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DISPLACEMENT SOLUTIONS FOR TIME DISCRETIZATION AND EVOLUTION PROBLEM RELATED TO MINIMAL SURFACES AND PLASTICITY: EXISTENCE, UNIQUENESS AND REGULARITY IN THE ONE-DIMENSIONAL CASE

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Introduction. This paper is devoted to the existence and uniqueness for some evolution equation of elliptic type, when the data, and the second member are in $W^{1,1}$ (and for special boundary conditions). A model problem is the evolution equation for minimal surfaces,

$$u_t - \operatorname{div}\big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\big) = f.$$

But we present also the case of plasticity and other examples coming from physics.

In a basic article of R. Temam [17], the author presented an abstract result for the evolution equation of minimal surfaces

$$\begin{cases} u_t - \operatorname{div}\big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\big) = f \\ u_{|\partial\Omega} = g, \quad u(0) = u_0 \end{cases} \tag{0.1}$$

in a very weak sense. The process employed was an abstract one, based upon the theory of semigroups.

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More recent theories allow us to understand the previous equation as a particular case of the following one:

$$\begin{cases} u_t - \operatorname{div} \sigma = f, \\ u_{|\partial\Omega} = g, \quad u(0) = u_0, \quad \sigma \in \partial\psi(\nabla u), \end{cases} \tag{0.2}$$

where ψ is a convex continuous function, which is linear at infinity on $\mathbf{R}^N$, and $\psi(\nabla u)$ is taken in the sense of convex function of a measure (cf. [7]). Indeed, for natural reasons, if $f \in L^\infty(]0,T[, BV(\Omega))$ and $u_0 \in BV(\Omega)$, the problem will probably be well posed in $L^\infty(]0,T[, BV(\Omega))$. We shall see in part 1 what we mean by "$\sigma \in \partial\psi(\nabla u)$."

In the present article we are especially interested in proving some sort of regularity of u. Indeed we know only a few things about regularity for the stationary problem, which are the following.

In the two dimensional case, if Ω has some regularity properties (which include for example the case of the Euclidian ball, but also more general open sets), and if $f = 0$, $g \in \mathcal{C}^{0,\alpha}(\partial\Omega)$, $\alpha > 0$, and ψ is strictly convex, Sternberg, William and Ziemer [15] proved existence, uniqueness and regularity of a particular solution of

$$\begin{cases} -\operatorname{div} \sigma = 0 \\ u_{|\partial\Omega} = g, \quad \sigma \in \partial\psi(\nabla u) \end{cases} \tag{0.3}$$

which may also be written as u achieves the minimum of

$$\inf_{u = g_{|\partial\Omega}} \left\{ \int_\Omega \psi(\nabla u) \right\}. \tag{0.4}$$

The solution found is explicitly constructed by using geometric measure theory, and it belongs to $\mathcal{C}^{0,\frac{1+\alpha}{2}}(\Omega)$.

Let us note that before this result the only known result was the existence of a solution u in $BV(\Omega)$, for the relaxed problem

$$\inf_{u \in BV(\Omega)} \left\{ \int_\Omega \psi(\nabla u) + \int_{\partial\Omega} \psi_\infty((g-u)\vec{n}) \right\} \tag{0.5}$$

(in the previous bracket $\vec{n}$ denotes the unit outward normal on $\partial\Omega$).

Another result concerning the regularity has been derived in the one dimensional case for the problem

$$\begin{cases} \sigma' + \lambda f = 0, \quad u(0) = \alpha, \quad u(1) = \beta \\ \sigma = \psi'(u'). \end{cases} \tag{0.6}$$

Here $f \neq 0$ and λ must be smaller than the limit load $\bar{\lambda}$ which is defined as

$$\bar{\lambda} = \sup_{\exists \sigma \in \mathrm{Dom}\ \psi^\star, \sigma\prime + \lambda f = 0} \{\lambda\},$$

where $\psi^\star$ denotes the conjugate in the sense of Fenchel of ψ . The domain of $\psi^\star$ is then a bounded set of $\mathbf{R}$. A regularity result was obtained by Thierry Astruc in his thesis using elementary methods (see [2]). He proved that for $f \in L^1(]0,1[)$, there exist "regular" limit loads $\underline{\lambda}_{\text{ reg}}$ and $\bar{\lambda}_{\text{reg}}$ which are such that if λ verifies $-\underline{\lambda}_{\text{reg}} < \lambda < \bar{\lambda}_{\text{reg}}$, problem (0.6) possesses a solution u in $W^{1,1}(]0,1[)$, which fulfills the boundary conditions.

The present article is concerned with the regularity of the solutions for the continuous problem

$$\begin{cases} u_t - \operatorname{div} \sigma = \lambda f, \quad u_{|\partial\Omega} = \alpha \\ u(0,x) = u_0(x),\ x \in]0,1[, \quad \sigma \in \partial\psi(\nabla u) \end{cases} \tag{0.7}$$

as well as for the time-discretized Euler scheme

$$\begin{cases} \frac{u_{n+1} - u_n}{h} - \ \operatorname{div}(\sigma_{n+1}) = \lambda f_{n+1} \\ u_{n+1|\partial\Omega} = \alpha_{n+1}, \quad \sigma_{n+1} \in \partial\psi(\nabla u_{n+1}). \end{cases} \tag{0.8}$$

In what follows and except for the generalities in Part 1, we restricted ourselves to the one dimensional case. We first prove (Theorem 2.2), that if $f_{n+1}, u_n \in W^{1,1}(]0,1[)$ and if ψ is strictly convex on $\mathbf{R}$, there exists a unique $u_{n+1} \in W^{1,1}(]0,1[)$ which solves (0.8). Moreover, if Ψ is $\mathcal{C}^{k+1}$, $u_{n+1} \in W^{k+2,1}(]0,1[)$. In the third part we prove that the solutions to the discretized problem tend to that of the continuous one as $h \to 0$, and that for $f \in L^\infty(]0,T[, W^{1,1}(]0,1[))$, $u_0 \in W^{1,1}(]0,1[)$, there exists a unique $u \in L^\infty(]0,T[, W^{1,1}(]0,1[))$ which verifies (0.7).

We will remark in Section 2 (Theorem 2.4, ii) that the discretized equation (0.8) has no regularizing effect, if f is not sufficiently regular. More precisely, we present the case of a function u_0 which is a Heaviside function, f is in $W^{1,1}$ and the first step in (0.8) exhibits a Dirac mass for the derivative of u_1. In addition, further research proves that we have no hope to prove that the measure $|u_1'|$ is absolutely continuous with respect to $|u_0'|$. Nevertheless it is not impossible that we have $\left|u_1'^S\right| << \left|u_0'^S\right|$ where μ^S denotes the singular part of μ. This is the object of a work in preparation as well as the study of the higher dimensional case.

In all that follows, Ω will denote a bounded open set of $\mathbf{R}^N$, whose boundary is piecewise-C^1, $BV(\Omega)$ is the space of functions in $L^1(\Omega)$ whose derivatives in the sense of distributions are real bounded measures on Ω. We will denote as usually by $W^{1,p}(\Omega)$ the Sobolev space of functions whose derivatives are in $L^p(\Omega)$, $p \in [1, \infty]$.

In the third section of this article, we will handle functions defined on $]0, T[\times]0, 1[$. We will be led to work in the spaces $L^p(]0, T[\times]0, 1[)$, $L^p(]0, T[\times W^{1,q}]0, 1[)$, and $L^\infty(]0, T[, BV(]0, 1[))$. We recall the definition of the latter one:

$$L^\infty(]0,T[, BV(]0,1[) = \big\{u \in L^\infty(]0,T[\times]0,1[), \text{ such that}$$
$$u_x(t,\cdot) \in BV(]0,1[), \text{ and } \int |u_x|\,(t) \le M, \text{ for some } M, \text{ independent on } t\big\}.$$

1. Existence of solutions in BV for the discretized formulation.

1.1. Survey of known results and notations. We now recall a few points about convex functions of a measure ([7], [8]). Assume that ψ is a convex continuous function on $\mathbf{R}^d$, with values in $\mathbf{R}$, which is at most linear at infinity, i.e.,

$$\text{there exists } c_1 > 0, \text{ such that for all } \xi \text{ in } \mathbf{R}^d,\ \psi(\xi) \le c_1(|\xi| + 1), \tag{1.1}$$

and coercive

$$\text{there exists } c_0 > 0, \text{ such that for all } \xi \text{ in } \mathbf{R}^d,\ \psi(\xi) \ge c_0(|\xi| - 1). \tag{1.2}$$

When ψ has such a behaviour, one can define its asymptotic function

$$\psi_\infty(x) = \lim_{t\to\infty} \frac{\psi(tx)}{t} \tag{1.3}$$

which is also convex, lower semi-continuous, and proper. We can remark here that since ψ_∞ is positively homogeneous of degree one, in the one dimensional case ψ_∞ is nothing else but

$$\psi_\infty(x) = \psi_\infty(+1)x^+ + \psi_\infty(-1)x^-, \tag{1.4}$$

where x^+ and x^- denote the positive and negative parts of x.

We will use in the following pages and through out this article, the conjugate $\psi^\star$ of ψ in the sense of Fenchel or Legendre (cf. [9], [14] and others), defined as

$$\psi^\star(\xi) = \sup_{\eta \in \mathbf{R}^d} (\xi.\eta - \psi(\eta)). \tag{1.5}$$

It is not difficult to see that the assumptions on ψ imply that $\psi^\star$ is always lower semi-continuous and proper, and that its domain K is bounded, and verifies

$$B(0, c_0) \subset K \subset B(0, c_1). \tag{1.6}$$

From the beginning of the second section, until the end of the article, we will assume that $\Omega =]0,1[$, and ψ will be defined on $\mathbf{R}$. We will also assume for simplicity that $K =]-1,+1[$. Other slight additional assumptions will be done, some of them could be removed without loss of generality.

We now give the definition of $\psi(\mu)$ when μ is a bounded measure on Ω, with values in $\mathbf{R}^d$. Assume that the Lebesgue decomposition of μ is

$$\mu = \mu^a dx + \mu^S, \tag{1.7}$$

where μ^a is absolutely continuous with respect to the Lebesgue measure dx on $\mathbf{R}^d$, and μ^S is a singular measure. We define $\psi(\mu)$ as

$$\psi(\mu) = \psi(\mu^a)dx + \psi_\infty(\mu^S), \tag{1.8}$$

where $\psi(\mu^a)$ makes sense as an L^1 function, due to the continuity of ψ from $L^1(\mathbf{R}^d)$ into $L^1(\mathbf{R}^d)$ and

$$\psi_\infty(\mu^S) = \psi_\infty(h)\nu \tag{1.9}$$

where $\mu^S = h\nu$ is every decomposition of μ^S as the product of $h \in L^\infty(\Omega, d\nu)$, with ν a positive measure such that $\left|\mu^S\right| << \nu$. This definition makes sense since formula (1.9) does not depend on the choice of h and ν (cf. [7]). We recall now some useful properties of $\psi(\mu)$, also proved in [7].

Proposition 1.1.

i) *$\psi(\mu)$ defined in* (1.8) *is a bounded measure, absolutely continuous with respect to μ if $\psi(0)$=0 (if $\psi(0) \neq 0$, $|\psi(\mu) - \psi(0)| << |\mu|$).*

ii) *If $(\mu_n)_{n\in\mathbf{N}} \in (M^1(\Omega))^{\mathbf{N}}$ is vaguely convergent toward $\mu \in M^1(\Omega)$, then*

$$\psi(\mu) \leq \underline{lim}_{n\to\infty} \psi(\mu_n).$$

iii) *For $\mu \in M^1(\Omega)$, there exists a sequence $u_n \in (\mathcal{C}_c^\infty(\Omega))^{\mathbf{N}}$, such that u_n tends toward μ in $M^1(\Omega)$ vaguely on Ω,*

$$\int_\Omega |u_n| \to \int_\Omega |\mu|, \quad \psi(u_n) \rightharpoonup \psi(\mu) \text{ vaguely on } \Omega \quad \int_\Omega \psi(u_n) \to \int_\Omega \psi(\mu).$$

We recall now the meaning of the measure $\nabla u : \sigma$. We denote by $(\mathbf{R}^{N^2})^S$ the space of symmetric matrices of order N. Let u be in $BV(\Omega)$ and $\sigma \in L^\infty(\Omega, (\mathbf{R}^{N^2})^S)$, such that $\operatorname{div} \sigma \in L^N(\Omega, \mathbf{R}^N)$ (cf. Strang-Temam [20]). We begin to define $(\nabla u : \sigma)$ as a distribution on Ω by the formula.

$$< (\nabla u : \sigma), \varphi >= - \int_\Omega u \operatorname{div} \sigma \varphi - \int_\Omega u\, \sigma \otimes \nabla \varphi. \tag{1.10}$$

for φ in $\mathcal{D}(\Omega)$. Using approximation's results established in [7], [20], [12], one can show that $(\nabla u : \sigma)$ extends as a bounded measure on Ω, which is absolutely continuous with respect to $|\nabla u|$

$$|(\nabla u : \sigma)| \leq |\sigma|_\infty |\nabla u|. \tag{1.11}$$

Moreover, Green's formula holds in the following sense: if $\partial\Omega$ is piecewise $\mathcal{C}^1$

$$\int_\Omega (\nabla u : \sigma) = - \int_\Omega u \ \operatorname{div} \sigma + \int_{\partial\Omega} \gamma_0(u) \sigma . n \tag{1.12}$$

where $\gamma_0(u)$ denotes the internal trace of $u \in BV(\Omega)$ (see [16], [20]). We finish this small section by recalling that the well known pointwise inequality

$$\psi(\xi)_+ \psi^\star(\eta) \geq \xi . \eta \ \text{ for all } (\xi, \eta) \in \mathbf{R}^d \tag{1.13}$$

extends to measures as follows: assume that $u \in BV(\Omega)$, $\sigma \in L^\infty(\Omega,\ \operatorname{dom} \psi^\star)$, $\operatorname{div} \sigma \in L^N(\Omega)$. Then

$$\psi(\nabla u) + \psi^\star(\sigma) \geq (\nabla u : \sigma) \tag{1.14}$$

holds true, in the sense of measure.

Let us note that if $k = N = 1$ (one dimensional case), we do not need to define $u'\sigma$, since here $\sigma \in L^\infty(]0,1[$ and $\sigma' \in L^1(]0,1[)$ imply that σ is continuous. Hence $u'\sigma$ is naturally defined as the product of a continuous function with a bounded measure.

1.2. A model discretized problem for evolution equations related to calculus of variations and plasticity. In this section we still assume that Ω is a bounded open set of $\mathbf{R}^N$, whose boundary $\partial\Omega$ is piecewise $\mathcal{C}^1$, h is a small positive parameter, $f \in L^N(\Omega, \mathbf{R})$, and v belongs to $W^{1,1}(\Omega, \mathbf{R}) \cap$

$L^2(\Omega, \mathbf{R})$, $\alpha \in L^1(\partial\Omega)$. We are looking for $(u, \sigma) \in (L^2 \cap W^{1,1})(\Omega, \mathbf{R}) \times L^\infty(\Omega, \mathbf{R}^N)$ such that

$$\begin{cases} \frac{u-v}{h} = \operatorname{div}\sigma + f \text{ a.e. } x \in \Omega, \\ u_{|\partial\Omega} = \alpha, \quad \sigma \in \partial\psi(\nabla u)(x) \text{ a.e. } x \in \Omega. \end{cases} \tag{1.15}$$

Problem (1.15) represents one step for the time-discretization of the following evolution problem.

To find (u, σ) defined on $[0, T] \times \Omega$, which satisfy

$$\begin{aligned} &u_t - \operatorname{div}\sigma = f, \quad u(0, x) = u_0(x) \\ &u(t, x)_{/\partial\Omega} = \alpha(t, x) \text{ a.e. } x \in \partial\Omega, \quad \sigma(t, x) \in \partial\psi(\nabla_x u(t, x)). \end{aligned}$$

We use here the implicit Euler scheme

$$\begin{aligned} &\frac{u_{n+1} - u_n}{h} - \operatorname{div}\sigma_{n+1} = f_{n+1} \\ &u_{n+1|\partial\Omega} = \alpha_{n+1}, \quad \sigma_{n+1} \in \partial\psi(\nabla u_{n+1}), \end{aligned}$$

where (α_{n+1}, f_{n+1}) are some approximations of (α, f) that we will precise in the sequel.

A first step to solve (1.15) is classically done by considering *it* as the Euler equation for the minimization problem

$$\inf_{\{u \in (W^{1,1}(\Omega, \mathbf{R}) \cap L^2); u = \alpha_{|\partial\Omega}\}} \left\{ \int_\Omega \psi(\nabla u) + \frac{1}{2h} \int |u - v|^2 - \int fu \right\}. \tag{1.16}$$

It is classical to verify by using generalized Green's formula and Convex analysis that if $u \in W^{1,1}(\Omega)$ is a solution of (1.16), which fills the boundary condition, then it solves (1.15), and conversely. Unfortunately classical methods in the calculus of variations (or plasticity) do not allow us to conclude to the existence of such solutions. More precisely, we shall prove in what follows the existence of solutions in a very weak sense: u belongs to $BV \cap L^2$ and solves a relaxed form of (1.16), that we will make precise in the sequel.

We use a method which is analogous to the one used by several authors, as Temam and Strang [20], Temam and Kohn [12], Demengel [6], Anzellotti and Giaquinta [4], Suquet [16], etc... Let us note that we will obtain results which are still valid (with some natural technical changes) for the following problem, inherited from the three dimensional perfect plasticity. Defining the space $U(\Omega)$ as

$$U(\Omega) = \{u \in L^1(\Omega, \mathbf{R}^N), \varepsilon_{ij}(u) = \frac{u_{i,j} + u_{j,i}}{2} \in M^1(\Omega, \mathbf{R}) \, \forall (i,j) \in [1,N]^2, \operatorname{div} u \in L^2(\Omega)\},$$

we look for (u, σ) such that $u \in U(\Omega)$, and

$$\begin{cases} \frac{u-v}{h} = (\operatorname{div} \sigma) + f, \quad u(0,x) = u_0(x), \\ u_{|\partial\Omega} = \alpha, \quad \sigma \in \partial\psi(\varepsilon(u)) \end{cases} \tag{1.17}$$

with $\psi(\xi) = \psi^D(\xi^D) + \alpha(\mathrm{tr}\xi)^2$, and ψ^D is at most linear at infinity and defined on $E^D = \{M \in (\mathbf{R}^{N^2}) \ M_{ij} = M_{ji}$ and $M_{ii} = 0\}$ and $\xi^D = \xi - \frac{\mathrm{tr}\xi}{N} Id \in E^D$. We begin to state a theorem which allows us to consider σ as the solution of a variational problem.

Theorem 1.1. i) *Let us consider the variational problem*

$$\sup_{\{\sigma \in L^\infty(\Omega,K), \operatorname{div}\sigma \in L^2(\Omega,\mathbf{R})\}} \{-\int_\Omega (\psi^\star(\sigma) + \frac{h}{2}|\operatorname{div}\sigma + f|^2) + \int_{\partial\Omega} \sigma.n\alpha - \int_\Omega (\operatorname{div}\sigma + f)v\}. \tag{1.18}$$

Then we have

$$\inf(1.16) = \mathrm{Sup}(1.18), \tag{1.19}$$

and (1.18) *possesses a solution* $\bar\sigma$. *Moreover if* $\bar u$ *is a solution of* (1.16), *then* $\bar\sigma = \partial\psi(\nabla \bar u)$.

Proof of Theorem 1.1. In order to prove (1.19) we first introduce the convenient spaces and operators to apply theorem (2.1) of [9].

We define the spaces V and Y as $V = W^{1,1} \cap L^2$, $Y = L^2 \times L^1(\Omega, \mathbf{R}^N)$ and the linear operator

$$\begin{aligned} \Lambda \ : \quad V &\to \quad Y \\ u &\mapsto \quad (u, \nabla u). \end{aligned}$$

Then the dual space of Y is $Y^\star = L^2 \times L^\infty(\Omega, \mathbf{R}^N)$. We denote by $\Lambda^\star$ the adjoint operator of Λ, $\Lambda^\star : \ Y^\star \to V^\star$. We finally define the functionals F

and G as

$$\begin{aligned} F : V &\to \mathbf{R} \\ u &\mapsto \begin{cases} -\int_\Omega fu & \text{if } \; u = \alpha_{|\partial\Omega} \\ +\infty & \text{if not} \end{cases} \\ G : Y &\to \mathbf{R} \\ (w,\tau) &\mapsto \int \psi(\tau) + \int \frac{|w-v|^2}{2h}. \end{aligned}$$

With these notations problem (1.16) reads also $\inf_{u\in V}\{F(u)+G(\Lambda u)\}$ and by using classical results in convex analysis (cf. [9]), it has the following dual problem $\sup_{(p,\sigma)\in Y^\star}\{-F^\star(\Lambda^\star(p,\sigma)) - G^\star(-(p,\sigma))\}$. To compute $G^\star$ is classical. One obtains, using either explicit computation, either a general result of Krasnoselskii [13]

$$G^\star(p,\sigma) = \begin{cases} \int_\Omega \psi^\star(\sigma) + \frac{h}{2}\int_\Omega |p|^2 + \int_\Omega pv, & \text{if } \sigma(x) \in \mathrm{Dom}\psi^\star \text{ a.e.} \\ +\infty & \text{if not.} \end{cases}$$

The computation of $F^\star(\Lambda^\star(p,\sigma))$ can be done by using classical methods in duality and convex analysis. One obtains that

$$F^\star(\Lambda^\star(p,\sigma)) = \begin{cases} \int(\sigma.n)\alpha & \text{if } \; -\operatorname{div}\sigma + p + f = 0 \\ +\infty & \text{if not.} \end{cases}$$

One finally get the announced result by changing (p,σ) in $(-p,-\sigma)$ and using the relation between p, σ and f. Moreover, Theorem (2.1) of [9] gives the Inf-Sup equality. The existence of $\bar\sigma$ is easy to prove and is left to the reader.

We now introduce, as announced above, a weak formulation of (1.16). We first remark that the theory of convex functions of measure allows us to extend Problem (1.16) to functions $u \in (BV\cap L^2)(\Omega)$, as

$$\inf_{\{u\in(L^2\cap BV)(\Omega),\ u=\alpha_{|\partial\Omega}\}} \Big\{\int_\Omega \psi(\nabla u) + \frac{1}{2h}\int_\Omega |u-v|^2 - \int_\Omega fu\Big\}, \tag{1.20}$$

where $\psi(\nabla u)$ makes sense by (1.8). Moreover, using an approximation result in [7], one can show that $\inf(1.20) = \inf(1.16)$. At this stage, the usual method to solve problems as (1.20) is to consider a minimizing sequence u_n

of (1.20), and to observe that u_n is bounded in $BV \cap L^2$. Then, one is allowed to extract from u_n a subsequence, still denoted u_n, such that

$$u_n \rightharpoonup u \text{ in } L^2, \qquad \nabla u_n \rightharpoonup \nabla u \text{ in } M^1.$$

Therefore, by using weak lower semi-continuity of $\mu \mapsto \int_\Omega \psi(\mu)$, as well as the weak-$L^2$ semi-continuity, one obtains

$$\begin{aligned} &\int_\Omega \psi(\nabla u) + \frac{1}{2h}\int_\Omega |u-v|^2 - \int_\Omega fu \\ &\leq \underline{\lim}_{n\to+\infty} \int_\Omega \psi(\nabla u_n) + \frac{1}{2h}\int_\Omega |u_n - v|^2 - \int_\Omega f u_n. \end{aligned}$$

Let us note that since the trace map γ_0 is not continuous for the weak topology , the weak convergence in BV of u_n toward u is not sufficient to ensure that $u_{n|\partial\Omega} \to u_{|\partial\Omega}$. To overcome this difficulty, we introduce a relaxed form of (1.20), in which the boundary condition $u = \alpha_{|\partial\Omega}$ has been inserted into the functional. This relaxed form of (1.20) reads

$$\inf_{u\in BV} \Big\{ \int_\Omega \psi(\nabla u) + \int_{\partial\Omega} \psi_\infty((\alpha - u)\vec{n}) + \frac{1}{2h}\int_\Omega |u-v|^2 - \int_\Omega fu \Big\}. \quad (1.21)$$

Moreover, one can show, using an approximation process in [7], that inf(1.21) = inf(1.20) = inf(1.16). (1.21) possesses a solution in BV. We omit the details of these facts, because it consists in bringing minor changes to the method used in [17], [6].

One can derive from the previous equalities some interesting consequences. Assume that u is a solution of (1.21) and σ a solution of (1.18). Then the following extremality relations hold

$$\psi(\nabla u) - (\nabla u : \sigma) + \psi^\star(\sigma) = 0 \text{ in } \Omega, \quad (1.22)$$

$$\frac{1}{2h}|u-v|^2 + \frac{h}{2}|\operatorname{div}\sigma + f|^2 - ((u-v)(\operatorname{div}\sigma + f)) = 0 \text{ in } \Omega, \quad (1.23)$$

$$\psi_\infty((\alpha - u)\vec{n}) = (\sigma.\vec{n})((\alpha - u)) \text{ on } \partial\Omega. \quad (1.24)$$

Indeed, we can derive from the equality sup(1.18) = inf(1.21) that

$$\begin{aligned} &\int_\Omega \psi(\nabla u) + \int_{\partial\Omega} \psi_\infty((\alpha-u)\vec{n}) + \frac{1}{2h}\int_\Omega |u-v|^2 - \int_\Omega fu \\ &= -\int_\Omega \psi^\star(\sigma) + \int_\Omega \sigma.n\ \alpha - \frac{h}{2}\int_\Omega |\operatorname{div}\sigma + f|^2 - \int_\Omega (\operatorname{div}\sigma + f)v. \end{aligned}$$

Now, using (1.12),

$$\int_\Omega \nabla u : \sigma = -\int_\Omega \operatorname{div} \sigma u + \int_{\partial\Omega} \sigma.\vec{n}u.$$

one obtains

$$\Big(\int_\Omega \psi(\nabla u) - \int_\Omega \nabla u : \sigma + \int_\Omega \psi^\star(\sigma)\Big) + \Big(\int_{\partial\Omega} \psi_\infty((\alpha - u)\vec{n}) - \int_{\partial\Omega} (\sigma.\vec{n}(\alpha - u))\Big)$$
$$+ \Big(\frac{1}{2h}\int_\Omega |u - v|^2 + \frac{h}{2}\int_\Omega (\operatorname{div} \sigma \ + f)^2 - \int_\Omega (\operatorname{div} \sigma + f)(u - v)\Big) = 0.$$

Now we remark that each of these three groups of integrals is the integral of a nonnegative function. This leads to (1.22), (1.23), (1.24). Let us observe that (1.23) is also

$$\frac{u - v}{h} = \operatorname{div} \sigma + f,$$

which is exactly the first line of (1.17).

In the following section we will restrict our attention to the one dimensional case. We can assume without loss of generality that $\Omega =]0, 1[$. The equations in (1.17) read

$$\frac{u - v}{h} = \sigma' + f, \quad u(0) = \alpha, \quad u(1) = \beta, \quad \sigma = \psi'(u') \tag{1.25}$$

and we assume that $f \in W^{1,1}(]0, 1[)$, $v \in W^{1,1}(]0, 1[))$, and $\alpha = (hf + v)(0), \beta = (hf + v)(1)$ (the values $(hf + v)(0)$ and $(hf + v)(1)$ make sense since $(f, v) \in W^{1,1}(]0, 1[)$ are continuous on $[0, 1]$).

The main result of the next section is about the smoothness of u. We prove that u belongs to $W^{1,1}(]0, 1[)$. (in place of BV(]0,1[)). As we shall see at the end of Section 2, u has no higher smoothness if we do not have higher regularity on (f, v).

Another result about this "no regularizing effect" is illustrated at the end of Section 2. We describe a case where $v \in BV$ has a jump and u has a jump. Nevertheless, we observe there that the jump decreases. This may be a chance to prove that, after a positive finite time, the continuous evolution equation has a regularizing effect.

2. Existence, uniqueness and regularity for the one-dimensional problem. As we announced it at the end of Section 1, the one-dimensional problem (1.25) reads

$$\frac{u - v}{h} = \sigma' + f, \quad \sigma = \psi'(u'), \quad u(0) = (v + hf)(0), \quad u(1) = (v + hf)(1), \tag{2.1}$$

where we assume that h is a positive parameter (not necessarily small at this time), and (f, v) belong to $W^{1,1}$. The assumptions on ψ are

i) ψ is strictly convex, ψ is $\mathcal{C}^2$ on $\mathbf{R}$,

ii) ψ is linear at infinity: there exist $c_0, c_1 > 0$, such that for all ξ in $\mathbf{R}$

$$c_0(|\xi| - 1) \leq \psi(\xi) \leq c_1(|\xi| + 1). \tag{2.2}$$

iii) $\psi'' > 0$ for simplicity (which excludes for example the case $\psi(x) = x^4$ in a neighborhood of 0). In fact, in the general case $\psi'' \geq 0$, the final regularity result still holds, as we shall remark at the end of the section.

iv) $\psi'(0) = 0$. This can be done, as one can see from the equation, without loss of generality

v) Dom $\psi^\star$ is the symmetric set $[-1, +1]$.

Remark 2.1. We shall use below that due to $\psi'(0) = 0$ and to the convexity of ψ, we have $x\psi'(x) \geq 0$.

Before giving the regularity result announced, we derive some consequences from the weak existence result obtained in the previous section. We have obtained that given v and f in $W^{1,1}(]0,1[)$, there exists $(u, \sigma) \in BV(]0,1[) \times L^\infty(]0,1[)$, such that

$$\begin{cases} \frac{u-v}{h} = \sigma' + f \text{ in } \ ]0,1[, \\ \psi(u') + \psi^\star(\sigma) = u'\sigma \text{ in } \ ]0,1[, \\ |\alpha - u(0^+)| = \sigma(0)(u(0) - \alpha), \\ |\beta - u(1)| = \sigma(1)(\beta - u(1)), \end{cases} \tag{2.3}$$

where $\alpha = (v + hf)(0)$, $\beta = (v + hf)(1)$.

The process to prove the existence of u consists in remarking that u, σ are respectively solutions of the two following minimization problems

$$\inf_{u \in BV(]0,1[)} \Big\{ \int_0^1 \psi(u') + \frac{1}{2h} \int_\Omega |u - v|^2 - \int_\Omega fu + |\alpha - u(0^+)| + |\beta - u(1^-)| \Big\} \tag{2.4}$$

$$\sup_{\sigma \in H^1(]0,1[)} \Big\{ -\int_\Omega \psi^\star(\sigma) - \frac{h}{2} \int_\Omega |\sigma' + f|^2 - \int_\Omega (\sigma' + f)v + \sigma(1)\beta - \sigma(0)\alpha \Big\}. \tag{2.5}$$

A first step to prove that u has some regularity consists in observing that u is continuous inside $]0,1[$. In other words, $\bar{u}$ cannot have a jump inside $]0,1[$. The proof we present below makes use of the continuity of v, which is necessary, as the counterexample at the end of Section 2 shows.

Lemma 2.1. *u is continuous on* $]0,1[$.

Proof of Lemma 2.1. To prove the continuity of u inside $]0,1[$, assume that u presents a positive jump on some point x_0 inside $]0,1[$. Then the extremality relation

$$\psi_\infty((u^+ - u^-)(x_0)) = \sigma(x_0)(u^+ - u^-)(x_0) \tag{2.6}$$

derived from (1.9), and the definition of ψ_∞ imply that $\sigma(x_0) = 1$. Hence $\sigma'(x)(x - x_0) \leq 0$ around x_0. By using the first equation in (2.3) and letting x go to x_0 for $x < x_0$ and $x > x_0$ one obtains

$$(u - v)(x_0^-) \geq 0 \qquad (2.7)$$

$$(u - v)(x_0^+) \leq 0 \qquad (2.7)$$

from which one gets that $(u(x_0^+) - u(x_0^-)) \leq 0$, which is a contradiction with the assumption.

By analogous arguments if $[u](x_0) < 0$, $\sigma(x_0) = -1$, σ has a minimum on x_0, $\sigma'(x_0^-) \leq 0$, $\sigma'(x_0^+) > 0$ and then $[u](x_0) \geq 0$, a contradiction.

We now prove that the boundary conditions are fulfilled. Assume that $\alpha > u(0^+)$; then (2.3) gives $\sigma(0) = -1$, σ achieves its minimum on 0, and $\sigma'(0^+) \geq 0$. Using the first equation in (2.3) one obtains, by passing to the limit $\left(\frac{u-v}{h} - f\right)(0) \geq 0$ and then $u(0^+) \geq \alpha$. By the same process one proves that $\alpha < u(0^+)$ implies by the third equation in (2.3) that $\sigma(0^+) = 1$, 0 is a maximum for σ, $\sigma'(0^+) \leq 0$, and by the first equation in (2.3), one finally gets $\frac{u-v}{h}(0^+) \neq 0$, a contradiction. Analogous considerations imply that $u(1) = \beta$.

In order to prove further regularity on u, we need a key argument concerning comparison between solutions. For the sake of simplicity, it will be useful to replace in the sequel the variable u by its derivative w. We are led to introduce the problem

For g in $L^1(]0,1[)$, find (w,σ) in $L^1(]0,1[\times W^{2,1}(]0,1[)$, such that

$$w - \sigma'' = g, \tag{2.8}$$

$$\sigma'(0) = \sigma'(1) = 0, \tag{2.9}$$

$$\sigma = \psi'(w). \tag{2.10}$$

Let us note that until now we have only been able to prove that $(w,\sigma) \in M^1(]0,1[\times HB(]0,1[)$,[1] ($\sigma'(0)$ and $\sigma'(1)$ are taken in the sense of the trace of BV functions).

[1] $HB(]0,1[) = \{\sigma \in W^{1,1}(]0,1[),\ \sigma'' \in M^1(]0,1[\}$.

To see that (2.1)–(2.4) are equivalent to (2.8)–(2.10), let w be a solution to (2.8)–(2.10) with $(hf+v)' = g$, and define $u(x) = (v+hf)(0)) + \int_0^x w$. Then u is a solution for (2.1)–(2.4).

In Theorem 2.1 below, we begin to prove existence, uniqueness and regularity result, assuming that g belongs to L^∞. Then, denoting by $T(g)$ the function w, we observe that T is an increasing mapping .

Theorem 2.1.

i) *Assume that* $w \in M^1(]0,1[)$ *is a solution of* (2.8)–(2.10). *Then* w *is unique. We define* $T(g) = w$.
ii) T *maps* L^∞ *into* $W^{2,\infty}(]0,1[)$.
iii) T *is non decreasing on* L^∞ *functions.*

Proof of Theorem 2.1. Let w_i, $i = 1,2$ be two solutions of (2.8)–(2.10). We make the difference $w = w_1 - w_2$, $\sigma = \sigma_1 - \sigma_2$ and obtain

$$w - \sigma'' = 0,\ \sigma'(0) = \sigma'(1) = 0. \tag{2.11}$$

We are allowed to multiply by $(\sigma_1 - \sigma_2)$ which is continuous. We obtain by integrating over $]0,1[$

$$\int_0^1 w\sigma - \int_0^1 \sigma''\sigma = 0.$$

Using a generalized Green's Formula on BV and (2.11), one gets

$$-\int_0^1 \sigma''\sigma = \int_0^1 \sigma'^2$$

and then

$$\int_0^1 w\sigma + \int_0^1 \sigma'^2 = 0.$$

In order to conclude here, we need to prove that, as a measure, $w\sigma \geq 0$. For that purpose , we recall from the theory of convex functions of a measure [7] that the pointwise inequalities

$$XY \leq \psi(X) + \psi^\star(Y) \quad \text{and} \quad X\partial\psi(X) = \psi(X) + \psi^\star(\partial\psi(X))$$

may be extended to measures and suitable functions σ. As a consequence, we get $w_i\sigma_i = \psi(w_i) + \psi^\star(\sigma_i)$, $i = 1,2$ and $w_i\sigma_j \leq \psi(w_i) + \psi^\star(\sigma_j)$. From this we derive that $(w_1 - w_2)(\sigma_1 - \sigma_2) \geq 0$. Finally $\sigma'^2 = 0$, and $\sigma_1 - \sigma_2 = cte$.

Either the constant is non zero and then $w_1 = w_2$, either $\sigma_1 = \sigma_2$ which also implies that $w_1 = w_2$.

ii) To prove that T maps L^∞ into itself, one can either use a Galerkin method, or a regularization by a viscous term, and get a priori L^∞ estimates independent on the viscous parameter $\epsilon > 0$. Assume that $w^\varepsilon, \sigma^\varepsilon$ are the solutions of

$$w^\varepsilon - \varepsilon w^\varepsilon_{xx} - \sigma^\varepsilon_{xx} = g, \quad w^\varepsilon_x(0) = w^\varepsilon_x(1) = 0, \quad \sigma_\epsilon = \psi'(w_\epsilon). \tag{2.12}$$

The existence of $(w_\epsilon, \sigma_\epsilon)$ can be obtained by considering w_ϵ as the derivative of the solution u^ϵ of the variational problem

$$\inf_{\{u\in H^1(]0,1[),u(0)=\alpha,u(1)=\beta\}}\{\frac{1}{2}\int_0^1 (u^\epsilon)^2 + \frac{\epsilon}{2}\int_0^1 (u^\epsilon_x)^2 + \int_0^1 \psi(u^\epsilon_{,x}) - \int_0^1 g u_\epsilon\}.$$

It is classical that w^ε belongs to $H^1(]0,1[)$. Multiplying by $|w^\varepsilon|^q w^\varepsilon$, one obtains, after integrating by parts

$$\int_0^1 |w^\varepsilon|^{q+2} + \varepsilon(q+1)\int_0^1 |w^\varepsilon|^q (w^\varepsilon)^2_x + (q+1)\int_0^1 \sigma^\varepsilon_x |w^\varepsilon|^q (w^\varepsilon_x) = \int_0^1 g|w^\varepsilon|^q w^\varepsilon.$$

Using $w^\varepsilon_x \sigma^\varepsilon_x = \psi''(w^\varepsilon) {w^\varepsilon_x}^2 \geq 0$, one gets

$$|w^\varepsilon|^{q+2}_{L^{q+2}} \leq |g|_{L^{q+2}} |w^\varepsilon|^{q+1}_{L^{q+2}};$$

hence

$$|w^\varepsilon|_{L^{q+2}} \leq |g|_{L^{q+2}}. \tag{2.13}$$

This implies that T sends L^{q+2} into L^{q+2}, for every $q \geq 0$, and letting q go to infinity we obtain the analogous result in L^∞. Moreover, $|T(g)|_{L^q} \leq |g|_{L^q}, \forall q < \infty$.

It remains to prove that $(\sigma_\epsilon, w_\epsilon) \to (\psi'(w), w)$, at least in a weak sense. For that purpose, we multiply the first equation in (2.12) by σ_ϵ to obtain

$$\int_0^1 w^\epsilon \sigma^\epsilon + \epsilon \int_0^1 w^\epsilon_x \sigma^\epsilon_x + \int_0^1 (\sigma^\epsilon_x)^2 = \int_0^1 g\sigma^\epsilon_x.$$

Using $w^\epsilon \sigma^\epsilon \geq 0$ and $w^\epsilon_x \sigma^\epsilon_x \geq 0$, one obtains that σ^ϵ_x is bounded in L^2. Since $\sigma^\epsilon_x = \psi''(w_\epsilon) w^\epsilon_x$, and since we have assumed that ψ'' is bounded from below on bounded sets, one obtains that w^ϵ is bounded in H^1 (independently of

ϵ). As a consequence, one may extract from w^ϵ a subsequence, still denoted w^ϵ, such that

$$w^\epsilon \to w \ \text{ in } \ H^1 \ \text{ weakly}, \qquad w^\epsilon \to w \ \text{ a.e. in } \ ]0,1[,$$

and then $\psi'(w^\epsilon) \to \psi'(w)$ a.e. We have finally obtained that w is a solution of

$$w - (\psi'(w))_{,xx} = g.$$

The previous estimates on w_ϵ imply that $w \in L^\infty$, and $|w|_\infty \leq |g|_\infty$. Now, using $\psi''(w)w_x = u - f \in W^{1,1}$ and the boundedness from below of ψ'' on compact sets, together with the boundedness of w, one gets that $w_x \in W^{1,1}$, and then $u \in W^{2,1}$. This ends the proof of assertion ii).

iii) To prove that T is non decreasing, let $w_i = T(g_i)$, $i = 1, 2$, and assume that $g_1 \leq g_2$. We subtract the two equations defining w_i, $i = 1, 2$, multiply by $(\sigma_1 - \sigma_2)^+$, and integrate by parts. We get

$$\int_0^1 (w_1 - w_2)(\sigma_1 - \sigma_2) + \int_0^1 ((\sigma_1 - \sigma_2)^+_{,x})^2 = \int_0^1 (g_1 - g_2)(\sigma_1 - \sigma_2)^+ \leq 0.$$

By the increasing behaviour of ψ', $(\sigma_1 - \sigma_2)^+(w_1 - w_2) = (\sigma_1 - \sigma_2)^+(w_1 - w_2)^+$, and all the quantities under the integrals on the left are non- negative. One obtains

$$(\sigma_1 - \sigma_2)(w_1 - w_2)^+ = 0$$

which implies that $w_1 \leq w_2$.

We are now able to prove the main result of this section.

Theorem 2.2. *Assume that $g \in L^1(]0,1[)$ and w is the solution of*

$$w - \sigma'' = g, \qquad \sigma'(0) = \sigma'(1) = 0, \quad \sigma = \psi'(w).$$

Then $w = T(g) \in L^1$. Moreover, T is a contraction mapping from L^1 into L^1.

Proof. Let g^p be in L^∞, which converges towards g in $L^1(]0,1[)$. By theorem 2.1 $w^p = T(g^p)$ belongs to $W^{1,1}$ and is then continuous. Let p and q be two integers. We write the two equations satisfied by w^p and w^q respectively, and subtract them. Let Y_ϵ be defined as

$$Y_\epsilon(x) = \begin{cases} \frac{x}{\epsilon} & \text{if } |x| < \epsilon \\ 1 & \text{if not .} \end{cases}$$

Let us note that if Y^+ denotes the positive Heaviside function, Y_ϵ tends to $Y^+ - Y^-$ in L^∞ weakly star and almost everywhere. Let us multiply the equation verified by $w^p - w^q$ by $Y_\epsilon(\sigma^p - \sigma^q)$ and integrate by parts the second integral, to obtain

$$\int_0^1 (w^p - w^q)(Y_\epsilon(\sigma^p - \sigma^q)) + \int_0^1 (\sigma^p - \sigma^q)^2_{,x} Y'_\epsilon(\sigma^p - \sigma^q) = \int_0^1 (g^p - g^q) Y_\epsilon(\sigma^p - \sigma^q).$$

The second integral on the left hand side reads

$$\frac{1}{\epsilon}\int_{\{x, |\sigma^p - \sigma^q| < \epsilon\}} (\sigma^p - \sigma^q)^2_{,x} \geq 0.$$

Moreover, $Y_\epsilon(\sigma^p - \sigma^q)$ tends towards $Y^+(w^p - w^q) - Y^-(w^p - w^q)$ in L^∞ weakly star, (because the functions are sufficiently regular and ψ' is invertible.) As a consequence, one obtains

$$\int_0^1 |w^p - w^q| \leq \int_0^1 |g^p - g^q|$$

and then w^p is a Cauchy sequence in L^1. Let w be its limit in L^1. Since we can assume by extracting a subsequence that w^p converges to w almost everywhere, we also have $\psi'(w^p) \to \psi'(w)$ almost everywhere. We have obtained that w is the solution of (2.1), and $w \in L^1$. To prove that T is a contraction mapping, we take g_1 and g_2 in L^1 and g_1^p and g_2^p, with $g_i^p \in L^\infty$, $i = 1, 2$, which converge in L^1 respectively to g_1 and g_2. Denoting by w_i^p the solutions $T(g_i^p)$, one obtains that

$$\int_0^1 |w_1^p - w_2^p| \leq \int_0^1 |g_1^p - g_2^p|.$$

Passing to the limit when p goes to infinity, and using the first part of the proof, we get

$$\int_0^1 |w_1 - w_2| \leq \int_0^1 |g_1 - g_2|$$

with $w_i = T(g_i), i = 1, 2$, which yields the desired result.

Theorem 2.3. *The increasing behaviour of T can be extended to L^1 functions.*

Proof. Take $g_1 \leq g_2$, $g_1, g_2 \in L^1(]0, 1[)$, and g_1^p and g_2^p be two sequences in L^∞ such that $g_1^p \leq g_2^p$, which converge in L^1 respectively to g_1 and g_2. We have, by Theorem 2.2, $T(g_1^p) \leq T(g_2^p)$ and by the content of the Proof of Theorem 2.2, $T(g_1) \leq T(g_2)$. This ends the proof of Theorem 2.3.

Theorem 2.4. (*Continuity of T for different topologies*) *Assume that $\psi \in \mathcal{C}^{k+1}$ and $g \in W^{k,p}(]0,1[)$, then $T(g) \in W^{k+2,p}(]0,1[)$.*

Proof of Theorem 2.4. We prove it by a recursive way on k. Suppose that $k=0$ and $g \in L^p$. Multiplying the equation by $|w|^{p-2}w$ and integrating by parts, one gets

$$\int_0^1 |w|^p + (p-1)\int_0^1 \sigma_{,x}|w|^{p-2}w_{,x} = \int_0^1 g|w|^{p-2}w \leq |g|_p|w|_p^{p-1}.$$

Using $\sigma_{,x}w_{,x} \geq 0$ one gets that $|w|_p \leq |g|_p$. Assume now that $k=1$. Then $g \in W^{1,p} \subset L^\infty$, hence $T(g) = w \in L^\infty$. The equation

$$w - \sigma'' = g$$

implies that $\sigma'' = w - g \in L^\infty$. From the primitive equation $u - \sigma' = f$, we get $\sigma' \in L^2$, and then $w' \in L^2$, since $\sigma' = \psi''(w)w'$ and ψ'' is bounded from below on bounded sets. From this remark and from the equation

$$\sigma'' = \psi''(w)w'' + \psi'(w)w'^2 \in L^\infty,$$

we derive the following $w'^2 \in L^1$ and $\sigma'' \in L^\infty \Longrightarrow \psi''(w)w'' \in L^1 \Longrightarrow w'' \in L^1 \Longrightarrow w' \in L^\infty$. Using $\sigma'' \in L^\infty$ once more, one finally gets that $w'' \in L^\infty$. We differentiate once again the equation to get

$$w' - \sigma^{(3)} = g' \Longrightarrow \sigma^{(3)} = -g' + w' = \psi''(w)w^{(3)} + P_3(w, w', w''),$$

where P_3 is a polynomial with coefficients in L^∞, in w, w', w''. This implies that if $g \in W^{1,p}$, then $\sigma^{(3)} \in L^p$ and $w \in W^{3,p}$. This gives the result for $k=1$. Suppose that we have proved that $g \in W^{k,p}$ implies $T(g) \in W^{2+k,p}$, and suppose that $g \in W^{k+1,p}$. Then $w = T(g) \in W^{2+k,p}$ and

$$w^{(k+1)} - \sigma^{(k+3)} = g^{(k+1)} \Longrightarrow \sigma^{(k+3)} = w^{(k+1)} - g^{(k+1)} \in L^p.$$

By a recursive way, one can prove the formula

$$\sigma^{(k+3)} = \psi''(w)w^{(k+3)} + c_{k+3}(w)w'w^{(k+2)} + P_{k+3}(w, w^1, ..., w^{(k+1)}),$$

where $c_{k+3}(w)$ is in L^∞ and P_k is a polynomial with coefficients in L^∞. We obtain, using $w' \in L^\infty$, that $\psi''(w)w^{(k+3)} \in L^p \Longrightarrow w \in W^{k+3,p}$. This implies the result.

A counterexample to the regularity of u when v is not in $W^{1,1}(\Omega)$. We conclude this section by exhibiting a counter example to the "regularity" of u when $g = f'$, or v' present a singular part. Let u be defined as

$$u(x) = \frac{-9}{2^{5/3}}\Big(\frac{1}{2} - x\Big)^{1/3} - x^2 \quad \text{for} \quad x < \frac{1}{2},$$
$$= \frac{9}{2^{5/3}}\Big(x - \frac{1}{2}\Big)^{1/3} + x^2 + \frac{1}{2} \quad \text{for} \quad x > \frac{1}{2}.$$

u is everywhere differentiable, except on $\frac{1}{2}$. We have a jump at this point, since $u\big(\frac{1}{2}\big)^- = -\frac{1}{4}$, $u\big(\frac{1}{2}\big)^+ = \frac{3}{4}$. Let $\sigma = \dfrac{u'}{\sqrt{1+u'^2}}$. σ may be extended as a $\mathcal{C}^1$ function on [0,1]. Indeed we have $\sigma\Big(\big(\frac{1}{2}\big)^{\pm}\Big) \sim 1$ and

$$\sigma'\Big(\frac{1}{2}\Big) \sim \frac{u''}{\big(1+u'^2\big)^{3/2}}\Big(\frac{1}{2}\Big) \sim c\Big(\frac{1}{2} - x\Big)^{1/3}$$

if we take

$$v = Y^+\Big(x - \frac{1}{2}\Big) = \begin{cases} 0 & \text{if } x < \frac{1}{2}, \\ 1 & \text{if } x > \frac{1}{2}, \end{cases}$$

and $f(x) = (u-v)(x) - \sigma'(x)$. Then f is continuous, $f(0) = (u-v)(0)$, $f(1) = (u-v)(1)$ and $f \in W^{1,1}$, since $v' = \delta_{1/2}$ and

$$u' = \delta_{1/2} + \frac{3}{2^{5/3}}|\tfrac{1}{2} - x|^{-2/3} - 2|x - 1/2| + \operatorname{sign}(x - 1/2).$$

Moreover, σ' is continuous. Let ψ be the function $\psi(x) = \sqrt{1+x^2}$. Using the uniqueness result, we see that u is the solution of

$$u - (\psi'(u'))' = v + f, \quad u(0) = v(0) + f(0), \quad u(1) = v(1) + f(1).$$

3. The evolution problem. We now consider the evolution problem

$$u_t - \sigma_x = f(t,x), \text{ in }]0,T[\times]0,1[\tag{3.1}$$
$$\sigma = \psi'(u_x), \tag{3.2}$$

with the boundary conditions

$$u(0,x) = u_0(x), \tag{3.3}$$
$$u(t,0) = \alpha(t), \tag{3.4}$$
$$u(t,1) = \beta(t). \tag{3.5}$$

The regularity assumptions on f, α, β, u_0 are the following: $u_0 \in W^{1,1}(]0,1[)$; $f \in L^1(]0,T[; W^{1,1}(]0,1[)), \alpha$ and β belong to $W^{1,\infty}(]0,T[)$. We also assume that the following compatibility relations hold: $u_0(0) = \alpha(0)$; $u_0(1) = \beta(0)$; $\alpha'(t) = f(t,0)$; $\beta'(t) = f(t,1)$.

The aim of this section is to prove that there exists a unique solution (u, σ) for (3.1)–(3.5), which enjoys the following regularity: u belongs to the space X^1 defined as

$$X^1 = \{u \in \mathcal{C}([0,T]; W^{1,1}(]0,1[)). \tag{3.6}$$

Remark 3.1. One can observe that (3.1)–(3.5) may be understood as a nonlinear evolution equation in σ with the homogeneous Neumann boundary conditions

$$\sigma_x(t,0) = \sigma_x(t,1) = 0. \tag{3.7}$$

We begin to state a result which implies uniqueness for the solution of (3.1)–(3.5).

Theorem 3.1. *Assume that u_0 is in $BV(]0,1[)$ and that f belongs to $L^\infty(]0,T[; BV(]0,1[))$. Then problem (3.1)–(3.5) admits at most one solution $u \in L^\infty(]0,T[, (BV]0,1[))$, such that $u_t \in L^2(]0,T[\times]0,1[)$.*

Proof of Theorem 3.1. Let (u^1, σ^1) and (u^2, σ^2) be two solutions of (3.1)–(3.5) which verify the assumptions in Theorem 3.1. We define $u = u^1 - u^2$ and $\sigma = \sigma^1 - \sigma^2$. Then u and σ verify

$$u_t - \sigma_x = 0, \tag{3.8}$$

with all boundary and initial conditions equal to 0. Let us multiply (3.8) by u and integrate with respect to x. On one hand

$$\int_0^1 u_t u = \frac{1}{2}\frac{d}{dt}\int_0^1 u^2. \tag{3.9}$$

holds. On the other hand, as for more classical assumptions on u and σ, the results stated in Proposition 1.1 allow us to write

$$-\int_0^1 \sigma_x u = \int_0^1 \sigma u_x. \tag{3.10}$$

At this stage, if we prove that

$$\sigma u_x = (\psi'(u_x^1) - \psi'(u_x^2))(u_x^1 - u_x^2) \tag{3.11}$$

is nonnegative as a measure, then we will derive from (3.8) that

$$\frac{d}{dt}\Big(\int_0^1 u^2\Big) = 0, \tag{3.12}$$

which yields the uniqueness result. For that purpose, we recall from the theory of convex functions of a measure (see Proposition 1.1) that the following equality and inequality hold

$$\sigma^i u_x^i = \psi(\sigma^i) + \psi^*(u_x^i);\ i = 1, 2, \tag{3.13}$$

$$\sigma^i u_x^j \leq \psi(\sigma^i) + \psi^\star(u_x^j);\ i \neq j,\ i = 1, 2,\ j = 1, 2. \tag{3.14}$$

From (3.13), (3.14), we easily deduce that σu_x, which has been defined in (3.11), is non-negative as a measure. This completes the proof of Theorem 3.1.

We now deal with the existence's result. The process is similar to the one employed in Section 2 for the stationary problem. We first prove an existence result for data that enjoy more regularity, namely for data in $W^{1,\infty}(]0,1[)$ instead of $W^{1,1}(]0,1[)$.

In order to prove existence of solutions for initial data and f regular enough, one can either use a regularization process, by adding a viscous term $-\epsilon w^{\epsilon}_{\epsilon xx}$, derive a priori estimates and pass to the limit , either use an approximation by a finite difference scheme in time, and the result of Section 1. We propose to present the second method.

Theorem 3.2. *Assume that*

$$f \in L^\infty(]0,T[, W^{1,\infty}(]0,1[)), \quad \textit{and} \quad \alpha, \beta \in W^{1,\infty}(]0,T[), \quad u_0 \in H^2(]0,1[).$$

Then there exists a unique $u \in \mathcal{C}([0,T], W^{1,\infty}(]0,1[)$ *which satisfies* (3.1)–(3.5).

Proof. We use the following discretization scheme

$$\frac{u^{n+1} - u^n}{h} - \sigma^{n+1}_{,x} = f^{n+1} \tag{3.15}$$

$$u^0(0,x) = u_0(x) \tag{3.16}$$

$$u^{n+1}(0) = \alpha((n+1)h) = \alpha^{n+1} \tag{3.17}$$

$$u^{n+1}(1) = \beta((n+1)h) = \beta^{n+1} \tag{3.18}$$

$$\sigma^{n+1} = \psi'(w^{n+1}) = \psi'((u^{n+1})'(x)) \tag{3.19}$$

with

$$f^{n+1}(x) = 1/h \int_{nh}^{(n+1)h} f(t,x)dt. \tag{3.20}$$

We denote by g^{n+1} the derivative $(f^{n+1})'$, and by f^h the linear interpolate function defined by

$$f^h(t,x) = \frac{f^{n+1} - f^n}{h}(t - nh) + f_n \quad \text{for} \quad nh \le t \le (n+1)h \tag{3.21}$$

$$g^h = \frac{\partial}{\partial x} f^h \tag{3.22}$$

and by (u^h, σ^h) the linear interpolate functions

$$u^h(t,x) = u^n + \frac{u^{n+1} - u^n}{h}(t - nh), \quad t \in [nh, (n+1)h[\tag{3.23}$$

$$\sigma^h(t,x) = \sigma^n + \frac{\sigma^{n+1} - \sigma^n}{h}(t - nh), \quad t \in [nh, (n+1)h[. \tag{3.24}$$

Using the first section, we obtain that $u^{n+1} \in W^{1,\infty}(]0,1[)$, for all n. We first establish a priori estimates in order to be able to pass to the limit when n goes to infinity. As in Section 2, we use the derivatives w^n, w^{n+1} instead of u^n, u^{n+1}. The equations are

$$\frac{w^{n+1} - w^n}{h} - \sigma^{n+1}_{,xx} = g^{n+1} \tag{3.25}$$

$$\sigma^{n+1}_{,x}(0) = \sigma^{n+1}_{,x}(1) = 0. \tag{3.26}$$

By the estimates obtained in Section 2, we have

$$|w^{n+1}|_\infty \le |w^n|_\infty + h|g^{n+1}|_\infty. \tag{3.27}$$

By summing with respect to n = 0 to n= N, one gets

$$|w^{N+1}|_\infty \le |w_0|_\infty + h\sum_0^N |g^{n+1}|_\infty \le |w_0|_\infty + ||f^h||_{L^1(]0,T[,W^{1,\infty}(]0,1[))}. \tag{3.28}$$

This implies that w^{N+1} is bounded in $L^\infty(]0,T\times]0,1[)$, independently of N. On the other hand, using

$$u^{n+1}(x) = \alpha^{n+1} + \int_0^x \frac{\partial u^{n+1}}{\partial x} dx \tag{3.29}$$

we derive that

$$u^h \text{ is bounded in } L^\infty(]0,T[\times]0,1[). \tag{3.30}$$

We need now an estimate on u_t^h or w_t^h. We multiply (3.25) by $\frac{\sigma^{n+1}-\sigma^n}{h}$, integrate by parts the second integral, and use the inequality

$$a|(w^{n+1}-w^n)| \le |\sigma^{n+1}-\sigma^n| \le b|(w^{n+1}-w^n)| \tag{3.31}$$

for some positive real numbers a and b. These inequalities can be obtained by using the mean value theorem for ψ', and the boundedness of w^n, independently on n. One obtains

$$\begin{aligned} &a\int_0^1 |\frac{w^{n+1}-w^n}{h}|^2 + \int_0^1 \sigma_{,x}^{n+1}\frac{(\sigma^{n+1}-\sigma^n)_{,x}}{h} \\ &= \int_0^1 g^{n+1}\frac{\sigma^{n+1}-\sigma^n}{h} \le b\int_0^1 |\frac{w^{n+1}-w^n}{h}||g^{n+1}|. \end{aligned} \tag{3.32}$$

Using the equality

$$\sigma_{,x}^{n+1}(\sigma^{n+1}-\sigma^n)_{,x} = \frac{1}{2}(\sigma_{,x}^{n+1}-\sigma_{,x}^n)^2 - 1/2(\sigma_{,x}^n)^2 + 1/2(\sigma_{,x}^{n+1})^2, \tag{3.33}$$

multiplying by h and summing from 0 to N, one obtains

$$a\int_0^1 w_t^{h2} + \sum_0^N \int_0^1 (\sigma^{n+1}-\sigma^n)_{,x}^2 + \int_0^1 \frac{(\sigma_{,x}^N)^2}{2} \le \frac{1}{2}\int_0^1 \sigma_{0,x}^2 + \frac{b^2}{a}||g^h||_{L^2(0,T\times(0,1))}^2. \tag{3.34}$$

Using $u_0 \in H^2(]0,1[)$, we finally get that

$$w_t^h \quad \text{is bounded in } L^2(]0,T[\times]0,1[) \tag{3.35}$$

and then

$$u_t^h(t) = \alpha_t^h(t) + \int_0^x w_t^h(t,y)dy \tag{3.36}$$

is bounded in $L^2((0,T),L^\infty(]0,1[))$. Coming back to the equation

$$u_t^h - \sigma_x^{n+1} = f^{n+1} \quad \text{for} \quad nh \le t \le (n+1)h$$

one gets that

$$\sigma_{,x}^{n+1} \text{ is bounded } \in L^2((0,T),L^\infty(]0,1[) \tag{3.37}$$

and then w^h is bounded in $H^1(]0,T[\times]0,1[)$. Passing if necessary to a subsequence, (still denoted w^h), we have

$$w^h \to w \ \text{ in } \ H^1(]0,T[\times]0,1[) \ \text{ weakly} \tag{3.38}$$

$$w^h \to w \ \text{ a.e. } \ (x,t) \in]0,1[\times]0,1]. \tag{3.39}$$

We infer from (3.28) and (3.35) that $\sigma_t^n = \psi''(w^n)w_t^n$, and then σ_t^h is bounded, independently of h, in $L^2(0,T\times(0,1))$. Hence, due to (3.37), we may assume that, up to a subsequence

$$\sigma^h \rightharpoonup \sigma \ \text{ weakly in } \ H^1(]0,T[\times]0,1[) \tag{3.40}$$

and

$$\sigma^h \to \sigma \ \text{ a.e. } \ (x,t) \text{ in } \ (0,1)\times(0,T). \tag{3.41}$$

We now prove that σ and $w = u_{,x}$ satisfy (3.2). For that purpose, we observe that due to (3.15), (3.23) and (3.24),

$$h|\sigma_h - \psi'(w^h)|^2 \le h|\sigma_{n+1} - \sigma_n|^2, \tag{3.42}$$

a.e. (x,t) in $(0,1)\times(nh,(n+1)h)$. Integrating (3.42) for $x \in [0,1]$, and summing from 0 to N, we obtain that

$$||\sigma^h - \psi'(w^h)||_{L^2(0,T\times 0,1)} \le h||\sigma_t^h||_{L^2((0,T)\times(0,1))}. \tag{3.43}$$

Hence, due to (3.39)–(3.41), letting h go to 0 in (3.43) gives (3.2).

We now prove that σ and u satisfy (3.1). By the definition of u^h, σ^h, f^h, one has that

$$(u_t^h - \sigma_x^h - f^h)(t,x) = (n+1-\frac{t}{h})(\sigma_x^{n+1} - \sigma_x^n + f^{n+1} - f^n)(x). \tag{3.45}$$

Same arguments as above lead to

$$\begin{aligned} &||u_t^h - \sigma_x^h - f^h||^2_{L^2(]0,T[\times]0,1[)} \\ \le& 2h\sum_0^{N-1}\big(||\sigma_x^{n+1} - \sigma_x^n||^2_{L^2(0,1)} + ||f^{n+1} - f^n||^2_{L^2(0,1))}\big). \end{aligned} \tag{3.46}$$

Due to (3.34), the first term on the r.h.s. of (3.46) converges towards 0 when $h \to 0$, and the second term also since f belongs to $L^2(0,T\times(0,1))$. One obtains that (u, σ) verify (3.1)–(3.5). Since the boundary conditions (3.4) (3.5) are easy to prove, we have constructed a solution (u, σ) which satisfy (3.1)–(3.5).

We now prove the regularity result for w. For that purpose, we differentiate (3.1) with respect to x, multiply the resulting equation by $\psi''(w)w_t$, and integrate with respect to x in $]0,1[$. Using

$$\psi''(w)w_x(t,0) = \psi''(w)w_x(t,1) = 0 \tag{3.47}$$

one obtains

$$\int_0^1 \psi''(w)(w_t)^2 + \frac{1}{2}\frac{d}{dt}\int_0^1 ((\psi'(w))_x)^2 = \int_0^1 \psi''(w) g w_t.$$

Now, since ψ'' is continuous and since w is bounded in $L^\infty(]0,T[\times]0,1[)$, there exists m_1 such that $0 < m_1 < \psi''(w) \le m_2$ a.e. We then obtain

$$m_1||w_t||^2 + \frac{1}{2}\frac{d}{dt}|(\psi'(w))_x|_2^2 \le m_2|g|_2|w|_2 \le \frac{m_2^2}{2m_1}|g|_{L^2} + \frac{m_1}{2}|w_t|_{L^2}^2. \tag{3.48}$$

Integrating this for $t \in [0,T]$ yields

$$w_t \in L^2(]0,T[\times]0,1[) \tag{3.49}$$

$$w_x \in L^\infty(]0,T[, L^2(]0,1[)). \tag{3.50}$$

Let us come back to the equation in w which reads as well

$$w_t - \psi''(w)w_{,xx} = g + \psi^{(3)}(w)(w_x)^2. \tag{3.51}$$

Using $w_t \in L^2(]0,T[\times]0,1[)$, $\psi^{(3)}(w) \in L^\infty(]0,T[\times]0,1[)$, and $w_x \in L^\infty(]0,T[\times]0,1[)$, one obtains that

$$-\psi''(w)w_{xx} \in L^2(]0,T[; L^1(]0,1[)) \tag{3.52}$$

and then

$$w_{,xx} \in L^2(]0,T[, L^1(]0,1[)). \tag{3.53}$$

Using the embedding $L^2(0,T; W^{1,1}(]0,1[)) \subset L^2(0,T; L^\infty(0,1))$, we derive from (3.53) that

$$w_x \in L^2(0,T; L^\infty(]0,1[)) \tag{3.54}$$

and by interpolating (3.50) and (3.54), that

$$w_x \in L^4(]0,T[\times]0,1[). \tag{3.55}$$

Therefore, if we go back to (3.51) using (3.55) instead of (3.50), we obtain

$$w_{,xx} \in L^2(]0,T[\times]0,1[). \tag{3.56}$$

Second a priori estimate: we differentiate (3.51) with respect to t to obtain

$$w_{tt} - (\psi''(w)w_{xt})_x = g_t + (\psi^{(3)}(w)w_t w_x)_x. \tag{3.57}$$

Multiplying (3.57) by w_t, integrating with respect to x in $[0,1]$, and using

$$(\psi''(w)w_{xt})(t,0) = (\psi''(w)w_{xt})(t,1) = 0 \tag{3.58}$$

one obtains

$$\frac{1}{2}\frac{d}{dt}|w_t|^2_{L^2} + \int_0^1 \psi''(w)(w_{xt})^2 = \int_0^1 g_t w_t - \int_0^1 \psi^{(3)}(w)_x w_t w_{xt}. \tag{3.59}$$

We majorize the right hand side as follows

$$\int_0^1 g_t w_t \leq |g_t|_{L^2}|w_t|_{L^2} \leq C + |w_t|^2_{L^2}, \tag{3.60}$$

where C is a constant which depends on the data of the equations.

(In all the previous estimates and the following one, C denotes a constant which depends only on the data w_0 and g of the equation. But of course C may differ from one line to another.)

We handle the second term in the right hand side of (3.59) as follows

$$\begin{aligned} -\int_0^1 \psi^{(3)}(w)w_x w_t w_{xt} &\leq m_3\,|w_t|_{L^2}\,|w_{xt}|_{L^2}\,|w_x|_{L^\infty} \\ &\leq \frac{\alpha}{2}\,|w_{xt}|^2_{L^2} + C\,|w_t|^2_{L^2}\,|w_x|^2_{L^\infty}\,, \end{aligned} \tag{3.61}$$

where $m_3 = \sup\{\left|\psi^{(3)}(\xi)\right|; |\xi| \leq \|w\|_{L^\infty}\}$. Therefore (3.59)–(3.61) yields

$$\frac{d}{dt}\,|w_t|^2_{L^2} + \alpha\,|w_{xt}|^2_{L^2} \leq C_1\big(1 + |w_x|^2_{L^\infty}\big)\,|w_t|^2_{L^2} + C_2. \tag{3.62}$$

By dropping in a first time the term $|w_{xt}|^2_{L^2}$ in (3.62), one obtains

$$\frac{d}{dt}\left[|w_t|^2_{L^2}\exp\bigl(-C_1\int_0^t\bigl(1+|w_x|^2_{L^\infty}\bigr)ds\bigr)\right]\leq C_2. \tag{3.63}$$

We easily derive from (3.57) and (3.63) that

$$w_t \in L^\infty(]0,T[;L^2(]0,1[)). \tag{3.64}$$

We now come back to (3.62). Integrating with respect to t in $[0,T]$ and using (3.56) and (3.64), we have

$$w_{xt} \in L^2(]0,T[\times]0,1[). \tag{3.65}$$

We now observe that (3.65) and the embedding $H^1(]0,1[) \subset L^\infty(]0,1[)$ imply that

$$w_t \in L^2(]0,T[,L^\infty(]0,1[)). \tag{3.66}$$

Hence, interpolating (3.64) and (3.66) yields to

$$w_t \in L^4(]0,T[\times]0,1[). \tag{3.67}$$

Now, (3.65) and (3.67) means

$$w \in W^{1,4}(]0,T[\times]0,1[), \tag{3.68}$$

and by Sobolev's imbedding Theorem,

$$w \in C(]0,T[\times]0,1[). \tag{3.69}$$

We are now able to prove the main result of that section. We assume that α and β belong to $W^{1,1}(]0,1[)$ and $g \in L^1(]0,T[\times]0,1[)$. Then we have the following result of existence and uniqueness.

Theorem 3.3. *Let α and β be in $W^{1,1}(]0,T[), g = f_{,x} \in L^1(]0,T[\times]0,1[)$ and $u_0 \in W^{1,1}(]0,1[)$. We still assume that ψ is convex, $\psi \in \mathcal{C}^2$ and $\psi'' > 0$. Then there exists a unique $u \in \mathcal{C}([0,T],W^{1,1}(]0,1[)$ which verifies*

$$u_{,t} - \sigma_x = f(t,x) \tag{3.70}$$

$$u(t,0) = \alpha(t) \tag{3.71}$$

$$u(t,1) = \beta(t) \tag{3.72}$$

$$u(0,x) = u_0(x). \tag{3.73}$$

If moreover $f \in L^2(]0,T[, L^2(]0,1[)$, *then* $u_{,t} \in L^2((0,T)\times(0,1))$.

Proof. We use the differentiated form of the equation

$$w_t - \sigma_{,xx} = g(t,x) \tag{3.74}$$

and we begin to approximate g by g^N where $g^N \to g \in L^1(]0,T[\times]0,1[)$ and g^N is regular enough to have $w^N = T(g^N) \in \mathcal{C}([0,T]\times[0,1])$. The equations verified by w^p and w^q read

$$w_t^p - \sigma_{,xx}^p = g^p(t,x) \tag{3.75}$$

$$w_t^q - \sigma_{,xx}^q = g^q(t,x). \tag{3.76}$$

Let us multiply the equation obtained after subtraction by $Y_\epsilon(\sigma^p - \sigma^q)$ and integrate by parts the second integral, to obtain

$$\int_0^1 (w_t^p - w_t^q)(Y_\epsilon(\sigma^p-\sigma^q)) + \int_0^1 (\sigma^p-\sigma^q)_{,x}^2 Y_\epsilon'(\sigma^p-\sigma^q) = \int_0^1 (g^p-g^q)Y_\epsilon(\sigma^p-\sigma^q).$$

The second integral reads

$$\frac{1}{\epsilon}\int_{\{x, |\sigma^p-\sigma^q|<\epsilon\}} (\sigma^p-\sigma^q)_{,x}^2 \geq 0.$$

Since $\sigma_p - \sigma_q$ are regular enough and ψ' is invertible, $Y_\epsilon(\sigma^p - \sigma^q)$ tends towards $Y^+(w^p - w^q) - Y^-(w^p - w^q)$ in L^∞ weakly star. Since w is regular enough to have $w_t \in L^2$ (see (3.49)), we obtain by passing to the limit when ϵ goes to zero

$$\frac{d}{dt}\int_0^1 |w^p - w^q| + (\geq 0) \leq \int_0^1 |g^p - g^q|.$$

Integrating with respect to t, one gets

$$\int_0^1 |w^p - w^q|(t,x)dx \leq \int_0^1 |w_0^p - w_0^q|(x)dx + \int_0^T\int_0^1 |g^p - g^q|.$$

In particular w^p is a Cauchy sequence in $\mathcal{C}(0,T,L^1(]0,1[))$. Then $w^p \to w$ in $\mathcal{C}([0,T],L^1(]0,1[)$. It is not difficult to see that w is a solution of the evolution problem. Suppose in addition that $f \in L^2$. Multiply the equation in u by u_t, integrating by parts in x, and finally integrating with respect to t, we obtain

$$\int_0^1 u_t^2 + \int_0^1 \sigma u_{tx} - \sigma(1)u_t(1) + \sigma(0)u_t(0) = \int u_t f,$$

hence,

$$\int u_t^2 + \frac{d}{dt}\int \psi(u_x) - \sigma(1)\beta'(t) + \sigma(0)\alpha'(t) = \int f u_t.$$

Then for every $T > 0$

$$\int_0^T \int_0^1 u_t^2 + \int_0^1 \psi(u_x)(T) \leq |f|_{L^2}|u_t|_{L^2} + Cte.$$

From this we get the desired result.

Conclusion. The interested reader can find more details about regularity and other technics, mainly based upon the increasing behaviour of T in [3].

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