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MONOTONE ITERATIVE METHODS FOR BOUNDARY VALUE PROBLEMS*

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(Submitted by: Jean Mawhin)

Dedicated to the memory of M. A. Krasnosel'skii

1. Introduction. Monotone iterative methods related to lower and upper solutions go back at least to E. Picard whose main contribution was published in two "mémoires", the first one [37] in 1890 and the second one [38] in 1893. In [38], the author considers the ODE problem

$$u'' + f(t,u) = 0, \quad u(a) = u(b) = 0, \tag{1}$$

in case $f(t,.)$ is increasing. Assuming further that $u = 0$ is a solution, i.e., $f(t,0) = 0$, the problem is to find a nontrivial solution. Under some more assumptions, he proves the existence of a positive function α_0 such that

$$\alpha_0'' + f(t,\alpha_0) > 0, \text{ on } (a,b), \quad \alpha_0(a) = \alpha_0(b) = 0.$$

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Such a function is now called a lower solution. Next, for $n \geq 1$, he considers the solutions of

$$-\alpha_n'' = f(t, \alpha_{n-1}), \quad \alpha_n(a) = \alpha_n(b) = 0;$$

he observes that the sequence $(\alpha_n)_n$ is monotone, $\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \dots$ and converges to a solution u of (1) greater than α_0. This provides an iterative method to compute solutions of (1), which we will call *the first monotone approximation scheme*. It is worth while to notice also that in the same paper, the author uses lower solutions with angles, i.e., which are piecewise $\mathcal{C}^2$.

In his first "mémoire" [37], E. Picard has made a study of the PDE problem in the opposite situation, where, transposed in the ODE setting (1), $f(t, \cdot)$ is decreasing. His main result concerns *bounds on the solution of* (1). With additional assumptions on f, he defines $\alpha_0 = 0$ and considers for $n \geq 1$ the solutions α_n and β_n of the problems

$$\begin{aligned} -\beta_n'' &= f(t, \alpha_{n-1}), \quad \beta_n(a) = \beta_n(b) = 0, \\ -\alpha_n'' &= f(t, \beta_n), \qquad \alpha_n(a) = \alpha_n(b) = 0. \end{aligned}$$

Here, $\alpha_n \leq \beta_n$, $(\alpha_n)_n$ is increasing and $(\beta_n)_n$ is decreasing. Hence, the sequences $(\alpha_n)_n$ and $(\beta_n)_n$ converge pointwise respectively to some functions u and v such that $u \leq v$. Further, the convergence is uniform, as it was proved in 1898 for the ODE problem [41] and in 1900 for the PDE one [42]; the functions u and v satisfy

$$\begin{aligned} u'' + f(t, v) = 0, \quad u(a) = u(b) = 0, \\ v'' + f(t, u) = 0, \quad v(a) = v(b) = 0. \end{aligned} \tag{2}$$

At last, the solution z of (1), which is unique with these assumptions, is such that

$$u \leq z \leq v.$$

This proves that the monotone sequences $(\alpha_n)_n$ and $(\beta_n)_n$ provide computable bounds on the solution z.

In his first paper [37], E. Picard also proved that, if $b - a$ is small enough, the functions u and v are equal and are solutions of (1). In that case, the monotone sequences $(\alpha_n)_n$ and $(\beta_n)_n$ provide approximations to the solution of (1). We will refer to this method as *the second monotone approximation scheme*. At last, he brought about in 1894 [39] (see also [40]) an ODE example where $u \neq v$ which proves that this second method only works under additional assumptions on f.

The method of upper and lower solutions was developed without reference to these iteration schemes (see e.g. J. Mawhin [32]). In 1931, G. Scorza Dragoni [46] used upper and lower solutions to prove existence of solution for the Dirichlet problem

$$u'' + f(t, u, u') = 0, \quad u(a) = u(b) = 0,$$

in case f is continuous. A major step concerning nonlinearities f which are derivative dependent is due to M. Nagumo [34] who introduced in 1937 the so-called Nagumo condition. This will be extended to the Carathéodory case by G. Scorza Dragoni [47].

Following S.A. Chaplygin [11] and taking into account the upper and lower solution technique, the russian school studied the monotone methods in some systematic way. In 1954, B.N. Babkin [5] considers the two point boundary value problem (1) with assumptions on $f(t, u)$ which imply the uniqueness of the solutions of (1). In his approach, he considers two approximation sequences. Given lower and upper solutions α_0 and β_0 and some $K \geq 0$, these approximations are obtained (for $n \geq 1$) as solutions of the linear problems

$$\begin{aligned} -\alpha_n'' + K\alpha_n &= f(t, \alpha_{n-1}) + K\alpha_{n-1}, \quad \alpha_n(a) = \alpha_n(b) = 0, \\ -\beta_n'' + K\beta_n &= f(t, \beta_{n-1}) + K\beta_{n-1}, \quad \beta_n(a) = \beta_n(b) = 0. \end{aligned}$$

The key assumption to prove that the sequences $(\alpha_n)_n$ and $(\beta_n)_n$ are monotone and converge to the unique solution of (1) is that the chosen $K > 0$ is such that the function $f(t, u) + Ku$ is increasing in u. As a by-product, this method provides an error bound $\beta_n - \alpha_n$ on the n^{th} approximations. This was already the case in E. Picard's second approximation scheme.

An important step is due to L. Kantorovich [25] in 1939. He noticed that the first monotone approximation scheme, used for the Cauchy problem associated with ODE as well as for other problems had a common structure related to positive operators. He then developed an abstract formulation of the method. This abstract formulation was further developed by M.A. Krasnosel'skii [29], [30] and H. Amann [3], [4]. In 1959, L. Collatz and J.

Schröder [14] gave an abstract formulation of the second monotone approximation scheme. Notice also that these two schemes have been unified in 1960 by J. Schröder [45] who showed that the second one can be reduced to the first one.

Existence of a minimal and a maximal solution between a lower and an upper solution for a nonlinear boundary value problem goes back to T. Satô in 1954 [43] but with some monotonicity assumption on f. In 1960, W. Mlak [33] gave a proof of this result without this extra monotonicity assumption but for a parabolic problem and in 1961, K. Akô [1] worked out the Dirichlet problem. In the same paper, W. Mlak proved also that the first monotone approximation scheme gives rise to sequences $(\alpha_n)_n$ and $(\beta_n)_n$ which converge to these minimal and maximal solutions.

In the english edition of their book [15] (1962), R. Courant and D. Hilbert describe the first monotone approximation scheme but without direct reference to upper and lower solutions. Following this work, monotone iterations have been extensively worked out by several authors such as S.V. Parter [36], Keller, Cohen and others [27], [12], [26], [51], Cohen [13], Shampine [49], Shampine and Wing [50], K. Schmitt [44], Heidel [24] ...

In every quoted papers, the monotonicity is obtained from a one-sided Lipschitz condition on the nonlinearity f. In 1974, H. Amann [2] assumes an Hölder condition on f. This is a particular case of the condition used in 1960 by W. Mlak [33] for a parabolic problem. On the other hand, in 1978, Stuart [52] assumes f is of bounded variations on compact intervals. Hence, he can use the decomposition $f = g - h$ where g and h are increasing functions. This assumption was also used in 1992 by S. Carl [8]. In such a case, the approximation sequences are obtained from nonlinear problems

$$\begin{aligned} -\alpha_n'' + h(t, \alpha_n) &= g(t, \alpha_{n-1}), \quad \alpha_n(a) = \alpha_n(b) = 0, \\ -\beta_n'' + h(t, \beta_n) &= g(t, \beta_{n-1}), \quad \beta_n(a) = \beta_n(b) = 0. \end{aligned}$$

In general, solutions cannot be made explicit which reduces considerably the interest of the approach.

The study of monotone iterative methods for nonlinearities depending on the derivative was initiated in 1964 by Gendzhoyan [21] who considers the problem

$$u'' + f(t, u, u') = 0, \quad u(a) = u(b) = 0. \tag{3}$$

Given a lower and an upper solution α_0 and β_0, he defines approximation sequences $(\alpha_n)_n, (\beta_n)_n$ as solutions of problems

$$\begin{aligned}
-\alpha_n'' + l(t)\alpha_n' + k(t)\alpha_n &= f(t, \alpha_{n-1}, \alpha_{n-1}') + l(t)\alpha_{n-1}' + k(t)\alpha_{n-1}, \\
\alpha_n(a) = \alpha_n(b) &= 0, \\
-\beta_n'' + l(t)\beta_n' + k(t)\beta_n &= f(t, \beta_{n-1}, \beta_{n-1}') + l(t)\beta_{n-1}' + k(t)\beta_{n-1}, \\
\beta_n(a) = \beta_n(b) &= 0,
\end{aligned}$$

where $k(t)$ and $l(t)$ are functions related to the assumptions on the function f. Here again the convergence is monotone and provides approximations of the solution together with some error bounds. The computability of the solution depends however on the possibility to solve the linear boundary value problem

$$-u'' + l(t)u' + k(t)u = h(t), \quad u(a) = u(b) = 0.$$

In 1974, J. Chandra and P.W. Davis [10] have considered a problem that depends linearly in the derivative. In 1977, S.R. Bernfeld and J. Chandra [6] generalize this result to a nonlinear dependence in u'. The first approximations are given lower and upper solutions, α_0 and $\beta_0 \geq \alpha_0$. Further approximations are obtained from the nonlinear problems

$$\begin{aligned}
-\alpha_n'' + K\alpha_n &= f(t, \alpha_{n-1}, \alpha_n') + K\alpha_{n-1}, \quad \alpha_n(a) = \alpha_n(b) = 0, \\
-\beta_n'' + K\beta_n &= f(t, \beta_{n-1}, \beta_n') + K\beta_{n-1}, \quad \beta_n(a) = \beta_n(b) = 0,
\end{aligned} \tag{4}$$

where K is related to the assumptions on f. Once again, computation of the approximations is not explicit.

In the present work, we consider the two approximation schemes for the phi-laplacian problem

$$-\frac{d}{dt}(\varphi \circ u') = f(t, u, u'), \quad u(a) = u(b) = 0 \tag{5}$$

in case f is L^1-Carathéodory. For this problem, existence of a solution between given lower and upper solutions is well known. It can be found for f L^1-Carathéodory and $\varphi(s) = s$ in G. Scorza Dragoni [47]. For a general phi-laplacian and f Carathéodory but without derivative dependence in C. De Coster [18], P. Omari and F. Zanolin [35] and in case of derivative dependence but with f continuous in J. Wang, W. Gao and Z. Lin [54]. The first iterative scheme for a similar PDE problem has been worked out by J. Diaz [19] who considers a generalized differential operator that satisfies Leray-Lions type conditions and by M. Cuesta [16] for the elliptic p-laplacian. Both these results concern nonlinearities without derivative dependence and use weak lower and upper solutions.

A major problem to extend monotone iteration method to Carathéodory functions with derivative dependence is that the problem (4) which defines the approximation α_n does not have a unique solution (see the remark after Theorem 7). This fact forced some authors to restrict the dependence of f on the derivative u' as in A. Cabada and J.J. Nieto [7] and M.X. Wang, J.J. Nieto and A. Cabada [55] (for an extension of this to the phi-laplacian, see Proposition 8 and Theorem 10). In the present work, we do not make restriction on f and define α_n to be the minimal solution of the corresponding problem. Maximal and minimal solutions for problem (5) with f continuous and without derivative dependence can be found in H. Dang, K. Schmitt and R. Shivaji [17]. For elliptic p-laplacian or operators with Leray-Lions type conditions and f Carathéodory, additional results can be found in S. Carl [9], M. Cuesta [16], T. Kura [31].

Existence of the maximal and minimal solutions is the content of Section 3. In Section 4, we investigate uniqueness of solutions. This gives conditions so that the maximal and minimal solutions coincide. Section 5 presents generalization of the first monotone approximation scheme. Here, we obtain sequences of approximations $(\alpha_n)_n$ and $(\beta_n)_n$ which converge to the minimal and maximal solutions of the problem (5). Except in simple cases, the computations cannot be made explicit. This motivates Section 6 which concern bounds on the solutions of the boundary value problem

$$\frac{d}{dt}(\varphi \circ u') + f(t, u, u) = 0, \quad u(a) = u(b) = 0 \tag{6}$$

and extends the case studied by E. Picard [37]. In (6), the notation $f(t, u, u)$ is used to take care of functions $f(t, u, u) = g(t, u) - h(t, u)$, where both functions $g(t, u)$ and $h(t, u)$ are increasing in u. In this section, we concentrate on the case without derivative dependence of f to keep the computation of the sequences $(\alpha_n)_n$ and $(\beta_n)_n$ explicit. Under some more restrictive conditions on the lower and upper solutions, we are able to construct monotone sequences $(\alpha_n)_n$ and $(\beta_n)_n$ which converge to some $\hat{u}_{min}$ and $\hat{u}_{max}$ such that $\hat{u}_{min} \le \hat{u}_{max}$, and every solution u of (6) is such that $\hat{u}_{min} \le u \le \hat{u}_{max}$. As already observed by E. Picard [39], we cannot hope to have $\hat{u}_{min} = \hat{u}_{max}$.

Section 7 is concerned with explicit conditions under which such a result holds. This is the second approximation scheme introduced by E. Picard. Theorem 14 is related to the work of Guo [22] (see also Guo and Lakshmikantham [23]) while Theorem 15 extends a result of Seda [48] and Theorem 16 extends, in the particular case of a single equation, a result of Khavanin and Lakshmikantham [28]. We can also quote in this direction the work of Vasin [53].

2. Preliminary results. Let Z be a Banach space. An order cone in Z is said to be *normal* if there exists $c > 0$ such that $0 \leq u \leq v$ implies $\|u\| \leq c\|v\|$. The following theorem is a particular case of [25]. It can also be obtain from small modifications of Theorem 7.A in [56].

Theorem 1. *Let Z be a Banach space with normal order cone, α and $\beta \in Z$, $\alpha \leq \beta$, $\mathcal{E} = \{u \in Z \mid \alpha \leq u \leq \beta\}$ and $T : \mathcal{E} \to Z$ be lower semi-continuous, monotone increasing and such that $T(\mathcal{E})$ is relatively compact. Assume α and β satisfy*

$$\alpha \leq T(\alpha) \quad \textit{and} \quad \beta \geq T(\beta).$$

Then, the sequence $(\alpha_n)_n$ defined by

$$\alpha_0 = \alpha,\, \alpha_n = T(\alpha_{n-1}),$$

converges to a fixed point u_{min} of T such that

$$\alpha \leq u_{min} \leq \beta.$$

Further, any fixed point $u \in \mathcal{E}$ of T is such that

$$u_{min} \leq u.$$

Proof. *Claim 1 : the sequence $(\alpha_n)_n$ converges in Z.* We deduce from the monotonicity of T that the sequence $(\alpha_n)_n$ is increasing

$$\alpha_1 = T(\alpha_0) \geq \alpha_0, \quad \alpha_n = T(\alpha_{n-1}) \geq T(\alpha_{n-2}) = \alpha_{n-1}$$

and bounded

$$\beta \geq T(\beta) \geq T(\alpha_0) \geq \alpha_0, \quad \beta \geq T(\beta) \geq T(\alpha_{n-1}) = \alpha_n.$$

As $T(\mathcal{E})$ is relatively compact, a subsequence converges and as the order cone is normal and the sequence is monotone, the sequence itself converges. Define

$$u_{min} := \lim_{n\to\infty} \alpha_n.$$

Claim 2 : u_{min} is a fixed point of T. Notice that

$$\lim_{n\to\infty} T(\alpha_n) = \lim_{n\to\infty} \alpha_{n+1} = u_{min}.$$

As T is lower semi-continuous, we deduce

$$u_{min} = \lim_{n\to\infty} T(\alpha_n) \geq T(\lim_{n\to\infty} \alpha_n) = T(u_{min}).$$

On the other hand, $u_{min} \geq \alpha_n$ and therefore $T(u_{min}) \geq T(\alpha_n)$. Going to the limit,

$$T(u_{min}) \geq \lim_{n\to\infty} T(\alpha_n) = u_{min}.$$

Claim 3 : any fixed point $u \in \mathcal{E}$ of T is such that $u_{min} \leq u$. Since $\alpha_0 = \alpha \leq u$, we deduce by recurrence $\alpha_n = T(\alpha_{n-1}) \leq T(u) = u$. Going to the limit, $u_{min} = \lim_{n\to\infty} \alpha_n \leq u$.

Remark 1. In a similar way, we obtain a maximal fixed point for upper semi-continuous operators.

3. Minimal and maximal solutions. In this section, we consider the method of upper and lower solutions for the problem

$$-\frac{d}{dt}(\varphi \circ u') = f(t, u, u'), \quad u(a) = u(b) = 0. \tag{7}$$

We shall use the following definitions. A function $f : D \subset [a, b]\times\mathbb{R}^n \to \mathbb{R}$, with $n = 1$, 2 or 3 is said to be a *Carathéodory function* if

(i) for a.e. $t \in [a, b]$, the function $f(t, \cdot)$ is continuous;
(ii) for any w, the function $f(\cdot, w)$ is measurable.

It is a L^1-*Carathéodory function* if it is a Carathéodory function and

(iii) for any bounded set $K \subset \mathbb{R}^n$, there exists a function $\psi \in L^1(a, b)$ such that for any $w \in K$ and a.e. $t \in [a, b]$ with $(t, w) \in D$, we have $|f(t, w)| \leq \psi(t)$.

We assume $\varphi : \mathbb{R} \to \mathbb{R}$, $f(t,u,v)$ is a L^1-Carathéodory function and look for solutions of (7) in

$$X := \{u \in \mathcal{C}^1([a,b],\mathbb{R}) \mid \varphi \circ u' \in W^{1,1}(a,b)\}. \tag{8}$$

Up to minor modifications, the following result is Lemma 2.2 in [20]. A set $B \subset L^1(a,b)$ is said to be equi-integrable if there exists $h \in L^1(a,b)$ such that for any $u \in B$, $|u(t)| \le h(t)$.

Lemma 2. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be an increasing homeomorphism. Then, the operator*

$$A : \mathrm{Dom}(A) \subset \mathcal{C}^1([a,b]) \to L^1(a,b), u \mapsto \mathcal{A}(u),$$

defined by

$$\mathcal{A}(u)(t) = -\frac{d}{dt}(\varphi \circ u')(t),$$

and

$$\mathrm{Dom}(A) = \{u \in X \mid u(a) = u(b) = 0\}$$

has a continuous inverse which sends equi-integrable sets into relatively compact sets.

Definition 1. Consider
(a) functions α and $\beta \in \mathcal{C}([a,b])$ such that for any $t \in [a,b]$, $\alpha(t) \le \beta(t)$, and let

$$D := \{(t,u,v) \in [a,b] \times \mathbb{R}^2 \mid \alpha(t) \le u \le \beta(t)\}, \tag{9}$$

(b) a continuous positive function $k : \mathbb{R}^+ \to \mathbb{R}_0^+$ and a function $\psi \in L^p(a,b)$, where $p \in [1,\infty]$, and let $q \in [1,\infty]$ be such that $\frac{1}{q} + \frac{1}{p} = 1$.

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be an increasing homeomorphism with $\varphi(0) = 0$ and $f : D \to \mathbb{R}$. The function f is said to satisfy a *Nagumo bound* with respect to (α,β) and (k,ψ), if for all $t \in [a,b]$, $u \in [\alpha(t),\beta(t)]$ and $v \in \mathbb{R}$,

$$|f(t,u,v)| \le \psi(t)k(|v|).$$

We say that $C > 0$ is *admissible* for φ, (α,β) and (k,ψ) if

$$\begin{aligned} \min\Big(\int_0^{\varphi(C)} \frac{(\varphi^{-1}(v))^{1/q}}{k(\varphi^{-1}(v))}\,dv, \int_{\varphi(-C)}^0 \frac{|\varphi^{-1}(v)|^{1/q}}{k(|\varphi^{-1}(v)|)}\,dv\Big) \\ > \big(\max_{a\le t\le b}\beta(t) - \min_{a\le t\le b}\alpha(t)\big)^{1/q}\|\psi\|_{L^p}. \end{aligned} \tag{10}$$

Remark 2. Existence of admissible C can be deduced from the condition

$$\min\Big(\int_0^\infty \frac{(\varphi^{-1}(v))^{1/q}}{k(\varphi^{-1}(v))}\,dv, \int_{-\infty}^0 \frac{|\varphi^{-1}(v)|^{1/q}}{k(|\varphi^{-1}(v)|)}\,dv\Big) > \big(\max_{a\le t\le b}\beta(t) - \min_{a\le t\le b}\alpha(t)\big)^{1/q}\|\psi\|_{L^p}.$$

Notice also that if φ is an odd $\mathcal{C}^1$ homeomorphism, we can write (10) as

$$\int_0^C \frac{u^{1/q}}{k(u)}\varphi'(u)\,du > \big(\max_{a\le t\le b}\beta(t) - \min_{a\le t\le b}\alpha(t)\big)^{1/q}\|\psi\|_{L^p}.$$

Such an admissible C always exists if

$$\int_0^\infty \frac{u^{1/q}}{k(u)}\varphi'(u)\,du = \infty.$$

In the sequel, we shall use the following set of assumptions.

Assumptions 1. (i) There exist α and $\beta \in W^{1,\infty}(a,b)$ such that for any $t \in [a,b]$, $\alpha(t) \le \beta(t)$, let D be defined in (9);

(ii) the function $\varphi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism, $\varphi(0) = 0$;

(iii) $f : D \to \mathbb{R}$ is a Carathéodory function which satisfies a Nagumo bound with respect to (α,β) and (k,ψ), with $k \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}_0^+)$, $\psi \in L^p(a,b)$ and $p \in [1,\infty]$.

Lemma 3. *Let Assumptions* 1 *be verified. Then, if $C > 0$ is admissible for φ, (α,β) and (k,ψ), any solution u of* (7)*, with $\alpha \le u \le \beta$, satisfies $\|u'\|_\infty < C$.*

Proof. There exists $t_0 \in [a,b]$ such that $u'(t_0) = 0$. Assume that for some $t_1 > t_0$, $u'(t_1) = C$ and for all $t \in]t_0,t_1]$, $u'(t) > 0$. Using the change of variables $v = \varphi(u'(t))$, we compute then

$$\begin{aligned}\int_0^{\varphi(C)} \frac{(\varphi^{-1}(v))^{1/q}}{k(\varphi^{-1}(v))}\,dv &= \int_{t_0}^{t_1} \frac{(u'(t))^{1/q}}{k(u'(t))}\frac{d}{dt}\varphi(u'(t))\,dt \\ &\le \int_{t_0}^{t_1} (u'(t))^{1/q}\psi(t)\,dt \le \|(u')^{1/q}\|_{L^q}\|\psi\|_{L^p} \\ &\le \big(\max_{a\le t\le b}\beta(t) - \min_{a\le t\le b}\alpha(t)\big)^{1/q}\|\psi\|_{L^p},\end{aligned}$$

which contradicts the admissibility of C.

We prove the cases $t_1 < t_0$ or $u'(t_1) = -C$ in a similar way. $\square$

Given α and $\beta \in \mathcal{C}([a,b])$ such that for any $t \in [a,b]$, $\alpha(t) \le \beta(t)$, we define the function

$$\begin{aligned} \gamma(t,u) &:= \alpha(t) && \text{if } u < \alpha(t), \\ &:= u && \text{if } \alpha(t) \le u \le \beta(t), \\ &:= \beta(t) && \text{if } \beta(t) < u. \end{aligned} \tag{11}$$

Let us first state a preliminary result (see Lemma 2.4 in [31] or Lemma 2 in [55]).

Lemma 4. *Let α and $\beta \in W^{1,\infty}(a,b)$ be such that for any $t \in [a,b]$, $\alpha(t) \le \beta(t)$ and γ be defined by (11). Then for any $u \in \mathcal{C}^1([a,b])$, the function $\gamma(t,u(t))$ is a.e. derivable. If further the sequence $(u_n)_n$ goes to u in $\mathcal{C}^1([a,b])$, then*

$$\begin{aligned} \gamma(\cdot,u_n(\cdot)) &\to \gamma(\cdot,u(\cdot)) && \textit{in } \mathcal{C}([a,b]), \\ \frac{d}{dt}\gamma(t,u_n(t)) &\to \frac{d}{dt}\gamma(t,u(t)) && \textit{for a.e. } t \in [a,b]. \end{aligned}$$

To state the next theorem we need to define first the notion of upper and lower solutions for (7).

Definition 2. A function $\alpha \in \mathcal{C}([a,b])$ is a *lower solution* of (7) if

(a) for any $t_0 \in]a,b[$, either $D_-\alpha(t_0) < D^+\alpha(t_0)$, or there exists an open interval $I_0 \subset]a,b[$ such that $t_0 \in I_0$, $\alpha \in \mathcal{C}^1(I_0)$, $\varphi \circ \alpha' \in W^{1,1}(I_0)$ and, for a.e. $t \in I_0$,

$$-\frac{d}{dt}\varphi(\alpha'(t)) \le f(t,\alpha(t),\alpha'(t));$$

(b) $\alpha(a) \le 0$, $\alpha(b) \le 0$.

In the same way, a function $\beta \in \mathcal{C}([a,b])$ is said to be an *upper solution* of (7) if,

(a) for any $t_0 \in]a,b[$, either $D^-\beta(t_0) > D_+\beta(t_0)$, or there exists an open interval $I_0 \subset]a,b[$ such that $t_0 \in I_0$, $\beta \in \mathcal{C}^1(I_0)$, $\varphi \circ \beta' \in W^{1,1}(I_0)$ and, for a.e. $t \in I_0$,

$$-\frac{d}{dt}\varphi(\beta'(t)) \ge f(t,\beta(t),\beta'(t));$$

(b) $\beta(a) \ge 0$, $\beta(b) \ge 0$.

Definition 3. Given α and $\beta \geq \alpha$, a solution $u_{min} \in \mathcal{E} := \{u \in \mathcal{C}([a,b]) \mid \alpha \leq u \leq \beta\}$ (resp. $u_{max} \in \mathcal{E}$) of (7) is said to be a *minimal solution* (resp. *maximal solution*) in $[\alpha, \beta]$ if any other solution u with $\alpha \leq u \leq \beta$ is such that $u_{min} \leq u$ (resp. $u \leq u_{max}$).

Theorem 5. *Let Assumptions* 1 *be verified. Assume α and β are lower and upper solutions of* (7) *and $C > 0$ is admissible for φ, (α, β) and (k, ψ). Then the problem* (7) *has a minimal and a maximal solution u_{min} and u_{max} in $[\alpha, \beta]$ and for any solution $u \in \mathcal{E}$, $\|u'\|_\infty \leq C$.*

Proof. Let $C_0 = \max(C, \|\alpha'\|_\infty, \|\beta'\|_\infty)$,

$$\begin{aligned} \delta(v) &:= -C_0 && \text{if } v < -C_0, \\ &:= v && \text{if } -C_0 \leq v \leq C_0, \\ &:= C_0 && \text{if } v > C_0 \end{aligned} \tag{12}$$

and $F(t, u, v) = f(t, u, \delta(v))$. Define $\gamma(t, u)$ as in (11) and consider the modified problem

$$\begin{gathered} -\frac{d}{dt}(\varphi \circ u') = F(t, \gamma(t,u), \frac{d}{dt}\gamma(t,u)), \\ u(a) = u(b) = 0. \end{gathered}$$

We can write this problem

$$Au = Nu, \tag{13}$$

where A is defined in Lemma 2 and $N : \mathcal{C}^1([a,b]) \to L^1(a,b), u \mapsto Nu$ is such that

$$Nu(t) := F(t, \gamma(t, u(t)), \frac{d}{dt}\gamma(t, u(t))).$$

Step 1—Claim : The equation (13) *has a solution* $u \in \mathrm{Dom}(A)$. From Lemma 2, we can write (13) as the fixed point problem

$$u = A^{-1}Nu,$$

where $A^{-1}N : \mathcal{C}^1([a,b]) \to \mathrm{Dom}(A) \subset \mathcal{C}^1([a,b])$ is completely continuous and globally bounded. Hence, the claim follows from Schauder's Theorem.

Step 2—Claim : Any solution u of (13) *is such that for all $t \in [a,b]$, $\alpha(t) \leq u(t) \leq \beta(t)$.* Define

$$t_0 = \max\{t \in [a,b] \mid u(t) - \alpha(t) = \min_{a \leq s \leq b}(u - \alpha)(s)\}.$$

If the claim is wrong, $u(t_0) - \alpha(t_0) < 0$, $t_0 < b$ and $D_-\alpha(t_0) \geq D^+\alpha(t_0)$. From the definition of a lower solution, there exists I_0 such that $\alpha \in \mathcal{C}^1(I_0)$ and for a.e. $t \in I_0$

$$-\frac{d}{dt}\varphi(\alpha'(t)) \leq f(t, \alpha(t), \alpha'(t)).$$

As $u - \alpha$ is minimum and derivable at t_0, $u'(t_0) = \alpha'(t_0)$. Further, we can restrict I_0 so that for all $t \in I_0 \cap [t_0, b]$, $u(t) < \alpha(t)$. Hence, for such values of t,

$$\frac{d}{dt}[\varphi(u'(t)) - \varphi(\alpha'(t))] \leq -F(t, \alpha(t), \alpha'(t)) + f(t, \alpha(t), \alpha'(t)) = 0.$$

We deduce then that, for $t \geq t_0$,

$$\varphi(u'(t)) - \varphi(\alpha'(t)) \leq \varphi(u'(t_0)) - \varphi(\alpha'(t_0)) = 0,$$

and as φ is increasing $u'(t) \leq \alpha'(t)$. This contradicts the definition of t_0.

We prove in the same way that $u(t) \leq \beta(t)$.

Step 3—Claim : Any solution u of (13) *is a solution of* (7). From Step 2, $\gamma(t, u(t)) = u(t)$ and $Nu(t) = F(t, u(t), u'(t))$. Define $\bar{k}(v) = k(\delta(v))$ and notice that F satisfies a Nagumo bound with respect to (α, β) and $(\bar{k}, \psi)$, that C is admissible for φ, (α, β) and $(\bar{k}, \psi)$. Hence, we deduce from Lemma 3 that $\|u'\|_\infty < C$ and $F(t, u(t), u'(t)) = f(t, u(t), u'(t))$. The claim follows.

Step 4—Definition of the minimal solution. Define

$$\mathcal{S} := \{u \in \mathcal{E} \mid u \text{ is a solution of } (7)\}$$

and

$$u_{min}(t) := \inf\{u(t) \mid u \in \mathcal{S}\}.$$

Step 5—Claim : For any $\epsilon > 0$, there exists an upper solution $\hat{\beta}$ such that $u_{min} \leq \hat{\beta} \leq u_{min} + 3\epsilon$. From Lemma 3, we know that for all $u \in \mathcal{S}$, $\|u'\|_\infty \leq C$. It is easy to see then that u_{min} is continuous. Further given $\epsilon > 0$, there exists $\delta > 0$, such that for any $t_i \in [a, b]$, any $u_i \in \mathcal{S}$ and all $t \in [t_i - \delta, t_i + \delta]$, $|u_{min}(t) - u_{min}(t_i)| \leq \epsilon$ and $|u_i(t) - u_i(t_i)| \leq \epsilon$. Consider the points $t_i = a + i\delta$ and functions $u_i \in \mathcal{S}$ satisfying for $i \geq 1$ $u_{min}(t_i) \leq u_i(t_i) \leq u_{min}(t_i) + \epsilon$ and for $i \geq 2$, $u_i(t_i) \leq u_{i-1}(t_i)$. It follows that

$$\forall t \in [t_i - \delta, t_i + \delta] \quad u_{min}(t) \leq u_i(t) \leq u_{min}(t) + 3\epsilon.$$

We shall build the upper solution $\hat{\beta}$ using some patching of these functions u_i. Consider the interval $[t_0, t_3]$ and define $s_1 := \inf\{s \in [a, t_2] \mid \forall t \in [s, t_2],\ u_1(t) \geq u_2(t)\}$. Next we write $\hat{\beta}_1(t) = u_1(t)$ for $t \in [a, s_1]$ and $\hat{\beta}_1(t) = u_2(t)$ for $t \in]s_1, b]$. This defines a function $\hat{\beta}_1$ which is an upper solution such that for all $t \in [t_0, t_3]$, $u_{min}(t) \leq \hat{\beta}_1(t) \leq u_{min}(t) + 3\epsilon$. Repeating the argument (with $\hat{\beta}_{k-1}$ and u_{k+1}), we build upper solutions $\hat{\beta}_k$, so that for all $t \in [t_0, t_{k+2}]$, $u_{min}(t) \leq \hat{\beta}_k(t) \leq u_{min}(t) + 3\epsilon$. After a finite number of steps, we obtain the claim.

Step 6—Claim : The function u_{min} is a minimal solution. From the previous claims, we know that for any $\epsilon > 0$, there exists a solution u of (7) such that $u_{min} \leq u \leq u_{min} + 3\epsilon$. Hence, we build a sequence $(u_n)_n \subset \mathcal{S}$ of solutions of (7) such that $u_n \to u_{min}$ in $\mathcal{C}([a, b])$. From Lemma 3, $|u_n'(t)| < C$ and $|\frac{d}{dt}(\varphi \circ u_n')| = |f(\cdot, u_n, u_n')| \leq \psi \max_{v \in [0,C]} k(v) \in L^1(a, b)$. Hence, we can prove a subsequence $(u_{n_i}')_i$ converges to u_{min}' and, going to the limit in the equation $-\frac{d}{dt}(\varphi \circ u_{n_i}') = f(t, u_{n_i}, u_{n_i}')$, we prove that u_{min} is a solution of (7). By construction, this solution is minimal in $[\alpha, \beta]$.

Step 7 : Existence of a maximal solution is proved from the same argument.

Remark. In case f does not depend on u', it is enough to assume α and β continuous.

4. Uniqueness of solutions. Let $\varphi : \mathbb{R} \to \mathbb{R}$. Suppose $h(t, u, v)$ is a L^1-Carathéodory function and consider the operator

$$\mathcal{B} : X \subset \mathcal{C}([a, b]) \to L^1(a, b), u \mapsto \mathcal{B}(u),$$

where X is defined in (8) and $\mathcal{B}(u) := -\dfrac{d}{dt}(\varphi \circ u') + h(\cdot, u, u')$.

In case h is continuous, we can write the following maximum principle.

Proposition 6. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a nondecreasing function, D be defined in (9) and $h : D \to \mathbb{R}$ be a continuous function such that $h(t, \cdot, v)$ is increasing. Assume u_1 and $u_2 \in X$ are such that*

$$\mathcal{B}(u_1) \leq \mathcal{B}(u_2), \quad u_1(a) \leq u_2(a),\ u_1(b) \leq u_2(b).$$

Then, $u_1 \leq u_2$.

Proof. Assume the claim were wrong. There exists $t_0 \in]a, b[$ such that

$$u_2(t_0) - u_1(t_0) = \min_{a \leq t \leq b}(u_2(t) - u_1(t)) < 0 \quad \text{and} \quad u_2'(t_0) = u_1'(t_0) = v_0.$$

We deduce then from the assumptions that

$$h(t_0, u_2(t_0), v_0) - h(t_0, u_1(t_0), v_0) < 0.$$

From the continuity of h, it follows that for $\tau > 0$ small enough and a.e. $t \in [t_0, t_0 + \tau]$,

$$\frac{d}{dt}\varphi(u_2'(t)) - \frac{d}{dt}\varphi(u_1'(t)) \leq h(t, u_2(t), u_2'(t)) - h(t, u_1(t), u_1'(t)) < 0.$$

Next, we can find $t_1 \in [t_0, t_0 + \tau]$ such that $u_2'(t_1) - u_1'(t_1) \geq 0$ and a direct integration on $[t_0, t_1]$ gives the contradiction

$$0 \leq \varphi(u_2'(t_1)) - \varphi(u_1'(t_1)) \leq \int_{t_0}^{t_1} [h(t, u_2(t), u_2'(t)) - h(t, u_1(t), u_1'(t))]\, dt < 0.$$

As a corollary, we obtain rightaway the following uniqueness result.

Theorem 7. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a nondecreasing function, D be defined in* (9) *and $h : D \to \mathbb{R}$ be a continuous function such that $h(t, \cdot, v)$ is increasing. Then, the boundary value problem*

$$-\frac{d}{dt}(\varphi \circ u') + h(t, u, u') = 0, \quad u(a) = u(b) = 0,$$

has at most one solution.

Proof. Let u_1 and u_2 be two such solutions. From Proposition 6, we have both $u_1 \leq u_2$ and $u_2 \leq u_1$.

Remark. Proposition 6 is not true if the function $h(t, u, v)$ is L^1-Carathéodory. A counter-example is given by the operator

$$\mathcal{B}(u) = -u'' + u + 3\left|\frac{u'}{t}\right|^{1/2},$$

where $u \in \mathcal{C}([-1, 1])$. The functions $u_1(t) = 0$ and $u_2(t) = t^2 - 1$ are such that

$$\mathcal{B}(u_1) = 0 < \mathcal{B}(u_2) = t^2 + 3(\sqrt{2} - 1), \quad u_1(-1) = u_2(-1),\, u_1(1) = u_2(1),$$

but $u_1(t) > u_2(t)$ if $t \neq \pm 1$.

To consider the L^1-Carathéodory case we will have to restrict the dependence of h on the derivative u'. A first result uses a strict increasing assumption together with some equicontinuity.

Proposition 8. *Assume $\varphi : \mathbb{R} \to \mathbb{R}$ is a nondecreasing function, D is defined in (9) and $h : D \to \mathbb{R}$ is a L^1-Carathéodory function such that for all $\delta > 0$ and $v_0 \in \mathbb{R}$ there exists $\epsilon > 0$ such that for almost every $t \in [a, b]$ and all u_1 and $u_2 \in \mathbb{R}$, $u_2 < u_1 - \delta$ implies $h(t, u_2, v_0) - h(t, u_1, v_0) < -\epsilon$. Assume further that for all $\epsilon > 0$, there exists a $\eta > 0$ such that for almost every (t, u, x), $(t, u, y) \in D$, $|x - y| \le \eta$ implies $|h(t, u, x) - h(t, u, y)| \le \epsilon$. Then, if u_1 and $u_2 \in X$ are such that*

$$\mathcal{B}(u_1) \le \mathcal{B}(u_2), \quad u_1(a) \le u_2(a),\, u_1(b) \le u_2(b),$$

we have $u_1 \le u_2$.

Proof. Assume the claim were wrong. There exists $\delta > 0$ and $t_0 \in]a, b[$ such that

$$u_2(t_0) - u_1(t_0) = \min_{a \le t \le b}(u_2(t) - u_1(t)) < -2\delta \quad \text{and} \quad u_2'(t_0) = u_1'(t_0) =: v_0.$$

We can then choose I_0, a small enough neighbourhood of t_0, such that

$$\forall t \in I_0, \quad u_2(t) - u_1(t) < -\delta.$$

We deduce then from the assumptions that there exists $\epsilon > 0$ such that for almost every $t \in I_0$,

$$h(t, u_2(t), v_0) - h(t, u_1(t), v_0) < -3\epsilon.$$

If we restrict further I_0, we have that for almost every $t \in I_0$,

$$h(t, u_2(t), u_2'(t)) - h(t, u_1(t), u_1'(t)) \le -\epsilon < 0.$$

Next, we can find $t_1 \in I_0$, $t_1 > t_0$ such that $u_2'(t_1) - u_1'(t_1) \ge 0$ and a direct integration on $[t_0, t_1]$ gives the contradiction

$$0 \le \varphi(u_2'(t_1)) - \varphi(u_1'(t_1)) \le \int_{t_0}^{t_1} [h(t, u_2(t), u_2'(t)) - h(t, u_1(t), u_1'(t))]\, dt < 0.$$

An other result uses a one-sided Lipschitz condition.

Proposition 9. *Assume $\varphi : \mathbb{R} \to \mathbb{R}$ is an increasing function, D is defined in (9) and $h : D \to \mathbb{R}$ is a L^1-Carathéodory function such that $h(t, \cdot, v)$ is increasing and for some function $L \in L^1(a,b)$ we have*

$$\forall (t,u,x) \in D,\ \forall (t,u,y) \in D,\ x \geq y$$
$$\Rightarrow h(t,u,x) - h(t,u,y) \leq L(t)(\varphi(x) - \varphi(y)).$$

If further u_1 and $u_2 \in X$ are such that

$$\mathcal{B}(u_1) \leq \mathcal{B}(u_2), \quad u_1(a) \leq u_2(a),\, u_1(b) \leq u_2(b),$$

then $u_1 \leq u_2$.

Proof. Assume the claim were wrong, i.e., there exists $t_0 \in]a,b[$ such that $u_2(t_0) - u_1(t_0) < 0$.
Step 1—Claim : There exists $[t_1,t_2] \subset]a,b[$ such that $u_2'(t_1) - u_1'(t_1) = 0$ and $\forall t \in [t_1,t_2]$, $u_2(t) - u_1(t) < 0$, $u_2'(t) - u_1'(t) \geq 0$. Define $t_* := \min\{t \geq t_0 \mid u_2(t) - u_1(t) \geq 0\}$. It follows that $t_* \in]t_0, b]$, $\forall t \in [t_0, t_*[$, $u_2(t) - u_1(t) < 0$, and $u_2(t_*) - u_1(t_*) = 0$. Hence, there exists $t_2 \in]t_0, t_*[$ such that $u_2'(t_2) - u_1'(t_2) > 0$. At last, we define $t_1 := \max\{t \in [t_0, t_2[\mid u_2'(t) - u_1'(t) \leq 0\}$ which is such that $t_1 \in [t_0, t_2[$.
Step 2—On $[t_1, t_2]$, we have

$$\begin{aligned}\frac{d}{dt}(\varphi(u_2'(t)) - \varphi(u_1'(t))) &= h(t, u_2(t), u_2'(t)) - h(t, u_1(t), u_1'(t)) \\ &< h(t, u_2(t), u_2'(t)) - h(t, u_2(t), u_1'(t)) \leq L(t)(\varphi(u_2'(t)) - \varphi(u_1'(t))).\end{aligned}$$

This gives the contradiction

$$\infty = \lim_{s \to t_1} \int_s^{t_2} \frac{\frac{d}{dt}(\varphi(u_2'(t)) - \varphi(u_1'(t)))}{\varphi(u_2'(t)) - \varphi(u_1'(t))}\, dt < \int_{t_1}^{t_2} L(t)\, dt.$$

As for Theorem 7, we deduce the following uniqueness result.

Theorem 10. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be an increasing function and $h : D \to \mathbb{R}$ be a L^1-Carathéodory function such that $h(t, \cdot, v)$ is increasing and for some function $L \in L^1(a,b)$ we have*

$$\forall (t,u,x) \in D,\ \forall (t,u,y) \in D,\ x \geq y$$
$$\Rightarrow h(t,u,x) - h(t,u,y) \leq L(t)(\varphi(x) - \varphi(y)).$$

Then, the boundary value problem

$$-\frac{d}{dt}(\varphi \circ u') + h(t, u, u') = 0, \quad u(a) = u(b) = 0,$$

has at most one solution.

5. The first monotone approximation scheme. In this section, we consider the boundary value problem

$$-\frac{d}{dt}(\varphi \circ u') = f(t, u, u, u'), \quad u(a) = u(b) = 0. \tag{14}$$

The notation $f(t, u, u, u')$ is used in order to take care of right members of the form $g(t, u, u') - h(t, u, u')$ for which we write $f(t, \eta, u, v) = g(t, \eta, v) - h(t, u, v)$. Let $\mathcal{C}([a, b])$ be the ordered Banach space of continuous functions $u : [a, b] \to \mathbb{R}$ with norm $\|u\|_\infty := \max_{t \in [a,b]} |u(t)|$ and order defined by

$$x \leq y \Leftrightarrow (\forall t \in [a, b],\, x(t) \leq y(t)).$$

We shall use the following assumptions.

Assumptions 2. (i) There exist α_0 and $\beta_0 \in W^{1,\infty}(a, b)$, which are lower and upper solutions of (14), such that for any $t \in [a, b]$, $\alpha_0(t) \leq \beta_0(t)$; let $D_0 := \{(t, \eta, u, v) \mid t \in [a, b],\ \eta, u \in [\alpha_0(t), \beta_0(t)],\ v \in \mathbb{R}\}$;

(ii) $\varphi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism, $\varphi(0) = 0$;

(iii) $f : D_0 \to \mathbb{R}$ is a Carathéodory function such that for some positive continuous function $k : \mathbb{R}^+ \to \mathbb{R}_0^+$, some $\psi \in L^p(a, b)$ with $p \in [1, \infty]$, and any $(t, \eta, u, v) \in D_0$, we have $|f(t, \eta, u, v)| \leq \psi(t) k(|v|)$ and $f(t, \cdot, u, v)$ is nondecreasing ;

(iv) $C > 0$ is admissible for φ, (α_0, β_0) and (k, ψ).

Lemma 11. *Let Assumptions* 2 *be verified. Then for any* $\eta \in \mathcal{E} := \{u \in \mathcal{C}([a, b]) \mid \alpha_0 \leq u \leq \beta_0\}$*, the boundary value problem*

$$-\frac{d}{dt}(\varphi \circ u') = f(t, \eta(t), u, u'), \quad u(a) = u(b) = 0, \tag{15}$$

has a unique minimal solution $T_{min}(\eta) \in \mathcal{E}$*. This defines an operator*

$$T_{min} : \mathcal{E} \subset \mathcal{C}([a, b]) \to \mathcal{E},$$

which is lower semi-continuous, monotone increasing, such that $T_{min}(\mathcal{E})$ is relatively compact and $\alpha_0 \leq T_{min}(\alpha_0)$, $\beta_0 \geq T_{min}(\beta_0)$.

Proof. *Claim 1 : T_{min} is well defined.* Let η be fixed and notice that α_0 and β_0 are lower and upper solutions for (15). As

$$|f(t, \eta(t), u, v)| \leq \psi(t)k(|v|),$$

we deduce from Theorem 5 the existence of a minimal solution $u_{min} \in \mathcal{E}$.

Claim 2 : The set $T_{min}(\mathcal{E}) \subset \mathcal{C}([a, b])$ is relatively compact. As $T_{min}(\mathcal{E}) \subset \mathcal{E}$, this set is equibounded. From Lemma 3, we know that for all $u \in T_{min}(\mathcal{E}) \subset \mathcal{S}$, $\|u'\|_\infty < C$. Hence, $T_{min}(\mathcal{E})$ is equicontinuous and the claim follows from Arzela-Ascoli Theorem.

Claim 3 : $T_{min} : \mathcal{E} \subset \mathcal{C}([a, b]) \to \mathcal{C}([a, b])$ is lower semi-continuous. Consider a sequence $(\eta_n)_n \subset \mathcal{C}([a, b])$, which converges to η and $u_n = T_{min}(\eta_n)$. Since $T_{min}(\mathcal{E})$ is relatively compact, there exists $(u_{n_k})_k \subset (u_n)_n$ such that $u_{n_k} \stackrel{\mathcal{C}}{\to} u$. As in Step 6 of Theorem 5, we prove that u is a solution of (15) and, by definition of T_{min}, we have $u = \lim_{n\to\infty} T_{min}(\eta_n) \geq T_{min}(\eta) = T_{min}(\lim_{n\to\infty} \eta_n)$.

Claim 4 : T_{min} is monotone increasing. Let $\eta_1 \leq \eta_2$. Notice that $T_{min}(\eta_2)$ is an upper solution for the problem (15) with $\eta = \eta_1$. It follows that this last problem has a solution u, with $\alpha_0 \leq u \leq T_{min}(\eta_2)$ and $T_{min}(\eta_1) \leq T_{min}(\eta_2)$.

Remark 3. Using the same ideas, we can prove that the maximal solution of (15) defines an operator T_{max} which is upper semi-continuous, monotone increasing, such that $T_{max}(\mathcal{E})$ is relatively compact and $\alpha_0 \leq T_{max}(\alpha_0)$, $\beta_0 \geq T_{max}(\beta_0)$.

Using Theorem 1, Lemma 11 and Remarks 1 and 3, we obtain immediately the following result.

Theorem 12. *Let Assumptions* 2 *be verified and T_{min}, T_{max} be defined from Lemma* 11 *and Remark* 3. *Then, the sequences $(\alpha_n)_n$ and $(\beta_n)_n$, defined for $n \geq 1$ by*

$$\alpha_n = T_{min}(\alpha_{n-1}), \quad \beta_n = T_{max}(\beta_{n-1}),$$

converge to solutions u_{min} and u_{max} of (14)*, such that*

$$\alpha_0 \leq u_{min} \leq u_{max} \leq \beta_0.$$

Further, any solution $u \in \mathcal{E}$ of (14) is such that

$$u_{min} \le u \le u_{max}.$$

Proof. Existence of u_{min} and u_{max} follows from Theorem 5. We also know from Theorem 1 that the sequence $(\alpha_n)_n$ converges to the minimal fixed point $\hat{u}_{min}$ of T_{min}. Let us prove $\hat{u}_{min} = u_{min}$. By definition of u_{min}, we have $\hat{u}_{min} \ge u_{min}$. To prove the converse, it suffices to show that u_{min} is a fixed point of T_{min}, i.e. u_{min} is the minimal solution of

$$-\frac{d}{dt}(\varphi \circ u') = f(t, u_{min}, u, u'), \quad u(a) = u(b) = 0. \tag{16}$$

Assume on the contrary, there exists a solution v of (16) such that $v(t_0) < u_{min}(t_0)$ for some $t_0 \in [a, b]$. Hence, there exist $t_1 < t_0 < t_2$ such that $v(t) < u_{min}(t)$ for $t \in]t_1, t_2[$, $v(t_1) = u_{min}(t_1)$ and $v(t_2) = u_{min}(t_2)$. The function $\hat{\beta}$, defined by $\hat{\beta}(t) = v(t)$ if $t \in [t_1, t_2]$ and $\hat{\beta}(t) = u_{min}(t)$ if $t \notin [t_1, t_2]$, is an upper solution for (14). It follows that this problem has a solution u with $\alpha_0 \le u \le \hat{\beta} \underset{\neq}{\le} u_{min}$. This contradicts the definition of u_{min}. A similar argument holds for u_{max}. □

Let us notice that the preceding theorem is really interesting in case the operators T_{min} or T_{max} are computable in some explicit way. First, if f satisfies the conditions of Section 4, $T_{min} = T_{max}$. The following examples are such that $T = T_{min} = T_{max}$ is computable.

Application 1. Consider the problem

$$u'' + h(t, u) = 0, \quad u(a) = u(b) = 0,$$

and assume $h(t, u)$ is a L^1-Carathéodory function such that for some $K > 0$, $g(t, u) := h(t, u) + Ku$ is increasing. We consider $f(t, \eta, u, v) := g(t, \eta) - Ku$, so that equations (15) read

$$-u'' + Ku = g(t, \eta), \quad u(a) = u(b) = 0.$$

This problem has a unique solution

$$u(t) = T(\eta)(t) := \int_a^b G(t, s) g(s, \eta(s))\, ds,$$

where $G(t,s)$ is the Green function associated with

$$-u'' + Ku = f(t), \quad u(a) = u(b) = 0.$$

It follows that $T_{min} = T_{max} = T$, these operators are explicitly computable and Theorem 12 provides explicit approximations of the maximal and minimal solutions.

Application 2. Consider the problem

$$-\frac{d}{dt}(\varphi \circ u') = g(t,u), \quad u(a) = u(b) = 0,$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism, $\varphi(0) = 0$, $g(t,u)$ is a L^1-Carathéodory function and $g(t,\cdot)$ is increasing. We choose $f(t,\eta,u,v) = g(t,\eta)$; problem (15) reads

$$-\frac{d}{dt}(\varphi \circ u') = g(t,\eta), \quad u(a) = u(b) = 0,$$

and has a unique solution

$$u(t) = T(\eta)(t) = \int_a^t \varphi^{-1}(A(\eta) - \int_a^s g(r,\eta(r))\,dr)\,ds.$$

Here $A(\eta)$ is the unique solution $x \in \mathbb{R}$ of the equation

$$\Psi(x,\eta) := \int_a^b \varphi^{-1}(x - \int_a^s g(r,\eta(r))\,dr)\,ds = 0.$$

Hence, the operators $T_{min} = T_{max} = T$ are identical, explicitly computable and approximations of the maximal and minimal solutions follow from Theorem 12.

Application 3. The same computability can be obtain for the problem

$$-u'' + h(t,u,u') = g(t,u), \quad u(a) = u(b) = 0,$$

where $f(t,\eta,u,v) = g(t,\eta) - h(t,u,v)$ is a L^1-Carathéodory function, $g(t,\cdot)$ is increasing and h satisfies a Lipschitz condition

$$|h(t,u,x) - h(t,v,y)| \le k(|u-v| + |x-y|),$$

with k small enough. In this case, problem (15) reads

$$-u'' + h(t,u,u') = g(t,\eta), \quad u(a) = u(b) = 0. \tag{17}$$

Solutions of this problem are fixed points of the operator $S : \mathcal{C}^1 \to \mathcal{C}^1$ defined by

$$S(u)(t) = \int_a^b G(t,s)(g(s,\eta(s)) - h(s,u(s),u'(s)))\,ds,$$

where $G(t,s)$ is the Green function of

$$-u'' = f(t), \quad u(a) = u(b) = 0.$$

If k is small enough, S is a contraction and problem (17) has a unique solution

$$u = T(\eta),$$

the unique fixed point of S, which is the limit of the sequence $u_n = S(u_{n-1})$, i.e., T is computable in some explicit way.

6. Bounds on solutions. In the previous section, we considered sequences $(\alpha_n)_n$ and $(\beta_n)_n$ which converge to the minimal and maximal solutions of (14). As we already noticed, finding explicitly α_n and β_n can be as difficult as solving directly problem (14). In this section, we give a method of approximations which is explicit, but provides only bounds on the solutions. In the next section, we will give assumptions which imply these bounds are equal, in which case they are solutions.

Consider the problem

$$-\frac{d}{dt}(\varphi \circ u') = f(t,u,u), \quad u(a) = u(b) = 0. \tag{18}$$

To simplify the argument, we assume here f is independent of the derivative u'. This is not essential and could be developed as in Theorem 12.

We shall use the following definition.

Definition 4. Functions α and $\beta \in \mathcal{C}([a,b])$ are *coupled lower and upper quasi-solutions* of (18) if

(a) for any $t \in [a,b]$, $\alpha(t) \le \beta(t)$;

(b) for any $t_0 \in]a,b[$, either $D_-\alpha(t_0) < D^+\alpha(t_0)$, or there exists an open interval $I_0 \subset]a,b[$ such that $t_0 \in I_0$, $\alpha \in \mathcal{C}^1(I_0)$, $\varphi \circ \alpha' \in W^{1,1}(I_0)$ and, for a.e. $t \in I_0$,

$$-\frac{d}{dt}\varphi(\alpha'(t)) \leq f(t, \alpha(t), \beta(t));$$

(c) for any $t_0 \in]a,b[$, either $D^-\beta(t_0) > D_+\beta(t_0)$, or there exists an open interval $I_0 \subset]a,b[$ such that $t_0 \in I_0$, $\beta \in \mathcal{C}^1(I_0)$, $\varphi \circ \beta' \in W^{1,1}(I_0)$ and, for a.e. $t \in I_0$,

$$-\frac{d}{dt}\varphi(\beta'(t)) \geq f(t, \beta(t), \alpha(t));$$

(d) $\alpha(a) \leq 0 \leq \beta(a)$, $\alpha(b) \leq 0 \leq \beta(b)$.

Consider the following auxiliary problem

$$\begin{aligned} -\frac{d}{dt}(\varphi \circ u') &= f(t,u,v), \quad u(a) = u(b) = 0, \\ -\frac{d}{dt}(\varphi \circ v') &= f(t,v,u), \quad v(a) = v(b) = 0. \end{aligned} \tag{19}$$

Theorem 13. *Let α_0 and β_0 be in $\mathcal{C}([a,b])$, $\varphi : \mathbb{R} \to \mathbb{R}$ be an increasing homeomorphism such that $\varphi(0) = 0$, $D_0 := \{(t,u,v) \mid t \in [a,b], u, v \in [\alpha_0(t), \beta_0(t)]\}$ and $f : D_0 \to \mathbb{R}$ be a L^1-Carathéodory function such that $f(t,u,v)$ is nondecreasing in u and nonincreasing in v. Assume α_0 and β_0 are coupled lower and upper quasi-solutions of* (18)*. Then, the sequences $(\alpha_n)_n$ and $(\beta_n)_n$ defined for $n \geq 1$ by*

$$\begin{aligned} -\frac{d}{dt}(\varphi \circ \alpha_n') &= f(t, \alpha_{n-1}, \beta_{n-1}), \quad \alpha_n(a) = \alpha_n(b) = 0, \\ -\frac{d}{dt}(\varphi \circ \beta_n') &= f(t, \beta_{n-1}, \alpha_{n-1}), \quad \beta_n(a) = \beta_n(b) = 0, \end{aligned} \tag{20}$$

converge to functions $\tilde{u}_{min}$ and $\tilde{u}_{max}$. The pair $(\tilde{u}_{min}, \tilde{u}_{max})$ is a solution of (19) *such that*

$$\alpha_0 \leq \tilde{u}_{min} \leq \tilde{u}_{max} \leq \beta_0.$$

Moreover, any solution (u,v) of (19) *with $\alpha_0 \leq u \leq \beta_0$, $\alpha_0 \leq v \leq \beta_0$ is such that*

$$\tilde{u}_{min} \leq u \leq \tilde{u}_{max}, \quad \tilde{u}_{min} \leq v \leq \tilde{u}_{max}. \tag{21}$$

Proof. Let $Z = \mathcal{C}([a,b]) \times \mathcal{C}([a,b])$ with order defined by $(u,v) \le (x,y)$ if and only if for all $t \in [a,b]$, $u(t) \le x(t)$ and $v(t) \ge y(t)$. Let also $\mathcal{E} = \{(u,v) \mid u,v \in \mathcal{C}([a,b]),\ (\alpha_0,\beta_0) \le (u,v) \le (\beta_0,\alpha_0)\}$. We apply Theorem 1 and define $T(u,v)$ to be the solution (x,y) of

$$\begin{aligned} -\frac{d}{dt}(\varphi \circ x') &= f(t,u,v), \quad x(a) = x(b) = 0,\\ -\frac{d}{dt}(\varphi \circ y') &= f(t,v,u), \quad y(a) = y(b) = 0. \end{aligned}$$

Notice that if u is a solution of the given problem (18), then (u,u) is a solution of the auxiliary problem (19), whence $\tilde{u}_{min}$ and $\tilde{u}_{max}$ are bounds on solutions of (18).

7. The second monotone approximation scheme. If the bounds $\tilde{u}_{min}$ and $\tilde{u}_{max}$ given in Theorem 13 are equal, $u := \tilde{u}_{min}$ is a solution of the initial problem (18) and this theorem provides an approximation scheme to a solution of (18). In particular, this will be the case if solutions of the auxiliary problem (19) are unique. Unfortunately, E. Picard [39] pointed out that, in general, these solutions are not unique and that $\tilde{u}_{min}$ can be different from $\tilde{u}_{max}$. He showed this is the case for the problem

$$-u'' + 2ke^u = 0, \quad u(0) = u(1) = 0,$$

if we use $\alpha_0(t) = -kt(1-t)$, $\beta_0 = 0$ and k is large enough. The next proposition proves under appropriate assumptions, uniqueness of solutions of the auxiliary problem (19), where φ is the p-Laplacian $\varphi_p(u) = |u|^{p-2}u$. It also proves convergence of the sequences defined in Theorem 13 to the unique solution of the given problem (18).

Theorem 14. *Suppose the assumptions of Theorem* 13 *hold with* $\varphi(v) = \varphi_p(v)$. *Assume moreover*

(i) *there exists* $\epsilon > 0$ *such that* $\alpha_0 \ge \epsilon\beta_0$ *;*
(ii) *for every* $s \in [\epsilon, 1[$, *almost every* $t \in [a,b]$ *and every* u, $v \in [\alpha_0, \beta_0]$ *with* $sv \le u \le v$,

$$f(t, \frac{u}{s}, sv) < (\frac{1}{s})^{p-1} f(t,u,v).$$

Then, the functions $\tilde{u}_{min}$ and $\tilde{u}_{max}$ defined in Theorem 13 *are equal.*

Proof. From assumption (i), we deduce

$$\epsilon \tilde{u}_{max} \le \tilde{u}_{min} \le \tilde{u}_{max}.$$

Let $s_0 = \sup\{s \mid s\tilde{u}_{max} \le \tilde{u}_{min}\}$. It is obvious that $s_0 \in [\epsilon, 1]$ and that $s_0\tilde{u}_{max} \le \tilde{u}_{min}$. From the definition of s_0, we deduce the existence of $t_0 \in [a,b]$ such that

$$\tilde{u}_{min}(t_0) - s_0\tilde{u}_{max}(t_0) = 0, \quad \tilde{u}'_{min}(t_0) - s_0\tilde{u}'_{max}(t_0) = 0.$$

If $t_0 \ne b$, we also have $t_1 > t_0$ such that $\tilde{u}'_{min}(t_1) - s_0\tilde{u}'_{max}(t_1) \ge 0$.

Assume now that $s_0 < 1$. Hence, we can write

$$\begin{aligned} -\frac{d}{dt}\varphi_p(\tilde{u}'_{max}) &= f(t, \tilde{u}_{max}, \tilde{u}_{min}) \le f(t, \frac{1}{s_0}\tilde{u}_{min}, s_0\tilde{u}_{max}) \\ &< (\frac{1}{s_0})^{p-1} f(t, \tilde{u}_{min}, \tilde{u}_{max}) = -\frac{d}{dt}\varphi_p(\tilde{u}'_{min})(\frac{1}{s_0})^{p-1} = -\frac{d}{dt}\varphi_p(\frac{1}{s_0}\tilde{u}'_{min}), \end{aligned}$$

which leads to the contradiction

$$\begin{aligned} 0 &\le \varphi_p(\frac{1}{s_0}\tilde{u}'_{min}(t_1)) - \varphi_p(\tilde{u}'_{max}(t_1)) \\ &= \int_{t_0}^{t_1} \big(\frac{d}{dt}\varphi_p(\frac{1}{s_0}\tilde{u}'_{min}(t)) - \frac{d}{dt}\varphi_p(\tilde{u}'_{max}(t))\big)\, dt < 0. \end{aligned}$$

A similar argument holds if $t_0 = b$. □

Conditions of Theorem 14 can be checked on any pair of lower and upper quasi-solutions. In some cases, it is useful to use some iterate (α_n, β_n) from the sequence defined in (20).

Other conditions can be obtained in case $\varphi(s) = s$. For example one can write the following theorem.

Theorem 15. *Assume α_0 and $\beta_0 \in \mathcal{C}([a,b])$, are coupled lower and upper quasi-solutions of*

$$-u'' = f(t,u,u), \quad u(a) = u(b) = 0. \tag{22}$$

Let $D_0 := \{(t,u,v) \mid t \in [a,b], u,v \in [\alpha_0(t), \beta_0(t)]\}$ and suppose $f : D_0 \to \mathbb{R}$ is a L^1-Carathéodory function such that for some $K \in \mathbb{R}$, $f(t,u,v) + Kv$ is *nondecreasing in u, nonincreasing in v and for almost every t and all u, v with* $\alpha_0(t) \le u < v \le \beta_0(t)$,

$$[f(t,u,v) - f(t,v,u)](u-v) < (\frac{\pi^2}{(b-a)^2} + 2K)(u-v)^2.$$

Then, both sequences $(\alpha_n)_n$ *and* $(\beta_n)_n$*, defined for* $n \ge 1$ *by*

$$\begin{aligned} &-\alpha_n'' + K\alpha_n = f(t,\alpha_{n-1},\beta_{n-1}) + K\beta_{n-1},\ \ \alpha_n(a) = \alpha_n(b) = 0, \\ &-\beta_n'' + K\beta_n = f(t,\beta_{n-1},\alpha_{n-1}) + K\alpha_{n-1},\ \ \beta_n(a) = \beta_n(b) = 0, \end{aligned} \tag{23}$$

converge to the same solution u of (22)*.*

Proof. As in Theorem 13, we can prove that the sequences $(\alpha_n)_n$ and $(\beta_n)_n$ converge respectively to functions $u = \tilde{u}_{min}$ and $v = \tilde{u}_{max} \ge u$ solutions of

$$\begin{aligned} &-u'' + Ku = f(t,u,v) + Kv, \quad u(a) = u(b) = 0, \\ &-v'' + Kv = f(t,v,u) + Ku, \quad v(a) = v(b) = 0. \end{aligned}$$

If $u \ne v$, we compute

$$\begin{aligned} &\int_a^b [(v''-u'') - K(v-u)](v-u)\,dt \\ &\quad = \int_a^b [f(t,u,v) - f(t,v,u) + K(v-u)](v-u)\,dt \\ &\quad > -(\frac{\pi^2}{(b-a)^2} + K)\int_a^b (v-u)^2\,dt \end{aligned}$$

and also

$$\begin{aligned} &\int_a^b [(v''-u'') - K(v-u)](v-u)\,dt \\ &\quad = -\int_a^b [(v'-u')^2 + K(v-u)^2]\,dt \\ &\quad \le -(\frac{\pi^2}{(b-a)^2} + K)\int_a^b (v-u)^2\,dt, \end{aligned}$$

which is a contradiction. □

An other result in this direction uses a one-sided Lipschitz condition on the function f.

Theorem 16. *Let α_0 and β_0 be in $\mathcal{C}([a,b])$, $D_0 := \{(t,u,v) \mid t \in [a,b], u,v \in [\alpha_0(t), \beta_0(t)]\}$ and assume $g : D_0 \to \mathbb{R}$ is a L^1-Carathéodory function such that $g(t,u,v)$ is nondecreasing in u and nonincreasing in v and for some $L > \frac{\pi^2}{(b-a)^2}$, $g(t,u,v) + Lv$ is nondecreasing in v.*

Assume α_0 and β_0 are coupled lower and upper quasi-solutions of (22) *with*

$$f(t,u,v) = \frac{1}{2}[g(t,u,u) + g(t,u,v) + L(u-v)].$$

Then, both sequences $(\alpha_n)_n$ and $(\beta_n)_n$, defined for $n \geq 1$ by

$$\begin{aligned} -\alpha_n'' &= f(t, \alpha_{n-1}, \beta_{n-1}), \quad \alpha_n(a) = \alpha_n(b) = 0, \\ -\beta_n'' &= f(t, \beta_{n-1}, \alpha_{n-1}), \quad \beta_n(a) = \beta_n(b) = 0, \end{aligned}$$

converge to the same solution u of

$$-u'' = g(t,u,u), \quad u(a) = u(b) = 0.$$

Proof. By Theorem 13, the limit functions $u = \lim_{n\to\infty} \alpha_n$ and $v = \lim_{n\to\infty} \beta_n \geq u$ exist and are such that

$$\begin{aligned} -u'' &= \frac{1}{2}[g(t,u,u) + g(t,u,v) + L(u-v)], \quad u(a) = u(b) = 0, \\ -v'' &= \frac{1}{2}[g(t,v,v) + g(t,v,u) + L(v-u)], \quad v(a) = v(b) = 0. \end{aligned}$$

Hence, $w = v - u$ is a nonnegative solution of

$$-w'' = Lw + h(t), \quad w(a) = w(b) = 0,$$

where $h(t) = \frac{1}{2}[g(t,v,v) - g(t,u,v) + g(t,v,u) - g(t,u,u)] \geq 0$. Such a nontrivial solution does not exists if $L > \frac{\pi^2}{(b-a)^2}$.

Remark. A similar result holds for L such that $g(t,u,v) - Lu$ is nonincreasing in u. In this case, we use

$$f(t,u,v) = \frac{1}{2}[g(t,v,v) + g(t,u,v) + L(u-v)].$$

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