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ELLIPTIC EQUATIONS WITH SINGULARITY ON THE BOUNDARY

Satoshi Hashimoto and Mitsuharu Ôtani†
Department of Applied Physics, School of Science and Engineering
Waseda University, 3-4-1, Okubo, Shinjuku-ku, Tokyo 169-8555, Japan

(Submitted by: Reza Aftabizadeh)

Dedicated to the memory of Professor Yukiyoshi Ebihara

Abstract. The existence and nonexistence of positive classical solutions is discussed for $-\Delta u = K(x)(1-|x|)^{-\alpha}u^{\beta}$ in the unit ball B with Dirichlet boundary condition $u|_{\partial B} = 0$. Our main tools are based on the variational method and Pohozaev's identity. The singularity of coefficients on the boundary will be handled with the symmetry of functions and some approximation procedures.

1. Introduction. Let B be the unit ball $\{x \in \mathbb{R}^N; |x| < 1\}$ in $\mathbb{R}^N$ and consider the nonlinear elliptic problem of the form:

$$\text{(E)} \qquad \begin{cases} -\Delta u(x) = \dfrac{K(x)}{(1-|x|)^{\alpha}} u^{\beta}(x) & x \in B, \\ u(x) \geq 0 & x \in B, \\ u(x) = 0 & x \in \partial B, \end{cases}$$

where $\alpha > 0$, $\beta > 1$ and $K(\cdot)$ is a given non-negative continuous function on $\overline{B}$. As a matter of course, this kind of problems which allow the case $\alpha \leq 0$ has been investigated so many peoples so far.

The peculiarity of our case $0 < \alpha$ lies in the fact that the coefficient of the nonlinear term possesses a singularity on the boundary ∂B.

This problem was already studied by Senba-Ebihara-Furusho [6] within the framework of the theory of ODE. They showed the existence of positive

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radial solutions in $C^2(B) \cap C^1(\overline{B})$ for the case $0 < \alpha < 2$ and $1 < \beta < (N+2)/(N-2)$. Hayashida-Nakatani [3] also studied some problems similar to ours and discussed some mathematical backgrounds for (E).

The main purpose of this paper is to discuss the existence and nonexistence of positive solutions of (E) from the viewpoint of the theory of nonlinear PDE. As for the existence of nontrivial solutions, we shall be concerned again with radial solutions. However, our method can cover more singular cases (i.e., $0 < \alpha < (\beta+1)/2+1$) than those in Senba-Ebihara-Furusho [6]. Our argument is based chiefly on the variational method (especially the mountain pass lemma) and contains some approximation procedures to avoid some difficulties caused by the singularity of coefficients on the boundary. In deriving a priori estimates for solutions of approximate equations, we shall make use of the symmetry of solutions.

In order to discuss the nonexistence of positive solutions, we rely on the well-known Pohozaev's identity, however, the way of its use is different from those ever known. Ebihara-Furusho [1] also discussed the nonexistence of radial solutions within the theory of ODE in a similar situation to ours. However, our argument does not require the symmetry of $K(\cdot)$ and solutions.

Our main results are stated in the next section, and their proofs will be given in §3 and §4.

2. Main results. Throughout this paper, the following condition will be imposed on $K(\cdot)$.

$$\text{(K)} \qquad \begin{cases} K(\cdot) \in C(\overline{B}), & \\ K(x) \geq 0 & \text{for all } x \in B, \\ K(x) > 0 & \text{for all } x \in \partial B. \end{cases}$$

Then our main results are stated as follows.

Theorem 1 (Existence)**.** *Let $K(\cdot)$ be radially symmetric and satisfy condition* (K)*. Let $0 < \alpha < (\beta+1)/2+1$, and $1 < \beta < (N+2)/(N-2)$, then* (E) *has at least one* (*radially symmetric*) *positive solution u belonging to $C^2(B) \cap C^1(\overline{B})$.*

Theorem 2 (Nonexistence)**.** *Let condition* (K) *be satisfied and let $1 < \beta + 1 \leq \alpha$. Then* (E) *does not admit any positive solutions belonging to $C^2(B) \cap C^1(\overline{B})$.*

Remark 1. (i) Consider the problem (E) with B replaced by the annulus $B_a = \{x \in \mathbb{R}^N \ ; \ a < |x| < 1\}$ $(0 < a < 1)$, which is denoted by $(\mathrm{E})_a$. Then Theorems 1 and 2 hold true with (E) and B replaced by $(\mathrm{E})_a$ and B_a respectively. Moreover we can drop out the subcritical condition $1 < \beta < (N+2)/(N-2)$ from the assumptions in Theorem 1.

(ii) Ebihara-Furusho [1] dealt with the same equation as (E) in the exterior of B and remarked that $\alpha < \beta + 1$ is a necessary condition for the existence of radially symmetric solutions. However, our result does not require the radial symmetry for solutions nor for $K(\cdot)$.

(iii) It would be interesting to investigate the existence (or nonexistence) of (not necessarily classical) solutions of (E) for the case $(\beta+1)/2+1 \le \alpha < \beta + 1$.

3. Proof of Theorem 1. The main tool for the existence of solutions here is the "*mountain pass lemma*". However, if one tries to apply this lemma directly to our problem, there arises some delicate difficulties caused by the singularity of coefficients on the boundary, such as the verification of the Palais-Smale condition etc. To avoid these difficulties, we introduce the following approximate equations:

$$(\mathrm{E})_\varepsilon \begin{cases} -\Delta u_\varepsilon(x) = \dfrac{K(r)}{(1+\varepsilon-r)^\alpha} u_\varepsilon^\beta(x) & x \in B, \\ u_\varepsilon(x) \ge 0 & x \in B, \\ u_\varepsilon(x) = 0 & x \in \partial B, \end{cases}$$

where $r = |x|$ and $\varepsilon \in (0,1)$ is a parameter. Since we are concerned with radially symmetric solutions, we shall work in the Banach space X defined by

$$X = \big\{ u \in H_0^1(B) \ ; \ u(x) \text{ is radially symmetric, i.e., } u(x) = \tilde{u}(r) \big\}$$

with norm

$$\|u\|_X^2 = \int_B |\nabla u|^2 dx = C_N \int_0^1 |\tilde{u}_r(r)|^2 r^{N-1} dr,$$

with

$$C_N = \int_{\partial B} 1 dS.$$

The functionals associated with approximate equations $(\mathrm{E})_\varepsilon$ are given by

$$J_\varepsilon(u) = \|u\|_X^2/2 - b_\varepsilon(u),$$
$$b_\varepsilon(u) = \frac{1}{\beta+1}\int_B \frac{K_0(r)}{(1+\varepsilon-r)^\alpha}|u|^\beta u dx,$$

where $K_0(r) = K(|x|)$. Then, by virtue of condition (K) and the fact that $\varepsilon > 0$ and $\beta < (N+2)/(N-2)$, standard arguments assure that $J_\varepsilon(u)$ belong to $C^1(X)$ and satisfy the Palais-Smale condition in X (see Rabinowitz [5]).

Furthermore we can show the following lemma which will play an important role in what follows.

Lemma 1. *Let $0 \le \alpha < (\beta+1)/2+1$, and $0 < \beta < (N+2)/(N-2)$. Then there exists a constant C_1 independent of ε satisfying*

$$|b_\varepsilon(u)| \le C_1\|u\|_X^{\beta+1}, \qquad \forall u \in X. \tag{1}$$

Proof. For any $s, t \in [1/2, 1]$ with $t < s$, we find

$$\begin{aligned} |\tilde u(s) - \tilde u(t)| &= \left|\int_t^s \tilde u_r(r)dr\right| \le \int_t^s |\tilde u_r(r)|dr, \\ &\le \left(\int_{1/2}^1 |\tilde u_r(r)|^2 dr\right)^{1/2} |s-t|^{1/2}, \\ &\le \left(\frac{2^{(N-1)}}{C_N}\right)^{1/2} \|u\|_X \; |s-t|^{1/2}. \end{aligned}$$

Hence,

$$|\tilde u(r)| = |\tilde u(r) - \tilde u(1)| \le \left(\frac{2^{(N-1)}}{C_N}\right)^{1/2} \|u\|_X (1-r)^{1/2}, \qquad \forall r \in [1/2, 1]. \tag{2}$$

Therefore, by Sobolev's embedding theorem, we can find a constant C_1 independent of ε such that

$$\begin{aligned}|b_\varepsilon(u)| \leq\ & (\beta+1)^{-1}\cdot\left\{\int_{|x|\leq 1/2}\frac{K_0(r)}{(1+\varepsilon-r)^\alpha}|u|^{\beta+1}dx\right.\\ &\qquad\left.+\int_{1/2\leq|x|\leq 1}\frac{K_0(r)}{(1+\varepsilon-r)^\alpha}|u|^{\beta+1}dx\right\}\\ \leq\ & (\beta+1)^{-1}\cdot\max_{0\leq r\leq 1}K_0(r)\left\{2^\alpha\int_B|u|^{\beta+1}dx\right.\\ &\left.+\left(\frac{2^{(N-1)}}{C_N}\right)^{(\beta+1)/2}\|u\|_X^{\beta+1}C_N\int_{1/2}^1(1-r)^{(\beta+1)/2-\alpha}dr\right\}\\ \leq\ & C_1\|u\|_X^{\beta+1}. \quad \square\end{aligned}$$

In view of (1), we can easily find positive numbers δ, C_δ independent of ε such that

$$\inf_{u\in S_\delta} J_\varepsilon(u)\geq C_\delta>0 \qquad \forall\varepsilon\in(0,1], \tag{3}$$

where $S_\delta=\{u\in X\ ;\ \|u\|_X=\delta\}$. Furthermore, by virtue of the assumption $K_0(\cdot)|_{\partial B}>0$, there exists an element $u_1\in X$ satisfying $b_1(u_1)>0$. Since $u_1(t)=t\cdot u_1$ satisfies

$$J_1(u_1(t))=t^2\ \frac{\|u_1\|_X^2}{2}-t^{\beta+1}b_1(u_1),$$

there exists a sufficiently large $t_0>0$ such that $J_1(t_0u_1)<0$. Noting that $J_\varepsilon(u)$ is monotone decreasing as $\varepsilon\downarrow 0$, we conclude that $J_\varepsilon(t_0u_1)<0$ for all $\varepsilon\in(0,1]$. Thus we can apply the usual mountain pass lemma (see Rabinowitz [5]) to $J_\varepsilon(\cdot)$. Hence, if we define $C(\varepsilon)$ by

$$C(\varepsilon)=\inf_{p\in\mathcal{P}}\max_{u\in p}J_\varepsilon(u)\geq C_\delta>0, \tag{4}$$

where $\mathcal{P}$ denotes the family of all continuous curves in X connecting 0 and t_0u_1, then $C(\varepsilon)$ is the critical value of J_ε, i.e., there exists $u_\varepsilon\in X$ such that $J_\varepsilon(u_\varepsilon)=C(\varepsilon)$ and $J'_\varepsilon(u_\varepsilon)=0$ in X. Therefore u_ε satisfies

$$\int_B\nabla u_\varepsilon(x)\cdot\nabla\varphi(x)dx+\int_B\frac{K_0(r)}{(1+\varepsilon-r)^\alpha}|u_\varepsilon(x)|^\beta\varphi(x)dx=0 \qquad \forall\varphi\in X. \tag{5}$$

By virtue of the fact that $\varepsilon > 0$ and $\beta < (N+2)/(N-2)$, the boot-strap method (Moser's iteration scheme) such as in Ôtani [4] assures the L^∞-estimate for u_ε. Hence $K_0(r)(1+\varepsilon-r)^{-\alpha}|u_\varepsilon|^\beta$ belongs to $L^2_s(B) = \{v \in L^2(B)\ ;\ v(x)\ \text{ is radially symmetric}\}$. Since Δ maps $L^2_s(B) \cap C_0^\infty(B)$ into itself, it is easy to see that $(\nabla u, \nabla v)_{L^2} = 0$ for all $u \in L^2_s(B) \cap H^1_0(B) = X$ and $v \in L^2_s(B)^\perp \cap H^1_0(B)$. Thus we find that u_ε satisfies (5) for all $\varphi \in H^1_0(B)$, whence follows

$$-\Delta u_\varepsilon(x) = \frac{K_0(r)}{(1+\varepsilon-r)^\alpha}|u_\varepsilon(x)|^\beta \qquad \text{in} \ \ D'. \tag{6}$$

Then the elliptic estimates in L^p and C^θ together with the L^∞-estimate for u_ε give $u_\varepsilon \in C^{2,\theta}(\overline{B})$.

Now the strong maximum principle applied for (6) implies that $u_\varepsilon(x) > 0$ for all $x \in B$. Consequently $u_\varepsilon(\cdot)$ yields a positive classical solution for our approximate equation $(\mathrm{E})_\varepsilon$.

A priori estimates. We are going to establish some a priori estimates for u_ε. In what follows we denote simply by C positive constants independent of ε, which will in general have different values in different places. We also often use the notation u to mean u_ε or $\tilde{u}_\varepsilon$ if no confusion arises.

Multiply $(\mathrm{E})_\varepsilon$ by u, then we easily get

$$\|u\|_X^2 = (\beta+1)b_\varepsilon(u).$$

Recalling $J_\varepsilon(u) = C(\varepsilon)$ and $0 < C_\delta \le C(\varepsilon) \le C(1)$, we obtain

$$0 < \frac{2(\beta+1)}{\beta-1}C_\delta \le \|u_\varepsilon\|_X^2 \le \frac{2(\beta+1)}{\beta-1}C(1) \qquad \forall\varepsilon \in (0,1]. \tag{7}$$

Hence, (2) yields

$$0 < \rho_\varepsilon := |\tilde{u}_\varepsilon(3/4)| \le C\|u_\varepsilon\|_X \le C\sqrt{\frac{2(\beta+1)}{\beta-1}C(1)}. \tag{8}$$

Moreover, $v_\varepsilon(x) := u_\varepsilon(x) - \rho_\varepsilon$ satisfies

$$(\mathrm{E})^1_\varepsilon \begin{cases} -\Delta v_\varepsilon(x) &= \dfrac{K_0(r)}{(1+\varepsilon-r)^\alpha}(v_\varepsilon(x)+\rho_\varepsilon)^\beta \qquad x \in B_1, \\ \quad v_\varepsilon(x) &= 0 \qquad\qquad\qquad\qquad\qquad\quad x \in \partial B_1, \end{cases}$$

where $B_1 := \{x \in \mathbb{R}^N \ ; \ |x| < 3/4\}$. Since

$$|K_0(r)(1+\varepsilon-r)^{-\alpha}(v+\rho_\varepsilon)^\beta| \le C(|v|^\beta+1),$$

the boot-strap method such as in the proof of Theorem II of Ôtani [4] gives the $L^\infty(B_1)$-estimate for v_ε. Thus, in view of (2), we can establish

$$\|u_\varepsilon\|_{L^\infty(B)} \le C \quad \forall \varepsilon \in (0,1]. \tag{9}$$

Therefore, Δu_ε is bounded in $L^\infty(B_1)$, and then by the elliptic estimate in L^p, we get

$$\|u_\varepsilon\|_{W^{2,p}(B_1)} \le C \quad \forall p \in [1,\infty), \quad \forall \varepsilon \in (0,1], \tag{10}$$

whence follows by Sobolev's embedding theorem,

$$\|u_\varepsilon\|_{C^1(\overline{B}_1)} \le C \quad \forall \varepsilon \in (0,1]. \tag{11}$$

Let $I = [1/2, 1]$ and integrate the identity $\{(u')^2/2\}' = u' \cdot u''$ over $[1/2, r]$, $r \in I$, then

$$|u'(r)|^2 \le 2\|u'\|_{L^2(I)} \cdot \|u''\|_{L^2(I)} + |u'(1/2)|^2 \quad \forall r \in I.$$

Therefore, (7), (11) and the fact that $\|u'\|_{L^2(I)} \le C\|u\|_X$ give

$$\|u'\|_{L^\infty(I)} \le C(\|u''\|_{L^2(I)}^{1/2} + 1). \tag{12}$$

Furthermore, noting that u satisfies

$$u''(r) + \frac{N-1}{r}u'(r) + \frac{K_0(r)}{(1+\varepsilon-r)^\alpha}u^\beta(r) = 0 \quad \forall r \in (0,1],$$

we have

$$\|u''\|_{L^2(I)} \le C\|u\|_X + C\|(1-r)^{-\alpha} \cdot u^\beta\|_{L^2(I)}. \tag{13}$$

By virtue of (2), (7), (12) and the relation $|u(r)| \le \|u'\|_{L^\infty(I)}(1-r)$ for all $r \in I$, we deduce

$$\begin{aligned} \|(1-r)^{-\alpha} \cdot u^\beta\|_{L^2(I)} &= \|(1-r)^{-\alpha} \cdot u^{2(1-\sigma)} u^{\beta-2(1-\sigma)}\|_{L^2(I)} \\ &\le C(\|u''\|_{L^2(I)}^{1/2} + 1)^{2(1-\sigma)} \cdot \|u\|_X^{\beta-2(1-\sigma)} \\ &\qquad \cdot \|(1-r)^{1-\sigma+\beta/2-\alpha}\|_{L^2(I)} \\ &\le C(\|u''\|_{L^2(I)}^{1-\sigma} + 1), \end{aligned} \tag{14}$$

where $\sigma = \min(1, \{(1 + (\beta + 1)/2) - \alpha\}/2)$. Then it follows from (13) and (14) that $\|u''\|_{L^2(I)} \leq C$, which together with (10) assure

$$\|\Delta u_\varepsilon\|_{L^2(B)} \leq C \quad \forall \varepsilon \in (0, 1], \tag{15}$$

$$\|\frac{K_0(r)}{(1 + \varepsilon - r)^\alpha} u_\varepsilon^\beta\|_{L^2} \leq C \quad \forall \varepsilon \in (0, 1]. \tag{16}$$

Convergence. Hence, by Rellich's compactness theorem, we can extract a subsequence u_{ε_n} denoted by u_n such that

$$u_n \longrightarrow u \quad \text{strongly in } H_0^1(B) \text{ and weakly star in } L^\infty(B), \tag{17}$$

$$u_n^\beta \longrightarrow u^\beta \quad \text{strongly in } L^2(B), \tag{18}$$

$$\Delta u_n \longrightarrow \Delta u \quad \text{weakly in } L^2(B), \tag{19}$$

$$\frac{K(r)}{(1 + \varepsilon_n - r)^\alpha} u_n^\beta \longrightarrow \chi \quad \text{weakly in} L^2(B). \tag{20}$$

Let $\varphi \in C_0^\infty(B)$, then $K(r)(1+\varepsilon_n - r)^{-\alpha} \cdot \varphi$ converges to $K(r)(1-r)^{-\alpha} \cdot \varphi$ strongly in $L^2(B)$. Then (18) implies that

$$\left(\frac{K(r)}{(1 + \varepsilon_n - r)^\alpha} u_n^\beta, \varphi \right)_{L^2} \longrightarrow \left(\frac{K(r)}{(1 - r)^\alpha} u^\beta, \varphi \right)_{L^2} \quad \text{as } n \longrightarrow \infty,$$

whence follows $\chi = K(r)(1 - r)^{-\alpha} u^\beta$.

Moreover, (7) and (17) assure that $u \not\equiv 0$. Thus we can conclude that u gives a nontrivial nonnegative solution of (E) belonging to $H^2(B) \cap L^\infty(B) \cap X$. For any $\delta > 0$, it is clear that $-\Delta u = K_0(r)(1 - r)^{-\alpha} u^\beta \in L^\infty(B^\delta)$ with $B^\delta = \{x \in \mathbb{R}^N \; ; \; |x| < 1 - \delta\}$. Then, by standard arguments, we can deduce $u \in C^2(\overline{B^\delta})$, whence follows $u \in C^2(B)$. Furthermore, since $\tilde{u}'' \in L^2(0, 1)$ implies $\tilde{u}' \in C^{1/2}([0, 1])$, we find $u \in C^1(\overline{B})$. To see $u(x) > 0$ for all $x \in B$, it suffices to apply the strong maximum principle. □

4. Proof of Theorem 2. To prove our nonexistence result, we prepare a couple of standard results.

Lemma 2 (Strong maximum principle)**.** *Let Ω be a bounded domain with smooth boundary $\partial\Omega$ and suppose that $u \in C^1(\overline{\Omega})$ satisfies*

$$\begin{cases} -\Delta u \geq 0 & in \;\; D'(\Omega), \\ u \geq 0 & in \quad \Omega. \end{cases}$$

Then $u(x) > 0$ for all $x \in \Omega$ and $\partial u(x)/\partial n < 0$ for all $x \in \Gamma = \{x \in \partial\Omega;\; u(x) < u(y) \text{ for all } y \in \Omega\}$, where $\partial u(x)/\partial n$ denotes the outer normal derivative of u at x.

Proof. The first assertion that $u(x) > 0$ for all $x \in \Omega$ follows from the Harnack's principle (see e.g. Trudinger [7]). The second assertion is nothing but the maximum principle of Hopf's type(see e.g. Gilbarg-Trudinger [2]). □

Corollary 1. *Let $B_a = \{x \in \mathbb{R}^N;\; a < |x| < 1\}$ and suppose that $u \in C^1(\overline{B_a})$ satisfies*

$$\begin{cases} -\Delta u \geq 0 & in \;\; D'(B_a), \\ u \geq 0 & in \quad B_a, \\ u = 0 & on \quad \Gamma_1, \end{cases}$$

where $\Gamma_1 = \{x \in \mathbb{R}^N \;;\; |x| = 1\}$. Then there exist numbers $\rho \in (a, 1)$ and $C_\rho > 0$ such that

$$u(x) \geq C_\rho(1 - |x|) \quad \forall x \in \overline{B_\rho} = \{x \in \mathbb{R}^N \;;\; \rho \leq |x| \leq 1\} \tag{21}$$

Proof. We can apply Lemma 2 with $\Gamma = \Gamma_1$ to deduce that $\partial u(x)/\partial r = \partial u(x)/\partial n < 0$ on Γ_1. Since $u \in C^1(\overline{B_a})$, there exists a positive number $\rho \in (a, 1)$ and C_ρ such that

$$\frac{\partial u}{\partial r}(x) \leq -C_\rho \quad \forall x \in \overline{B_\rho} = \{x \in \mathbb{R}^N \;;\; \rho \leq |x| \leq 1\}. \tag{22}$$

Then, for all $x \in \overline{B_\rho}$, we have

$$\begin{aligned} -u(x) &= u(x/|x|) - u(x) &&= \int_1^{1/|x|} \frac{du}{dt}(tx)dt \\ &= \int_1^{1/|x|} \nabla u(tx) \cdot x dt &&= \int_1^{1/|x|} \frac{\partial u}{\partial r}(tx) \cdot |x| dt \\ &\leq -\int_1^{1/|x|} C_\rho |x| dt &&= -C_\rho(1 - |x|), \end{aligned}$$

whence follows (21). □

Lemma 3 (Pohozaev Identity)**.** *Let Ω be a radially symmetric domain. Suppose that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies*

$$-\Delta u = F(x,u) \qquad in \;\; \Omega \tag{23}$$

with $F(x,u) \in C(\overline{\Omega} \times \mathbb{R}^1)$. Then the following identity holds

$$\frac{2-N}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_{\partial\Omega} |\nabla u|^2 (x \cdot n) dS - \int_{\partial\Omega} \frac{\partial u}{\partial n}\frac{\partial u}{\partial r}|x| dS = \int_\Omega F(x,u)\frac{\partial u}{\partial r}|x| dx, \tag{24}$$

where n denotes the unit outward normal vector at $x \in \partial\Omega$.

Proof. Multiply (23) by $\sum_{i=1}^N x_i \cdot \partial u/\partial x_i$ and integrate over Ω. Then the standard calculations give

$$\frac{2-N}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_{\partial\Omega} |\nabla u|^2 (x \cdot n) dS - \int_{\partial\Omega} (x\cdot\nabla u)(n\cdot\nabla u) dS = \int_\Omega (x\cdot\nabla u) F(x,u) dx.$$

Hence, to derive (24), it suffices to note that $(x \cdot \nabla u) = \partial u/\partial r \cdot |x|$ and $(n \cdot \nabla u) = \partial u/\partial n$. □

Proof of Theorem 2. Since $K(x) > 0$ on $|x| = 1$, there exist numbers $\rho_1 \in [\rho, 1)$ and $\delta > 0$ such that $K(x) \geq \delta$ for all $x \in \overline{B_{\rho_1}} = \{x \in \mathbb{R}^N \;;\; \rho_1 \leq |x| \leq 1\}$. Hence, by (21) and (22), we get

$$K(x)u^\beta(x) \cdot \frac{\partial u}{\partial r}(x) \cdot |x| \leq -\delta C_\rho^{\beta+1}(1-|x|)^\beta \rho_1 \qquad \forall x \in \overline{B_{\rho_1}}.$$

Then application of Lemma 3 with $\Omega = B^\varepsilon = \{x \in \mathbb{R}^N \;;\; \rho_1 < |x| < 1-\varepsilon\}$ yields

$$\begin{aligned}
I_\varepsilon \;&:= \frac{2-N}{2}\int_{B^\varepsilon} |\nabla u|^2 dx + \frac{1}{2}\int_{\partial B^\varepsilon} |\nabla u|^2 (x\cdot n) dS - \int_{\partial B^\varepsilon} \frac{\partial u}{\partial n}\frac{\partial u}{\partial r}|x| dS \\
&= \int_{B^\varepsilon} \frac{K(x)}{(1-|x|)^\alpha} u^\beta(x)|x|\frac{\partial u}{\partial r}(x) dx \\
&\leq -\int_{B^\varepsilon} \delta C_\rho^{\beta+1}\rho_1(1-|x|)^{\beta-\alpha} dx \\
&= -C_N \delta C_\rho^{\beta+1}\rho_1 \int_{\rho_1}^{1-\varepsilon} (1-r)^{\beta-\alpha} r^{N-1} dr \\
&\leq -C_N \delta C_\rho^{\beta+1}{\rho_1}^N \int_{\rho_1}^{1-\varepsilon} (1-r)^{\beta-\alpha} dr.
\end{aligned}$$

The fact $u \in C^1(\overline{B_{\rho_1}})$ assures the existence of $\lim_{\varepsilon \to 0} I_\varepsilon = I_0$. Therefore

$$\int_{\rho_1}^{1} (1-r)^{\beta-\alpha} dr \leq \frac{I_0}{C_N \delta C_\rho^{\beta+1} {\rho_1}^N} < +\infty$$

whence follows

$$\alpha < \beta + 1. \quad \square$$

Remark 2. (i) In deriving the identity (24), we do not require any boundary condition for u.

(ii) Multiplication of (E) by u gives

$$\int_B |\nabla u|^2 dx = \int_B K(x)(1-r)^{-\alpha} u^{\beta+1}(x) dx.$$

Then it easily follows from (21) that (E) has no positive solutions in $C^1(\overline{B})$ for $\alpha \geq \beta + 2$.

(iii) If B is replaced by the annulus B_a, then the X-norm is equivalent to $\|\tilde{u}_r\|_{L^2(a,1)}$ and the estimate similar to (2) assures that X is continuously embedded in $C^{1/2}(\overline{B_a})$. Hence we can repeat the same verifications as for the proof of Theorem 1 without assuming the subcritical condition $1 < \beta < (N+2)/(N-2)$.

REFERENCES

[1] Y. Ebihara and Y. Furusho, *On the exterior Dirichlet problem for semilinear elliptic equations with coefficients unbounded on the boundary*, Hiroshima math. J. **Vol. 19** (1989), 379–396.

[2] D. Gilbarg and N. S. Trudinger, "Elliptic Partial Differential Equations of Second Order, 2nd ed.," Springer-Verlag, Berlin, Heidelberg and New York, 1983.

[3] K. Hayashida and M. Nakatani, *On radially symmetric positive solutions of a semilinear elliptic equation in hyperbolic space*, Math. Japonica **Vol. 40** (1994), 561–584.

[4] M. Ôtani, *Existence and nonexistence of nontrivial solutions of some nonlinear degenerate elliptic equations*, J. Funct. Anal. **Vol. 76** (1988), 140–159.

[5] P. H. Rabinowitz, "Minimax Method in Critical Point Theory with Applications to Differential Equations," Amer. Math. Soc. Colloq. Publ., Rhode Island, 1986.

[6] T. Senba, Y. Ebihara and Y. Furusho, *Dirichlet problem for a semilinear equation with singular coefficients*, Nonlinear Anal. **Vol. 15** (1990), 299–306.

[7] N. S. Trudinger, *On Harnack type inequalities and their application to quasilinear elliptic equations*, Comm. Pure Appl. Math. **Vol. 20** (1967), 721–747.