Differential and Integral Equations Volume 12, Number 3, May 1999, 351–371.

SINGULAR DIFFERENTIAL EQUATIONS WITH DELAY

Angelo Favini
Dipartimento di Matematica, Università di Bologna, Bologna, Italy

Luciano Pandolfi
Dipartimento di Matematica, Politecnico di Torino, Torino, Italy

Hiroki Tanabe
Department of Economics, Otemon Gakuin University, Osaka, Japan

(Submitted by: Herbert Amann)

Abstract. The differential equation $d(Mu(t))/dt = -Lu(t)+L_1u(t-1)$, $t \geq 0$, $u(t) = \varphi(t)$, $-1 \leq t \leq 0$, for a given strongly continuous X-valued function φ on $[-1, 0]$ is studied, where M, L, L_1 are closed linear operators from the complex Banach space X into itself, and L is invertible. Though already in the finite dimensional case in general existence of continuous solutions on $[-1,\infty)$ may fail or it is possible to have continuous solutions only on a finite interval, we indicate classes of operators for which existence results analogous to the ones for regular equations $M = I$ hold. In particular, solutions are given explicitly by a recovery formula.

1. Introduction. In various applications involving linear processes there appear differential equations containing a delay of the type

$$\frac{d}{dt}(Mu(t)) = -Lu(t) + L_1u(t-1), \quad t \geq 0, \tag{1.1}$$

$$u(t) = \varphi(t), \quad -1 \leq t \leq 0, \tag{1.2}$$

where M, L, L_1 are square matrices and $\varphi(\cdot)$ is a $\mathbf{C}^M$-valued continuous function. A study of (1.1), (1.2) can be found in the paper [1] by S.L. Campbell, where the author, among other things, points out that even if $zM + L$ is invertible for some $z \in \mathbf{C}$ (and hence the problem

$$\frac{d}{dt}(Mu(t)) = -Lu(t), \quad t \geq 0$$
$$Mu(0) = Mu_0$$

has a unique solution for each consistent vector), it can happen that no continuous solution exists. Moreover, it is also possible that continuous solutions exist only

Received for publication October 1997.
AMS Subject Classifications: 35R10, 35K65, 35K22, 34G10.

on a finite interval. These phenomena occur when the index of ML^{-1}, that is the least nonnegative integer ν such that the kernel or the null spaces $N((ML^{-1})^{\nu})$ and $N((ML^{-1})^{\nu+1})$ coincide, is larger than 1; indeed, if $\nu = 1$, a compatibility relation between $\varphi(0)$ and $\varphi(-1)$ guarantees that (1.1), (1.2) has one strict solution.

We shall work in a complex Banach space X with norm $\| \ \|_X$ and we always shall assume that M, L, L_1 are closed linear operators from X into itself such that $D(L) \subset D(L_1), D(L) \subset D(M), 0 \in \rho(L)$, so that $T = ML^{-1} \in \mathcal{L}(X)$, where $\mathcal{L}(X)$ denotes the spaces of all bounded linear operators from X to X endowed with the uniform norm.

We shall denote the range of T by $R(T)$. If $Y \subset X$, then Y^a is the closure in the topology of X.

In Section 2 we shall study problem (1.1), (1.2) when $z = 0$ is a polar singularity of order 1 for the resolvent $(z+T)^{-1} = L(zL+M)^{-1}$. In this case the index of ML^{-1} is 1. Under suitable compatibility conditions we shall show that a continuous or differentiable solution exists in $[0, \infty)$. The case as was described in S.L. Campbell [1] can only happen if the index is larger than 1. Section 3 concerns the case where $\|(zT+1)^{-1}\|_{\mathcal{L}(X)} \leq$ const on $\operatorname{Re} z \geq 0$, i.e., T is an abstract potential operator in the reflexive Banach space X. In section 5 we give some concrete applications in partial differential equations.

2. The delay equation in the polar singularity case. Let us assume that $z = 0$ is a simple polar singularity of $(z+T)^{-1} = L(zL+M)^{-1}$. Then it is well known (A.E. Taylor [13]: Theorem 5.8-A, p. 306) that $R(T)$ is a closed subspace of X and the direct sum representation $X = N(T) \oplus R(T)$ holds. Moreover, if P denotes the projection operator onto $N(T)$ associated to this representation, T commutes with P and the restriction $\tilde{T}$ of T to $R(T)$ is an isomorphism from $R(T)$ onto $R(T)$.

Before establishing our first existence and uniqueness result, we specify what a strict solution to (1.1), (1.2) means. We always assume $\varphi \in C([-1, 0]; D(L))$.

Definition 1. A strict solution u of (1.1), (1.2) on $(0, T), 0 < T \leq \infty$, with initial datum $\varphi(\cdot)$ is a function $u \in C([-1, T[; D(L))$ such that $Mu \in C^1([0, T[; X)$ and (1.1), (1.2) are satisfied with $0 \leq t < T$ in place of $t \geq 0$.

Of course $C^k(I; X), I$ an interval of R, X a Banach space, k a nonnegative integer, denotes the space of all k times continuously differentiable X-valued functions on I, with $C^0(I, X) = C(I, X)$. $D(L)$ is endowed with the norm $\|u\|_{D(L)} := \|Lu\|_X$. It is well known that then $(D(L), \| \cdot \|_{D(L)})$ is a Banach space.

In order to solve (1.1), (1.2) we observe that the position $Lu = v$ transforms (1.1), (1.2) into the equivalent problem

$$\frac{d}{dt}(Tv(t)) = -v(t) + L_1 L^{-1} v(t-1), \quad t \geq 0, \tag{2.1}$$

$$v(t) = L\varphi(t), \quad -1 \leq t \leq 0. \tag{2.2}$$

Taking into account that $X = N(T) \oplus R(T)$, in its turn (2.1), (2.2) split into

$$\frac{d}{dt}(\tilde{T}(1-P)v(t)) = -(1-P)v(t) + (1-P)L_1L^{-1}v(t-1), \quad t \geq 0, \tag{2.3}$$

$$Pv(t) = PL_1L^{-1}v(t-1), \quad t \geq 0, \tag{2.4}$$

$$(1-P)v(t) = (1-P)L\varphi(t), \quad -1 \leq t \leq 0, \tag{2.5}$$

$$Pv(t) = PL\varphi(t), \quad -1 \leq t \leq 0. \tag{2.6}$$

We have

Theorem 1. *If $z = 0$ is a simple pole of $(z+T)^{-1}$, then for all $\varphi \in C([-1,0]; D(L))$ satisfying the compatibility relation*

$$L\varphi(0) - L_1\varphi(-1) \in R(T) \tag{2.7}$$

problem (1.1), (1.2) *has a* (*necessarily unique*) *strict solution u.*

Proof. First of all we observe that (2.4), (2.6) for $t = 0$ and (2.2) for $t = -1$ furnish $PL\varphi(0) = PL_1\varphi(-1)$, that is precisely (2.7).

Let $0 \leq t \leq 1$. Then $Pv(t) = PL_1\varphi(t-1)$ and (2.3), together with (2.5) at $t = 0$, becomes

$$\begin{aligned} &\frac{d}{dt}((1-P)v(t)) = -\tilde{T}^{-1}(1-P)v(t) + \tilde{T}^{-1}(1-P)L_1\varphi(t-1), \quad 0 \leq t \leq 1, \\ &\quad (1-P)v(0) = (1-P)L\varphi(0). \end{aligned}$$

Then, if $v_0(\cdot)$ denotes the solution to (2.3)–(2.6) on the interval $[0,1]$, we have

$$v_0(t) = PL_1\varphi(t-1) + e^{-t\tilde{T}^{-1}}(1-P)L\varphi(0) + \int_0^t e^{-(t-s)\tilde{T}^{-1}}\tilde{T}^{-1}(1-P)L_1\varphi(s-1)ds,$$

and correspondingly

$$u_0(t) = L^{-1}v_0(t) = \begin{cases} L^{-1}PL_1\varphi(t-1) + L^{-1}e^{-t\tilde{T}^{-1}}(1-P)L\varphi(0) \\ \quad + L^{-1}\int_0^t e^{-(t-s)\tilde{T}^{-1}}\tilde{T}^{-1}(1-P)L_1\varphi(s-1)ds, \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 < t \leq 1, \\ \varphi(t) \quad -1 \leq t \leq 0, \end{cases}$$

satisfies, in fact, for $0 < t \leq 1$

$$\begin{aligned} &Mu_0(t) = \tilde{T}e^{-t\tilde{T}^{-1}}(1-P)L\varphi(0) + \int_0^t e^{-(t-s)\tilde{T}^{-1}}(1-P)L_1\varphi(s-1)ds, \\ &\frac{d}{dt}(Mu_0(t)) = -e^{-t\tilde{T}^{-1}}(1-P)L\varphi(0) \\ &\quad + (1-P)L_1\varphi(t-1) - \int_0^t e^{-(t-s)\tilde{T}^{-1}}\tilde{T}^{-1}(1-P)L_1\varphi(s-1)ds \\ &\quad = (1-P)L_1\varphi(t-1) - Lu_0(t) + PL_1\varphi(t-1) = -Lu_0(t) + L_1\varphi(t-1) \\ &\quad = -Lu_0(t) + L_1u_0(t-1), \quad 0 \leq t \leq 1. \end{aligned}$$

Note that in virtue of (2.7)

$$Lu_0(t) \to PL_1\varphi(-1) + (1-P)L\varphi(0) = PL\varphi(0) + (1-P)L\varphi(0) = L\varphi(0) \text{ as } t \to 0+,$$

i.e. that the function u_0 on $[-1,1]$ is an element of $C([-1,1];D(L))$ and satisfies (1.1) on the interval $[0,1]$. In general, u_0 is not differentiable.

Let $1 \le t \le 2$ and define

$$\begin{aligned} v_1(t) :&= PL_1L^{-1}v_0(t-1) + e^{-(t-1)\tilde{T}^{-1}}(1-P)v_0(1) \\ &+ \int_1^t \tilde{T}^{-1}e^{-(t-s)\tilde{T}^{-1}}(1-P)L_1L^{-1}v_0(s-1)ds, \quad 1 \le t \le 2. \end{aligned}$$

Then

$$Tv_1(t) = \tilde{T}e^{-(t-1)\tilde{T}^{-1}}(1-P)v_0(1) + \int_1^t e^{-(t-s)\tilde{T}^{-1}}(1-P)L_1L^{-1}v_0(s-1)ds$$

yields for any $t \in [1,2]$

$$\begin{aligned} \frac{d}{dt}(Tv_1(t)) &= -e^{-(t-1)\tilde{T}^{-1}}(1-P)v_0(1) + (1-P)L_1L^{-1}v_0(t-1) \\ &- \int_1^t \tilde{T}^{-1}e^{-(t-s)\tilde{T}^{-1}}(1-P)L_1L^{-1}v_0(s-1)ds \\ &= -v_1(t) + PL_1L^{-1}v_0(t-1) + (1-P)L_1L^{-1}v_0(t-1) \\ &= -v_1(t) + L_1L^{-1}v_0(t-1); \end{aligned}$$

in particular,

$$\begin{aligned} \frac{d}{dt}Tv_1(1) &= -v_1(1) + L_1L^{-1}v_0(0) = -v_1(1) + L_1u_0(0) \\ &= -PL_1L^{-1}v_0(0) - (1-P)v_0(1) + L_1u_0(0) \\ &= -PL_1\varphi(0) - v_0(1) + PL_1\varphi(0) + L_1u_0(0) \text{ (by the definition of } v_0(\cdot)) \\ &= -v_0(1) + L_1u_0(0) = -Lu_0(1) + L_1\varphi(0) = \frac{d}{dt}Mu_0(1) = \frac{d}{dt}Tv_0(1). \end{aligned}$$

Moreover,

$$v_1(1) = PL_1L^{-1}v_0(0) + (1-P)v_0(1) = Pv_0(1) + (1-P)v_0(1) = v_0(1),$$

in view of the relation

$$\begin{aligned} Pv_0(1) &- PL_1L^{-1}v_0(0) = Pv_0(1) - PL_1u_0(0) \\ &= PL_1\varphi(0) - PL_1L^{-1}PL_1\varphi(-1) - PL_1L^{-1}(1-P)L\varphi(0) = \text{(by (2.7))} \\ &= PL_1\varphi(0) - PL_1L^{-1}PL\varphi(0) - PL_1L^{-1}(1-P)L\varphi(0) \\ &= PL_1\varphi(0) - PL_1\varphi(0) = 0. \end{aligned} \tag{2.8}$$

The relation (2.8) implies the compatibility condition in the interval $[0, 1]$:

$$PLu_0(1) = PL_1u_0(0).$$

On the interval $[n, n+1]$ we are allowed to define a function $v_n(\cdot)$ by means of

$$\begin{aligned} v_n(t) &:= PL_1L^{-1}v_{n-1}(t-1) + e^{-(t-n)\tilde{T}^{-1}}(1-P)v_{n-1}(n) \\ &+ \int_n^t \tilde{T}^{-1}e^{-(t-s)\tilde{T}^{-1}}(1-P)L_1L^{-1}v_{n-1}(s-1)ds, \ n \le t \le n+1, \ n \ge 2, \end{aligned}$$

and to deduce that

$$\begin{aligned} &\frac{d}{dt}(Tv_n(t)) = -v_n(t) + L_1L^{-1}v_{n-1}(t-1), \quad n \le t \le n+1, \\ &v_n(n) = PL_1L^{-1}v_{n-1}(n-1) + (1-P)v_{n-1}(n) \\ &\qquad = Pv_{n-1}(n) + (1-P)v_{n-1}(n) \text{ (by induction)} \\ &\qquad = v_{n-1}(n). \end{aligned}$$

Therefore, the function $v(\cdot)$ given by

$$v(t) = v_n(t), \text{ if } t \in [n+, n+1], \ n = -1, 0, 1, \ldots$$

is the (necessarily) unique strict solution to (2.1), (2.2), as claimed.

Theorem 1 has the following corollary extending finite-dimensional theory (R.F. Curtain and H.J. Zwart [3], Corollary 2.4.2, p. 54).

Corollary 1. *Under the above assumptions, if $\varphi \in C([-1, 0]; D(L))$ satisfies* (2.7), *then for arbitrary $0 \le t_0 < t$ the solution $u(\cdot)$ to* (1.1),(1.2) *satisfies*

$$\begin{aligned} u(t) &= L^{-1}PL_1u(t-1) + L^{-1}e^{-(t-t_0)\tilde{T}^{-1}}(1-P)Lu(t_0) \\ &+ L^{-1}\int_{t_0}^t \tilde{T}^{-1}e^{-(t-s)\tilde{T}^{-1}}(1-P)L_1u(s-1)ds. \end{aligned}$$

Proof. We have

$$\begin{aligned} Lu(t) &= PL_1u(t-1) + e^{-(t-t_0)\tilde{T}^{-1}}(1-P)Lu(t_0) \\ &+ \int_{t_0}^t \tilde{T}^{-1}e^{-(t-s)\tilde{T}^{-1}}(1-P)L_1u(s-1)ds \end{aligned}$$

and

$$Mu(t) = \tilde{T}e^{-(t-t_0)\tilde{T}^{-1}}(1-P)Lu(t_0) + \int_{t_0}^t e^{-(t-s)\tilde{T}^{-1}}(1-P)L_1u(s-1)ds,$$

so that

$$\begin{aligned}\frac{d}{dt}(Mu(t)) &= e^{-(t-t_0)\tilde{T}^{-1}}(1-P)Lu(t_0) + (1-P)L_1u(t-1) \\ &\quad - \int_{t_0}^{t} \tilde{T}^{-1}e^{-(t-s)\tilde{T}^{-1}}(1-P)L_1u(s-1)ds \\ &= e^{-(t-t_0)\tilde{T}^{-1}}(1-P)Lu(t_0) - PL_1u(t-1) \\ &\quad - \int_{t_0}^{t} \tilde{T}^{-1}e^{-(t-s)\tilde{T}^{-1}}(1-P)L_1u(s-1)ds + L_1u(t-1),\end{aligned}$$

and the affirmation is proven.

We have previously observed that our existence result guarantees no differentiability property to the solution u. It only suffices to consider the trivial example

$$u(t) = u(t-1) - v(t-1), \quad t \geq 0, \tag{2.9}$$

$$\dot{v}(t) = \frac{d}{dt}v(t) = -v(t) + u(t-1), \quad t \geq 0, \tag{2.10}$$

$$u(t) = \varphi_1(t), \quad -1 \leq t \leq 0, \tag{2.11}$$

$$v(t) = \varphi_2(t), \quad -1 \leq t \leq 0, \tag{2.12}$$

where $\varphi_i \in C([0,1]), i = 1,2$. Then Theorem 1 applies with condition (2.7) reducing to $\varphi_1(0) = \varphi_1(-1) - \varphi_2(-1)$. On the other hand, if we take $\varphi_1(t) = \varphi_2(t) = t, -1 \leq t \leq 0$, the corresponding solution $(u(t), v(t))$ is not differentiable on $[0,\infty)$, for $u(t) \equiv 0, v(t) = t - 2 + 2e^{-t}, 0 \leq t \leq 1$ and $u(t) = 3 - t - 2e^{1-t}, v(t) = (2/e-1)e^{1-t}, 1 \leq t \leq 2$, and therefore $u'_-(1) = 0 \neq u'_+(1) = 1$.

Next result provides a condition sufficient to ensure that the strict solution u to (1.1), (1.2) is differentiable on $[0,\infty)$ and satisfies

$$M\frac{du(t)}{dt} = -Lu(t) + L_1u(t-1), \quad t \geq 0. \tag{2.13}$$

Theorem 2. *Under the assumptions above for M, L, L_1, if $\varphi \in C^1([-1,0]; D(L))$ satisfies the compatibility relations* (2.7)*,*

$$L\varphi'(0) - L_1\varphi'(-1) \in R(T), \tag{2.14}$$

$$M\varphi'(0) = -L\varphi(0) + L_1\varphi(-1), \tag{2.15}$$

then the problem (2.13), (1.2) *has a unique strict solution and this solution is differentiable.*

Proof. Let $v(\cdot)$ be the solution of the initial value problem

$$\frac{d}{dt}(Mv(t)) = -Lv(t) + L_1v(t-1), \quad t \geq 0, \tag{2.16}$$

$$v(t) = \varphi'(t), \quad -1 \leq t \leq 0. \tag{2.17}$$

The existence of the solution $v(\cdot)$ is guaranteed by Theorem 1 and we have $v \in C([-1,\infty);D(L))$ and $Mv \in C^1([0,\infty);X)$. If we set

$$u(t) = \begin{cases} \int_0^t v(s)ds + \varphi(0) & t > 0, \\ \varphi(t) & -1 \le t < 0. \end{cases} \tag{2.18}$$

Integrating both sides of (2.16) yields

$$Mv(t) - Mv(0) = -L\int_0^t v(s)ds + L_1\int_0^t v(s-1)ds, \quad t \ge 0. \tag{2.19}$$

Since $\frac{d}{dt}(Mu(t)) = Mv(t) = M\frac{du(t)}{dt}$, $v(0) = \varphi'(0)$, it follows from (2.19) that

$$\frac{d}{dt}(Mu(t)) = -Lu(t) + L\varphi(0) + M\varphi'(0) + L_1\int_0^t v(s-1)ds. \tag{2.20}$$

If $0 \le t < 1$, we have

$$\begin{aligned} L_1\int_0^t v(s-1)ds &= L_1\int_0^t \varphi'(s-1)ds \\ &= L_1(\varphi(t-1) - \varphi(-1)) = L_1(u(t-1) - \varphi(-1)), \end{aligned}$$

and hence with the aid of (2.15), (2.20) we obtain (1.1). Similarly if $t > 1$,

$$\begin{aligned} L_1\int_0^t v(s-1)ds &= L_1\int_0^1 v(s-1)ds + L_1\int_1^t v(s-1)ds \\ &= L_1\int_0^1 \varphi'(s-1)ds + L_1\int_1^{t-1} v(s)ds \\ &= L_1(\varphi(0) - \varphi(-1)) + L_1(u(t-1) - \varphi(0)) \\ &= L_1u(t-1) - L_1\varphi(-1), \end{aligned}$$

and we conclude (1.1), and the proof is complete.

Let us observe that in the previous example (2.9)–(2.12) with

$$\varphi_1(0) = \varphi_1(-1) - \varphi_2(-1),$$

the further assumptions that

$$\varphi_1'(0) = \varphi_1'(-1) - \varphi_2'(-1), \quad \varphi_2'(0) = -\varphi_2(0) + \varphi_1(-1)$$

guarantee that for all $t \ge 0$ the derivative $\dot{u}(t)$ exists. For instance,

$$\varphi_1(t) = At, \quad \varphi_2(t) = -A$$

are consistent initial values, for

$$u(t) = At, \qquad v(t) = Ae^{-t} + At - 2A, \quad 0 \le t \le 1,$$
$$u(t) = 2A - Ae^{1-t}, \quad v(t) = Ae^{-t} + At - 2A, \quad 1 \le t \le 2,$$

etc. are functions differentiable on $[0, \infty)$.

3. The abstract potential operator case. In this section we shall assume that X is a reflexive Banach space, but we shall weaken the pole hypothesis of Section 2 to the following one: there exist constants $k, \delta > 0$ such that

$$\|L(zM + L)^{-1}\|_{\mathcal{L}(X)} \le k, \quad \forall \mathrm{Re} z \ge -\delta, \tag{3.1}$$

or, equivalently, with $T = ML^{-1}$ as above

$$\|(zT + I)^{-1}\|_{\mathcal{L}(X)} \le k, \quad \forall \mathrm{Re} z \ge -\delta.$$

It is a consequence of the ergodic theorem that $X = N(T) \oplus R(T)^a$ and that the restriction $\tilde{T}$ of T to $R(T)^a$ is an abstract potential operator, according to which $-\tilde{T}$ is the generator of an analytic semigroup in the space $R(T)^a$.

Again we shall use the notation P to denote the projection operator onto $N(T)$. We refer to A. Favini [4] for further details; see A. Favini and A. Yagi [6], too.

In what follows we shall need the real interpolation spaces $(X_1, X_2)_{\theta,\infty}, 0 < \theta < 1$. See A. Lunardi [10], where $X_1 = R(T)^a, X_2 = D(\tilde{T}^{-1}) = R(T)$.

It is well known (A. Lunardi [10]) that $(X, D(\tilde{T}^{-1}))_{\theta,\infty} = (R(T)^a, R(T))_{\theta,\infty}$, and it is proven in A. Favini and A. Yagi [6] that this space in fact coincides with

$$\{a \in X | \sup_{t>0} t^\theta \|L(tM + L)^{-1} a\|_X < \infty\} = D_{-\tilde{T}^{-1}}(\theta, \infty).$$

Notice that

$$\begin{aligned} L(tM + L)^{-1} a &= (tT + 1)^{-1}\{Pa + (1 - P)a\} \\ &= Pa + \tilde{T}^{-1}(t + \tilde{T}^{-1})^{-1}(1 - P)a \end{aligned}$$

says that necessarily an element $a \in X$ with the property

$$\sup_{t>0} t^\theta \|L(tM + L)^{-1} a\|_X < \infty,$$

belongs to $R(T)^a$, so that $a = (1 - P)a$.

Concerning the initial values $\varphi(t)$ we shall suppose that

$$\varphi \in C^\theta([-1, 0]; D(L)), \; L\varphi(0) - L_1\varphi(-1) \in R(T)^a, \tag{3.2}$$
$$L\varphi(0) - L_1\varphi(-1) \in D_{-\tilde{T}^{-1}}(\theta, \infty). \tag{3.3}$$

Assumptions (3.2), (3.3) are motivated by the well known maximal regularity result according to which if $-A$ is a sectorial operator in $X, 0 < \theta < 1, u_0 \in D(A), f \in C^\theta([0,T];X), -Au_0 + f(0) \in D_{-A}(\theta,\infty)$, then the mild solution u:

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A} f(s)ds$$

of $u' = -Au + f, u(0) = u_0$, is in fact a strict solution on $[0,T]$ with $u', Au \in C^\theta([0,T];X), u'$ being strongly measurable and bounded from $[0,T]$ into $D_{-A}(\theta,\infty)$, see A. Lunardi [10], Theorem 4.3.1, p. 134. After these preliminaries we can establish next result as follows.

Theorem 3. *Let X be a reflexive Banach space and let L, L_1, M be closed linear operators from X into itself such that $D(L) \subset D(M) \cap D(L_1), 0 \in \rho(L)$ and (3.1) holds. Assume that φ fulfills (3.2) ,(3.3). Then problem (1.1), (1.2) has a unique strict solution u and $\frac{d}{dt}(Mu) \in C^\theta([0,\tau];X)$ for all $\tau > 0$.*

Proof. Let us introduce functions $v_n(\cdot), n = -1, 0, 1, \dots$, by means of

$$\begin{aligned} v_{-1}(t) &= L\varphi(t), \quad -1 \le t \le 0, \\ v_n(t) &= PL_1L^{-1}v_{n-1}(t-1) + e^{-(t-n)\tilde{T}^{-1}}(1-P)v_{n-1}(n) \\ &+ \int_n^t \tilde{T}^{-1}e^{-(t-s)\tilde{T}^{-1}}(1-P)L_1L^{-1}v_{n-1}(s-1)ds, \\ &\qquad n = 0, 1, \dots ,\ t \in [n, n+1], \end{aligned}$$

as in the proof of Theorem 1, but in the present case more caution must be taken because $\tilde{T}^{-1}$ is not bounded from $R(T)^a$ into itself.

The assumptions imply that $v_{-1} \in C^\theta([-1,0];X)$. Moreover, if

$$z_0(t) = \tilde{T}e^{-t\tilde{T}^{-1}}(1-P)L\varphi(0) + \int_0^t e^{-(t-s)\tilde{T}^{-1}}(1-P)L_1\varphi(s-1)ds \equiv \tilde{T}(1-P)v_0(t),$$

since $L_1\varphi \in C^\theta([-1,0];X)$ and

$$\begin{aligned} &-\tilde{T}^{-1}(\tilde{T}(1-P)L\varphi(0)) + (1-P)L_1\varphi(-1) = -(1-P)[L\varphi(0) - L_1\varphi(-1)] \\ &\quad = -[L\varphi(0) - L_1\varphi(-1)](\text{by } (3.3)) \in D_{-\tilde{T}^{-1}}(\theta,\infty), \end{aligned}$$

the above mentioned maximal regularity theorem says that for $0 \le t \le 1$

$$\begin{aligned} \frac{d}{dt}z_0(t) &= -\tilde{T}^{-1}z_0(t) + (1-P)L_1\varphi(t-1) \\ &= -\tilde{T}^{-1}z_0(t) + (1-P)L_1L^{-1}v_{-1}(t-1) \\ &= -(1-P)v_0(t) + (1-P)L_1L^{-1}v_{-1}(t-1), \end{aligned}$$

and $z_0'(\cdot), \tilde{T}^{-1}z_0(\cdot) = (1-P)v_0(\cdot) \in C^\theta([0,1];X)$. On the other hand, we have

$$\begin{aligned} Tv_0(t) &= z_0(t), \quad 0 \le t \le 1, \\ Pv_0(t) &= PL_1\varphi(t-1) = PL_1L^{-1}v_{-1}(t-1), \quad 0 \le t \le 1, \\ v_0(0) &= PL_1\varphi(-1) + (1-P)L\varphi(0) = PL\varphi(0) + (1-P)L\varphi(0) = L\varphi(0). \end{aligned} \tag{3.4}$$

Therefore, v_0 satisfies

$$\frac{d}{dt}(Tv_0(t)) = -v_0(t) + L_1L^{-1}v_{-1}(t-1), \quad 0 \le t \le 1,$$

and $v_0(0) = L\varphi(0) = v_{-1}(0)$. Furthermore,

$$\begin{aligned} \frac{dz_0}{dt}(1) &= -\tilde{T}^{-1}z_0(1) + (1-P)L_1\varphi(0) \\ &= -(1-P)v_0(1) + (1-P)L_1\varphi(0) \\ &= (1-P)[L_1\varphi(0) - v_0(1)] \in D_{-\tilde{T}^{-1}}(\theta,\infty) \end{aligned}$$

implies that if $u_0(t) = L^{-1}v_0(t)$, then noting $Pv_0(1) = PL_1L^{-1}v_{-1}(0) = PL_1\varphi(0)$, we have

$$\begin{aligned} Lu_0(1) - L_1u_0(0) &= Lu_0(1) - L_1\varphi(0) \\ &= (1-P)v_0(1) + Pv_0(1) - (1-P)L_1\varphi(0) - PL_1\varphi(0) \\ &= -(1-P)[L_1\varphi(0) - v_0(1)] + P[v_0(1) - L_1\varphi(0)] \\ &= -(1-P)[L_1\varphi(0) - v_0(1)] \in D_{-\tilde{T}^{-1}}(\theta,\infty). \end{aligned}$$

Hence we are in the same situation on the interval [1, 2]. Indeed

$$\begin{aligned} v_1(t) =& PL_1L^{-1}v_0(t-1) + e^{-(t-1)\tilde{T}^{-1}}(1-P)v_0(1) \\ &+ \int_1^t \tilde{T}^{-1}e^{-(t-s)\tilde{T}^{-1}}(1-P)L_1L^{-1}v_0(s-1)ds \end{aligned}$$

leads to define

$$\begin{aligned} &z_1(t) : (= \tilde{T}(1-P)v_1(t) = Tv_1(t)) \\ &\quad = e^{-(t-1)\tilde{T}^{-1}}\tilde{T}^{-1}(1-P)v_0(1) + \int_1^t e^{-(t-s)\tilde{T}^{-1}}(1-P)L_1L^{-1}v_0(s-1)ds. \end{aligned}$$

Then $v_0(t) = PL_1\varphi(t-1) + \tilde{T}^{-1}z_0(t) \in C^\theta([0,1];X)$ implies that

$$\begin{aligned} \frac{dz_1(t)}{dt} &= -\tilde{T}^{-1}z_1(t) + (1-P)L_1L^{-1}v_0(t-1), \quad 1 \le t \le 2, \\ z_1(1) &= \tilde{T}(1-P)v_0(1) \end{aligned} \tag{3.5}$$

in view of

$$\begin{aligned}&-(1-P)v_0(1)+(1-P)L_1L^{-1}v_0(0)\\&\qquad=-(1-P)[v_0(1)-L_1\varphi(0)]=\frac{dz_0}{dt}(1)\in D_{-\tilde{T}^{-1}}(\theta,\infty).\end{aligned}$$

This property guarantees that

$$\frac{dz_1}{dt}(2)=-\tilde{T}^{-1}z_1(2)+(1-P)L_1L^{-1}v_0(1)\in D_{-\tilde{T}^{-1}}(\theta,\infty),$$

too.

With the aid of (3.5) and (3.4) one verifies that

$$\begin{aligned}\frac{d}{dt}(Tv_1(t))&=\frac{dz_1(t)}{dt}=-(1-P)v_1(t)+(1-P)L_1L^{-1}v_0(t-1)\\&=-v_1(t)+PL_1L^{-1}v_0(t-1)+(1-P)L_1L^{-1}v_0(t-1)\\&=-v_1(t)+L_1L^{-1}v_0(t-1),\quad 1\le t\le 2,\\v_1(1)&=PL_1L^{-1}v_0(0)+(1-P)v_0(1)=PL_1\varphi(0)+(1-P)v_0(1)\\&=PL_1L^{-1}v_{-1}(0)+(1-P)v_0(1)=Pv_0(1)+(1-P)v_0(1)=v_0(1).\end{aligned}$$

If we set $u_1(t)=L^{-1}v_1(t)$, then

$$\begin{aligned}&P[Lu_1(2)-L_1u_1(1)]\\&\qquad=P[v_1(2)-L_1L^{-1}v_1(1)]=PL_1L^{-1}v_0(1)-PL_1L^{-1}v_1(1)=0,\end{aligned}$$

and hence

$$\begin{aligned}Lu_1(2)&-L_1u_1(1)=(1-P)[Lu_1(2)-L_1u_1(1)]\\&=(1-P)[v_1(2)-L_1L^{-1}v_1(1)]=\tilde{T}^{-1}z_1(2)-(1-P)L_1L^{-1}v_0(1)\\&=-\frac{dz_1}{dt}(2)\in D_{-\tilde{T}^{-1}}(\theta,\infty).\end{aligned}$$

Since the procedure to obtain v_n from v_{n-1} is the same for all $n\in\mathbf{N}$, this proves our affirmation. It suffices to define $v(t)=v_n(t), n\le t\le n+1, u(t)=L^{-1}v(t)$.

Of course, it may be a difficult matter to characterize the projection P and the interpolation spaces $D_{-\tilde{T}^{-1}}(\theta,\infty)$. A condition sufficient for (3.3) reads $L\varphi(0)-L_1\varphi(-1)=Mx$ for a certain $x\in D(L)$, for it says us that $L\varphi(0)-L_1\varphi(-1)\in R(T)=D(\tilde{T}^{-1})$.

4. Representation formula. In this section we give a variant of variation of constants formula. Related results are found in J.K. Hale and P. Martinez-Amores [8] and P. Martinez-Amores [11].

Theorem 4. *Under the assumptions as in Theorem* 1*, there exists a function* $Y(t)$ *with values in* $\mathcal{L}(X)$ *such that* $Y(t)T$ *is strongly continuously differentiable in* $[0,\infty)$ *and satisfying*

$$\frac{d}{dt}Y(t)M = -Y(t)L + Y(t-1)L_1, \quad 0 \le t < \infty, \text{ on } D(L), \tag{4.1}$$

$$Y(0)M = M \text{ on } D(L), \quad Y(t) = 0, \quad t < 0. \tag{4.2}$$

If $u(\cdot) \in C([-1,T];X)$ *is a function satisfying*

$$\frac{d}{dt}(Mu(t)) = -Lu(t) + L_1 u(t-1) + f(t), \quad 0 < t \le T, \tag{4.3}$$

where $f \in C([0,T];X)$*, then we have*

$$Mu(t) = Y(t)Mu(0) + \int_0^1 Y(t-s)L_1 u(s-1)ds + \int_0^t Y(t-s)f(s)ds, \quad 0 \le t \le T. \tag{4.4}$$

Remark 1. The existence of the solution of the inhomogeneous equation (4.3) is shown by the same method as the proof of Theorem 1.

Proof. Suppose that we have an operator valued function $Y(\cdot)$ satisfying (4.1), (4.2). Set $H = L_1 L^{-1}$ for simplicity. Then for a function $v = Lu$ with $u \in C([-1,T];D(L))$ satisfying (4.3) we have for $0 < s < t$

$$\begin{aligned}
&\frac{d}{ds}(Y(t-s)Tv(s)) \\
&\quad = -(-Y(t-s) + Y(t-s-1)H)v(s) + Y(t-s)(-v(s) + Hv(s-1) + f(s)) \\
&\quad = -Y(t-s-1)Hv(s) + Y(t-s)Hv(s-1) + Y(t-s)f(s).
\end{aligned}$$

Integrating both sides from 0 to t

$$\begin{aligned}
&Y(0)Tv(t) - Y(t)Tv(0) \\
&\quad = -\int_0^t Y(t-s-1)Hv(s)ds + \int_0^t Y(t-s)Hv(s-1)ds + \int_0^t Y(t-s)f(s)ds \\
&\quad = -\int_1^{t+1} Y(t-s)Hv(s-1)ds + \int_0^t Y(t-s)Hv(s-1)ds + \int_0^t Y(t-s)f(s)ds \\
&\quad = \int_0^1 Y(t-s)Hv(s-1)ds + \int_0^t Y(t-s)f(s)ds
\end{aligned}$$

since $Y(t) = 0$ for $t < 0$. Thus we conclude that (4.4) holds.

Now we construct $Y(\cdot)$. We write $x = \begin{pmatrix} (I-P)x \\ Px \end{pmatrix} \in X = R(I-P) \oplus R(P)$ and $Y(t) = \begin{pmatrix} Y_{11}(t) & Y_{12}(t) \\ Y_{21}(t) & Y_{22}(t) \end{pmatrix}$, where

$$\begin{aligned}
Y_{11}(t) &= (I-P)Y(t)|_{R(I-P)} \in \mathcal{L}(R(I-P), R(I-P)),\\
Y_{12}(t) &= (I-P)Y(t)|_{R(P)} \in \mathcal{L}(R(P), R(I-P)),\\
Y_{21}(t) &= PY(t)|_{R(I-P)} \in \mathcal{L}(R(I-P), R(P)),\\
Y_{22}(t) &= PY(t)|_{R(P)} \in \mathcal{L}(R(P), R(P)),
\end{aligned}$$

and similarly for $H(t)$. If we set $T_1 = T|_{R(I-P)}$, then $T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}$. The differential equation for Y represented componentwise is

$$\begin{aligned}
\frac{d}{dt} Y_{11}(t)T_1 &= -Y_{11}(t) + Y_{11}(t-1)H_{11} + Y_{12}(t-1)H_{21}\\
\frac{d}{dt} Y_{21}(t)T_1 &= -Y_{21}(t) + Y_{21}(t-1)H_{11} + Y_{22}(t-1)H_{21}\\
0 &= -Y_{12}(t) + Y_{11}(t-1)H_{12} + Y_{12}(t-1)H_{22}\\
0 &= -Y_{22}(t) + Y_{21}(t-1)H_{12} + Y_{22}(t-1)H_{22},
\end{aligned}$$

and the initial condition is $Y_{11}(0) = I, Y_{21}(0) = 0$. In the interval $0 < t < 1$ we have

$$\begin{aligned}
\frac{d}{dt} Y_{11}(t)T_1 &= -Y_{11}(t), \quad Y_{11}(0) = I,\\
\frac{d}{dt} Y_{21}(t)T_1 &= -Y_{21}(t), \quad Y_{21}(0) = 0,\\
0 &= -Y_{12}(t), \qquad 0 = -Y_{22}(t).
\end{aligned}$$

Therefore, one has $Y_{21}(t) = 0, Y_{12}(t) = 0, Y_{22}(t) = 0$. In the interval $1 < t < 2$ one has

$$\begin{aligned}
&\frac{d}{dt} Y_{11}(t)T_1 = -Y_{11}(t) + Y_{11}(t-1)H_{11}, \quad Y_{11}(1+0) = Y_{11}(1-0),\\
&\frac{d}{dt} Y_{21}(t)T_1 = -Y_{21}(t), \qquad Y_{21}(1+0) = 0,\\
&0 = -Y_{12}(t) + Y_{11}(t-1)H_{12}, \qquad 0 = -Y_{22}(t).
\end{aligned}$$

Hence $Y_{21}(t) = 0, Y_{12}(t) = Y_{11}(t-1)H_{12}, Y_{22}(t) = 0$. Continuing this way we obtain that for $2 \le n < t < n+1$

$$\frac{d}{dt} Y_{11}(t)T_1 = -Y_{11}(t) + Y_{11}(t-1)H_{11} + \sum_{k=2}^{n} Y_{11}(t-k)H_{12}H_{22}^{k-2}H_{21},$$

$$Y_{12}(t) = \sum_{k=1}^{n} Y_{11}(t-k)H_{12}H_{22}^{k-1}, \quad Y_{21}(t) = 0, \quad Y_{22}(t) = 0.$$

It is easy to show that the above obtained operator valued function $Y(\cdot)$ satisfies (4.1), (4.2).

Remark 2. $Y(\cdot)$ itself is not continuous at $t = n$ in general.

Remark 3. In the above we chose $Y(0) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ as the initial condition to be satisfied by $Y(\cdot)$. In the regular case only one choice $Y(0) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$ is possible. In degenerate cases, however, the choice of the special $Y(\cdot)$ will depend on the practical methods that are chosen to solve the problem. See (19) in L. Pandolfi [12] for a different choice.

5. Applications.

Application 1. In several concrete examples of partial differential equations the operators L and M commute according to $L^{-1}(M+\lambda_0)^{-1} = (M+\lambda_0)^{-1}L^{-1}$ for some $\lambda_0 \in \rho(-M), \lambda = 0$ is a simple pole of the resolvent $(z-M)^{-1}$ and $D(L) = D(M)$. Then our operator $T(= ML^{-1})$ has the properties $N(T) = N(M), R(T) = R(M)$ and thus $X = N(M) \oplus R(M) = N(T) \oplus R(T)$.

It is an easy matter to recognize this and we confirme to prove that $N(M) = N(T)$. Indeed, if $x \in N(M)$, then $Tx = (M + \lambda_0 - \lambda_0)L^{-1}x = L^{-1}(M+\lambda_0)x - \lambda_0 L^{-1}x = 0$. Conversely, if $Tx = 0$, then

$$(M+\lambda_0)L^{-1}x = \lambda_0 L^{-1}x$$

and so

$$L^{-1}x = \lambda_0(M+\lambda_0)^{-1}L^{-1}x = \lambda_0 L^{-1}(M+\lambda_0)^{-1}x,$$

that is, $x = \lambda_0(M+\lambda_0)^{-1}x$ or $Mx = 0$, as claimed.

Let us give a simple one-dimensional (in space) example of this situation. Let us consider the delay equation

$$\left(\frac{\partial^2}{\partial x^2}+1\right)\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + a_0(x)\frac{\partial^2 u}{\partial x^2}(t-1,x), \quad t \geq 0, \quad 0 \leq x \leq \pi,$$

with boundary conditions

$$u(t,0) = u(t,\pi) = \frac{\partial^2 u}{\partial x^2}(t,0) = \frac{\partial^2 u}{\partial x^2}(t,\pi) = 0$$

and initial condition

$$u(t,x) = \varphi(t,x), \quad -1 \leq t \leq 0, \quad 0 \leq x \leq \pi,$$

where $a_0(\cdot) \in C[0,\pi], \varphi \in C([-1,0]; C^2[0,\pi])$ and

$$\varphi(t,0) = \varphi(t,\pi) = \frac{\partial^2 \varphi}{\partial x^2}(t,0) = \frac{\partial^2 \varphi}{\partial x^2}(t,\pi) = 0, \quad -1 \leq t \leq 0.$$

Take $X = \{f \in C[0,\pi]; f(0) = f(\pi) = 0\}$ endowed with the supremum norm and let L, L_1, M be defined by

$$\begin{aligned} D(L) &= \{u \in C^2[0,\pi]; u(0) = u(\pi) = u''(0) = u''(\pi) = 0\},\\ Lu &= -u'', \quad u \in D(L),\\ D(L_1) &= D(L), \quad L_1 u = a_0 u'', \quad u \in D(L_1),\\ D(M) &= D(L), \quad Mu = (-L+1)u, \quad \forall u \in D(M). \end{aligned}$$

As it is well known, 0 is a simple pole for $(z-M)^{-1}, L$ and M commute and the projection P is given by

$$Pf(x) := \frac{2}{\pi}\sin x \int_0^\pi f(y)\sin y dy, \quad f \in X.$$

In order to apply Theorem 2 we need to assume that $\varphi(t)$, where $\varphi(t)(x) := \varphi(t,x)$, has the regularity $\varphi \in C^1([-1,0]; D(L))$, and

$$\begin{aligned} &\frac{\partial^3\varphi}{\partial x^2\partial t}(0,x) + a_0(x)\frac{\partial^3\varphi}{\partial x^2\partial t}(-1,x)\\ &\quad = u_0''(x) + u_0(x),\ \forall x \in [0,\pi],\ \text{for some } u_0 \in D(L),\\ &\frac{\partial^2\varphi}{\partial x^2}(0,x) + a_0(x)\frac{\partial^2\varphi}{\partial x^2}(-1,x) = \frac{\partial^3\varphi}{\partial x^2\partial t}(0,x) + \frac{\partial\varphi}{\partial t}(0,x),\ \forall x \in [0,\pi]. \end{aligned}$$

Of course, different choices of the space X are allowed, too, for instance, $X = L^2(0,\pi)$ and $D(L) = D(L_1) = D(M)$ is the closure in the Sobolev space $H^2(0,\pi)$ of the class of functions f in $C^2[0,\pi]$ which satisfy $f(0) = f(\pi) = 0$. For more details to this regard see J. Lagnese [9], pp. 630–631.

Application 2. Let Ω be a bounded open set in R^n with a smooth boundary $\partial\Omega$ and let $m(\cdot)$ be ≥ 0 and $m \in L^\infty(\Omega)$ so that the multiplication operator by $m(\cdot)$, denoted by M, is bounded from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$. Let $L = -\Delta$ with Dirichlet boundary condition in $H^{-1}(\Omega)$, so that $D(L) = H_0^1(\Omega)$.

It is seen in Favini and Yagi [5, pp. 378–379] that condition (3.1) is satisfied in $X := H^{-1}(\Omega)$. Hence Theorem 3 allows to handle a typical problem like

$$\begin{aligned} &\frac{\partial}{\partial t}(m(x)u(t,x)) = \Delta u(t,x) + a_0(x)\Delta u(t-1,x), \quad t \geq 0,\ x \in \Omega,\\ &u(t,x) = 0, \quad x \in \partial\Omega,\ t \geq 0,\\ &u(t,x) = \varphi(t,x), \quad -1 \leq t \leq 0,\ x \in \Omega, \end{aligned}$$

when the function $\varphi(t)$ defined by $\varphi(t)(x) = \varphi(t,x)$ satisfies $\varphi \in C^\theta([-1,0]; H_0^1(\Omega)), \Delta\varphi(0,x) + a_0(x)\Delta\varphi(-1,x) = m(x)v(x), v(\cdot) \in H_0^1(\Omega)$, and $a_0(\cdot) \in C^1(\bar{\Omega})$.

Application 3. Let V, W, H be Hilbert spaces such that $V \subset W \subset H$ densely and continuously. The norm and inner product of H are denoted by $(\cdot,\cdot)$ and $|\cdot|$ respectively. Let $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ be continuous, positive definite symmetric sesquilinear forms defined on $V \times V$ and $W \times W$ respectively. The operators associated with them are denoted by A and B:

$$\begin{aligned} D(A) &= \{u \in V;\ v \mapsto a(u,v) \text{ is continuous in the topology of } V \text{ induced by } H\} \\ (Au,v) &= a(u,v) \text{ for } u \in D(A) \text{ and } v \in V, \\ D(B) &= \{u \in W;\ v \mapsto b(u,v) \text{ is continuous in the topology of } W \text{ induced by } H\} \\ (Bu,v) &= b(u,v) \text{ for } u \in D(B) \text{ and } v \in W. \end{aligned}$$

The operators A and B are positive definite selfadjoint in H. Suppose that $D(A) \subset D(B)$ and the operator BA^{-1} is completely continuous in H. We also assume that $0 \neq \lambda \in \sigma(BA^{-1})$. Let A_1 be a closed linear operator such that $D(A_1) \supset D(A)$. Under these assumptions we consider the equation

$$\frac{d}{dt}(B - \lambda A)u = -Au + A_1 u(t-1).$$

We are going to show that the operators $M = B - \lambda A, L = A$ and $L_1 = A_1$ satisfy our assumptions. Following W.M. Greenlee [7] we consider the operator $\mathcal{A}$ associated with the sesquilinear form $a(\cdot,\cdot)$ in the Hilbert space W which we endow with the inner product $b(\cdot,\cdot)$ here and in what follows:

$$\begin{aligned} D(\mathcal{A}) &= \{u \in V;\ v \mapsto a(u,v) \text{ is continuous in the topology of } V \text{ induced by } W\} \\ b(\mathcal{A}u,v) &= a(u,v) \text{ for } u \in D(\mathcal{A}) \text{ and } v \in V. \end{aligned}$$

Then $\mathcal{A}$ is a positive definite selfadjoint operator in W. It is easy to see that

$$A = B\mathcal{A}. \tag{5.1}$$

By the Riesz-Schauder theory 0 is an eigenvalue of $T = BA^{-1} - \lambda$ of finite multiplicity and a pole of $(z+T)^{-1}$, whose order we denote by n. Since $(z+T)^{-1}$ has the Laurent series expansion

$$(z+T)^{-1} = \sum_{k=0}^{\infty} A_k z^k + \sum_{k=1}^{\infty} (-1)^{k-1} T^{k-1} P z^{-k},$$

where

$$P = \frac{1}{2\pi i} \oint_{|\zeta|=\epsilon} (\zeta - T)^{-1} d\zeta,$$

ϵ being a positive number so small that 0 is the unique element of $\sigma(T)\cap\{\zeta;|\zeta|<\epsilon\}$, we have

$$T^n P = 0. \tag{5.2}$$

Suppose that $T^2u = 0$. In view of (5.1) we have

$$T^2 u = B(\mathcal{A}^{-1} - \lambda)^2 B^{-1} u = 0.$$

Hence we have $(\mathcal{A}^{-1} - \lambda)^2 B^{-1} u = 0$. Since $\mathcal{A}$ is selfadjoint, this implies $(\mathcal{A}^{-1} - \lambda)B^{-1}u = 0$ which in turn $Tu = 0$. Repeating this argument we see that $T^n u = 0$ implies $Tu = 0$. Combining this with (5.2) we conclude $TP = 0$, and hence that 0 is a simple pole of the resolvent $(z+T)^{-1}$.

The first example is as follows:

$$V = H_0^2(\Omega), \quad W = H_0^1(\Omega), \quad H = L^2(\Omega), \tag{5.3}$$

$$a(u,v) = \sum_{|\alpha|=2} \int_\Omega D^\alpha u \overline{D^\alpha v} dx, \quad b(u,v) = \sum_{|\alpha|=1} \int_\Omega D^\alpha u \overline{D^\alpha v} dx, \tag{5.4}$$

where Ω is a bounded domain in R^n with smooth boundary. The associated operators A and B are the realizations of Δ^2 and $-\Delta$ under the Dirichlet boundary condition in $L^2(\Omega)$ respectively. It is easy to show that the same result remains valid in the spaces $L^p(\Omega), 1 < p < \infty$, since we can show that 0 is a simple pole of the resolvent of T also in these spaces by reducing the problem to that in $L^2(\Omega)$ with the aid of Sobolev's imbedding theorem.

The second example is

$$a(u,v) = \int_\Omega \sum_{i,j=1}^n \frac{\partial u}{\partial x_i} \overline{\frac{\partial v}{\partial x_j}} dx, \; u, v \in V = H_0^1(\Omega),$$

$$b(u,v) = \int_\Omega m u \bar{v} dx, \; u, v \in W,$$

where m is a real valued measurable function such that $\operatorname{ess\,inf} m > 0$ and $W = \{u : \sqrt{m}u \in L^2(\Omega)\}$. Therefore, we are considering the operators $A = -\Delta|_{H^2(\Omega)\cap H_0^1(\Omega)}$ and $Bu = mu$. Let $2 < q < \infty$ be such that $\frac{1}{2} - \frac{2}{n} < \frac{1}{q}$. Then by Rellich's theorem the imbedding $H^2(\Omega) \subset L^q(\Omega)$ is compact, and

$$\|mu\|_2 \le \|m\|_{2q/(q-2)} \|u\|_q \le \|m\|_{2q/(q-2)} C \|u\|_{2,2},$$

where $\| \ \|_q$, and $\| \ \|_{2,2}$ are the norms of $L^q(\Omega)$ and $H^2(\Omega)$ respectively. Hence if $m \in L^{2q/(q-2)}(\Omega)$, we have

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega) \subset \{u; mu \in L^2(\Omega)\} = D(B).$$

From Heinz' inequality it follows that

$$V = D(A^{1/2}) \subset D(B^{1/2}) = W.$$

It is easy to show that V is dense in W. Hence all the assumptions are satisfied.

Remark 4. If the condition

$$D(\mathcal{A}) \subset D(B) \tag{5.5}$$

is satisfied, the compatibility relation (2.7) which is equivalent to

$$\mathcal{A}\varphi(0) - B^{-1}L_1\varphi(-1) \in (\mathcal{A}^{-1} - \lambda)D(B)$$

in the present case can be weakened to

$$\mathcal{A}\varphi(0) - B^{-1}L_1\varphi(-1) \in (\mathcal{A}^{-1} - \lambda)W \tag{5.6}$$

as will be shown below. In the case of the first example the condition (5.5) is satisfied since

$$D(\mathcal{A}) \subset V = H_0^2(\Omega) \subset H^2(\Omega) \cap H_0^1(\Omega) = D(B).$$

The initial value problem (1.1), (1.2) is rewritten as

$$\frac{d}{dt}(1 - \lambda\mathcal{A})u(t) + \mathcal{A}u(t) = B^{-1}L_1u(t-1), \tag{5.7}$$

$$u(t) = \varphi(t), \quad -1 \le t \le 0. \tag{5.8}$$

Let $\mathcal{P}$ be the orthogonal projection onto $N(\lambda^{-1} - \mathcal{A})$ (with respect to the inner product $b(\cdot,\cdot)$). Then noting

$$\mathcal{A}\mathcal{P} = \lambda^{-1}\mathcal{P} \tag{5.9}$$

one has

$$\frac{d}{dt}(1 - \lambda\mathcal{A})(1 - \mathcal{P})u(t) + \mathcal{A}(1 - \mathcal{P})u(t) = (1 - \mathcal{P})B^{-1}L_1u(t-1), \tag{5.10}$$

$$\mathcal{P}u(t) = \lambda\mathcal{P}B^{-1}L_1u(t-1), \tag{5.11}$$

$$(1 - \mathcal{P})u(t) = (1 - \mathcal{P})\varphi(t), \quad \mathcal{P}u(t) = \mathcal{P}\varphi(t), \quad -1 \le t \le 0. \tag{5.12}$$

Also noting (5.9) we see that the compatibility condition (5.6) is equivalent to

$$\mathcal{P}\varphi(0) = \lambda\mathcal{P}B^{-1}L_1\varphi(-1). \tag{5.13}$$

Let $\tilde{\mathcal{A}}$ be the restriction of $\mathcal{A}$ to $R(1-\mathcal{P})$, and $\tilde{\mathcal{A}}_\lambda = \tilde{\mathcal{A}}(1-\lambda\tilde{\mathcal{A}})^{-1}$. Then from (5.10) we get

$$\frac{d}{dt}(1-\mathcal{P})u(t) + \tilde{\mathcal{A}}_\lambda(1-\mathcal{P})u(t) = (1-\lambda\tilde{\mathcal{A}})^{-1}(1-\mathcal{P})B^{-1}L_1 u(t-1).$$

From this and (5.11), (5.12) it follows that for $0 < t < 1$

$$\begin{aligned} u(t) &= \lambda\mathcal{P}B^{-1}L_1\varphi(t-1) + e^{-t\tilde{\mathcal{A}}_\lambda}(1-\mathcal{P})\varphi(0) \\ &\quad + \int_0^t e^{-(t-s)\tilde{\mathcal{A}}_\lambda}(1-\lambda\tilde{\mathcal{A}})^{-1}(1-\mathcal{P})B^{-1}L_1\varphi(s-1)ds. \end{aligned} \tag{5.14}$$

In order to show that the right hand side of (5.14) and its derivative belong to $D(L)$ and $D(M)$ respectively we make the following observation. Let ψ be an arbitrary element of $D(B)$ and $v(t) = e^{-t\tilde{\mathcal{A}}_\lambda}(1-\mathcal{P})\psi$. Then

$$v'(t) = -\tilde{\mathcal{A}}_\lambda v(t) = -\frac{1}{\lambda}(1-\lambda\tilde{\mathcal{A}})^{-1}v(t) + \frac{1}{\lambda}v(t).$$

Integrating this we get

$$v(t) = e^{t/\lambda}(1-\mathcal{P})\psi - \frac{1}{\lambda}(1-\lambda\tilde{\mathcal{A}})^{-1}\int_0^t e^{(t-s)/\lambda}v(s)ds. \tag{5.15}$$

In view of the assumption (5.5) we have the relation $R(\mathcal{P}) \subset D(\mathcal{A}) \subset D(B)$. Hence we get from (5.15) that

$$Bv(t) = e^{t/\lambda}B(1-\mathcal{P})\psi - \frac{1}{\lambda}B(1-\lambda\tilde{\mathcal{A}})^{-1}\int_0^t e^{(t-s)/\lambda}v(s)ds.$$

Thus we obtain that

$$Be^{-t\tilde{\mathcal{A}}_\lambda}(1-\mathcal{P})B^{-1} \text{ is strongly continuous from } [0,\infty) \text{ to } \mathcal{L}(H). \tag{5.16}$$

Therefore, from (5.14) it follows that

$$\begin{aligned} Mu(t) &= B(1-\lambda\mathcal{A})u(t) = Be^{-t\tilde{\mathcal{A}}_\lambda}(1-\lambda\tilde{\mathcal{A}})(1-\mathcal{P})\varphi(0) \\ &\quad + \int_0^t Be^{-(t-s)\tilde{\mathcal{A}}_\lambda}(1-\mathcal{P})B^{-1}L_1\varphi(s-1)ds, \end{aligned} \tag{5.17}$$

$$\begin{aligned} \frac{d}{dt}Mu(t) &= -B\tilde{\mathcal{A}}_\lambda e^{-t\tilde{\mathcal{A}}_\lambda}(1-\lambda\tilde{\mathcal{A}})(1-\mathcal{P})\varphi(0) + B(1-\mathcal{P})B^{-1}L_1\varphi(t-1) \\ &\quad - \int_0^t B\tilde{\mathcal{A}}_\lambda e^{-(t-s)\tilde{\mathcal{A}}_\lambda}(1-\mathcal{P})B^{-1}L_1\varphi(s-1)ds. \end{aligned} \tag{5.18}$$

The second term of the right hand side of (5.18) is a continuous function of t with values in H since $B(1-\mathcal{P})B^{-1}$ is a bounded operator in H by the assumption (5.5). As is easily seen

$$\begin{aligned} &B\tilde{\mathcal{A}}_\lambda e^{-t\tilde{\mathcal{A}}_\lambda}(1-\lambda\tilde{\mathcal{A}})(1-\mathcal{P})\varphi(0) = Be^{-t\tilde{\mathcal{A}}_\lambda}\tilde{\mathcal{A}}(1-\mathcal{P})\varphi(0) \\ &\quad = Be^{-t\tilde{\mathcal{A}}_\lambda}(1-\mathcal{P})\mathcal{A}\varphi(0) = Be^{-t\tilde{\mathcal{A}}_\lambda}(1-\mathcal{P})B^{-1}L\varphi(0), \end{aligned} \tag{5.19}$$

$$\begin{aligned} &B\tilde{\mathcal{A}}_\lambda e^{-(t-s)\tilde{\mathcal{A}}_\lambda}(1-\mathcal{P})B^{-1} = B\tilde{\mathcal{A}}(1-\lambda\tilde{\mathcal{A}})^{-1}e^{-(t-s)\tilde{\mathcal{A}}_\lambda}(1-\mathcal{P})B^{-1} \\ &\quad = \frac{1}{\lambda}B(1-\lambda\tilde{\mathcal{A}})^{-1}e^{-(t-s)\tilde{\mathcal{A}}_\lambda}(1-\mathcal{P})B^{-1} - \frac{1}{\lambda}Be^{-(t-s)\tilde{\mathcal{A}}_\lambda}(1-\mathcal{P})B^{-1} \\ &\quad = \frac{1}{\lambda}Be^{-(t-s)\tilde{\mathcal{A}}_\lambda}(1-\mathcal{P})B^{-1}\cdot B(1-\lambda\tilde{\mathcal{A}})^{-1}(1-\mathcal{P})B^{-1} \\ &\quad - \frac{1}{\lambda}Be^{-(t-s)\tilde{\mathcal{A}}_\lambda}(1-\mathcal{P})B^{-1}. \end{aligned} \tag{5.20}$$

It follows from (5.16), (5.19), (5.20) that the first and third terms of the right hand side of (5.18) are continuous functions with values in H. Analogously,

$$\begin{aligned} &Lu(t) = Au(t) = B\mathcal{A}u(t) = B\mathcal{P}B^{-1}L_1\varphi(t-1) \\ &\quad + Be^{-t\tilde{\mathcal{A}}_\lambda}\tilde{\mathcal{A}}(1-\mathcal{P})\varphi(0) + \int_0^t B\tilde{\mathcal{A}}_\lambda e^{-(t-s)\tilde{\mathcal{A}}_\lambda}(1-\mathcal{P})B^{-1}L_1\varphi(s-1)ds. \end{aligned} \tag{5.21}$$

From (5.18), (5.19), (5.21) it follows that

$$\frac{d}{dt}Mu(t) + Lu(t) = L_1u(t-1), \quad 0 < t < 1. \tag{5.22}$$

As $t \to 0$

$$\begin{aligned} Lu(t) &= B\mathcal{P}B^{-1}L_1\varphi(t-1) + Be^{-t\tilde{\mathcal{A}}_\lambda}(1-\mathcal{P})B^{-1}L\varphi(0) \\ &\quad + \int_0^t B\tilde{\mathcal{A}}_\lambda e^{-(t-s)\tilde{\mathcal{A}}_\lambda}(1-\mathcal{P})B^{-1}L_1\varphi(s-1)ds \\ &\to B\mathcal{P}B^{-1}L_1\varphi(-1) + B(1-\mathcal{P})B^{-1}L\varphi(0) \\ &= B\mathcal{P}B^{-1}L_1\varphi(-1) + B\mathcal{A}\varphi(0) - B\mathcal{P}\mathcal{A}\varphi(0) \\ &= B\big\{\mathcal{P}B^{-1}L_1\varphi(-1) + \mathcal{A}\varphi(0) - \frac{1}{\lambda}\mathcal{P}\varphi(0)\big\} \end{aligned} \tag{5.23}$$

in H. Hence if the compatibility condition (5.13) is satisfied, the last side of (5.23) is equal to $B\mathcal{A}\varphi(0) = L\varphi(0)$ and the initial condition

$$\lim_{t\to 0} Lu(t) = L\varphi(0) \quad \text{in} \quad H$$

is satisfied.

We could refer to some other applications in Chapter 3 of the monograph [2] by S.L. Campbell, concerning electrical circuits which contain operational amplifiers. The resulting system is described by a singular, linear, finite dimensional differential equation.

If a feedback loop is created around a device with an operational amplifier, then the resulting system is described by a singular differential difference equation as soon as the loop contains transmission delays which cannot be ignored.

REFERENCES

[1] S.L. Campbell, *Singular linear systems of differential equations with delays,* Appl. Anal., 11 (1980), 123–136.

[2] S.L. Campbell, "Singular Systems of Differential Equations II," Pitman, San Francisco-London-Melbourne 1982.

[3] R.F. Curtain and H.J. Zwart, *An introduction to infinite-dimensional linear systems theory,* Springer-Verlag, New York, 1995.

[4] A. Favini, *Abstract potential operators and spectral methods for a class of degenerate evolution problems,* J. Diff. Eqs., 39 (1981), 212–225.

[5] A. Favini and A. Yagi, *Multivalued linear operators and degenerate evolution equations,* Ann. Mat. Pure Appl., 163(IV), 353–380.

[6] A. Favini and A. Yagi, "Degenerate Differential Equations in Banach Spaces," Pure and Applied Math., 215, Marcel Deker, 1998.

[7] W.M. Greenlee, *Rate of convergence in singular perturbations,* Ann. Inst. Fourier, Grenoble, 18(2) (1968), 135–191.

[8] J.K. Hale and P. Martinez-Amores, *Stability in neutral equations,* Nonlinear Anal., TMA, 1 (1977), 161–173.

[9] J. Lagnese, *Singular differential equations in Hilbert space,* SIAM J. Math. Anal., 4 (1973), 623–637.

[10] A. Lunardi, "Analytic Semigroups and Optimal Regularity in Parabolic Problems," Birkhäuser, 1995.

[11] P. Martinez-Amores, *Periodic solutions for coupled systems of differential-difference and difference equations,* Annali Mat. Pura Appl., 121(4) (1979), 171–186.

[12] L. Pandolfi, *Hankel operators of systems with feedthrough delays,* IMA J. Math. Control Information, 14 (1997), 1–12.

[13] A.E. Taylor, "Introduction to Functional Analysis," J. Wiley and Sons, 1958.