big(1 + \frac{p}{N}\big) > 1, \tag{54}$$

hence by the usual rules on fast geometric convergence, $Y_n \to 0$ if Y_o is sufficiently small. However,

$$Y_o = \fint\!\!\fint_{Q_\rho} \big(u - \frac{k}{2}\big)_+^m \, dx\, dt \tag{55}$$

so if $k = k(\rho, \sigma_o, c_i, N, p, \delta, s, \mu, \|\phi_o\|_{L_{\mu,loc}(\Omega_T)}, \|\phi_1, \phi_2\|_{L_{s,loc}(\Omega_T)})$ is sufficiently large, we may guarantee that $Y_n \to 0$; consequently

$$Y_\infty = \fint\!\!\fint_{Q_{\sigma\rho}} (u - k)_+^m \, dx\, dt = 0 \tag{56}$$

so that u is bounded above. Similar considerations for $(u + k)_-$ show that u is bounded below.

4.2. Case 2: $\frac{N+2}{N} p \leq 2$. Let $\lambda > \max\{2, p, m, \frac{s}{s-1}\}$ be chosen later and set

$$Y_n = \fint\!\!\fint_{Q^n} (u - k_n)_+^\lambda \, dx\, dt; \tag{57}$$

this is well defined thanks to Proposition 3. Now for any finite $\Lambda > \lambda > m$, we can apply the usual convexity inequality to obtain

$$Y_{n+1} = \fint\!\!\fint_{Q^{n+1}} (u - k_{n+1})_+^\lambda \, dx\, dt \leq \frac{1}{\operatorname{meas} Q^{n+1}} \Big(\iint_{Q^{n+1}} (u - k_{n+1})_+^\Lambda \, dx\, dt\Big)^{\frac{\lambda}{\Lambda}\theta} \times \Big(\iint_{Q^{n+1}} (u - k_{n+1})_+^m \, dx\, dt\Big)^{\frac{\lambda}{m}(1-\theta)} \tag{58}$$

where

$$\theta = \frac{\frac{1}{m} - \frac{1}{\lambda}}{\frac{1}{m} - \frac{1}{\Lambda}} = \frac{\Lambda}{\lambda}\frac{\lambda - m}{\Lambda - m}. \tag{59}$$

Thus

$$Y_{n+1} \le \frac{1}{\operatorname{meas} Q^{n+1}} \|u\|_{L_\Lambda(Q_\rho)}^{\frac{\lambda-m}{\Lambda-m}\Lambda} \Big(\iint_{Q^{n+1}} (u - k_{n+1})_+^m \, dx\, dt \Big)^{\frac{\Lambda-\lambda}{\Lambda-m}} \tag{60}$$

and hence

$$\iint_{Q^{n+1}} (u - k_{n+1})_+^m \, dx\, dt \ge [\operatorname{meas} Q^{n+1}]^{\frac{\Lambda-m}{\Lambda-\lambda}} \frac{1}{\|u\|_{L_\Lambda(Q_\rho)}^{\frac{\lambda-m}{\Lambda-\lambda}\Lambda}} Y_{n+1}^{\frac{\Lambda-m}{\Lambda-\lambda}} \tag{61}$$

which estimates the left side of (37). The right side of this equation is estimated in the same fashion as case 1, indeed

$$\frac{\gamma 2^{np}}{(1-\sigma)^p \rho^p} \iint_{Q^n} (u - k_{n+1})_+^2 \, dx\, dt \le \frac{\gamma \rho^N 2^{(p+\lambda-2)n}}{(1-\sigma)^p k^{\lambda-2}} Y_n, \tag{62}$$

$$\frac{\gamma 2^{np}}{(1-\sigma)^p \rho^p} \iint_{Q^n} (u - k_{n+1})_+^p \, dx\, dt \le \frac{\gamma \rho^N 2^{\lambda n}}{(1-\sigma)^p k^{\lambda-p}} Y_n, \tag{63}$$

$$\gamma \iint_{Q^n} |u|^\delta \chi[u > k_{n+1}] \, dx\, dt \le \frac{\gamma \rho^{N+p} 2^{\lambda n}}{k^{\lambda-\delta}} Y_n, \tag{64}$$

$$\frac{\gamma 2^n}{(1-\sigma)\rho} \Big[\iint Q^n (u - k_{n+1})_+^{\frac{s}{s-1}} \, dx\, dt \Big]^{1-\frac{1}{s}} \le \frac{\gamma \rho^{(N+p)(1-\frac{1}{s})} 2^{[\lambda(1-\frac{1}{s})-1]n}}{(1-\sigma) k^{\lambda(1-\frac{1}{s})-1}} Y_n^{1-\frac{1}{s}}, \tag{65}$$

$$\gamma \big(\operatorname{meas} A_{n+1}\big)^{1-\frac{1}{\mu}} \le \frac{\gamma \rho^{(N+p)\left(1-\frac{1}{\mu}\right)} 2^{\lambda\left(1-\frac{1}{\mu}\right)n}}{k^{\lambda\left(1-\frac{1}{\mu}\right)}} Y_n^{1-\frac{1}{\mu}}. \tag{66}$$

Thus

$$\begin{aligned} \rho^{(N+p)\left(\frac{\Lambda-m}{\Lambda-\lambda}\right)} Y_{n+1} \le \|u\|_{L_\Lambda(Q_\rho)}^{\frac{\lambda-m}{\Lambda-m}\Lambda} \Bigg\{ & \frac{\gamma \rho^{N+p} 2^{(p+\lambda-2)\left(1+\frac{p}{N}\right)}}{(1-\sigma)^{p+N} k^{(\lambda-2)\left(1+\frac{p}{N}\right)}} Y_n^{1+\frac{p}{N}} \\ & + \frac{\gamma \rho^{N+p} 2^{(p+\lambda)\left(1+\frac{p}{N}\right)n}}{(1-\sigma)^{p+N} k^{(\lambda-p)\left(1+\frac{p}{N}\right)}} Y_n^{1+\frac{p}{N}} + \frac{\gamma \rho^{(N+p)\left(1+\frac{p}{N}\right)} 2^{\lambda\left(1+\frac{p}{N}\right)n}}{k^{(\lambda-\delta)\left(1+\frac{p}{N}\right)}} Y_n^{1+\frac{p}{N}} \end{aligned} \tag{67}$$

$$+\frac{\gamma\rho^{\left[(N+p)\left(1-\frac{1}{s}\right)-1\right]\left(1+\frac{p}{N}\right)}2^{\left[\lambda\left(1-\frac{1}{s}\right)-1\right]\left(1+\frac{p}{N}\right)n}}{(1-\sigma)^{1+\frac{p}{N}}k^{\left[\lambda\left(1-\frac{1}{s}\right)-1\right]\left(1+\frac{p}{N}\right)}}Y_n^{\left(1-\frac{1}{s}\right)\left(1+\frac{p}{N}\right)}$$

$$+\frac{\gamma\rho^{(N+p)\left(1-\frac{1}{\mu}\right)\left(1+\frac{p}{N}\right)}2^{\lambda\left(1-\frac{1}{\mu}\right)\left(1+\frac{p}{N}\right)n}}{k^{\lambda\left(1-\frac{1}{\mu}\right)\left(1+\frac{p}{N}\right)}}Y_n^{\left(1-\frac{1}{\mu}\right)\left(1+\frac{p}{N}\right)}\Bigg\}^{\frac{\Lambda-\lambda}{\Lambda-m}}.$$

Let λ be so large that $\lambda\big(1-\frac{1}{s}\big)-1>0$. Working as we did before, if we fix ρ and require $k\geq 1$ and $0<\sigma<\sigma_o<1$, then we will obtain a pair of constants $\tilde\gamma=\tilde\gamma(\Lambda,\lambda,\|u\|_{L_\Lambda(Q_\rho)},\sigma_o,\rho,c_i,N,p,\delta,s,\mu,\|\phi_o\|_{L_{\mu,loc}(\Omega_T)},\|\phi_1,\phi_2\|_{L_{s,loc}(\Omega_T)})$ and $B=B(\Lambda,\lambda,N,p,\delta,s,\mu)$ so that

$$Y_{n+1}\leq\tilde\gamma B^nY_n^{\left(1+\frac{p}{N}\right)\left(\frac{\Lambda-\lambda}{\Lambda-m}\right)}+\tilde\gamma B^nY_n^{\left(1+\frac{p}{N}\right)\left(1-\frac{1}{s}\right)\left(\frac{\Lambda-\lambda}{\Lambda-m}\right)}+\tilde\gamma B^nY_n^{\left(1+\frac{p}{N}\right)\left(1-\frac{1}{\mu}\right)\left(\frac{\Lambda-\lambda}{\Lambda-m}\right)}. \quad (68)$$

Now since $s,\mu>\frac{N+p}{p}$, we know $\big(1+\frac{p}{N}\big)\big(1-\frac{1}{s}\big)>1$ and $\big(1+\frac{p}{N}\big)\big(1-\frac{1}{\mu}\big)>1$. Further, since

$$\lim_{\Lambda\to\infty}\frac{\Lambda-\lambda}{\Lambda-m}=1 \quad (69)$$

we can choose $\Lambda=\Lambda(\lambda,m,s,p,N)$ so that both

$$\big(1+\frac{p}{N}\big)\big(1-\frac{1}{s}\big)\big(\frac{\Lambda-\lambda}{\Lambda-m}\big)>1 \quad (70)$$

and

$$\big(1+\frac{p}{N}\big)\big(1-\frac{1}{\mu}\big)\big(\frac{\Lambda-\lambda}{\Lambda-m}\big)>1. \quad (71)$$

The usual rules on fast geometric convergence then allow us to proceed as we did in case 1, and find $u\in L_{\infty,loc}(\Omega_T)$ as required.

5. Proof of Proposition 2. Let $(x_o,t_o)\in\Omega_T$; modulo a translation we shall assume $(x_o,t_o)=(0,0)$. Set $Q_R=B_R\times(-R^p,0)$, and suppose that $Q_R\subset\subset\Omega_T$. Let $-R^p\leq\tau\leq 0$ and set $Q_R^\tau=B_R\times(-R^p,\tau)$. Let $0<\sigma<1$ be fixed. Let $\zeta\in C^\infty(Q_R)$ be a cutoff function so that $0\leq\zeta\leq 1$, with $\zeta(x,t)=1$ for $(x,t)\in Q_{\sigma R}$, so that $\zeta(x,t)=0$ near $|x|=R$ or $t=-R^p$, and so that $|\nabla\zeta|^p + |\zeta_t| \le 2/(1-\sigma)^p R^p$. Let $\zeta_j \in C_0^\infty(Q_R^\tau)$ be a sequence of cutoff functions that tend to ζ, so that $\zeta_j(x,t) = 0$ if $t \ge \tau - 1/j$ and $\zeta_j(x,t) = \zeta(x,t)$ if $t \le \tau - 2/j$. Let J_η be a symmetric mollifying kernel in space and time, and denote the space-time convolution $J_\eta * f$ by f_η.

Fix $\eta > 0$, and $k > 0$, and consider the function

$$\psi(x,t) = \left\{(u_\eta - k)_+ \zeta_j^p\right\}_\eta . \tag{72}$$

If η is sufficiently small, then $\psi \in \mathcal{D}(\Omega_T)$, thus

$$\begin{aligned}\iint_{\Omega_T} -u\frac{\partial}{\partial t}\left\{(u_\eta - k)_+\zeta_j^p\right\}_\eta \,dx\,dt \\ + \iint_{\Omega_T} a^i(x,t,u,\nabla u)\frac{\partial}{\partial x_i}\left\{(u_\eta - k)_+\zeta_j^p\right\}_\eta \,dx\,dt \\ = \iint_{\Omega_T} b(x,t,u,\nabla u)\left\{(u_\eta - k)_+\zeta_j^p\right\}_\eta \,dx\,dt \\ I_1 + I_2 = I_3.\end{aligned} \tag{73}$$

We shall investigate each term separately.

Since the mollifier is symmetric,

$$\begin{aligned}I_1 &= -\iint_{\Omega_T} u_\eta \frac{\partial}{\partial t}\left\{(u_\eta - k)_+\zeta_j^p\right\} dx\,dt \\ &= \frac{1}{2}\iint_{\Omega_T}\left\{\frac{\partial}{\partial t}(u_\eta - k)_+^2\right\}\zeta_j^p\,dx\,dt - \frac{p}{2}\iint_{\Omega_T}(u_\eta - k)_+^2\zeta_j^{p-1}\frac{\partial\zeta_j}{\partial t}\,dx\,dt.\end{aligned} \tag{74}$$

Send $j \to \infty$ and integrate by parts to obtain

$$\lim_{j\to\infty} I_1 = \frac{1}{2}\int_{B_R}(u_\eta - k)_+^2\zeta^p\Big|_\tau\,dx - \frac{p}{2}\iint_{Q_R^\tau}(u_\eta - k)_+^2\zeta^{p-1}\zeta_t\,dx\,dt \tag{75}$$

and since $u \in C_{loc}(0,T;L_{2,loc}(\Omega))$, we have

$$\lim_{\eta\downarrow 0}\lim_{j\to\infty} I_1 = \frac{1}{2}\int_{B_R}(u-k)_+^2\zeta^p\Big|_\tau dx - \frac{p}{2}\iint_{Q_R^\tau}(u-k)_+^2\zeta^{p-1}\zeta_t\,dx\,dt. \tag{76}$$

As for I_2, we have

$$I_2 = \iint_{\Omega_T} a_\eta^i(x,t,u,\nabla u)\frac{\partial}{\partial x_i}\left\{(u_\eta - k)_+\zeta_j^p\right\} dx\,dt. \tag{77}$$

The integrability of u and the structure conditions (H3) and (H5) guarantee that $a^i(x,t,u,\nabla u) \in L_{\frac{p}{p-1},loc}(\Omega_T)$, thus we can send $j \to \infty$ and $\eta \downarrow 0$ to obtain

$$\lim_{\eta\downarrow 0}\lim_{j\to\infty} I_2 = \iint_{Q_R^\tau} a^i(x,t,u,\nabla(u-k)_+)\left\{\frac{\partial}{\partial x_i}(u-k)_+\right\}\zeta^p\,dx\,dt \\ + p\iint_{Q_R^\tau} a^i(x,t,u,\nabla(u-k)_+)(u-k)_+\zeta^{p-1}\zeta_{x_i}\,dx\,dt. \quad (78)$$

Applying (H2), (H3) and Young's inequality, we obtain

$$\lim_{\eta\downarrow 0}\lim_{j\to\infty} I_2 \geq \gamma\iint_{Q_R^\tau} |\nabla(u-k)_+|^p\zeta^p\,dx\,dt - \gamma\iint_{Q_R^\tau} |u|^\delta\chi[u>k]\zeta^p\,dx\,dt \\ - \gamma\iint_{Q_R^\tau}(u-k)_+^p|\nabla\zeta|^p\,dx\,dt - \iint_{Q_R^\tau}\phi_o\chi[u>k]\zeta^p\,dx\,dt \\ - \iint_{Q_R^\tau}\phi_1(u-k)_+\zeta^{p-1}|\nabla\zeta|\,dx\,dt. \quad (79)$$

Lastly, we turn to I_3,

$$I_3 = \iint_{\Omega_T} b_\eta(x,t,u,\nabla u)(u_\eta-k)_+\zeta^p\,dx\,dt. \quad (80)$$

The integrability of u and the structure conditions (H4) and (H6) guarantee that $b(x,t,u,\nabla u) \in L_{\frac{m}{m-1},loc}(\Omega_T)$, where $m = \frac{N+2}{N}p$; since Sobolev embedding implies $u \in L_{m,loc}(\Omega_T)$ we can send $j \to \infty$ and $\eta \downarrow 0$ to obtain

$$\lim_{\eta\downarrow 0}\lim_{j\to\infty} I_3 = \iint_{Q_R^\tau} b(x,t,u,\nabla u)(u-k)_+\zeta^p\,dx\,dt, \quad (81)$$

so by (H4)

$$\lim_{\eta\downarrow 0}\lim_{j\to\infty} I_3 \leq c_2\iint_{Q_R^\tau}|\nabla(u-k)_+|^{p\left(1-\frac{1}{\delta}\right)}(u-k)_+\zeta^p\,dx\,dt \\ + c_5\iint_{Q_R^\tau}|u|^\delta\chi[u>k]\zeta^p\,dx\,dt + \iint_{Q_R^\tau}\phi_2(u-k)_+\zeta^p\,dx\,dt. \quad (82)$$

If we put together our estimates for I_1, I_2, and I_3, together with Young's inequality, we obtain

$$\begin{aligned}\operatorname*{ess\,sup}_{-R^p<\tau<0}\int_{B_R}|(u-k)_+\zeta^\kappa|^2\Big|_\tau\,dx+\iint_{Q_R}|\nabla(u-k)_+\zeta^\kappa|^p\,dx\,dt\\ \le\frac{\gamma}{(1-\sigma)^pR^p}\iint_{Q_R}\left\{(u-k)_+^2+(u-k)_+^p\right\}dx\,dt\\ +\gamma\iint_{Q_R}|u|^\delta\chi[u>k]\,dx\,dt+\gamma\iint_{Q_R}\phi_o\chi[u>k]\,dx\,dt\\ +\frac{\gamma}{(1-\sigma)R}\iint_{Q_R}\phi_1(u-k)_+\,dx\,dt+\gamma\iint_{Q_R}\phi_2(u-k)_+\,dx\,dt\end{aligned}\tag{83}$$

where $\kappa=\max\{1,p/2\}$. The Sobolev embedding theorem then yields the result; at least for $(u-k)_+$. The case for $(u+k)_-$ is handled similarly.

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