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EXISTENCE AND UNIQUENESS OF SOLUTIONS OF SOME ABSTRACT DEGENERATE NONLINEAR EQUATIONS

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Abstract. The nonlinear abstract differential equation

$$\frac{d}{dt}(Ay) + By(t) = F(t, Ky), \quad 0 \leq t \leq \tau,$$

where A, B, K are linear closed operators from a complex Banach space Y into a Banach space X is considered. The main assumption reads that the point $\xi = 0$ is a polar singularity of the resolvent $(T - \xi I)^{-1}$, where $T = A(\lambda A + B)^{-1}$, λ being a regular point of the operator pencil $\lambda A + B$. Mainly the case of a simple pole and of a second order pole are considered. Some examples of application to concrete partial differential equations are given. In particular, we show that the results work for mathematical models of nonlinear electrical networks.

1. Introduction. We consider the nonlinear abstract differential equation

$$\frac{d(Ay)}{dt} + By(t) = F(t, Ky), \quad 0 \leq t \leq \tau \tag{1.1}$$

where A, B, K are closed linear operators from a Banach space Y into a Banach space X. Here $D_A \cap D_B \subset D_K$. Let the point λ be a regular point of the operator pencil $\lambda A + B$: $(\lambda A + B)^{-1} \in L(X, Y)$. Make the substitution $v(t) = e^{-\lambda t}(\lambda A + B)y(t)$ and introduce the notation $T = A(\lambda A + B)^{-1}$, $N = K(\lambda A + B)^{-1}$, $f(t, x) = e^{-\lambda t}F(t, e^{\lambda t}x)$. Then equation (1.1) is transformed into equation (1.2) with a more simple linear part:

$$\frac{d}{dt}(Tv) + v(t) = f(t, Nv), \quad 0 \leq t \leq \tau. \tag{1.2}$$

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If the operators A, T are degenerate, then equations (1.1), (1.2) are called degenerate. In the degenerate case the point $\zeta = 0$ is a spectral point of the operator T and a singular point of the resolvent $(T - \zeta I)^{-1}$. We shall suppose that $\zeta = 0$ is a pole of multiplicity m for the resolvent. Let us denote by $X_2 = KerT^m$ the spectral subspace and by $P_2 : X \to X_2$ the spectral projector [7]. The operator $T_2 = T_{|_{X_2}} \in L(X_2)$ induced by T in X_2 is eigennilpotent (see [7]), i.e., $T_2^m = 0$, $T_2^{m-1} \neq 0$. The resolvent has the Laurent expansion

$$(T - \zeta I)^{-1} = -\frac{P_2}{\zeta} - \sum_{n=1}^{m-1} \frac{T_2^n P_2}{\zeta^{n+1}} + \sum_{n=0}^{\infty} \zeta^n S^n. \tag{1.3}$$

The operator $P_1 = I - P_2 \in L(X)$ is a projector on the subspace $X_1 = P_1(X) = \Re(T^m)$, which is an invariant subspace of the operator T. We have the decompositions

$$X = X_1 \dot{+} X_2, \quad T = T_1 \dot{+} T_2; \quad T_k = T_{|_{X_k}} \tag{1.4}$$

The operator T_1 induced by T in X_1 has a bounded inverse $T_1^{-1} \in L(X_1)$.

We also observe that the decompositions (1.4) hold for an unbounded operator T if its resolvent has a pole at $\zeta = 0$. Then T is closed in X, T_1 is closed in X_1 ($D_{T_1} = P_1 D_T$), and T_2 is bounded in X_2.

Some results concerning solvability of equation (1.1) are well-known in the literature and we refer to the book [3] by R.W. Carroll and R.E. Showalter both for main theorems and a large bibliography. However, in those statements only the case of a simple pole could be treated and their methods heavily depend on the theory of monotone operators. On the other hand, some degenerate semi-linear equations like (1.2) appear in various applied sciences when $\lambda = 0$ is a multiple pole for the resolvent $(T - \lambda I)^{-1}$. See the monographs [1, 2] by S.L. Campbell, devoted mainly to equations in finite-dimensional spaces X.

In this paper we consider equation (1.2) in the case of the multiple pole ($m > 1$) of the resolvent (1.3). For the sake of brevity we completely investigate the case of $m = 2$, give some particular theorem for $m > 2$ and postpone the general case to a later paper. New results for the case of $m = 1$ from Sections 3,4 are necessary to solve our problem in the case of $m > 1$. In the simple pole case we also succeed in weakening the assumptions in [5, 6] guaranteeing that (1.1) has a local solution. Moreover, in Theorems 4.1–4.3

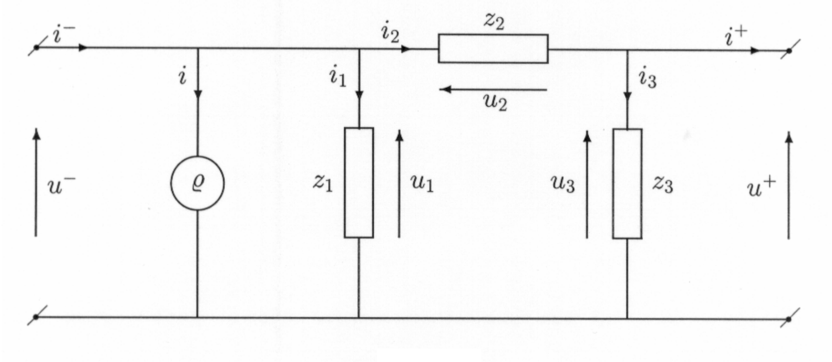

Figure 1.

we formulate some local conditions for f under which (1.2) has at least one solution. The second order pole case is more complicated. In Section 5 we examine a particular case where the subspace X_2 $(=P_2(X))$ is generated by one eigenvector and one associated vector. One example of application is described, too. Section 6 deals with the general case, even when the dimension of X_2 is infinite. Finally, Section 7 contains some concrete applications of the previous results to partial differential equations and to the nonlinear electrical networks outlined in Section 2.

2. An example from physics. Mathematical models of nonlinear electrical networks usually contain equations of the form (1.1), (1.2) in a finite-dimensional space X. The electrical transfer four-port represented in Figure 1 has three inner branches with known impedances z_k, unknown currents i_k and tensions u_k $(k = 1, 2, 3)$. An input current $i^-(t)$ and an input tension $u^-(t)$ are given functions. The four-port contains also a compensating branch ϱ, in which a current i is controlled by currents i_1, i_2 according to a nonlinear law $i = \varrho(i_1, i_2)$. We use the following Kirchhoff equations:

$$i + i_1 + i_2 = i^-, \quad u_1 = u^-, \quad u_2 + u_3 = u^-. \tag{2.1}$$

2.1. Let the element z_1 be an inductance with a nonlinear resistor, z_2 be a nonlinear resistor, z_3 be a linear capacitor:

$$u_1 = \frac{d}{dt}(Li_1) + \psi(t, i_1), \quad u_2 = ri_2 + \varphi(t, i_2), \quad i_3 = \frac{d}{dt}(Cu_3). \tag{2.2}$$

Here, ri_2 is a linear part of the Ohm law for the branch z_2; L, r, C are real constants. Substitute $i = \varrho$ and u_1, u_2 (2.2) in Kirchhoff equations (2.1).

We obtain the following system of equations in the variables i_1, i_2, u_3:

$$i_2 + i_1 = i^- - \varrho(i_1, i_2); \quad \frac{d}{dt}(Li_1) = u^- - \psi(t, i_1); \quad ri_2 + u_3 = u^- - \varphi(t, i_2).$$

This system is written in the vector form (1.1), where $y = (i_2, i_1, u_3)^{tr}$, $X = Y = \mathbf{C}^3$, $K = I$,

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ r & 0 & 1 \end{bmatrix}, \quad F(t,y) = \begin{bmatrix} i^-(t) - \varrho(i_1, i_2) \\ u^-(t) - \psi(t, i_1) \\ u^-(t) - \varphi(t, i_2) \end{bmatrix}.$$

Note that when $\varrho = 0, \psi = 0, \varphi = 0$ we have a linear network described by the degenerate equation $Ay'(t) + By(t) = f(t)$. This linear four-port was considered in [8].

Choose $\lambda = 1$ and pass to (1.2). We have

$$(A+B)^{-1} = N = \begin{bmatrix} 1 & -L^{-1} & 0 \\ 0 & L^{-1} & 0 \\ -r & rL^{-1} & 1 \end{bmatrix}; \quad T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

$$f(t, Nv) = e^{-t} F(t, e^t Nv).$$

At the point $\zeta = 0$ the spectral subspace X_2 of the operator T coincides with its eigen subspace, $dim X_2 = 2$. The resolvent $(T - \zeta I)^{-1}$ has a simple pole at the point $\zeta = 0$. The spectral projector P_2 is

$$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

2.2. Now let the element z_1 be a capacitor C with a nonlinear conduction G_1 and the elements z_2, z_3 be inductors with nonlinear resistors:

$$i_1 = C\frac{du_1}{dt} + G_1(t, u_1); \quad u_k = \frac{d}{dt}(L_k i_k) + \varphi_k(t, i_k), \quad k = 2, 3. \tag{2.3}$$

Here C, L_k are constants. In the compensating branch the current i is controlled by one current i_2 with the help of a known function G_2: $i = G_2(t, i_2)$.

Substituting i_1, u_k (2.3) into Kirchhoff equations (2.1), we obtain the equation of form (1.1) for the vector $y = (y_1, y_2, y_3)^{tr} = (u_1, i_2, i_3)^{tr}$. Here we have $K = I$,

$$A = \begin{bmatrix} 0 & 0 & 0 \\ C & 0 & 0 \\ 0 & L_2 & L_3 \end{bmatrix}; B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

$$F(t, y) = \begin{bmatrix} u^-(t) \\ i^-(t) - G_1(t, y_1) - G_2(t, y_2) \\ u^-(t) - \varphi_2(t, y_2) - \varphi_3(t, y_3) \end{bmatrix}.$$

In replacing $y \to v$ we choose $\lambda = 1$. The function $v(t) = e^{-t}(A + B)y(t)$ satisfies the equation

$$\frac{d}{dt}(Tv) + v(t) = f(t, Nv) \equiv e^{-t}F(t, e^t Nv) \tag{2.4}$$

with

$$T = \begin{bmatrix} 0 & 0 & 0 \\ C & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad N = (A + B)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -C & 1 & 0 \\ \frac{CL_2}{L_3} & -\frac{L_2}{L_3} & \frac{1}{L_3} \end{bmatrix}.$$

The spectral subspace X_2 is the linear span of the eigenvector $\varphi_0 = (0, 1, 0)^{tr}$ and the associated (rooted) vector $\varphi_1 = (C^{-1}, 0, 0)^{tr}$ of the operator T at the point $\zeta = 0$. The resolvent $(T - \zeta I)^{-1}$ has a pole of multiplicity $m = 2$ at the point $\zeta = 0$. The matrix coefficients in (1.3) are

$$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 0 & 0 \\ C & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad T_2^2 = 0.$$

3. The simple pole case. Equation (1.2) with the initial condition

$$\lim_{t \to +0} Tv(t) = x_0 \tag{3.1}$$

was studied in [5, 6] in the case when $\zeta = 0$ is a simple pole of the resolvent $(T - \zeta I)^{-1}$ $(m = 1)$. In Sections 3, 4 of this paper we'll obtain other existence and uniqueness theorems in the simple pole case. In (1.4) we have $T_2 = 0$, $X_1 = \Re(T)$, $X_2 = KerT$, $T_1 = T_{|X_1} \in L(X_1)$, $T_1^{-1} \in L(X_1)$, P_k is the corresponding projector of X onto X_k. Here the operator T is bounded; the case of an unbounded operator T will be considered in Sections 6, 7. We shall denote $v(t) = v_1(t) + v_2(t), v_k(t) = P_k v(t)$. The assumption $x_0 \in \Re(T)$ is required for correctness of initial condition (3.1). If the function $v(t)$ is continuous on $[0, \tau]$, then (3.1) is equivalent to the Cauchy condition for the projection $v_1(t)$:

$$v_1(0) = v_{10} = T_1^{-1} P_1 x_0. \tag{3.2}$$

Definition. By a solution of equation (1.2) on a segment $[0, \tau]$ we mean a function $v(t) \in C([0, \tau], X)$ satisfying equation (1.2) in the classical sense at every point $t \in [0, \tau]$. In particular, $Tv(t)$ is a differentiable function.

Theorem 3.1. *Let $f : [0, \tau] \times X \to X$ be continuous and suppose that the projections $P_k f(t, x)$ satisfy the conditions (for all $x', x'' \in X$)*

1^0. $\|P_1 f(t, x') - P_1 f(t, x'')\| \leq b\|x' - x''\|$

2^0. $\|P_2 f(t, x') - P_2 f(t, x'')\| \leq a(t)\|x' - x''\|; \quad \lim_{t \to +0} a(t) = 0.$

Then for all $x_0 \in \Re(T) = X_1$ the initial value problem (1.2), (3.1) has a unique solution $v(t)$ on some interval $0 \leq t \leq \tau_0$ $(0 < \tau_0 \leq \tau)$. The projection $P_1 v(t)$ is continuously differentiable.

Proof. Equation (1.2) is equivalent to the following equations

$$\frac{d}{dt}(T_1 v_1(t)) + v_1(t) = P_1 f(t, N(v_1 + v_2)) \tag{3.3}$$

$$v_2(t) = P_2 f(t, N(v_1 + v_2)). \tag{3.4}$$

Integrate (3.3) under condition (3.2). Then we obtain the equation

$$v_1 = \Phi_1(v_1, v_2), \tag{3.5}$$

where $\Phi_1(v_1, v_2)(t) \equiv T_1^{-1}\{P_1 x_0 + \int_0^t [P_1 f(s, N(v_1(s) + v_2(s))) - v_1(s)]ds\}$.

Introduce the function classes

$$S_1 = \{v_1(t) : v_1 \in C([0, \tau], X_1), v_1(0) = T_1^{-1} P_1 x_0\} \tag{3.6}$$

$$S_2 = \{v_2(t) : v_2 \in C([0, \tau], X_2)\}. \tag{3.7}$$

In these classes we consider the norm $\|v_k\|_{S_k} = \sup_{0 \leq t \leq \tau} \|v_k(t)\|_X$. Clearly, $\Phi_1(S_1, v_2) \subset S_1$ for all $v_2 \in S_2$. Condition 1^0 implies that the estimate

$$\|\Phi_1(v_1', v_2)(t) - \Phi_1(v_1'', v_2)(t)\|_X \leq t\|T_1^{-1}\|(1 + b\|N\|)\|v_1' - v_1''\|_{S_1}$$

(for all $v_2 \in S_2$) holds true. Consider problem (1.2), (3.1) on a segment $[0, \tau_1]$, where $\tau_1 < \min\{\tau, K_1^{-1}\}$, $K_1 = \|T_1^{-1}\|(1 + b\|N\|)$. Then the mapping Φ_1 in (3.5) is a contraction on $S_1 = \{v_1\}$ for every fixed function $v_2 \in S_2$. There exists a unique solution $v_1(t) = v_1(t, v_2)$ of equation (3.5). This solution satisfies the Lipschitz condition

$$\|v_1(t, v_2') - v_1(t, v_2'')\|_{S_1} \le M\|v_2' - v_2''\|_{S_2}, \quad \text{for all } v_2', v_2'' \in S_2.$$

The Lipschitz constant depends on τ_1 and is equal to

$$M = M(\tau_1) = \frac{\tau_1 b\|T_1^{-1}\| \cdot \|N\|}{1 - \tau_1 K_1}.$$

The Lipschitz property follows from the estimate

$$\begin{aligned}&\|\Phi_1(v_1(t, v_2'), v_2')(t) - \Phi_1(v_1(t, v_2''), v_2'')(t)\|_X \\ &\quad \le \tau_1 K_1\|v_1(t, v_2') - v_1(t, v_2'')\|_{S_1} + \tau_1 b\|T_1^{-1}\| \cdot \|N\| \cdot \|v_2' - v_2''\|_{S_2}.\end{aligned}$$

Substitute the solution $v_1(t, v_2)$ in (3.4) and obtain the equation in v_2:

$$v_2 = \Phi_2(v_2), \quad \Phi_2(v_2) \equiv P_2 f(t, N(v_1(t, v_2) + v_2)).$$

Clearly, $\Phi_2(S_2) \subset S_2$. Taking into account the Lipschitz property of $v_1(t, v_2)$ and Condition 2^0 in the theorem, we have

$$\|\Phi_2(v_2')(t) - \Phi_2(v_2'')(t)\|_X \le a(t)\|N\| \ (1 + M(\tau_1))\|v_2' - v_2''\|_{S_2}. \tag{3.8}$$

Now we consider problem (1.2), (3.1) on a segment $[0, \tau_0]$, where $0 < \tau_0 \le \tau_1$, $\sup_{0 \le t \le \tau_0} a(t) < \varepsilon$, $\varepsilon^{-1} = \|N\|(1 + M(\tau_1))$. It follows from the estimate of type (3.8) on the segment $[0, \tau_0]$ that $\|\Phi_2(v_2') - \Phi_2(v_2'')\|_{S_2(\tau_0)} \le q\|v_2' - v_2''\|_{S_2(\tau_0)}$, where $q = \|N\|[1 + M(\tau_0)] \sup_{0 \le t \le \tau_0} a(t)$. The Lipschitz constant $M(\tau_1)$ monotonically depends on τ_1: $\tau_0 < \tau_1$ implies $M(\tau_0) < M(\tau_1)$. Then $q < 1$ and in the function class $S_2(\tau_0)$ there exists a unique solution v_2 of the equation $\Phi_2(v_2) = v_2$. Hence, the function $v(t) = v_1(t, v_2) + v_2(t)$ is the solution of (1.2), (3.1) on $[0, \tau_0]$ to be found.

Corollary 3.1. *Theorem* 1 *is valid if the condition* 2^0 *on* $P_2 f$ *is replaced by the Lipschitz condition* $\|P_2 f(t, x') - P_2 f(t, x'')\| \le a_0\|x' - x''\|$, $a_0 < \frac{1}{\|N\|}$.

Indeed, the mapping $\Phi_2 : S_2 \to S_2$ satisfies the inequality

$$\|\Phi_2(v_2')(t) - \Phi_2(v_2'')(t)\|_X \le a_0\|N\|(1 + M(\tau_1))\|v_2' - v_2''\|_{S_2(\tau_1)}.$$

Since $\lim_{\tau_1 \to +0} M(\tau_1) = 0$, there exists $0 < \tau_0$ ($\tau_0 \le \tau_1$) such that $q = a_0\|N\|[1 + M(\tau_0)] < 1$. Then, Φ_2 is a contraction from S_2 into itself. This completes the proof of Corollary 3.1.

The conditions involving the projections $P_k f(t, x)$, $k = 1, 2$, may be replaced by conditions on the function $f(t, x)$, as the following theorem shows.

Theorem 3.2. *Let $f : [0, \tau] \times X \to X$ be a continuous mapping, the point $\zeta = 0$ be a simple pole of the resolvent $(T - \zeta I)^{-1}$ and suppose $\alpha > \|P_2\|$, where P_2 is the spectral projector onto $KerT$. If the function f satisfies the Lipschitz condition*

$$\|f(t, x_1) - f(t, x_2)\| \le \frac{1}{\alpha\|N\|}\|x_1 - x_2\|, \quad \forall x_k \in X,$$

then the initial value problem (1.2), (3.1) is uniquely solvable on some segment $0 \le t \le \tau_0$ for every vector $x_0 \in \Re(T)$.

The theorem is valid because the conditions of Corollary 3.1 are satisfied.

4. Local restrictions in the simple pole case.

4.1. Up to now the conditions on the function $f(t, x)$ were global in the space X. Now we consider conditions which are fulfilled in some neighbourhood of the initial vector v_{10} given in (3.2). Denote

$$V_1 = \{v_1 \in X_1 : \|v_1 - T_1^{-1}P_1x_0\| \le \rho_1\}, V_2 = \{v_2 \in X_2 : \|v_2\| \le \rho_2\}. \quad (4.1)$$

Theorem 4.1. *Let the function $f_N(t, v) \equiv f(t, Nv)$ be continuous on the set $[0, \tau] \times (V_1 \dot{+} X_2)$. Let the projections P_kf satisfy the Lipschitz condition in this set:*

$$\|P_kf(t, Nv') - P_kf(t, Nv'')\|_X \le b_k\|v' - v''\|_X, \quad k = 1, 2, \qquad (4.2)$$

where $b_2 < 1$. Then for $x_0 \in \Re(T)$ problem (1.2), (3.1) has a unique solution $v(t)$ on some interval $0 \le t \le \tau_0$, where $0 < \tau_0 \le \tau$.

Theorem 4.2. *Let the function $f_N(t, v) \equiv f(t, Nv)$ satisfy the conditions of Theorem 4.1 on the bounded closed set $[0, \tau] \times (V_1 \dot{+} V_2)$ and the values of the function P_2f belong to the ball V_2 defined by (4.1) : $\|P_2f(t, N(v_1 + v_2))\| \le \rho_2$; $\forall t \in [0, \tau]$, $v_k \in V_k$. Then for $x_0 \in \Re(T)$, problem (1.2), (3.1) has a unique continuous solution $v(t)$ on some non-trivial interval $0 \le t \le \tau_0$.*

Here, the component $v_1(t)$ of the solution is continuously differentiable. We will prove Theorems 4.1 and 4.2 at the same time. In the function classes S_k (3.6), (3.7) we consider the bounded closed sets S_{ρ_k}:

$$\begin{aligned} S_{\rho_1} &= \{v_1 \in S_1 : \|v_1 - T_1^{-1}P_1x_0\|_{S_1} \le \rho_1\}; \\ S_{\rho_2} &= \{v_2 \in S_2 : \|v_2\|_{S_2} \le \rho_2\}. \end{aligned} \qquad (4.3)$$

Fix $v_1 \in S_{\rho_1}$ and write equation (3.4) in the form $v_2 = \Phi_{v_1}(v_2)$, where $\Phi_{v_1}(v_2) \equiv P_2 f(t, N(v_1 + v_2))$. Under the conditions of Theorem 4.1 we have $\Phi_{v_1}(S_2) \subset S_2$ and under the conditions of Theorem 4.2 we have $\Phi_{v_1}(S_{\rho_2}) \subset S_{\rho_2}$. In virtue of (4.2), $\|\Phi_{v_1}(v_2')(t) - \Phi_{v_1}(v_2'')(t)\|_X \le b_2 \|v_2'(t) - v_2''(t)\|_X$, $b_2 < 1$, $\forall t \in [0, \tau]$. For this reason $\Phi_{v_1}(v_2)$ is a contractive mapping in the norm of S_2. It acts in the whole S_2 under the conditions of Theorem 4.1 or in S_{ρ_2} under the conditions of Theorem 4.2. For the functional equation (3.4) there exists a unique continuous solution $v_2(t) = v_2(t, v_1)$ such that $v_2 \in S_2$ under the conditions of Theorem 4.1 or $v_2 \in S_{\rho_2}$ under the conditions of Theorem 4.2. The solution $v_2(v_1) : S_1 \to S_2$ (respectively $v_2 : S_{\rho_1} \to S_{\rho_2}$) possesses the Lipschitz property:

$$\|v_2(v_1') - v_2(v_1'')\|_{S_2} \le \beta \|v_1' - v_1''\|_{S_1}, \quad \beta = \frac{b_2}{1 - b_2}. \tag{4.4}$$

Hence, the class of functions $v_2(v_1)$ is uniformly bounded for all $v_1 \in S_{\rho_1}$. Substitute the solution $v_2(v_1)$ in equation (3.5) and seek the solution v_1 of the equation $v_1 = \Phi(v_1) \equiv \Phi_1(v_1, v_2(v_1))$ in the function class S_{ρ_1} defined by (4.3). Continuity of $f(t, Nv)$ and Lipschitz properties (4.2), (4.4) imply the estimate

$$\|P_1 f(t, N(v_1(t) + v_2(v_1)(t)))\| \le d, \quad \forall t \in [0, \tau], \quad \text{for all } v_1 \in S_{\rho_1}$$

with $d < \infty$ some constant. For every $v_1, v_1', v_1'' \in S_{\rho_1}$ we obtain

$$\|\Phi(v_1)(t) - T_1^{-1} P_1 x_0\|_X \le \tau K; \quad K = \|T_1^{-1}\|(\rho + \|T_1^{-1} P_1 x_0\| + d);$$
$$\|\Phi(v_1')(t) - \Phi(v_1'')(t)\|_X \le \tau M \|v_1' - v_1''\|_{S_1};\ M = \|T_1^{-1}\|(1 + b_1)(1 + \beta).$$

Choose $\tau_0 < \min\{\rho K^{-1}, M^{-1}\}$, $0 < \tau_0 \le \tau$, and consider our problem on the segment $0 \le t \le \tau_0$. Then $\Phi : S_{\rho_1} \to S_{\rho_1}$ is a contraction in the norm of S_1. There exists a unique solution $v_1(t)$ of (3.5) in the class S_{ρ_1}. In this case $v_1 \in C^1([0, \tau_0], X_1)$. The proof is complete.

Consider the example of electrical networks from Section 2.1.

Let $i^-, u^- : [0, \tau] \to \mathbf{C}^1$, $\varrho : \mathbf{C}^2 \to \mathbf{C}^1$, $\psi : [0, \tau] \times \mathbf{C}^1 \to \mathbf{C}^1$, $\varphi : [0, \tau] \times \mathbf{C}^1 \to \mathbf{C}^1$ be continuous functions and ϱ, ψ, φ possess the Lipschitz property:

$$|\varrho(\xi', \eta') - \varrho(\xi'', \eta'')| \le a_\varrho \|(\xi', \eta') - (\xi'', \eta'')\|_{\mathbf{C}^2},$$
$$|\varphi(t, \xi') - \varphi(t, \xi'')| \le a_\varphi |\xi' - \xi''|, \text{ for all } t \in [0, \tau]; \xi, \eta \in \mathbf{C}^1.$$

Assume that the Lipschitz constants a_ϱ, a_φ satisfy the estimate $\max\{a_\varrho, a_\varphi\} < \frac{1}{2M}$, $M^2 = 1 + (1+r)^2 + (2+r)^2 L^{-2}$.Then the system of differential-algebraic equations $\{(2.1), (2.2), i = \varrho(i_1, i_2)\}$ of the electrical network has a unique solution with the initial condition $i_1(0) = i_{10}$ (for all $i_{10} \in \mathbf{C}^1$). Actually, the matrices N, T and the function f, obtained in Section 2.1, satisfy the conditions of Theorem 4.1. Since $v = e^{-t}(i_1 + i_2, Li_1, ri_2 + u_3)^{tr}$, the initial value (3.2) has the form $P_1 v(0) = v_1(0) = (0, Li_1(0), 0)^{tr}$.

4.2. Now we examine problem (1.2), (3.1) with the help of the Fréchet derivative of the function $f(t,x)$ or its projection $P_2 f$.

Theorem 4.3. *Assume that the stationary equation*

$$v_{20} = P_2 f(0, Nv_{10} + Nv_{20}) \tag{4.5}$$

in the Banach space X has a solution $v_{20} \in X_2$ $(= KerT)$ for given $v_{10} = T_1^{-1} P_1 x_0$ (3.2). Let $S(\xi, r) = \{x \in X : \|x - \xi\| \le r\}$ be a closed neighbourhood of the point $\xi = N(v_{10} + v_{20})$ and the function $f(t,x) : [0,\tau] \times S(\xi, r) \to X$ be continuous. Let one of the two conditions hold true:

1) *the function $f(t,x)$ has the continuous Fréchet derivative $\frac{\partial}{\partial x} f(t,x)$ on the set $[0,\tau] \times S(\xi, r)$;*
2) *the component $P_1 f(t,x)$ is a Lipschitz function and the component $P_2 f(t,x)$ has the continuous Fréchet partial derivative $\frac{\partial}{\partial x} P_2 f(t,x)$ on $[0,\tau] \times S(\xi, r)$.*

If the initial value of the derivative at the center of the ball S is a contractive[1] *operator $Q = Q(\xi) = \frac{\partial}{\partial x}[P_2 f(0,\xi)] N_{|X_2} \in L(X_2)$, then problem* (1.2), (3.1) *has at least one solution $v(t)$ on a non-trivial interval $0 \le t \le \tau_0$. The component $P_2 v(t)$ is continuous, and the component $P_1 v(t)$ is continuously differentiable.*

Proof. The vector $v_0 = v_{10} + v_{20}$ is the initial value for the unknown solution $v(t)$ of (1.2). Equation (3.4) will be solved with respect to an implicit function $v_2(t, v_1)$. To this purpose we apply the implicit function theorem to the equation $v_2 - P_2 f(t, Nv_1 + Nv_2) = 0$ in the space $X = X_1 \dot{+} X_2$. There exists a continuous solution $v_2 = \psi(t, v_1)$ with the initial condition $v_{20} = \psi(0, v_{10})$ and a continuous Fréchet derivative $\frac{\partial \psi}{\partial v_1}(t, v_1)$ on some set $\Omega \equiv [0, \tau_1] \times S(v_{10}, \varrho) \subset [0,\tau] \times X_1$, $0 < \varrho \le r$. Substitute the solution v_2 in

[1]Instead of contractivity of Q, it is sufficient that the spectrum of the operator Q don't contain the point $\xi = 1$.

equation (3.3) and obtain

$$\frac{dv_1(t)}{dt} + T_1^{-1}v_1(t) = \varphi(t, v_1), \quad \varphi \equiv T_1^{-1}P_1 f(t, Nv_1 + N\psi(t, v_1)). \tag{4.6}$$

The right-hand side has the continuous Fréchet derivative $\frac{\partial\varphi}{\partial v_1}(t, v_1)$ on the set Ω if Condition 1) of our theorem is fulfilled. By virtue of the Cauchy type theorem ([4], 10.4.5), there exists a unique solution $v_1(t)$ of equation (4.6), satisfying the initial condition $v_1(0) = v_{10}$. If Condition 2) of the theorem is valid, then the function $\frac{\partial\psi}{\partial v_1}$ is also continuous on Ω. Therefore $\psi(t, v_1)$ and $\varphi(t, v_1)$ are local Lipschitz functions ([4], 10.4.6). We can apply the existence and uniqueness theorem of Picard type to equation (4.6) with the initial condition $v_1(0) = v_{10}$.

The function $v(t) = v_1(t) + \psi(t, v_1)$ is a solution of our problem on some non-trivial interval $0 \le t \le \tau_0$. The theorem is proved.

Remark 4.1. If the initial equation (4.5) has more than one solution v_{20} for given v_{10}, then the initial value problem (1.2), (3.1) has more than one solution (see Example 4.1). But the Cauchy problem for equation (1.2) with the initial condition $v(0) = v_{10} + v_{20}$ has a unique solution. In this case equality (4.5) is a relation between components v_{k0} of the initial vector v_0. In order that a solution v_{20} of (4.5) be unique, it is sufficient that

$$\begin{gathered}\|P_2 f(0, x') - P_2 f(0, x'')\| \le a_2\|x' - x''\|, \quad a_2 < \frac{1}{\|N_2\|}, \\ \text{for all } x', x'' \in Nv_{10} + N(X_2); \quad X_2 = KerT; \quad N_2 = N_{|X_2}.\end{gathered} \tag{4.7}$$

Example 4.1. (Illustration of Theorem 4.3.) Suppose we are given the system of differential-algebraic equations with the initial condition:

$$\frac{d}{dt}u(t) + u(t) = 1 - u^2 + y^2; \quad u(0) = 0 \tag{4.8}$$

$$y = u^2 + y^2 + t. \tag{4.9}$$

Write the system in the form (1.2), (3.1) in $\mathbf{R}^2$:

$$\begin{gathered}\frac{d}{dt}(Tv) + v(t) = f(t, v); \quad Tv(0) = (0, 0)^{tr}, \\ v = \begin{pmatrix} u \\ y \end{pmatrix}, \quad T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 1 - u^2 + y^2 \\ t + u^2 + y^2 \end{bmatrix}.\end{gathered} \tag{4.10}$$

Under the notation of Theorem 4.3 we have $X_2 = \{(0, x_2)^{tr}\} \sim \mathbf{R}^1$,

$$N = I,\ P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ T_1 = 1,\ v_{10} = 0,\ \frac{\partial}{\partial x}[P_2 f(t,x)]N_{|X_2} = 2y.$$

The equation $y(0) = y^2(0)$ of the form (4.5) has two solutions, namely $y_1(0) = 0, y_2(0) = 1$. Correspondingly, there exist two points $\xi_1 = (0,0)^{tr}$, $\xi_2 = (0,1)^{tr}$ from which solutions $v(t)$ start. At these points $Q(\xi_1) = 0, Q(\xi_2) = 2$. The mappings $[1 - Q(\xi_k)] : X_2 \to X_2$ are invertible, $Q(\xi_2)$ not being a contraction. The conditions of Theorem 4.3 are fulfilled for problem (4.10) with respect to each point ξ_k. For this reason there exist two solutions $v_k(t)$ of (4.10). The Cauchy condition $v_k(0) = \xi_k$ uniquely determines the function $v_k(t)$. Equation (4.9) is of form (3.4). It has two solutions $y_{1,2}(t,u) = \frac{1}{2} \mp \frac{1}{2}\sqrt{1 - 4u^2 - 4t}$.

Note that the uniqueness condition (4.7) isn't globally fulfilled in the whole space X_2. The corresponding inequality being locally satisfied in a ball $|y| \le r, r < \frac{1}{2}$ doesn't ensure uniqueness.

5. The second order pole case. Consider first a simple situation where the subspace $X_2 = P_2(X)$ is a linear span of one eigenvector and one associated vector: $X_2 = span\{\varphi_0, \varphi_1\}, T\varphi_0 = 0, T\varphi_1 = \varphi_0; m = 2$. Using the representation $P_2 v = \pi_{20}(P_2 v)\varphi_0 + \pi_{21}(P_2 v)\varphi_1$ in the space X_2, we introduce the projectors $\Pi_{2i} v = \pi_{2i}(P_2 v)\varphi_i$ $(i = 0, 1)$ in the space X $(P_2 = \Pi_{20} + \Pi_{21})$. For convenience we write $\pi_{2i} v = \pi_{2i}(P_2 v)$. Let us introduce the following notation: $f_{21}(t,x) = \pi_{21} f(t,x) : [0,\tau] \times X \to \mathbf{C}$. We have $f_{21}(t,x)\varphi_1 = \Pi_{21} f(t,x)$.

Theorem 5.1. *Let a function $f(t,x) : [0,\tau] \times X \to X$ be continuous and for each function $g(t) \in C^1[0,\tau]$ the component f_{21} generate the continuously differentiable function $f_{21}(t, g(t)N\varphi_1) \equiv \psi_g(t) \in C^1[0,\tau]$. Suppose that*

1^0. $\|P_1 f(t,x') - P_1 f(t,x'')\| \le b\|x' - x''\|$
2^0. $\|\Pi_{20} f(t,x') - \Pi_{20} f(t,x'')\| \le a(t)\|x' - x''\|$
3^0. $\Pi_{21} f(t, Nv) = \Pi_{21} f(t, N\Pi_{21} v)$, *for every* $v \in X$
4^0. $\pi_{21} f(0, N(\pi_{20} x_0)\varphi_1) = \pi_{20} x_0$
5^0. $\|\psi_{g'} - \psi_{g''}\|_{C^1[0,\tau]} \le b_1 \|g' - g''\|_{C^1[0,\tau]}$, $\quad b_1 < 1$.

Here b, b_1 are constants; the function $a(t) \ge 0$ is bounded for $t \in [0,\tau]$, $\lim_{t \to +0} a(t) = 0$. Then for all $x_0 \in \Re(T)$ $(= X_1 \dot{+} Ker T)$ there exists a unique solution $v(t)$ of problem (1.2), (3.1) *on some interval $0 \le t \le \tau_0, 0 < \tau_0 \le \tau$.*

Proof. We denote by $P_1 v = v_1, \Pi_{20} v = g_0(t)\varphi_0, \Pi_{21} v = g_1(t)\varphi_1$ the components of the vector $v(t)$. Then equation (1.2) and condition (3.1) are equivalent to the following problems:

$$\frac{d}{dt}(T_1 v_1(t)) + v_1(t) = P_1 f(t, Nv(t)); \qquad \lim_{t\to+0} T_1 v_1(t) = P_1 x_0; \quad (5.1)$$

$$\frac{d}{dt} g_1(t) + g_0(t) = \pi_{20} f(t, Nv(t)); \quad \lim_{t\to+0} g_1(t) = \pi_{20} x_0; \quad (5.2)$$

$$g_1(t) = \pi_{21} f(t, Nv(t)); \qquad 0 = \pi_{21} x_0. \quad (5.3)$$

The last equality is equivalent to $x_0 \in \Re(T)$. The remaining equalities are rewritten in the following form:

$$v_1(t) = \Phi_1(v_1, g_0, g_1), \quad \Phi_1 \equiv T_1^{-1}\{P_1 x_0 + \int_0^t [P_1 f(s, Nv(s)) - v_1(s)]ds; \quad (5.4)$$

$$g_0(t) = \Phi_{20}(g_0, v_1, g_1), \quad \Phi_{20} \equiv -\frac{d}{dt} g_1(t) + \pi_{20} f(t, Nv(t)); \quad (5.5)$$

$$g_1(t) = \Phi_{21}(g_1(t)), \qquad \Phi_{21} \equiv \pi_{21} f(t, g_1(t) N\varphi_1). \quad (5.6)$$

In order to pass from (5.3) to (5.6) we used Condition 3^0 of the theorem.

We shall seek a solution v_1, g_0, g_1 of the system (5.4)–(5.6) in the function classes S_1 (see (3.6)), S_{20}, S_{21} respectively, where

$$S_{20} = \{g_0 \in C([0,\tau], \mathbf{C})\}; \quad S_{21} = \{g_1 \in C^1([0,\tau], \mathbf{C}), g_1(0) = \pi_{20} x_0\}. \quad (5.7)$$

It follows from Condition 4^0 and continuity of f that $\Phi_{21}(S_{21}) \subset S_{21}$ (see (5.6)). It follows from Condition 5^0 that $\Phi_{21} : S_{21} \to S_{21}$ is a contraction with respect to the norm of the space $C^1[0,\tau]$. Then there exists a unique solution $g_1(t) \in S_{21}$ of equation (5.6). Substituting this solution in the system (5.4), (5.5), we have two equations with respect to functions $v_1(t), v_2(t) \equiv g_0(t)\varphi_0$ with values in the space $X_1 \dot{+} \Pi_{20}(X_2)$. These equations are analogous to equations (3.5), (3.4). Hence, system (5.4), (5.5) has a unique solution $v_1(t), g_0(t)$. The function $v(t) \equiv v_1(t) + g_0(t)\varphi_0 + g_1(t)\varphi_1$ is the solution to be found.

Note that the Hypothesis 4^0 of Theorem 5.1 on the relation between $f(t,x)$ and x_0 is a necessary condition to solve problem (1.2), (3.1). The Hypothesis 3^0 means that the component $f_{21}(t, Nv)$, corresponding to the associated vector φ_1, doesn't depend on the projection of the argument v on the supplement of $\operatorname{span}\{\varphi_1\}$.

Example 5.1. Let us consider the system of three equations

$$\frac{dv_1}{dt} + v_1 = \frac{t^2 - 2}{\sqrt{1-t^2}} + \arccos t + \sin v_1;$$
$$\frac{d}{dt} g_1 + g_0 = t^2 \sin v_1; \quad g_1 = \frac{1}{4}\int_0^t \sin g_1(\tau) d\tau.$$

The system may be written in the form (1.2) in the space $X = Y = \mathbf{C}^3$, where

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ \varphi_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ \varphi_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \ v = \begin{pmatrix} v_1 \\ g_0 \\ g_1 \end{pmatrix}.$$

We look for a solution $v(t)$ under the condition $\lim_{t\to+0} Tv(t) = (\frac{\pi}{2}, 0, 0)^{tr} = x_0$. The function $\psi_g(t) = f_{21}(t, Ng\varphi_1)$ and the projectors appearing in the theorem are as follows

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ P_{20} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ P_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ N = I,$$

$$f_{21}(t, Nv) = \frac{1}{4}\int_0^1 \sin g_1(\tau) d\tau, \quad \pi_{2k} v = g_k \quad (k = 0, 1).$$

The conditions of Theorem 5.1 are valid on any segment $[0, \tau] \subset [0, 1)$. Moreover, $b = 1$, $a(t) = t^2$, $\pi_{20}(P_2 x_0) = 0$, $b_1 = \frac{1}{2}$. The solution $v(t)$, existing according to Theorem 5.1, is expressed by elementary functions and satisfies the system (on the segment $0 \le t < 1$):

$$v_1 = \arccos t, \quad g_0(t) = t^2\sqrt{1-t^2}, \quad g_1(t) \equiv 0.$$

We can weaken some conditions of Theorem 5.1.

Corollary 5.1. *Theorem* 5.1 *holds if* 2^0 *is replaced by the Lipschitz condition:*

$$2. \quad \|\Pi_{20} f(t, x') - \Pi_{20} f(t, x'')\| \le a_0 \|x' - x''\|, \quad a_0 < \frac{1}{\|N\|}.$$

Let us now consider local restrictions on the component $P_1 f$ of the function $f(t, x)$ in the ball V_1 (see (4.1)).

Theorem 5.2. *Let all conditions of Theorem* 5.1 *except* $1^0, 2^0$ *be fulfilled. Assume also that for every* $t \in [0,\tau]$ *the following conditions hold:*

1. $\|P_1 f(t,Nv') - P_1 f(t,Nv'')\| \leq b\|v' - v''\|$, *for all* $v', v'' : P_1 v', P_1 v'' \in V_1$;
2. $\|\Pi_{20} f(t,Nv') - \Pi_{20} f(t,Nv'')\| \leq a_0 \|v' - v''\|, \quad a_0 < 1$, *for all* $v', v'' \in X_2$.

Then for all $x_0 \in \Re(T)$ *there exists a unique solution* $v(t)$ *of problem* (1.2), (3.1) *on some non-trivial interval* $0 \leq t \leq \tau_0$. *The projection* $P_1 v(t) = v_1(t)$ *lies in the ball* V_1: $\|v_1(t) - T_1^{-1} P_1 x_0\| \leq \rho_1$, $0 \leq t \leq \tau_0$.

Using arguments like those used in the proof of Theorem 5.1, we infer that there exists a unique solution $g_1 \in S_1$ of equation (5.6). Substitute this solution in equations (5.4), (5.5). Let us now make use of the proof of Theorem 4.1. For this purpose we fix v_1 and denote $v_2 = v_2(v_1) \equiv g_0(t, v_1)\varphi_0$, $\Phi_{v_1} \equiv \Phi_{20}(g_0; v_1) : S_{20} \to S_{20}$. Then for the equation $g_0 = \Phi_{20}(g_0; v_1)$ we get a solution $g_0(v_1)$ in the class S_{20} of functions continuous on the interval $0 \leq t \leq \tau$. At last we return to equation (3.5). There exists some subinterval $[0,\tau_0] \subset [0,\tau]$ on which the equation $v_1 = \Phi(v_1) \equiv \Phi_1(v_1; g_0(v_1))$ is uniquely solvable in the function class S_{ρ_1} (see (4.3)).

6. The case of a spectral subspace of arbitrary dimension. We start from a finite-dimensional subspace X_2 and one-block eigennilpotent T_2 (1.4). Then in X_2 there exists a generating vector $\varphi = \varphi_n$ $(n = m-1)$ such that $T^n \varphi_n = \varphi_0$ is an eigenvector and the system of vectors $\{\varphi_k = T^{n-k}\varphi_n\}_{k=0}^n$ forms a basis for X_2. With the help of the decomposition $P_2 v = \sum_{k=0}^n \pi_{2k}(P_2 v)\varphi_k$ we introduce the one-dimensional projectors $\Pi_{2k} v = \pi_{2k}(P_2 v)\varphi_k$ in X. For convenience, we write $\pi_{2k}(P_2 v) = \pi_{2k} v$. Notice that the additive representation of the spectral projector $P_2 = \sum_{k=0}^n \Pi_{2k}$ by the one-dimensional projectors Π_{2k} is not unique. We look for a solution $v = v(t)$ of equation (1.2) in the form

$$v = v_1 + g_0\varphi_0 + g_1\varphi_1 + \ldots + g_n\varphi_n; \quad v_1 = P_1 v; \quad g_k = \pi_{2k} v. \tag{6.1}$$

Using the sketch of proof of Theorem 5.1, we obtain the following proposition.

Theorem 6.1. *Let* $f : [0,\tau] \times X \to X$ *be a continuous function. Suppose:*

1). *The components* $\pi_{2k} f$ *do not depend on the variables* $v_1, g_1, g_2, \ldots, g_{k-1}$:

$$\pi_{2k} f(t, Nv) = \pi_{2k} f(t, N \sum_{j=k}^{n} \Pi_{2j} v), \quad k = n, n-1, \ldots, 1.$$

2). *The projections* $P_1 f$, $\Pi_{20} f$ *satisfy the Lipschitz conditions: for all* $x', x'' \in X, 0 \le t \le \tau$

$$\|P_1 f(t,x') - P_1 f(t,x'')\|_X \le a\|x' - x''\|_X, \quad a \in \mathbf{R}_+;$$
$$|\pi_{20} f(t,x') - \pi_{20} f(t,x'')| \le b_0 \|x' - x''\|_X, \quad b_0 < \frac{1}{\|N\|}.$$

3). *For each* $k = 1, 2, \dots, n$ *the function* $\psi_k(t) \equiv \pi_{2k} f(t, N \sum_{j=k}^{n} g_j(t)\varphi_j)$ *belongs to the class* $C^k[0,\tau]$ *for* $g_j \in C^j[0,\tau]$, $j = k, ..., n$. *The mapping* $\pi_{2k} f \equiv \Phi_k(t, g_k, g_{k+1}, ..., g_n)$ *possesses the Lipschitz property with respect to* $g_k \in C^k[0,\tau]$ *uniformly in the variables* $t \in [0,\tau]$, $g_j \in C^j[0,\tau]$, $j = k+1, ..., n$:

$$\|\Phi_k(t, g_k', g_{k+1}, ..., g_n) - \Phi_k(t, g_k'', g_{k+1}, ..., g_n)\|_{C^k} \le b_k \|g_k' - g_k''\|_{C^k}, \; b_k < 1.$$

Then the projection $P_1 x_0$ *of the initial vector* x_0 *in* (3.1) *defines a unique solution* $v(t)$ *of problem* (1.2), (3.1) *on some subinterval* $[0,\tau_0]$ *of the interval* $[0,\tau]$. *Furthermore,* $v_1 \in C^1[0,\tau_0], g_k \in C^k[0,\tau_0]$.

Clearly, in the case when the eigennilpotent contains more than one block, a similar theorem may be formulated. In so doing, there are no new difficulties. However, another approach is needed if the spectral subspace X_2 is infinite-dimensional. Such an existence and uniqueness theorem has been obtained in [9]. Here we shall present another approach on the subject.

It is assumed that $dim X_2 \le \infty$ and resolvent (1.3) has a pole of multiplicity $m = 2$. Thus $X_2 = Ker T^2$, $T_2^2 = 0$. Let the eigennilpotent T_2 be *normally decomposed* (see [9]). This means that the kernel $E_1 = Ker T$ has a direct closed complement E_2 with respect to the spectral subspace X_2; the T-image $E_{11} = T(E_2)$ is closed and has a direct closed complement E_{10} with respect to the kernel E_1:

$$X_2 = E_1 \dot{+} E_2, \quad E_1 \equiv Ker T = E_{11} \dot{+} E_{10}. \tag{6.2}$$

Introduce the projectors $\Pi_k : X_2 \to E_k$ $(k = 1,2)$; $\Pi_{1j} : E_1 \to E_{1j}$ $(j = 0,1)$. Define them on the whole space X: $\Pi_k = \Pi_k P_2$, $\Pi_{1j} = \Pi_{1j}\Pi_1$. In view of these definitions, $P_2 = \Pi_1 + \Pi_2$ $(\Pi_1\Pi_2 = 0)$, $\Pi_1 = \Pi_{11} + \Pi_{10}(\Pi_{11}\Pi_{10} = 0)$. The kernel of the linear mapping $T_{|_{E_2}} = S : E_2 \to E_{11}$ is trivial $(Ker S = 0)$. Hence, by virtue of the Banach theorem, there exists the bounded inverse $S^{-1} \in L(E_{11}, E_2)$. It is clear that $P_2 T = \Pi_{11} T P_2 = S\Pi_2$. Introduce the projectors $P_1 = I - P_2$, $\Pi = \Pi_{10} + \Pi_2$. Any vector v is represented in the form $v = z + h_{11} + h_{10} + h_2$, where $z = P_1 v$, $h = P_2 v$, $h_k = \Pi_k h$, $h_{1,k-1} = \Pi_{1,k-1} h$ $(k = 1, 2)$.

It follows from the initial relation (3.1) that

$$z^0 = z(0) = P_1 v(0) = T_1^{-1} P_1 x_0; \quad T_1 = T_{|X_1}. \tag{6.3}$$

A necessary condition for the vector x_0 to satisfy (3.1) is

$$P_2 x_0 = \Pi_{11} x_0 \quad (\leftrightarrow \Pi x_0 = 0 \leftrightarrow P_2 x_0 \in E_{11}). \tag{6.4}$$

If condition (6.4) is valid, then one may uniquely obtain

$$h_2^0 \equiv h_2(0) = S^{-1} P_2 x_0 (= \Pi_2 v(0)). \tag{6.5}$$

It follows from equation (1.2) for $t = 0$ that the initial value $h_1(0) \equiv h_1^0$ must satisfy the following initial relation

$$\Pi f(0, N(z^0 + h_1^0 + h_2^0)) = \Pi(h_1^0 + h_2^0). \tag{6.6}$$

Theorem 6.2. *Let the eigennilpotent $T_2 = T_{|X_2}$ be normally decomposed*[2] *and $T_2^2 = 0$. Suppose that x_0 satisfies (6.4) and there exists at least one vector $h_1^0 \in E_{10}$ satisfying the initial relation (6.6) with z^0, h_2^0 from (6.3), (6.5). Let for some neighbourhood $\bar{S}(v^0)$ of the point $v^0 = z^0 + h_1^0 + h_2^0$ the mapping $f(t, Nv) \equiv f_N(t, v) : [0, \tau] \times \bar{S}(v^0) \to X$ be continuous and have the continuous Fréchet derivative $\frac{\partial f}{\partial v}$. If the operator*

$$\{\Pi \frac{\partial f}{\partial v}(0, Nv^0) - \Pi\}_{|E_1} \equiv G : E_1 \to E (= E_{10} \dot{+} E_2) \tag{6.7}$$

is invertible and $G^{-1} \in L(E, E_1)$, then there exists a continuously differentiable solution $v(t)$ of (1.2), (3.1) on some segment $[0, \delta]$. If the initial relation (6.6) has a unique solution h_1^0, then the initial value problem (1.2), (3.1) is uniquely solvable.

Proof. The initial value problem (1.2), (3.1) is equivalent to the system of three problems

1.a) $\dfrac{d}{dt}(T_1 z(t)) + z(t) = P_1 f(t, Nv)$, b) $T_1 z(0) = P_1 x_0$;

2.a) $\dfrac{d}{dt}(S h_2(t)) + h_{11}(t) = \Pi_{11} f(t, Nv)$, b) $S h_2(0) = \Pi_{11} x_0$;

3.a) $\Pi v(t) = \Pi f(t, Nv)$, b) $\Pi x_0 = 0$.

[2] A finite-dimensional nilpotent is always normally decomposed.

Condition 3.b) on the vector x_0 is equivalent to (6.4). The initial values z^0, h_2^0 of functions $z(t), h_2(t)$ are obtained from 1.b), 2.b) and have the form (6.3), (6.5). Applying the implicit function theorem to 3.a), one may express the function $h_1(t) = h_{11}(t) + h_{10}(t)$ in terms of $z(t), h_2(t)$. Thus there exists a segment $[0, \tau_1] \subset [0, \tau]$ and a neighbourhood $\bar{S}_1(v^0) \subset \bar{S}(v^0)$ such that $h_1 = F(t, z, h_2)$, for all $(t, v) \in [0, \tau_1] \times \bar{S}_1(v^0)$. Moreover, $F(t, z, h_2)$ is continuous and the partial Fréchet derivatives $\frac{\partial F}{\partial z}, \frac{\partial F}{\partial h_2}, \frac{\partial F}{\partial t}$ are continuous too. Substitute h_1 in 1.a), 2.a):

$$\frac{d}{dt} z(t) = T_1^{-1} P_1 f(t, N(z + F(t, z, h_2) + h_2)) - T_1^{-1} z(t), \tag{6.8}$$

$$\frac{d}{dt} h_2(t) = S^{-1} \Pi_{11} f(t, N(z + F(t, z, h_2) + h_2)) - S^{-1} \Pi_{11} F(t, z, h_2). \tag{6.9}$$

The right-hand sides of these equations have continuous partial Fréchet derivatives with respect to the variables z and h_2 (and hence in $u = z + h_2$). For this reason the right-hand sides are locally lipschitz functions of t, u in some neighbourhood of the point $(t, u) = (0, z^0 + h_2^0)$ (see [4], Section 10.4). Applying the classic local existence and uniqueness theorem, we find a unique solution $z(t), h_2(t)$ of Cauchy problem (6.3), (6.5), (6.8), (6.9). We have obtained two components of the solution $v(t)$.The last component is $h_1(t) = F(t, z, h_2)$.

Remark 6.1. Theorem 6.2 remains true in the case when T is an unbounded closed operator.

This is proved in the same way as Theorem 6.2 except that the equation

$$\frac{d}{dt} x_1(t) = -T_1^{-1} x_1 + P_1 f(t, N(T_1^{-1} x_1(t) + F(t, T_1^{-1} x_1, h_2) + h_2))$$

in $x_1(t) = T_1 z(t)$ is used instead of the equation (6.8) in $z(t)$.

7. Applications.

7.1. Consider the problem

$$\begin{aligned} &\frac{\partial}{\partial t}(1 + \frac{\partial^2}{\partial x^2})g - \frac{\partial^2 g}{\partial x^2} = f_0(t, g), \quad t \geq 0, \quad x \in [0, \pi]; \\ &g(t, 0) = g(t, \pi) = 0, \quad \lim_{t \to +0}(g + \frac{\partial^2 g}{\partial x^2}) = w_0(x). \end{aligned} \tag{7.1}$$

Here $f_0(t,\xi)$ is a scalar continuously differentiable function on $[0,\tau]\times\mathbf{R}^1$. Put $X=Y=C[0,\pi]$, $K=I$, $Au(x)=u+\frac{d^2u}{dx^2}$, $Bu=-\frac{d^2u}{dx^2}$. The domain $D=D_A=D_B$ consists of functions $u(x)\in C^2[0,\pi]$ such that $u(0)=u(\pi)=0$. Define a mapping $F(t,u):[0,\tau]\times X\to X$ by the rule $F(t,u)(x)=f_0(t,u(x))$. Put $y(t)=g(t,x)$. Then problem (7.1) is equivalent to the problem in the space X for equation (1.1) and the initial condition $\lim_{t\to+0}Ay(t)=w_0$. Passing to equation (1.2) with $\lambda_0=0$ we have

$$N=B^{-1},\quad B^{-1}u=-\int_0^x d\xi\int_0^\xi u(s)ds+\frac{x}{\pi}\int_0^\pi d\xi\int_0^\xi u(s)ds,$$
$$T=AB^{-1}=B^{-1}-I,\quad f(t,u)=F(t,u);\quad v(t)=By(t).$$

The initial condition in (7.1) is transformed into the form (3.1): $\lim\limits_{t\to+0}Tv(t)=w_0$. The spectral subspace X_2 and the spectral projector P_2 are

$$X_2=KerT=span\{\sin x\},\quad P_2u=\frac{2}{\pi}\sin x\int_0^\pi u(\xi)\sin\xi d\xi.$$

Clearly, the mapping $f:[0,\tau]\times X\to X$ is continuous and has the continuous Fréchet derivative $\frac{\partial}{\partial u}f(t,u)$. Let the function $f_0(t,\xi)$ in (7.1) satisfy the condition $|\frac{\partial}{\partial\xi}f_0(0,\xi)|\le q<\frac{1}{2}$, for all $\xi\in\mathbf{R}^1$. This happens for functions f_0 like $f_0(t,x)=t^p\varphi(t,x)+M$, where M is a constant. Since $N_2=B^{-1}_{|X_2}=I_{X_2},\|P_2\|\le 2$, condition (4.7) holds. Theorem 4.3 and Remark 4.1 imply that if the initial function $w_0(x)\in C[0,1]$ satisfies the condition $\int_0^\pi w_0(x)\sin xdx=0$, then problem (7.1) has a unique solution $g(t,x)$ $(0\le t\le\tau_0, 0\le x\le\pi, 0<\tau_0\le\tau)$.

7.2. As it has been shown in Section 2.2, the electrical network in Fig. 1 is described by equation (2.4). Let $v_k(0),x_{0k}$ be the components of vectors $v(0),x(0)\in\mathbf{C}^3=X$. Then the initial condition $Tv(0)=x_0$ of form (3.1) is equivalent to the equalities $v_1(0)=x_{02},v_3(0)=x_{03},x_{01}=0$. In this case the projectors and functionals introduced in Section 5 are $P_1z=(0,0,z_3)^{tr},\pi_{20}z=z_2,\pi_{21}z=Cz_1,z_1=v_1,z_2=v_2-Cv_1,z_3=\frac{CL_2}{L_3}v_1-\frac{L_2}{L_3}v_2-\frac{1}{L_3}v_3,\pi_{21}f=Ce^{-t}u^-(t),\pi_{20}f=e^{-t}\{i^-(t)-G_1(t,e^tz_1)-G_2(t,e^tz_2)\},P_1f=(0,0,f_3)^{tr},f_3=\{u^-(t)-\varphi_2(t,e^tz_2)-\varphi_3(t,e^tz_3)\}$. Let the continuous functions $\varphi_j(t,\xi),G_k(t,\xi)$ satisfy the conditions

$$|\varphi_j(t,\xi_1)-\varphi_j(t,\xi_2)|\le b_j|\xi_1-\xi_2|,\quad\forall t\in[0,\tau];\xi_1,\xi_2\in\mathbf{C},\quad j=2,3;$$
$$|G_k(t,\xi_1)-G_k(t,\xi_2)|\le a_k(t)|\xi_1-\xi_2|,\quad\forall t\in[0,\tau];\xi_1,\xi_2\in\mathbf{C},\quad k=1,2, \tag{7.2}$$

where $\lim_{t\to+0} a_k(t) = 0$. Then clearly, conditions $1^0, 2^0$ of Theorem 5.1 are valid. Since the function $\psi_g(t) \equiv \pi_{21} f(t, g(t)N\varphi_1)$ is equal to $e^{-t}u^-(t)$, Condition 5^0 is fulfilled for every $b_1 \in (0,1)$ if the input tension $u^-(t)$ is sufficiently smooth: $u^-(t) \in C^1[0,\tau]$. Condition 3^0 holds since the right- and left-hand sides of 3^0 are equal to the vector $(e^{-t}u^-(t), 0, 0)^{tr}$. At last, Condition 4^0 is equivalent to the equality $Cu^-(0) = x_{02}$. Applying Theorem 5.1, we obtain

Proposition. *Let the nonlinear elements of the electrical network in Section* 2.2 *be continuous and satisfy conditions* (7.2). *If the initial relations*

$$Cu_1(0) = x_{02}, \quad L_2 i_2(0) + L_3 i_3(0) = x_{03}$$

hold, then in the electrical network there exists a unique distribution of currents and tensions $i_k(t), u_k(t)$.

7.3. Consider the mixed problem

$$\begin{aligned} &\frac{\partial}{\partial t}\frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t) = f_0(t,u), \quad \frac{\partial^2 u}{\partial x^2}u(x,0) = w_0(x) \\ &u(0,t) - u(1,t) + \frac{1}{2}\frac{\partial u(1,t)}{\partial x} = 0, \quad \frac{\partial u}{\partial x}(0,t) = 0 \end{aligned} \tag{7.3}$$

in the domain $0 \le x \le 1, t \ge 0$. If $f_0(t,\xi)$ is a continuously differentiable function on $[0,\tau] \times \mathbf{R}^1$, then this problem can be written in abstract form (1.2), (3.1) in the space $C[0,1] = X = Y$. For this purpose we put $u(x,t) = v(t)$ and introduce the operator T in $C[0,1]$ in the following way:

$$Tu = \frac{d^2u(x)}{dx^2}, \ \ u(x) \in C^2[0,1], \quad u(0) - u(1) + \tfrac{1}{2}u'(1) = 0, \quad u'(0) = 0. \tag{7.4}$$

Now problem (7.3) can be written as

$$\frac{d}{dt}(Tv(t)) + v(t) = f(t,v), \quad (Tv)(0) = w_0, \tag{7.5}$$

where $f(t,v) : [0,\tau] \times C[0,1] \to C[0,1]$, $f(t,u)(x) = f_0(t,u(x))$.

The operator T in (7.5) is closed and non-selfajoint. The resolvent $(T - \xi I)^{-1}$ has a pole of order $m = 2$ at the point $\xi = 0$. The spectral subspace X_2 of the operator T is generated by the eigenfunction $e_1(x) = 1$ and the associated function $e_2(x) = \frac{\sqrt{5}}{2}(3x^2 - 1)$; $\quad Te_2 = 3\sqrt{5}e_1, Te_1 = 0$. The invariant subspace X_1 in (1.4) is a complement of the subspace $X_2 = span\{e_1, e_2\}$. The following result was obtained in [9]. The subspace X_1 is also a complement of the linear span of two functions $v_1(x) = \sqrt{3}(2x-1), v_2(x) = \frac{1}{2}\sqrt{\frac{7}{11}}(10x^3-15x^2+6x+2)$. The spectral projector $P_2 : X \to X_2$ ($P_2X_1 = 0$) acts by the rule:

$$P_2v(x) = \Pi_1 v + \Pi_2 v, \quad \Pi_k v = a_k(v)e_k(x),$$
$$a_1(v) = \frac{4}{5}\sqrt{\tfrac{11}{7}}\int_0^1 v(x)v_2(x)dx,\ a_2(v) = \frac{4}{\sqrt{15}}\int_0^1 v(x)v_1(x)dx.$$

Under the same notation as in Section 6 (Theorem 6.2), for problem (7.5), we have

$$z^0 = z^0(s) = (T_1^{-1}x_1)(s) = z^0(0) + \int_0^s d\xi \int_0^\xi x_1(\xi)d\xi,\ \forall x_1 \in X_1,$$
$$S^{-1}\alpha = \frac{1}{3\sqrt{5}}\alpha e_2(x), \quad \forall \alpha = \alpha e_1 \in E_1 = KerT,$$
$$\Pi_1 = \Pi_{11}, \quad \Pi_{10} = 0, \quad \Pi = \Pi_2, \quad v = z + h_1 + h_2.$$

Condition (6.4) is of the form $\Pi_2 w_0 = 0$ or

$$\int_0^1 w_0(x)(2x-1)dx = 0. \tag{7.6}$$

For the initial function $v_0 = v_0(x) = v_{|t=0}$ we know the projection

$$(P_1 + \Pi_2)v_0 = z^0 + h_2^0 = T_1^{-1}P_1w_0 + S^{-1}\Pi_1 w_0 \doteq \mu(x). \tag{7.7}$$

Equation (6.6) has the integral form

$$\int_0^1 f_0(0, d + \mu(x))(2x-1)dx = \frac{1}{30}\rho, \tag{7.8}$$

$$\rho = \rho(w_0) = \int_0^1 w_0(x)(10x^3 - 15x^2 + 6x + 2)dx, \tag{7.9}$$

in the unknown $d = \Pi_1 v_0$. Invertibility of the operator G in (6.7) is equivalent to the following condition being satisfied:

$$\int_0^1 \frac{\partial f_0}{\partial v}(0, v_0(x))(2x-1)dx \neq 0, \quad v_0 = d + \mu(x). \tag{7.10}$$

Applying Theorem 6.2 and Remark 6.1, we obtain the following proposition.

Proposition. *The mixed problem* (7.3) *has a continuously differentiable solution* $u(x,t)$ *(*$0 \le t \le \tau_0 \le \tau, 0 \le x \le 1$*) if*

1) $f_0(t,u) : [0,\tau] \times \mathbf{R}^1 \to \mathbf{R}^1$ *is a continuously differentiable function;*
2) *the initial function* $w_0(x)$ *is orthogonal to the function* $2x-1$ *in the sense of* (7.6)*;*
3) *integral equation* (7.8) *is solvable with respect to* d*;*
4) *the function* $\frac{\partial f_0}{\partial v}(0, v_0(x))$ *isn't orthogonal to* $(2x-1)$ *in the sense of* (7.10)*.*

As an example, consider the function $f(t,v) = \beta(t)v^2$ $(\beta(0) = \beta_0 \neq 0)$. Condition 1) is fulfilled if $\beta(t) \in C^1[0,\tau]$. Condition 2) is only a requirement on the initial function w_0. If $\rho(w_0) \neq 0$ (see (7.9)), then there exists a unique solution $d = \frac{1}{2\beta_0} - \frac{15}{\rho}\int_0^1 \mu^2(x)(2x-1)dx$ of equation (7.8), where the function $\mu(x)$ is defined by (7.7). One can immediately verify Condition 4). Note that the solution $u(t,x)$ is unique as d is unique.

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