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STOCHASTIC PDE FOR NONLINEAR VIBRATION OF ELASTIC PANELS*

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(Submitted by: G. Da Prato)

Abstract. This paper is concerned with a nonlinear integro-partial differential equation perturbed by a state dependent white noise arising from the aeroelastic panel vibration. With the aid of a stochastic energy equation, the existence, uniqueness and regularity of solutions are proved.

1. Introduction. In nonlinear aeroelasticity, the fluttering or large-amplitude vibration of an elastic panel excited by aerodynamic forces is a well known problem of practical importance. On a simplified formulation the mathematical model for the nonlinear penal vibration is governed by the quasi-linear wave like integro-differential equation with the initial boundary condition

$$
\begin{aligned}
\partial_t^2 u\,(t,x) - \Big[\alpha + \beta \int_0^\ell |\partial_y u\,(t,y)|^2\,dy\Big]\partial_x^2 u + \gamma \partial_x^4 u &= F\,(t,x,\partial_t u,\partial_x u) \\
u\,(t,0) = u\,(t,\ell) = 0, \quad \partial_x u(t,0) &= \partial_x u\,(t,\ell) = 0, \\
u\,(0,x) = \varphi_0(x), \quad \partial_t u\,(0,x) &= \varphi_1\,(x)\,,
\end{aligned}
\tag{1.1}
$$

for $0 \le t \le T, 0 \le x \le \ell$, where $\partial_t = \frac{\partial}{\partial_t}, \partial_x = \frac{\partial}{\partial_x}, \alpha, \beta, \gamma$ are positive constants; φ_0 and φ_1 are given functions and F is the total force consisting of the air pressure $f\,(t,x)$ and the aerodynamic force $g\,(t,x,\partial_t u,\partial_x u)$ due to the flow speed U. At high speed the aerodynamic force is approximately of the form

$$g\,(t,x,\partial_t u,\partial_x u) = -C\,(\partial_t u + U\partial_x u) \tag{1.2}$$

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for some constant $C > 0$. A detailed derivation of this model equation and many relevant references on this subject can be found in the book by Dowell [1].

For simplicity most studies are based on the assumption that the force F is deterministic. In reality, due to air turbulence at high speed, both the pressure f and the aerodynamic force g are perturbed by random fluctuations. In this paper we will consider the special case for which the random perturbations are spatially dependent white noises in time. More precisely the force F in (1.1) takes the form

$$F(t,x,\partial_t u,\partial_x u) = f(t,x) + \sum_{k=1}^{\infty} \sigma^k(t,x)\, \dot{w}_k(t) \tag{1.3}$$
$$+g(t,x,\partial_t u,\partial_x u) + \sum_{k=1}^{\infty} \breve{\sigma}^k(t,x,\partial_t u,\partial_x u)\, \dot{\breve{w}}_k(t),$$

where f and g are the mean forces, $\{\sigma^k(t,x)\}$ and $\{\tilde{\sigma}^k(t,x,y,z)\}$ are given sequences of functions; $\{w^k(t)\}$ and $\{\tilde{w}^k(t)\}$ are independent sequences of standard Wiener processes in one dimension. We write

$$\sigma_t(x)\, \dot{w}_t = \sum_{k=1}^{\infty} \sigma^k(t,x)\, \dot{w}_k(t), \tag{1.4}$$
$$\breve{\sigma}_t(x,\dot{u},\partial_x u)\, \dot{\breve{w}}_t = \sum_{k=1}^{\infty} \breve{\sigma}(t,x,\dot{u},\partial_x u)\, \dot{w}_k(t),$$

where $\dot{u} = \partial_t u$. Then the equation (1.1) can be written as a second order Itô equation

$$d\dot{u}(t,x) = \Big[\Big(\alpha + \beta \int_0^\ell |\partial, u|^2\, dy\Big)\partial_x^2 u - \gamma \partial_x^4 u +$$
$$+f(t,x) + g(t,x,\dot{u}\partial_x u)\Big] dt + \sigma_t(x)\, dw_t + \tilde{\sigma}_t(x,\dot{u},\partial_x u) d\, \tilde{w}_t, \tag{1.5}$$
$$u(t,0) = \partial_x u(t,0) = 0, \quad u(t,\ell) = \partial_x u(t,\ell) = 0,$$
$$\dot{u}(0,x) = \varphi_1(x), \quad u(0,x) = \varphi_0(x).$$

The solutions of stochastic PDEs of parabolic type have been studied by many authors (see many references cited in the book by Da Prato and

Zabczyk [2]). For nonlinear stochastic wave equations, the existence of strong solutions and the uniqueness question were first treated comprehensively by Pardoux [3] for a class of monotone operators. In the present case, referring to (1.5), the cubic nonlinear term, representing the additional tension caused by a large deflection, is not monotone. Therefore, the proof by the Galerkin approximation as adopted in [3] does not apply to our problem directly. Alternatively we will employ the method of truncation and a fixed-point argument in our existence proof. However, as done by Strauss [4] in the deterministic case, the energy equation for a linear stochastic wave equation derived by Pardoux [3] will also play an indispensable role in our study.

The text of this paper is organized as follows. In section 2 we formulate the problem and Introduce the related operators in a proper function space setting. Section 3 is concerned with the stochastic energy equation and estimator for a linear equation and the uniqueness question for the nonlinear PDE perturbed by an additive noise. The corresponding existence theorem is proved in section 4. Then, in section 5, the nonlinear PDE with a general noise is treated. By combining the results for the additive and the multiplicative noises, we can deduce the existence and uniqueness theorem for the general case given by equation (1.5).

2. Nonlinear Stochastic PDE. Introduce the Hilbert spaces $L^2 = L^2\left(]0,\ell[\right), H_0^1 = H_0^1\left(]0,\ell[\right)$ and $H_0^2 = H_0^2\left(]0,\ell[\right)$, where H_0^1 and H_0^2 are Sobolev spaces of order one and two, respectively, satisfying the homogeneous boundary conditions. Denote by H^{-k} the dual space of H_0^k with $k = 1, 2$. Let $(\cdot,\cdot)$ denote the L^2- inner product, and $< \cdot,\cdot >$ the duality pairing between H_0^2 and H^{-2}. The norms $\|\cdot\|$ on L^2, $\|\cdot\|_{(k)}$ on H_0^k and $\|\cdot\|_{(-k)}$ on H^{-k}, for $k = 1, 2$, are defined as usual. Clearly we have the following dense and compact inclusions: $H_0^2 \subset H_0^1 \subset L^2 \subset H^{-1} \subset H^{-2}$.

Let us consider the operator

$$Au = \alpha\partial_x^2 u - \gamma\partial_x^4 u, \tag{2.1}$$

$$B\left(u\right) = \beta\Big(\int_0^\ell \left|\partial_x u\right|^2 dx\Big)\partial_x^2 u, \tag{2.2}$$

which are defined in H_0^2 and H_0^1, respectively. In fact we have $A : H_0^2 \to H^{-2}$ being linear and continuous and $B : H_0^1 \to H^{-1}$ being nonlinear and locally

Lipschitz continuous. Actually, it is easy to verify the following estimates:

$$\|A\varphi\|_{(-2)} \leq \big(\gamma + \frac{\alpha\ell}{2}\big)\, \|\varphi\|_{(2)}\, , \varphi \in H_2^0, \tag{2.3}$$

$$\begin{aligned}\|B\,(\varphi) - B\,(\psi)\|_{(-1)} &\leq \beta\big(\, \|\varphi\|_{(1)}^2 + \|\psi\|_{(1)}^2 + \|\varphi\|_{(1)}\, \|\psi\|_{(1)}\,\big) \\ &\quad \times \|\varphi - \psi\|_{(1)}\, , \varphi, \psi \in H_0^1.\end{aligned} \tag{2.4}$$

Alternatively we need to consider $B : H_0^2 \to L^2$, which is also a locally Lipschitz continuous nonlinear operator. Indeed, from the fact

$$\begin{aligned}\|B(\varphi) - B(\psi)\| = \beta\Big\|\Big[\int_0^\ell (|\partial_y\varphi|^2 - |\partial_y\psi|^2)\, dy\Big]\partial_x^2\varphi \\ + \Big[\int_0^\ell |\partial_y\psi|^2 dy\Big](\partial_x^2\varphi - \partial_x^2\psi)\Big\| \\ \leq \beta\|\partial_x^2\varphi\| \int_0^\ell |(\partial_y\varphi)^2 - (\partial_y\psi)^2| dy + \beta \int_0^\ell |\partial_y\psi|^2 dy \|\partial_x^2\varphi - \partial_x^2\psi\|,\end{aligned}$$

it follows that

$$\begin{aligned}\|B\,(\varphi) - B\psi\| \leq \beta\, \big\|\partial_x^2\varphi\big\|\ \ \|\partial_x\varphi + \partial_x\psi\|\ \ \|\partial_x\varphi - \partial_x\psi\| \\ +\beta\, \|\partial_x\psi\|^2\, \big\|\partial_x^2\varphi - \partial_x^2\psi\big\|\, , \varphi, \psi \in H_0^2.\end{aligned} \tag{2.5}$$

For the random force terms, let $\{w_k\,(t)\}$ and $\{\widetilde{w}_k\,(t)\}$ be two independent, identically distributed (standard) Wiener processes in $\mathbb{R}^1$ defined on a complete probability space (Ω, F, P). One may regard $w_t = (w_1\,(t)\,, w_2\,(t)\,, ...)$ and $\widetilde{w}_t = (\,\widetilde{w}_1\,(t)\,, \widetilde{w}_2\,(t), ...)$ as two independent cylindrical Brownian motions in the sequence space ℓ_2 (see §4.3 in [2]). Let

$$\dot{M}_t = \sigma_t\dot{w}_t = \sum_{k=1}^{\infty} \sigma^k\,(t, \cdot)\, \dot{w}_k\,(t) \tag{2.6}$$

or in the form of the stochastic integral,

$$M_t = \int_0^t \sigma_s dw_s = \sum_{k=1}^{\infty} \int_0^t \sigma^k\,(s, \cdot)\, dw_k\,(s)\, , \tag{2.7}$$

which is well defined if

$$\int_0^T w\left(\sigma_t^2\right) dt = \sum_{k=1}^{\infty} \int_0^t \left\|\sigma^k\,(t, \cdot)\right\|^2 dt < \infty. \tag{2.8}$$

Similarly, for $\dot{u} \in L^2\left(\Omega; C\left([0,T]; L^2\right)\right)$ and $u \in L^2\left(\Omega, C\left([0,T]; H_0^1\right)\right)$ both non-anticipating, if

$$E\int_0^T \widetilde{M}[\widetilde{\sigma}_s\,(\dot{u},u)]^2 dr = \sum_{k=1}^{\infty} E\int_0^T ||\,\widetilde{\sigma}_s\,(\dot{u}u)||^2 ds < \infty \tag{2.9}$$

where $\widetilde{\sigma}_s^k\,(\dot{u},u)(x) \cong \widetilde{\sigma}^k\,(t,x,\dot{u}(t,x),u(t,x))$, then $\dot{\widetilde{M}}_t$ and $\widetilde{M}_t$ can be defined as (2.6) and (2.7), respectively. Moreover, we have $M, \widetilde{M} \in M_T^2\left(L^2\right)$, the space of L^2-valued, square-integrable martingale over $[0,T]$.

By writing $u_t = u(t,\cdot), \dot{u}_t = \partial_t u(t,\cdot)$ etc., we now interpret the formal stochastic PDE (1.5) as a second-order Itô's equation in the variational form:

$$\begin{aligned} d(\dot{u}_t,\psi) =&< Au_t + B(u_t) + f_t + g_t(\dot{u}_t,u_t), \psi > dt \qquad (2.10)\\ &+ (\psi, \sigma_t dw_t) + (\psi\,, \widetilde{\sigma}_t\,(\dot{u}_t,u_t) d\,\widetilde{w}_t), \\ (u_0\psi) = (\varphi_o,\psi)&, \quad (\dot{u}_0,\psi) = (\varphi,\psi), \qquad (2.11)\end{aligned}$$

which hold a.s. for any $\psi \in H_0^2, 0 \le t \le T$.

3. Energy Estimates and Uniqueness. Let $\lambda(t)$ and ξ_t be progressively measurable processes with values in $\mathbb{R}^1$ and L^2, respectively. We associate equation (1.5) with the following linear problem:

$$d\dot{u}_t = [A_\lambda(t)\,u_t + f_t + \xi_t]\,dt + dM_t, \quad 0 < t < T, \tag{3.1}$$

with $\dot{u}_0 = \varphi_1$ and $u_0 = \varphi_0$, where

$$A_\lambda(t)\,\varphi = A\varphi + \lambda(t)\,\partial_x^2\varphi, \quad \varphi \in H_0^2. \tag{3.2}$$

Assume that the following conditions hold:

$$\lambda \in L^2\left(\Omega; C^1\left(]0,T[;\mathbb{R}\right)\right) \text{ and } \xi \in L^1\left(\Omega \times]0,T[; L_2\right), \tag{3.3}$$

$$\varphi_1 \in H_0^2, \quad \varphi_0 \in L^2, \text{ and } f \in L^2\left(]0,T[\times]0,\ell[\right), \tag{3.4}$$

$M \in M_T^2\left(L^2\right)$ with the quadratic variation $\ll M \gg_t$ satisfying

$$E\,Tr \ll M \gg_t < \infty, \text{ for } t \le T \tag{3.5}$$

For a given λ, introduce the associated energy process

$$e_\lambda(t,u) = \frac{1}{2}\left\{\|\dot{u}_t\|^2 + [\alpha + \lambda(t)]\|\partial_x u_t\|^2 + \gamma\left\|\partial_x^2 u_t\right\|^2\right\} \tag{3.6}$$

The following is the key lemma for the linear equation (3.1) which is essential to the subsequent analysis.

Lemma 3.1. *Under the assumption* (3.3)–(3.5), *there exists a unique solution* u *of the initial-value problem* (3.1) *such that*

$$u \in L^2\left(\Omega; L^\infty\left(]0,T[\,; H_0^2\right)\right), \tag{3.7}$$

and

$$\dot{u} \in L^2\left(\Omega; L^\infty\left(]0,T[\,; H\right)\right). \tag{3.8}$$

In fact the sample path u_t *is smooth, that is,*

$$u \in C\left([0,T]\,; H_0^2\right) \text{ and } \dot{u} \in C\left([0,T]\,; H\right) \text{ a.s.}, \tag{3.9}$$

and there holds the energy equation:

$$e_\lambda(t,u) = e_\lambda(0,u) - \frac{1}{2}\int_0^t < \dot{A}_\lambda(s)\,u_s, u_s > ds + \int_0^t (f_s + \xi_s, \dot{u}_s)ds \tag{3.10}$$
$$+ \int_0^t (\dot{u}_s, dw_s) + \frac{1}{2} Tr \ll M \gg_t \;\; for\, t \in [0,T]\,, a.s.$$

Proof. Under the assumption (3.3)-(3.5), the lemma is a direct consequence of Pardoux's results (see chapter 2, part III in [3]), which generalized the deterministic results by Strauss [4] for hyperbolic equations. The only difference here is that $\dot{A}_x(t)$ is a random linear operator. But, by condition (3.3), it is easily seen that, for $\varphi, \psi \in H_0^2$, the map $t \to< \dot{A}_\lambda(t), \varphi, \psi >= \dot{\lambda}(t)\left(\partial_x^2\varphi, \psi\right)$ is a.s. continuous. By following Pardoux's proofs as mentioned above, we can conclude that the initial-value problem has a unique solution with the regularity (3.9) and it satisfies the equation

$$||\dot{u}_t||^2 = ||\dot{u}_0||^2 + \langle A_\lambda(t)\,u_t, u_t\rangle - \langle A_\lambda(0)\,u_0, u_0\rangle - 2\int_0^t \langle \dot{A}_\lambda(s)\,u_s, u_s\rangle ds$$
$$+ 2\int_0^t (f_s + \xi_s, \dot{u}_s)ds + 2\int_0^t (\dot{u}_s, dM_s) + Tr \ll M \gg_t, \; a.s. \tag{3.11}$$

In view of (3.6), the equation (3.11) can be rewritten as the stochastic energy equation (3.10). □

With the aid of this key lemma, we will prove the regularity and uniqueness of a solution to the nonlinear initial-value problem (2.9) and (2.10) in the special case of additive noise ($g = 0$ and $\breve{\sigma} = 0$). To this end we define the energy process

$$e(t,u) = \tfrac{1}{2}\{||\dot{u}_t||^2 + (\alpha + \frac{\beta}{2}||\partial_x u_t||^2)||\partial_x u_t||^2 + \gamma||\partial_x^2 u_t||^2\}. \tag{3.12}$$

In case of additive noise, we have the following uniqueness theorem.

Theorem 3.1. *Assume that the conditions* (2.6) *and* (3.4) *hold. If the initial-value problem* (2.9) *and* (2.10) *for the nonlinear stochastic equation, with* $g = 0$ *and* $\breve{\sigma} = 0$, *has a solution* u *such that*

$$u \in L^2\left(\Omega; L^\infty\left(]0,T[\,; H_0^2\right)\right), \qquad \dot{u} \in L^2\left(\Omega; L^\infty\left(]0,T[\,; L^2\right)\right),$$

then the solution is unique and has the properties:

$$u \in C\left([0,T]\,; H_0^2\right), \quad \dot{u} \in C\left([0,T]\,; L^2\right). \tag{3.13}$$

Moreover, the corresponding energy process satisfies

$$e(t,u) = e(0,u) + \int_0^t [(f_s, \dot{u}_s)ds + \frac{1}{2}Tr\left(\sigma_s^2\right)]ds + \int_0^t (\dot{u}_s, \sigma_s dw_s). \tag{3.14}$$

Proof. First let $B(u_t) = g_t$ and $dM_t = \sigma_t dw_t$ in (2.9) which yields Eq. (3.1) with $\lambda \equiv 0$. By assumptions, it is easy to check that the conditions for Lemma 3.1 are met. Therefore the regularity properties (3.13) follow. As a result, if we set

$$\lambda(t) = \beta\,\|\partial_x u_t\|^2\,, \tag{3.15}$$

then $\lambda \in C^1([0,T])$ *a.s.*. With λ as defined, consider Eq. (2.9) as a special form of *Eq.* (3.1) with $\xi \equiv 0$. By applying Lemma 3.1, the energy equation (3.10) reads

$$e_\lambda(t,u) = e_\lambda(0,u) - \frac{1}{2}\int_0^{\dot{x}} \lambda(s)\left\langle \partial_x^2 u_s, u_s\right\rangle ds \tag{3.16}$$
$$+ \int_0^t [(f_s, \dot{u}_s) + \frac{1}{2}T_r\left(\sigma_s^2\right)]ds + \int_0^t (\dot{u}_s, \sigma_s dw_s).$$

Note that, by (3.15),

$$\int_0^t \dot{\lambda}(s) \left\langle \partial_x^2 u_s, u_s \right\rangle ds = \frac{1}{2}\beta(||\partial_x u_0||^4 - ||\partial_x u_t||^4) \tag{3.17}$$

Taking (3.6), (3.11), (3.15) and (3.17) into account, the equation (3.16) yields the desired energy equation (3.14).

To show uniqueness, let u_t and $\widetilde{u}_t$ be two solutions. Then $v_t = u_t - \widetilde{u}_t$ satisfies

$$d\dot{v}_t = Av_t dt + \left[B(u_t) - B\left(\breve{u}_t\right)\right]dt, \quad \dot{v}_0 = 0,\ v_0 = 0.$$

Recall that, by (2.5), $B : H_0^2 \to L^2$ is locally Lipschitz-continuous. If B were globally Lipschitz-continuous, the uniqueness would have followed as in the deterministic case. In the present case we have to use the localization technique to show uniqueness by introducing a stopping time. This technique will also be used later in the proof of Theorem 4.1, where the detail will be provided. □

As a consequence of the formula (3.14), we can derive the following energy estimate.

Corollary 3.1. *For any $\varepsilon > 0$, the following a prior estimate holds for some constant $C_\varepsilon > 0$*

$$E\{\sup_{0\le t\le T}\left[e(t,u)\, e^{-\varepsilon t}\right]\} \le 2e(0,u) + C_\varepsilon \int_0^t \left(\|f_s\|^2 + Tr\sigma_s^2\right) ds. \tag{3.18}$$

Proof. By (3.14) and the Itô formula, we obtain

$$e(t,u)\, e^{-\varepsilon t} = e(0,u) + \int_0^t [(f_s, \dot{u}_s) + \frac{1}{2}T_r\sigma_s^2 - \varepsilon e(s,u)]e^{-es}ds + \int_0^t e^{-\varepsilon s}(\dot{u}_s, \sigma_s dw_s). \tag{3.19}$$

Making use of the fact that $||\dot{u}_t||^2 \le 2e(t,u)$ and the submartingale inequality (p.6, [3])

$$E\{\sup_{0\varepsilon t\le T} | \int_0^t e^{-\varepsilon s}(\dot{u}_s, \sigma_s dw_s)|\} \le 3E\{\int_0^T e^{-2\varepsilon s}(\sigma_s^2 \dot{u}_s, \dot{u}_s,)ds\}^{\frac{1}{2}},$$

the equation (3.19) yields

$$E\{\sup_{0\le+\le T}\left[e\left(t,u\right)e^{-\varepsilon t}\right]\}\le e\left(0,u\right)+\int_0^T\frac{1}{2}[\frac{1}{\varepsilon}\left\|f_s\right\|^2+T_r\sigma_s^2]e_{ds}^{-\varepsilon s}$$

$$+\,3E\{\sup_{0\le t\le T}[||\dot u_t||^2e^{-\varepsilon t}]\int_0^T e^{-\varepsilon s}Tr\sigma_s^2ds\}^{\frac{1}{2}}$$

$$\le e(0,u)+\tfrac{1}{2}E\{\sup_{0\le t\le T}\left[e\left(t,u\right)e^{-\varepsilon t}\right]\}+\tfrac{1}{2}\int_0^T\left[\frac{1}{\varepsilon}\left\|f_s\right\|^2+145\times Tr\sigma_s^2\right]e^{-\varepsilon s}ds,$$

which implies the inequality (3.18) with $C_\varepsilon=\left(\frac{1}{\varepsilon}+145\right)$. □

4. Existence of solution. Instead of finding a solution by some finite-dimensional approximation, we will adopt the method of truncation and a fixed point argument to prove the following existence theorem in the case of additive noise.

Theorem 4.1. *Let the conditions* (2.6) *and* (3.4) *on the data be satisfied. Then there exists a solution u of the initial-value problem* (2.10) *and* (2.11) *for the nonlinear stochastic PDE with $g\cong 0$ and $\widetilde{\sigma}\cong 0$. Moreover, the solution satisfies the regularity*

$$u\in L^2(\Omega;C([0,T]\,;H_0^2),\qquad \dot u\in L^2\left(\Omega;C([0,T]\,;L^2\right).$$

and the energy equation holds;

$$||\dot u_t||^2=||\dot u_0||^2+\langle Au_t,U_t\rangle-\langle Au_0,u_0\rangle+2\int_0^t<\beta(u_s,\dot u_s)>ds \tag{4.1}$$

$$+\,2\int_0^t[(f_s,\dot u_s)+\frac{1}{2}Tr\sigma_s^2(u_s,\dot u_s)\,ds]+2\int_0^t(\dot u_s,\sigma_s(u_s,\dot u_s)dw_s).$$

Proof. For a fixed $r>0$ and for a progressively measurable process u_t with values in H_0^2, we define a new process $u_t=R_r\left(u_t\right)$, as the solution of the following linear equation

$$d\dot v_t=\left[Av_t+B_r\left(u_t\right)+f_t\right]dt+\sigma_sdw_s \tag{4.2}$$

with the data $\dot v_0, v_0$ and f satisfying condition (3.4), where

$$B_r\left(\varphi\right)=\begin{cases}\beta\left\|\partial_x\varphi\right\|^2\partial_x^2\varphi, & \text{if } \left\|\partial_x\varphi\right\|\le r,\\ \beta_2^r\partial_x^2\varphi, & \text{if otherwise, } \varphi\in H_0^2,\end{cases}$$

that is a Lipschitz-truncation of B within the ball of radius r in H_0^1. Then we consider the linear equation (4.2) in the Banach space of progressively measurable processes:

$$V = \{v \in L^2\Omega; L^\infty\left(]0,T[\,;H_0^2\right) : \dot{v} \in L^2\Omega; L^\infty\left(]0,T[\,;L^2\right)\},$$

endowed with the (linear) energy norm

$$\|v\|_V = \{E \sup_{0\leq + \leq T}[e_L(t,v)\,e^{-\eta^2 t}]\}^{\frac{1}{2}} \tag{4.3}$$

for some $\eta > 0$, where

$$e_L(t,v) = \frac{1}{2}\left\{||\dot{v}_t||^2 + \alpha\,\|\partial_x v_t\|^2 + \gamma\left\|\partial_x^2 v_t\right\|^2\right\}. \tag{4.4}$$

Given $u \in V$, define $\xi_t = B_r(u_t)$ and, similar to Lemma 3.1, the resulting linear equation (4.2) is known to have a unique solution $u \in V$. Therefore the mapping $R_r : V \to V$ is well defined. In fact, let $v^i = R_r\left(u^i\right), i = 1.2$, and $\widetilde{v} = v^1 - v^2$. Then, by means of the energy equation for $\widetilde{v}$ from (4.2), we get

$$\begin{aligned}\left\|R_r\left(u^1\right) - R_r\left(u^2\right)\right\|_V^2 &= E\{\sup_{0\leq\in+\leq t}[e_L(t,\widetilde{v})e^{-\eta^2 t}]\} \qquad (4.5)\\ &\leq E\int_0^T\left[\left(B_r\left(u_s^1\right) - B_r\left(u_s^2\right), \dot{\widetilde{v}}\right) - \eta^2 e_L(s,\widetilde{v})\right]e^{-\eta^2 s}ds\\ &\leq \frac{1}{2\eta^2}E\int_0^T\left\|B_r\left(u_s^1\right) - B_r\left(u_s^2\right)\right\|^2 e^{-\eta^2 s}ds.\end{aligned}$$

In view of (2.5) and (4.2), it is clear that

$$\begin{aligned}\left\|B_r\left(u_t^1\right) - B_r\left(u_t^2\right)\right\|^2 &\leq \beta^2 r^4\left(\left\|\partial_x^2\left(u_t^1 - u_t^2\right)\right\| + 2\left\|\partial_x\left(u_t^1 - u_t^2\right)\right\|\right)^2\\ &\leq c^2 r^4 e_L\left(t, u^1 - u^2\right), \qquad (4.6)\end{aligned}$$

where the constant C depends on $\partial, \beta, \partial$ but not on r. With the aid of this inequality, (4.5) yields

$$\left\|R_r\left(u^1\right) - R_r,\left(u^2\right)\right\|_V^2 \leq \frac{c^2 T r^4}{2\eta^2} E\sup_t\left[e_L\left(t, u^1 - u^2\right)e^{-\eta^2 t}\right]$$

or

$$\left\|R_r\left(u^1\right)-R_r,\left(u^2\right)\right\|_V \leq \sqrt{\tfrac{T}{2}}\frac{cr^2}{\eta}\left\|u^1-u^2\right\|_V, u^1, u^2 \in_V,$$

which shows that $R_r : V \to V$ is a contraction mapping for a sufficiently large η. We have then a fixed point for R_r, that is, there exists $u^r \in V$ such that $u^r = R_r\left(u^r\right)$, or

$$du_t^r = \left[Au_t^r + B_r\left(\mu_t^r\right) + f_t\right]dt + \sigma_t dw_t \tag{4.7}$$

satisfying the same initial conditions for (4.2). By Theorem 3.1, this fixed-point solution has the regularity

$$u^r \in L^2(\Omega; C\left([0,T]; H_0^2\right), \dot{u}r \in L^2\Omega; C\left([0,T]; L^2\right).$$

Introduce a stopping time $\tau_r = \tau_r\left(u^r\right)$ defined as

$$\tau_r\left(\mu^r\right) = \inf\left\{t>0 : ||\dot{u}_t^r|| \geq r, \text{ or } \left\|\partial_x u_t^r\right\| \geq r \text{ or } \left\|\partial_x^2 u_t^r\right\| \geq r\right\}.$$

For $t \in [0, \tau_r]$, from (4.2) and (4.7), $u_t = u_t^r$ is the unique solution of the initial-value problem (2.9) and (2.10).

Now, due to the following property of the stopping time: $\tau_r \leq \tau_r'$ if $r' \geq r$, there exists the limit $\tau_r = \lim_{r\to\infty} \tau_r \leq \infty$ a.s. and it satisfies $u_t^r = u_t^{r'}$ if $t \in (0, \tau_r)$, for all $r' \geq r$, $a.s.$. So the solution of the initial-value problem can be defined as $u_t = \lim_{r\to\infty} u_t^r$ for $t \in [0,T]\, a.s.$ provided that $P\left\{\tau \geq T\right\} = 1$. To verify this fact, we deduce from the energy equation (3.14) that

$$\begin{aligned} e\left(t\wedge\tau, u\right)e^{-\varepsilon(t\wedge\tau)} \leq e\left(0,u\right) + \int_0^{(t\wedge\tau)} \left[\frac{1}{2\varepsilon}\left\|f_s\right\|^2 + \frac{1}{2}t_r\left(\sigma_s^2\right)\right]e^{-\varepsilon s}\,ds \\ + \int_0^{(t\wedge\tau)} e^{-\varepsilon s}(\dot{u}_s, \sigma_s dw_s). \end{aligned} \tag{4.8}$$

Since $e^{-\varepsilon t} \leq e^{-\varepsilon\tau r}, \forall\varepsilon > 0$, the nonlinear energy process $e\left(t,u\right)$ has the estimate

$$\left(\beta r^2 \wedge 2\gamma\Lambda 2\right)\frac{r^2}{4}e^{-\varepsilon\tau}1_{(\tau<T)} \leq e\left(\tau_r, u^r\right)e^{-\varepsilon\tau_r}1_{(\tau_r<T)}, \tag{4.9}$$

where $1_{(\cdot)}$ denotes the indicator function. Hence, in view of (4.8) and (4.9), we get

$$\begin{aligned}&\left(\beta r^2 \wedge 2\gamma \wedge 2\right) \frac{r^2}{4} E\left\{e^{-\varepsilon\tau} 1_{(\tau_r<T)}\right\}\\&\leq e(0,u) + E\Big\{\int_o^{\tau_r} \big[\frac{1}{2\varepsilon}\|f_s\|^2 + \frac{1}{2}^{\tau_r}\left(\sigma_s^2\right)\big] e^{-\varepsilon s} ds\Big\}\\&\leq e(0,u)\int_0^T \big[\frac{1}{2\varepsilon}\|f_s\|^2 + \frac{1}{2}\tau_r\left(\sigma_s^2\right)\big] ds < \infty.\end{aligned}$$

So, as $r \to \infty$, we can deduce that $E\{e^{-\varepsilon\tau} 1_{(\tau_r<T)}\} = 0 \ \forall \varepsilon > 0$, or $P\{\tau \geq T\} = 1$ as to be shown. Finally, by means of Lemma 3.1, the energy equation (4.1) can be readily verified. □

5. Solutions in the General Case. We first consider the case of multiplicative noise

$$du_t = \left[Au_t + B(u_t) + g_t(\dot{u}_t, u_t)\right] dt + \tilde{\sigma}(\dot{u}_t, u_t)\, d\tilde{w}_t \tag{5.1}$$

subject to the initial condition (2.11), where, for

$$\varphi \in L^2, \psi \in H_0^{\sim}, \tilde{\sigma}_t(\varphi,\psi) h = \sum_{k=1}^{\infty} \tilde{\sigma}_t^k(\varphi,\psi) h_k$$

with

$$\tilde{\sigma}_t^k(\varphi,\psi)(x) = \tilde{\sigma}_k(t, x, \varphi(x), \psi(x)), h = (h_1, h_2, ...) \in \ell^2.$$

Let g and $\tilde{\sigma}$ satisfy the following conditions:g$(\cdot,\cdot) : [0,T] \times L^2 \times H_0^1 \to L^2$ is such that, for $\varphi \in L^2$ and $\psi \in H_0^2, g(\varphi,\psi) \in C([0,T]; H)$ and it satisfies the Lipschitz and the linear growth conditions

$$\left\|g_t(\varphi,\psi) - g_t(\varphi',\psi')\right\|^2 \leq L\big(\left\|\varphi - \varphi'\right\|^2 + \left\|\psi - \psi'\right\|_{(1)}\big), \tag{5.2}$$

$$\left\|g_t(\varphi,\psi)\right\|^2 \leq C\big(1 + \|\varphi\|^2 + \|\psi\|_{(1)}^2\big), \tag{5.3}$$

for $t \in [0,T], \varphi, \varphi' \in L^2; \psi' \in H_0^1$, where L and C are some positive constants.

For $\varphi \in L^2, \psi \in H_0^1, \widetilde{\sigma}_t\ (\varphi,\psi) : \ell^2 \to L^2$ is continuous in $t \in [0,T]$ such that

$$Tr[\widetilde{\sigma}_t\ (\varphi,\psi)\ \widetilde{-\sigma}_t\ (\varphi^1,\psi^1)]^2 = \sum_{k=1}^{\infty}\left\|\widetilde{\sigma}_t^k\ (\varphi,\psi) - \widetilde{\sigma}_t^k\ (\varphi^1,\psi^1)\right\|^2 \tag{5.4}$$

$$\leq L\left(\left\|\varphi-\varphi'\right\|^2 + \left\|\psi-\psi'\right\|^2_{(1)}\right)$$

$$Tr_r[\widetilde{\sigma}_t\ (\varphi,\psi)]^2 = \sum_{k=1}^{\infty}\left\|\widetilde{\sigma}_t^k\ (\varphi,\psi)\right\|^2 \leq c\big(1+\|\varphi\|^2+\|\psi\|^2_{(1)}\big). \tag{5.5}$$

Then the following existence and uniqueness theorem holds true.

Theorem 5.1. *Suppose that the conditions* (3.4) *and* (5.2)–(5.5) *are fulfilled. Then the initial-value problem for* (5.1) *has a unique solution* u *with the regularity*

$$u \in L^2\left(\Omega; C\left([0,T]\,; H_0^2\right)\right) \quad and \quad \dot{u} \in L^2\left(\Omega; C\left([0,T]\,; L^2\right)\right). \tag{5.6}$$

Proof. Given $u_t \in H_0^2$, consider the linear stochastic equation

$$d\dot{v}_t = [Av_t + B_r\,(u_t) + g_t\,(\dot{u}_t, u_t)]\,dt + \widetilde{\sigma}_t\ (\dot{u}_t, u_t)\,d\ \widetilde{w}_t, \tag{5.7}$$

while $B_r\,(\cdot)$ is truncated $B\,(\cdot)$ as defined by (4.2). Introduce a Banach space V_0, the space V with $\eta = 0$ in the norm (4.3), that is,

$$\|u\|_{V_o} = \left\{E \sup_{0\leq t\leq T} e_L\,(t,v)\right\}^{\frac{1}{2}}. \tag{5.8}$$

For $u \in V_0$, let Q_r denote the solution operator for the initial-value problem (5.1). By assumptions, Lemma 3.1 can be applied to show that the unique solution $u = Q_r\,(u)$ has the regularity (5.6), or the map $Q_r = V_0 \to V_0$ is well defined. In fact, as to be shown, Q_r is a contraction mapping. In contrast with the proof of Theorem 4.1, we will first prove that Q_r is a contraction for a sufficiently small T and then show that the fixed-point solution $u = Q_r\,(u)$ can be continued to any finite $T > 0$. To this end, let $u^i \in V_0$ and $u^i = Q_r\left(u^i\right)$ for $i = 1.2$. Similar to the inequality (4.6), by applying the energy equation to $\widetilde{v} = v^1 - v^2$, we can obtain

$$
\begin{aligned}
&\left\|Q_r\left(u^1\right) - Q_r\left(u^2\right)\right\|_{V_0} \le E\int_0^T \Big\{\left(B_r\left(u_s^1\right) - B_r\left(u_s^2\right), \dot{\widetilde{v}}_s\right) \\
&+ (g_s\dot{u}_s^1, u_s^1) - g_s(\dot{u}_s^2, u_s^2), \dot{\widetilde{v}}_s) + \frac{1}{2}Tr[\widetilde{\sigma}_s\,(\dot{u}_s^1, u_s^1) - \widetilde{\sigma}_s\,(\dot{u}_s^2, u_s^2)]^2\Big\}ds \\
&+ 3E\Big\{\int_0^T ||[\widetilde{\sigma}_s\,(\dot{u}_s^1, u_s^1) - \widetilde{\sigma}_s\,(\dot{u}_s^2, u_s^2)]\ \dot{\widetilde{u}}_s\ ||^2 ds\Big\}^{\frac{1}{2}} \qquad (5.9) \\
&\le \frac{\varepsilon}{2}E[\sup_{0\le t\le T} ||\ \dot{\widetilde{v}}_t\ ||^2] + \frac{16}{\varepsilon}E\Big\{\int_0^T (||\beta_r\left(u_s^1\right) - B_r(\dot{u}_s^1, u_s^1) - B_r(\dot{u}_s^2, u_s^2)||) \\
&+ ||g_s(\dot{u}_s^1, u_s^1) - g_s(\dot{u}_s^2, u_s^2)||ds\Big\}^2 \\
&+ (\frac{1}{2} + \frac{144}{\varepsilon})E\int_0^T Tr[\widetilde{\sigma}_s\,(\dot{u}_s^1, u_s^1) - \widetilde{\sigma}_s\,(\dot{u}_s^2, u_s^2)]^2 ds,
\end{aligned}
$$

for a fixed $\varepsilon \in (0,1)$.

By taking (4.6), (5.3), (5.4) into account and noting that

$$\frac{1}{2}E[\sup_{0\le t\le T} ||\ \dot{\widetilde{v}}_t\ ||^2] \le \left\|Q_r\left(u^1\right) - Q_r\left(u^2\right)\right\|_{V_0}^2,$$

we can deduce from (5.9) that there exist positive constants C_1 and C_2, depending on γ, ∂, L and ε, such that

$$\left\|Q_r\left(u^1\right) - Q_r\left(u^2\right)\right\|_{V_0} \le \frac{T}{(1-\varepsilon)}\left(C_1T + C_2\right)\left\|u_1 - u_2\right\|_{V_o}.$$

This shows that Q_r is a contraction mapping for $T \le T_0$ sufficiently small. Therefore the fixed-point of Q_r is the unique solution : $u = Q_r(u)$ of the equation (5.1) for $t \in [0, T_0]$. Since the existence of solution for non-anticipating initial data can be similarly proved, the solution can be continued beyond T_0 as in the deterministic case. By means of the energy equation, similar to the last part of proof for Theorem 4.1, it can be shown that the fixed-point solution u^r satisfying

$$d\dot{u}_t^r = \left[Au_t^r + B_r\left(u_t^r\right) + g_t\left(\dot{u}_t^r, u_t^r\right)\right]dt + \widetilde{\sigma}_t\left(\dot{u}_t^r, u_t^r\right)d\,\widetilde{w}_t, \qquad (5.10)$$

remains bounded $a.s.$ for t in any finite interval $[0, T]$ and

$$u_t = \lim_{r\to\infty} u_t^r,\ \forall t \in [0,T]\,a.s. \qquad (5.11)$$

is a solution of (5.1). The regularity properties (5.6) can be verified by using Lemma 3.1. Since B_r, g_t, and $\widetilde{\sigma}_t$ are Lipschitz-continuous, it is clear that the solution u_t^r of the truncated problem (5.9) is unique. To remove the cut-off, noting (5.10), the solution u_t of (5.1) is unique up to the stopping time τ_r defined by

$$\tau_r = \inf\left\{t > 0 : ||\dot{u}_t|| \vee ||\gamma_x u_t|| \vee ||\gamma_x^2 u_t|| > r\right\}.$$

As in Theorem 4.1, it can be shown by the energy estimate that $P\{\tau_r > T\} = 1$ for any finite $T > 0$. Hence the solution u_t is unique for $t \in [0, T]$ as asserted.□

Now, in view of the proofs of Theorems 4.1 and 5.1, the existence and uniqueness of a solution to the general initial value problem (2.10) and (2.11) can be proved in a similar fashion with minor modifications.

Theorem 5.2. *Under the assumptions* (3.4) *and* (5.2)–(5.5), *the initial-value problem* (2.10) *and* (2.11) *has a unique solution u with the regularity properties*

$$u \in L^2\left(\Omega; \left(C\left[0,T\right]; H_0^2\right)\right), \qquad \dot{u} \in L^2\left(\Omega; \left(C\left[0,T\right]; L^2\right)\right).$$

Moreover, the following energy equation holds

$$e\left(t,u\right) = e\left(0,u\right) + \int_0^t (f_s + g_s(\dot{u}_s, u_s), \dot{u}_s) ds \tag{5.12}$$
$$+ \tfrac{1}{2}\int_0^t \{Tr\sigma_s^2 + T_r[\widetilde{\sigma}_s\ (\dot{u}_s, u_s)]^2\} ds + \int_0^t [(\dot{u}_s, \sigma_s\ dw_s) + (\dot{u}_s, \widetilde{\sigma}_s\ (\dot{u}_s, u_s) d\ \widetilde{w}_s)].$$

Remarks. 1. As seen from the proofs of this and the previous theorems, the stochastic energy equation of Pardoux-Strauss plays a crucial role. In view of the fact that the Itô formula does not hold for a hyperbolic type of equations in general, such an energy equation becomes indispensable in the related stochastic analysis.

2. The present analysis applies with slight modifications to the stochastic hyperbolic equations with locally Lipschitz nonlinearity which is non-monotone, such as

$$d\dot{u}_t = \left[\partial_x^2 u_t + u_t\left(1 - u_t^2\right)\right] dt + \sigma_t dw_t,$$

with $u_t = 0$ at $x = 0$ and ℓ, and the appropriate initial conditions.

3. When applied to a concrete problem, such as (1.1), the boundary conditions can be changed to other kind, e.g. $u = 0$ and $\partial_x^2 u = 0$ at $x = 0$ and ℓ.

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