Differential and Integral Equations Volume 12, Number 5, September 1999, 613–636.

SEMILINEAR PARABOLIC EQUATIONS WITH SINGULAR INITIAL DATA IN ANISOTROPIC WEIGHTED SPACES

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(Submitted by: Y. Giga)

Abstract. We consider the Cauchy problem for semilinear parabolic equations with strongly singular initial data and nonlinear terms with superlinear or sublinear growth at infinity. We show, under a certain link between the growth at infinity of the nonlinear term and the order of the maximal singularity of the initial data, existence and uniqueness theorems for local and global solutions. For this we introduce anisotropic weighted Hölder type spaces, following T. Kato in [16]. We examine the regularity up to the initial plane of these solutions.

1. Introduction. This paper is concerned with the Cauchy problem for the semilinear parabolic equation

$$\partial_t u - \triangle u + g(u) = 0, \quad t > 0, \, x \in \mathbb{R}^n \tag{1.1}$$

where $g(u)$ is a (locally) Lipschitz real-valued function. The initial data are allowed to be strongly singular, for example

$$u(0, \cdot) = \mu \in \mathcal{M}^k(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n), \quad k \in \mathbb{Z}_+, \tag{1.2}$$

where $\mathcal{M}^k(\mathbb{R}^n) = (C_b^k(\mathbb{R}^n))^*$ is the strong dual of the Banach space $C_b^k(\mathbb{R}^n)$ of all $C^k(\mathbb{R}^n)$ functions with bounded derivatives up to order k. We note

Received for publication January 1998.

†Partially supported by CNPq, Brazil and GNAFA of the CNR, Italy.

‡Partially supported by a University of Cagliari Coordinated Research Project and 40% of MURST, Italy.

§Partially supported by FAPESP, Brazil, a University of Cagliari Coordinated Research Project, 40% of MURST, Italy and NFSR of Bulgaria.

AMS Subject Classifications: 35K55, 35R05.

that $\mathcal{M}^0(\mathbb{R}^n)$ coincides with the space of finite Radon measures $\mathcal{M}(\mathbb{R}^n)$. A typical example of initial data $\mu \in \mathcal{M}^k(\mathbb{R}^n)$ is given by

$$\mu = \sum_{j=1}^{\infty} \sum_{|\alpha| \le k} b_{j\alpha} \partial_x^\alpha \delta(x - \xi^j), \quad b_{j\alpha} \in \mathbb{R},\ \xi^j \in \mathbb{R}^n \tag{1.3}$$

where $\delta(x-\xi)$ stands for the Dirac measure massed at $\xi \in \mathbb{R}^n$ and $\{b_{j\alpha}\}_1^\infty \in \ell^1$, i.e., $\sum_{j=1}^\infty |b_{j\alpha}| < \infty$ for $|\alpha| \le k$.

We want to examine whether, under certain hypotheses on the growth s of the nonlinear term $g(u)$ and the order k of the maximal singularity of the initial data μ in (1.2), we can find $T = T_\mu$, $0 < T \le +\infty$, such that there exists a unique weak solution $u(t,x)$ to the Cauchy problem (1.1), (1.2) in the strip $0 < t < T$, $x \in \mathbb{R}^n$ which becomes regular for $t > 0$.

The main novelty of our paper on the model initial data (1.3) may, roughly speaking, be summarized as follows: given $k \in \mathbb{Z}_+$ and assuming that the growth s satisfies $s < (n+2)/(n+k)$ with the additional condition

$$\max_{|\alpha| \le k} \sum_{j=1}^{\infty} |b_{j\alpha}|^s < \infty,$$

if $s < 1$, we show that the Cauchy problem (1.1), (1.2) admits a local weak solution which becomes classical for $t > 0$. The uniqueness question is more delicate. While for a growth $s \ge 1$ we show uniqueness in suitable weighted Hölder type spaces $C_\theta(L^p(\mathbb{R}^n) : T)$ used by T. Kato [16], in the case of sublinear growth $s < 1$ we can show the uniqueness among the solutions which are in the form of a suitable perturbation of the solution to the heat equation with the same initial data.

In particular, if $g(u) = \ln(1 + u^2) - 1$, we can solve the aforementioned Cauchy problem for all $k \in \mathbb{Z}_+$ and μ given by (1.3). The result for the sublinear growth seems to be completely new. Moreover, if $s > 1$, we consider also singular initial data in anisotropic spaces of the following type if $1 \le d \le n-1$,

$$\mu = |D'|^k \mu_1 \otimes f, \quad \mu_1 \in \mathcal{M}(\mathbb{R}^d_{x'}),\ f \in L^q(\mathbb{R}^{n-d}_{x''}), \tag{1.4}$$

for $x = (x', x'') \in \mathbb{R}^n$ and $|D'| = (-\triangle')^{1/2}$ where $\triangle'$ stands for the Laplacian in $\mathbb{R}^d_{x'}$. Furthermore, we show $L^1 - L^\infty$ estimates on the remainder $u(t) - e^{t\Delta}\mu$ down to $t = 0$. Our approach is based on the introduction of an anisotropic version of the weighted Hölder type spaces $C_\theta(L^p(\mathbb{R}^n) : T)$ and suitable nonlinear superposition estimates. The main advantage of such spaces turns out to be the possibility to capture the behavior of (1.1) for the critical indices of the L^p singularity of the initial data.

Let us review briefly the results on semilinear parabolic equations with singular initial data. The Cauchy problem for (1.1) with initial data in $L^p(\mathbb{R}^n)$ has been extensively studied cf. F.B. Weissler [22], [23], A. Haraux and F.B. Weissler [15], Y. Giga [14], see also H. Fujita [13] for blow-up results and H. Brézis and T. Cazenave [10] for the Dirichlet initial boundary value problem for (1.1).

In the case where μ equals a positive Dirac measure $\delta(x-\xi)$ and $g(u) = u|u|^{s-1}$, $s > 1$, H. Brézis and A. Friedman [8] proved that the solution to (1.1), (1.2) exists if and only if $0 < s < (n+2)/n$. Later, P. Baras and M. Pierre [3] gave necessary and sufficient conditions involving the growth of the nonlinear term and the capacity of the initial data measure μ. The paper of H. Brézis, D. Peletier and D. Termam [9] exhibits a positive solution to (1.1) with a strong singularity at $(0,0)$.

Existence results have been obtained by Y. Niwa [19] for the Cauchy problem with general nonlinear terms $g(u)$ provided the initial data are measures. On the other hand, J.-F. Colombeau and M. Langlais [11] ($n = 1$, $g(u) = u^3$) and M. Langlais [18] have studied monotone semilinear parabolic equations in bounded domains in the setting of generalized functions of Colombeau type recovering the classical solutions when the initial data are L^p functions. H. Kozono and M. Yamazaki [17] examined the Cauchy problem for (1.1) and the Navier-Stokes equation. The initial data belong to functional spaces generalizing the Besov and Morrey spaces and they contain distributions more singular than measures. The existence results in [17] are shown under a smallness condition on the initial data in the corresponding spaces. If the growth s is greater than $(n+2)/n$ and the initial data are in certain L^p spaces, $p > 1$, this smallness condition is necessary in view of the results of F.B. Weissler [22]. But in the case of growth $1 < s < (n+2)/n$ they recover the previous results for the semilinear heat equation with measures as initial data only if the measures have small total variation.

Recently, F. Ribaud [20], [21] studied semilinear parabolic equations with superlinear nonlinear terms of the type $\kappa(D)g(u)$, $\kappa(D)$ being a p.d.o. of order < 2 and initial data $\mu \in H^r_p(\mathbb{R}^n)$, $1 < p < \infty$, $r \leq 0$. In view of the embeddings of the Sobolev spaces, initial data more singular than measures are allowed. As it concerns parabolic equations with nonlinear conservative terms and singular initial data we cite H.A. Biagioni and T. Gramchev [6] (for the 2D Navier-Stokes equation with forcing term) and D. Bekhiranov [4], D. Dix [12] and H.A. Biagioni, L. Cadeddu and T. Gramchev [5] on Burgers' type equations.

Finally, for boundary value problems for nonlinear parabolic equations with measures as initial or right-hand side data we refer to L. Boccardo, A. Dall'Aglio, T. Gallouët and L. Orsina [7].

We can summarize our new results as follows:

A) Let the nonlinear term $g(u)$ have a superlinear growth at infinity, namely, at most like $|u|^s$, $s > 1$. Then we show existence-uniqueness results for (1.1) with initial data μ modeled by (1.4) with k being a nonnegative real number, $k < 2$ and $s < \frac{d+2}{d+k}$. The case $k = 0$ is contained in [3]. Moreover, if μ_1 is a nonnegative finite Radon measure and $f \geq 0$, the necessary condition obtained by D. Andreucci and E. DiBenedetto [2], D. Andreucci [1] for the existence of a positive solution to (1.1), (1.2) is equivalent to $s \leq \frac{d+2}{d}$. One cannot apply [17, Theorem 2.5] for initial data μ as in (1.4) if $k \notin \mathbb{Z}_+$ since the symbol $|\xi'|^k$ is not smooth as a function of $(\xi', \xi'') \in \mathbb{R}^n \setminus \{0\}$. Next, if $\mu \in \mathcal{M}^k(\mathbb{R}^n)$, we estimate the life-span of the solution $u(t, \cdot)$ in terms of the $\mathcal{M}^k$ norm of μ and study the regularity down to $t = 0$ of $u(t, \cdot) - e^{t\triangle}\mu$ in the weighted spaces $C_\theta(L^p(\mathbb{R}^n) : T)$, $1 \leq p \leq \infty$. We show also stability and continuous dependence results for the Cauchy problem (1.1), (1.2) in these spaces as well as persistence results in $\mathcal{M}^k(\mathbb{R}^n)$.

Finally, we recover old and/or obtain new results for μ modeled by $\mu = |D|^k\psi$, $\psi \in L^p(\mathbb{R}^n)$ with $p \geq \max\{1, \frac{n(s-1)}{2}\}$ while $p > 1$ when $s = (n+2)/n$ and $0 \leq k \leq k(s, n, p)$ where $k(s, n, p)$ is an explicitly defined nonnegative real number which is zero only for $p = n(s-1)/2$. Moreover, for growth $s > (n+2)/n$ we show global results for small data of the type (1.4) in the anisotropic case.

B) Let the nonlinear term $g(u)$ grow at most like $|u|^s$ at infinity with $s < 1$. For given $k \in \mathbb{Z}_+$, require $s < (n+2)/(n+k)$ and consider initial data given by (1.3) with the additional requirement $\sum_1^\infty |b_{j,\alpha}|^s < \infty$. Then we show the existence of a global in $t > 0$ solution to the Cauchy problem (1.1), (1.2). Uniqueness holds for a particular class of solutions in suitable weighted spaces $C_\theta(L^p(\mathbb{R}^n) : T)$. We estimate again regularity down to $t = 0$ of $u(t, \cdot) - e^{t\triangle}\mu$ in the weighted L^p Hölder type spaces, $1 \leq p \leq \infty$.

C) Finally, we consider the sequence $\{u^\varepsilon\}_{\varepsilon>0}$ of solutions to (1.1) with smooth initial data $u^\varepsilon(0, x) = \mu * \varphi^\varepsilon(x)$, $\mu \in \mathcal{M}^k(\mathbb{R}^n)$, where $\varphi^\varepsilon(x) := \varepsilon^{-n}\varphi(\varepsilon^{-1}x)$, $\varepsilon > 0$, φ being a standard Friedrichs mollifier. Then we study the weak limit $\lim_{\varepsilon\searrow 0} u^\varepsilon$ in the spaces $C_\theta(L^p(\mathbb{R}^n):T)$. While in the superlinear case we are able to prove the convergence for each $1 \le p \le \infty$, in the sublinear case the convergence holds for $p \ge 1/s > 1$.

As usual, first we try to find suitable functional spaces where the existence and the uniqueness results should hold. Having in mind the weighted Hölder type spaces $C_\theta(L^p(\mathbb{R}^n):T)$, $1 \le p \le \infty$, $\theta \in \mathbb{R}$, $T > 0$, used by T. Kato [16] we introduce anisotropic weighted Hölder spaces $C_\theta(L^s(\mathbb{R}^d : L^\infty(\mathbb{R}^{n-d})) : T)$ in order to deal with the case when singularity of the initial data μ is on a plane of codimension d. This is done in Section 2.

Sections 3 and 4 are dedicated to the superlinear and sublinear growth cases, respectively.

2. Weighted Hölder type spaces. If X is a Banach space, we shall denote by $C_\alpha(X:T)$, $\alpha \in \mathbb{R}$, the space of all X-valued continuous functions f on $(0,T)$ such that

$$\| f \|_{X,\alpha,T} = \sup_{t\in(0,T)} \{ t^\alpha \|f(t)\|_X \} < \infty. \tag{2.1}$$

Evidently, $\alpha \le \beta$, $T' \le T$ imply $C_\alpha(X:T) \hookrightarrow C_\beta(X:T')$. For $\alpha \ne 0$, let $\dot{C}_\alpha(X)$ be the subspace of $f \in C_\alpha(X:T)$ such that $\lim_{t\searrow 0} t^\alpha \|f(t)\|_X = 0$. For $\alpha = 0$, we define $\dot{C}_0(X) = C_0(X)$. Note that if $\alpha < 0$ and $v \in C_\alpha(X)$, then $\lim_{t\searrow 0} \|v(t)\|_X = 0$ and clearly we have

$$C_\alpha(X:T) \subset \dot{C}_0(X) \quad \text{provided} \quad \alpha < 0. \tag{2.2}$$

We denote by $U^0_\mu(t,x)$ the unique solution to the linearized Cauchy problem for the heat operator

$$\partial_t u - \triangle u = 0,\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \quad u(0,\cdot) = \mu. \tag{2.3}$$

We can write $U^0_\mu(t,\cdot) = e^{t\triangle}\mu$ via the heat semigroup or, setting

$$E_n(t,x) = \frac{1}{(4\pi t)^{n/2}} \exp(-\frac{|x|^2}{4t}), \quad t > 0,\ x \in \mathbb{R}^n$$

to be the fundamental solution of the heat operator, we get that

$$U^0_\mu(t,x) = (E_n(t,\cdot) * \mu)(x) = \int_{\mathbb{R}^n} E_n(t,x-y)\mu(y)dy. \tag{2.4}$$

First we need some auxiliary assertions.

Proposition 2.1. *Let $\mu \in \mathcal{M}^k(\mathbb{R}^n)$, $k \in \mathbb{Z}_+$. Then there exist $N = \#\{\alpha \in \mathbb{Z}_+^n, |\alpha| \le k\}$ finite Radon measures μ_α, $|\alpha| \le k$ having the following properties*

$$\mu = \sum_{|\alpha|\le k} \partial_x^\alpha \mu_\alpha \quad in \ \ \mathcal{M}^k(\mathbb{R}^n), \tag{2.5}$$

and

$$\| \mu \|_k = \max_{|\alpha|\le k} \| \mu_\alpha \|, \tag{2.6}$$

where $\|\cdot\|$ denotes the total variation and $\|\cdot\|_k$ denotes the norm in $\mathcal{M}^k$.

Proof. Let L be the linear subspace of $X := (C_b(\mathbb{R}^n))^N$ defined by

$$L = \{\psi = \{\psi_\alpha\}_{|\alpha|\le k} : \psi_\alpha = (-1)^{|\alpha|}\partial_x^\alpha \varphi, \alpha \in \mathbb{Z}_+^n, |\alpha| \le k, \varphi \in C_b^k(\mathbb{R}^n)\}$$

and let Φ be the linear functional on L given by $\Phi(\psi) = \mu(\varphi)$, $\psi \in L$. Clearly, $\|\Phi\|_{L^*} = \|\mu\|_k$. The Hahn-Banach theorem implies that there exists an extension $\tilde{\Phi}$ of Φ on X such that

$$\|\tilde{\Phi}\|_{X^*} = \|\Phi\|_{L^*}, \quad \tilde{\Phi}(\psi) = \Phi(\psi), \quad \psi \in L. \tag{2.7}$$

Since $\tilde{\Phi}$ is linear we can write $\tilde{\Phi} = \sum_{|\beta|\le k} \mu_\beta \circ \Pi_\beta$, where $\mu_\beta \in \mathcal{M}(\mathbb{R}^n)$ is defined by

$$\mu_\beta(f) = \tilde{\Phi}(\{\psi_\alpha^f\}_{|\alpha|\le k}), \quad \psi_\alpha^f = 0 \text{ if } \alpha \ne \beta, \ \psi_\beta^f = f, \quad f \in C_b(\mathbb{R}^n) \tag{2.8}$$

and Π_β stands for the projection $\Pi_\beta : X \longrightarrow C_b(\mathbb{R}^n)$, $\Pi_\beta(\{\psi_\alpha\}_{|\alpha|\le k}) = \psi_\beta$, $|\beta| \le k$. Clearly, we get $\|\mu\|_k = \|\tilde{\Phi}\|_{X^*} = \max_{|\alpha|\le k} \|\mu_\alpha\|$ and

$$\mu(\varphi) = \sum_{|\alpha|\le k} \mu_\alpha((-1)^{|\alpha|}\partial_x^\alpha \varphi) = (\sum_{|\alpha|\le k} \partial_x^\alpha \mu_\alpha, \varphi)$$

for all $\varphi \in C_b^k(\mathbb{R}^n)$. $\square$

Straightforward calculations yield that for every $\alpha \in \mathbb{Z}_+^n$

$$\partial_x^\alpha E_n(t,x) = t^{-\frac{n+|\alpha|}{2}} \mathcal{P}_\alpha(\frac{x}{2\sqrt{t}}) \exp(-\frac{|x|^2}{4t}), \tag{2.9}$$

where $\mathcal{P}_\alpha(z)$ is a polynomial of degree $|\alpha|$.

Let $d \in \mathbb{Z}_+$, $1 \leq d \leq n$. If $d \leq n-1$ we write $x = (x', x'')$, $x' = (x_1, \ldots, x_d) \in \mathbb{R}^d$, $x'' = (x_{d+1}, \ldots, x_n) \in \mathbb{R}^{n-d}$. For every $1 \leq p, q, \leq \infty$ we introduce the anisotropic spaces $L_d^{p,q}(\mathbb{R}^n) = L^p(\mathbb{R}^d : L^q(\mathbb{R}^{n-d}))$ with the norm

$$\|u\|_{p,q} = \Big(\int_{\mathbb{R}^d} (\|u(x', \cdot)\|_q)^p dx' \Big)^{1/p}, \tag{2.10}$$

where $L_n^{p,q}(\mathbb{R}^n) := L^p(\mathbb{R}^n)$, $1 \leq p, q, \leq \infty$. Further, if $X = L_d^{p,q}(\mathbb{R}^n)$ (respectively, $X = L^p(\mathbb{R}^n)$) we denote by $\|\cdot\|_{\alpha,(p,q),T}$ (respectively, $\|\cdot\|_{\alpha,p,T}$) the norm $\|\cdot\|_{X,\alpha,T}$.

We now estimate the $L_d^{p,q}$, $1 \leq d \leq n$, $1 \leq p, q \leq \infty$, norms of $E_n(t, \cdot)$.

Proposition 2.2. *Let $1 \leq r_j \leq q_j \leq \infty$, $j = 1, 2$ and let $\gamma_1 = d/2(1/r_1 - 1/q_1)$, $\gamma_2 = (n-d)/2(1/r_2 - 1/q_2)$. We put $\gamma = \gamma_1 + \gamma_2$ and $1 \leq p_j \leq \infty$ defined by $1 + \frac{1}{q_j} = \frac{1}{p_j} + \frac{1}{r_j}$, $j = 1, 2$.*

i) *If $f \in L_d^{p_1,p_2}(\mathbb{R}^n)$, $g \in L_d^{r_1,r_2}(\mathbb{R}^n)$, then*

$$\|f * g\|_{q_1,q_2} \leq \|f\|_{p_1,p_2} \|g\|_{r_1,r_2}. \tag{2.11}$$

ii) *E_n is bounded on $L_d^{r_1,r_2}(\mathbb{R}^n)$ to $\dot{C}_\gamma(L_d^{q_1,q_2}(\mathbb{R}^n))$ on $(0,\infty)$ with bound*

$$\|E_n\|_{(r_1,r_2)\to(q_1,q_2),\gamma} \leq (4\pi)^{-\gamma} \prod_{i=1}^{2} (1 - 2\gamma_i/d_i)^{1-\gamma_i} =: \Theta(\gamma). \tag{2.12}$$

iii) *∂E_n is bounded on $L_d^{r_1,r_2}(\mathbb{R}^n)$ to $\dot{C}_{1/2+\gamma}(L_d^{q_1,q_2}(\mathbb{R}^n))$ if $\gamma > 0$ (resp. to $C_{\frac{1}{2}}(L_d^{q_1,q_2}(\mathbb{R}^n))$ if $\gamma = 0$) with bound*

$$\|\partial E_n\|_{(r_1,r_2)\to(q_1,q_2),1/2+\gamma} \leq \chi(\gamma), \tag{2.13}$$

where

$$\chi(\gamma) = 2^{-1-\gamma}\pi^{-\gamma} \prod_{i=1}^{2} (1 - 2\gamma_i/d_i)^{1/2} (\Gamma(1/(2 - \gamma_i/d_i q_i)))^{1-\gamma_i}, \tag{2.14}$$

and $d_1 = d$, $d_2 = n - d$ in (2.12), (2.14).

iv) *If $k_1 \geq 0$, $k_2 \geq 0$, $k := k_1 + k_2 > 0$, then $(-\triangle')^{k_1/2}(-\triangle'')^{k_2/2} E_n$ (resp. $\partial_{x'}^{\beta'} \partial_{x''}^{\beta''} E_n$ for $|\beta'| = k_1$, $|\beta''| = k_2$) is bounded on $L_d^{r_1,r_2}(\mathbb{R}^n)$ to $\dot{C}_{\frac{k}{2}+\gamma}(L_d^{q_1,q_2}(\mathbb{R}^n))$ if $\gamma > 0$ (resp. $C_{\frac{k}{2}}(L_d^{q_1,q_2}(\mathbb{R}^n))$, if $\gamma = 0$) with bound depending on k and γ.*

v) *For all* $m \in \mathbb{Z}_+$, $\beta \in \mathbb{Z}_+^n$, $k \in \mathbb{Z}_+$, $1 \leq p \leq \infty$, *there exist* $k+1$ *positive constants* $C_0(m, \beta, p, n), \dots, C_k(m, \beta, p, n)$ *such that for all* $\mu \in \mathcal{M}^k(\mathbb{R}^n)$ *and* $t > 0$,

$$\|\partial_t^m \partial_x^\beta U_\mu^0(t, \cdot)\|_p \leq t^{-m-\frac{k+|\beta|}{2}-\frac{n}{2}(1-\frac{1}{p})} \sum_{j=0}^{k} C_j(m, \beta, p, n) t^{\frac{j}{2}} |\mu|_k. \tag{2.15}$$

Proof. Clearly, (2.11) is an anisotropic generalization of the Young inequality. Indeed, using the Minkowski inequality and the Young inequality with respect to the x'' variables, we have

$$\|(f * g)(x', \cdot)\|_{q_2} \leq \int_{\mathbb{R}^d} \|(f(x'-y', \cdot)\|_{p_2} \|g(y', \cdot)\|_{r_2} dy'$$

and applying once more the Young inequality for the $L^{q_1}(\mathbb{R}^d)$ norm of the above convolution, we show (2.11).

In order to show ii)-iv) we note that (2.11) implies, for $f \in L_d^{r_1, r_2}(\mathbb{R}^n)$,

$$\|E_n(t, \cdot) * f\|_{q_1, q_2} \leq \|E_n(t, \cdot)\|_{p_1, p_2} \|f\|_{r_1, r_2}$$

with $1 + \frac{1}{q_j} = \frac{1}{p_j} + \frac{1}{r_j}$, $j = 1, 2$. On the other hand,

$$\|E_n(t, \cdot)\|_{p_1, p_2} = \|E_d(t, \cdot)\|_{p_1} \|E_{n-d}(t, \cdot)\|_{p_2}$$

and then we can use the classical results. As to v), we take into account Proposition 2.1 and the fact that $\partial_t^m E_n(t, x) = \triangle^{2m} E_n(t, x)$ and use iv). $\square$

We set $\mathcal{M}_{delta}^{k,s}(\mathbb{R}^n)$ as the linear subspace of $\mathcal{M}^k(\mathbb{R}^n)$ consisting of all elements μ in the form (1.3) such that

$$\{b_{j\alpha}\}_{j=1}^{\infty} \in \ell^s, \text{ i.e., } \|\{b_{j\alpha}\}_{j=1}^{\infty}\|_{\ell^s} := (\sum_{j=1}^{\infty} |b_{j\alpha}|^s)^{1/s} < +\infty, \ |\alpha| \leq k. \tag{2.16}$$

We note that if $s \geq 1$ the restriction (2.16) is superfluous since $\ell^1 \subset \ell^s$ while in the case $0 < s < 1$ we have that ℓ^s is strictly contained in ℓ^1.

Proposition 2.3. *Let $g \in C(\mathbb{R} : \mathbb{R})$, $g(0) = 0$. Then we have:*

i) *if there exist $A > 0$, $s > 1$ such that*

$$|g(u)| \leq A|u|^s, \quad u \in \mathbb{R}, \tag{2.17}$$

then the following estimate holds, for all $u \in L_d^{ps,qs}(\mathbb{R}^n)$,

$$\|g(u)\|_{p,q} \leq A(\|u\|_{ps,qs})^s; \tag{2.18}$$

ii) *if there exist $A > 0$, $0 < s < 1$ such that*

$$|g(u)| \leq A|u|^s, \quad u \in \mathbb{R}, \tag{2.19}$$

then for each $\mu \in \mathcal{M}_{delta}^{k,s}(\mathbb{R}^n)$ we claim that for some $C_{k,s} > 0$ the following estimate holds

$$\|g(U_\mu^0(t,\cdot))\|_p \leq C_{k,s} \sum_{|\alpha|\leq k} (\|\{b_{j\alpha}\}_{j=1}^\infty\|_{\ell^s})^s \, t^{\frac{n}{2p}-(\frac{n+|\alpha|}{2})s}. \tag{2.20}$$

Proof. The proof of i) is immediate. In order to prove ii) we use (2.19) and the inequality $|\sum_{j=1}^\infty b_j|^s \leq \sum_{j=1}^\infty |b_j|^s$, $0 < s \leq 1$ and obtain that

$$|g(U_\mu^0(t,x))| \leq A \sum_{j=1}^\infty \sum_{|\alpha|\leq k} |b_{j\alpha}|^s |\partial_x^\alpha E_n(t, x - \xi^j)|^s \tag{2.21}$$

for all $t > 0$, $x \in \mathbb{R}^n$. Hence

$$\|g(U_\mu^0(t,\cdot))\|_p \leq A \sum_{j=1}^\infty \sum_{|\alpha|\leq k} |b_{j\alpha}|^s \||\partial_x^\alpha E_n(t,\cdot)|^s\|_p. \tag{2.22}$$

Then (2.9) allows us to calculate $\||\partial_x^\alpha E_n(t,\cdot)|^s\|_p$ and to get the desired estimate. ☐

We introduce the space

$$X_r(T) := C_r(L^1(\mathbb{R}^n) : T) \cap C_{r+n/2}(L^\infty(\mathbb{R}^n) : T) \tag{2.23}$$

with the norm $\|u\|_{r,T} := \|u\|_{r,1,T} + \|u\|_{r+n/2,\infty,T}$. Clearly, for every $T > 0$, $r \in \mathbb{R}$, $X_r(T)$ is a Banach space.

3. Nonlinear term with superlinear growth. We require that $g \in C(\mathbb{R} : \mathbb{R})$, $g(0) = 0$ and there exist two constants $A > 0$ and $s > 1$ such that

$$|g(u) - g(v)| \leq A|u - v|(1 + (|u| + |v|)^{s-1}), \quad u, v \in \mathbb{R}, \tag{3.1}$$

or the stronger requirement

$$|g(u) - g(v)| \leq A|u - v|((|u| + |v|)^{s-1}, \quad u, v \in \mathbb{R}. \qquad (3.1)'$$

Now we state the first assertion on the Cauchy problem (1.1), (1.2).

Theorem 3.1. *Let $k = 0$ or $k = 1$ and suppose that*

$$1 < s < \frac{n+2}{n+k}. \tag{3.2}$$

Then we claim that if (3.1) (respectively, (3.1)′) holds for any given $\mu \in \mathcal{M}^k(\mathbb{R}^n)$ one can find $T = T_\mu > 0$ such that

i) *there exists a unique weak solution $u \in X_{k/2}(T) \cap C([0,T) : \mathcal{M}^k(\mathbb{R}^n))$ (respectively, $u \in C_{k/2+n/2(1-1/s)}(L^s(\mathbb{R}^n) : T) \cap C([0,T) : \mathcal{M}^k(\mathbb{R}^n)))$ of the Cauchy problem (1.1), (1.2);*

ii) *the following properties hold:*

$$u \in \bigcap_{p=1}^{\infty} C_{k/2+n/2(1-1/p)}(L^p(\mathbb{R}^n) : T), \tag{3.3}$$

$$\partial_x^\alpha v \in \bigcap_{p=1}^{\infty} C_{\theta_k(p)+|\alpha|/2}(L^p(\mathbb{R}^n) : T), \quad \alpha \in \mathbb{Z}_+^n,\ |\alpha| \leq 1, \tag{3.4}$$

where $v(t,x) = u(t,x) - U_\mu^0(t,x)$ and

$$\theta_k(p) = \frac{k}{2} + \frac{n}{2}(1 - \frac{1}{p}) - \rho_k(s),\ \rho_k(s) = 1 - \frac{n+k}{2}(s-1) \tag{3.5}$$

(we observe that (3.2) leads to $\rho_o(s) > 0$ and $\rho_1(s) > 1/2$);

iii) *if (3.1)′ is satisfied, the life-span T_μ is bounded from below by*

$$\overline{T}_\mu = C \,|\, \mu \,|_k^{-\frac{2(s-1)}{2\rho_k(s)+k(s-1)}}, \quad |\, \mu_k \,| \leq 1; \tag{3.6}$$

if (3.1) is satisfied, we get $\overline{T}_\mu$ as a universal constant independent of $|\,\mu\,|_k$ when $|\,\mu\,|_k \leq 1$; assuming, in particular, that g is of the form

$$g(u) = cu + G(u), \tag{3.7}$$

c a constant and G satisfying (3.1)′; this is true, for example, if $g \in C^2(\mathbb{R} : \mathbb{R})$ is such that

$$\sup |g''(u)| < \infty, \tag{3.8}$$

then, in the case $c \geq 0$, $\overline{T}_\mu$ is given by (3.6), otherwise,

$$\overline{T}_\mu = C \ln\left(\frac{1}{|\mu|_k}\right), \quad |\mu|_k \to 0; \tag{3.9}$$

iv) *if $(3.1)'$ holds we have the following local stability estimate:*

$$\|u_1 - u_2\|_{k/2+n/2(1-1/s),s,T} \leq \|U^0_{\mu_1} - U^0_{\mu_2}\|_{k/2+n/2(1-1/s),s,T}, \tag{3.10}$$

for $0 < T \leq \min\{\overline{T}_{\mu_1}, \overline{T}_{\mu_2}\}$.
Finally we note that $u(t,x) \in C^{2,1}((0,T) \times \mathbb{R}^n)$.

Proof. Suppose first that $(3.1)'$ is true. We rewrite the Cauchy problem (1.1), (1.2) as an integral equation

$$\begin{aligned} u(t,\cdot) &= K(u)(t,\cdot) := e^{t\triangle}\mu + K_o(u)(t,\cdot), \\ K_o(u)(t,\cdot) &:= \int_0^t e^{(t-\tau)\triangle} g(u(\tau,\cdot))d\tau. \end{aligned} \tag{3.11}$$

Using (2.12) with $n = d,\ \gamma_1 = \gamma = n/2(1-1/s),\ \gamma_2 = 0$ and $(3.1)'$ we get:

$$\begin{aligned} &t^{k/2+n/2(1-1/s)}\|K_o(u)(t,\cdot)\|_s \\ &\leq A\Theta(\gamma)t^{k/2+n/2(1-1/s)} \int_0^t \frac{(\|u\|_{k/2+n/2(1-1/s),s,\tau})^s}{(t-\tau)^{n/2(1-1/s)}\tau^{ks/2+n(s-1)/2}} d\tau \\ &\leq A\Theta(\gamma)B_k T^{\rho_k(s)}(\|u\|_{\theta_k(s)+\rho_k(s),s,T})^s, \end{aligned} \tag{3.12}$$

where $B_k = B(1 - n/2(1-1/s), 1 - ks/2 - n(s-1)/2)$ is the Beta function, which, together with (2.15), leads to

$$\begin{aligned} \|K(u)\|_{\theta_k(s)+\rho_k(s),s,T} \leq \sum_{j=0}^k C_j(s)T^{j/2}|\mu|_k \\ + A\Theta(\gamma)B_k T^{\rho_k(s)}(\|u\|_{\theta_k(s)+\rho_k(s),s,T})^s. \end{aligned} \tag{3.13}$$

where $C_j(s) = C_j(0,0,s,n)$ is given in (2.15). Next we have:

$$\begin{aligned} &\|K(u_1) - K(u_2)\|_{\theta_k(s)+\rho_k(s),s,T} \\ \leq &2^{s-1}A\Theta(\gamma)B_k T^{\rho_k(s)}\|u_1 - u_2\|_{\theta_k(s)+\rho_k(s),s,T} \max_{i=1,2}(\|u_i\|_{\theta_k(s)+\rho_k(s),s,T})^{s-1}. \end{aligned} \tag{3.14}$$

Let us explicitly find $T > 0$ and $R > 0$ such that the Fixed Point Theorem (FPT) applies to K on

$$B_R(T) = \{u \in C_{\theta_k(s)+\rho_k(s)}(L^s(\mathbb{R}^n) : T) : \|u\|_{\theta_k(s)+\rho_k(s),s,T} \leq R\}.$$

In view of (3.13) and (3.14) it is enough for R and T to satisfy

$$\sum_{j=0}^{k} C_j(s) T^{j/2} |\mu|_k + A\Theta(\gamma) B_k T^{\rho_k(s)} R^s \leq R \tag{3.15}$$

and

$$\delta := 2^{s-1} A\Theta(\gamma) B_k T^{\rho_k(s)} R^{s-1} < 1. \tag{3.16}$$

Set $H_k = A\Theta(\gamma)B_k$. Replacing (3.16) in (3.15) gives:

$$\sum_{j=0}^{k} C_j(s) T^{j/2} |\mu|_k - (1 - \frac{\delta}{2^{s-1}}) R \leq 0. \tag{3.17}$$

From (3.16) we get

$$R = (\frac{\delta}{2^{s-1} H_k T^{\rho_k(s)}})^{1/(s-1)} \tag{3.18}$$

and (3.17), (3.18) imply

$$(1 - \frac{\delta}{2^{s-1}})\delta^{1/(s-1)} \geq \sum_{j=0}^{k} 2C_j(s) T^{j/2} |\mu|_k H_k^{1/(s-1)} T^{\rho_k(s)/(s-1)}. \tag{3.19}$$

Let

$$h(\delta) = (1 - \frac{\delta}{2^{s-1}})\delta^{1/(s-1)}, \quad \delta > 0; \tag{3.20}$$

it has a maximum at $\delta_o = 2^{s-1}/s$ which, for $s > 1$, belongs to $]0, 1[$ only if $s < 2$. Then

$$\sup_{0<\delta<1} h(\delta) = \begin{cases} \frac{2(s-1)}{s^{s/(s-1)}} & \text{if } 1 < s < 2 \\ h(1) = 1 - \frac{1}{2^{s-1}} & \text{if } s \geq 2 \ (h \text{ is increasing if } \delta < \frac{2^{s-1}}{s}). \end{cases}$$

Let $T_{s,k}(n) > 0$ be the unique solution of the equation obtained by replacing the left hand side of (3.19) by its supremum.

Now we consider separately the cases $k=0$ and $k=1$:
1) if $k=0$ and $n>1$ (which implies $s<2$), $T_{s,0}(n)$ is given by

$$T_{s,0}(n) = \frac{(s-1)^{\frac{s-1}{\rho_o(s)}}}{s^{\frac{s}{\rho_o(s)}} H_o^{\frac{1}{\rho_o(s)}} (C_o(s)|\mu|)^{\frac{s-1}{\rho_o(s)}}}. \tag{3.21}$$

If $n=1$ we have $s<3$. In this case,

$$T_{s,0}(1) = \Big(\frac{2^{s-1}-1}{s^s H_o^{1/(s-1)} C_o(s)|\mu|}\Big)^{(s-1)/\rho_o(s)}. \tag{3.22}$$

For all $T \le T_{s,0}(n),\ n>1$ (respectively, $T < T_{s,0}(1)$), the inequality (3.17) is true with $\delta = \delta_o$ (respectively, for some $0<\delta_1<1$) and FPT holds.

2) if $k=1$, by Proposition 2.1, μ is written as $\mu = \mu_o + \sum_{j=1}^n \partial_{x_j}\mu_j$ where $\mu_j \in \mathcal{M}(\mathbb{R}^n),\ j=0,\dots,n$ and $|\mu|_1 = \max_{0\le j\le n}|\mu_j|$. In this case a more precise estimate for $t^{1/2+n/2(1-1/s)}\|e^{t\triangle}\mu\|_s$ is

$$t^{1/2+n/2(1-1/s)}\|e^{t\triangle}\mu\|_s \le \Theta(\gamma)T^{1/2}|\mu_o| + \chi(\gamma)|\mu|_1,$$

where $\gamma = n/2(1-1/s)$; thus, we consider the cases $\mu_o = 0$ and $\mu_o \ne 0$:

2.a) $\mu_o = 0$: $T_{s,1}(n)$ is given by

$$T_{s,1}(n) = \Big(\frac{s-1}{s^{s/(s-1)} H_1^{1/(s-1)} \chi(\gamma)|\mu|_1}\Big)^{\frac{s-1}{\rho_1(s)}}; \tag{3.23}$$

2.b) $\mu_o \ne 0$: $T_{s,1}(n)$ is given implicitly by

$$\frac{2(s-1)}{s^{s/(s-1)}} = H_1^{1/(s-1)} T_{s,1}(n)^{\rho_1(s)/(s-1)} (\Theta(\gamma) T_{s,1}(n)^{1/2} + \chi(\gamma))|\mu|_1. \tag{3.24}$$

Now we deal with ii). In order to estimate $v(t,\cdot)$ in $L^1(\mathbb{R}^n)$ we note that

$$\|v(t,\cdot)\|_1 \le A\int_0^t (\|u(\tau,\cdot)\|_s)^s d\tau \le A t^{-\theta_k(1)} (\|u\|_{k/2+n/2(1-1/s),s,t})^{s-1}, \tag{3.25}$$

for all $0<t\le T$. Then we use (3.12), (3.25) and the convexity property of the L^p norms to get $v \in C_{\theta_k(p)}(L^p(\mathbb{R}^n):T)$ for $1\le p\le s$.

The estimates in $L^\infty(\mathbb{R}^n)$ are more delicate. We have

Lemma 3.1. *Let $g(u)$ satisfy $(3.1)'$. Then for every $u \in X_{k/2}(T)$ we have*

$$\|K_o(u)\|_{\theta_k(\infty),\infty,T} \leq AB' \, \|u\|_{k/2,T}(\|u\|_{(n+k)/2,\infty,T})^{s-1} \tag{3.26}$$

with certain $B' > 0$ depending on n, s and k.

Proof. In view of (3.11) and the definitions of v and $\theta_k(\infty)$ we can write the following chain of inequalities for $0 < t \leq T$

$$\begin{aligned}\|K_o(u)(t,\cdot)\|_\infty &\leq A\Big(\int_0^{\frac{t}{2}} \frac{\|u(\tau,\cdot)\|_1(\|u(\tau,\cdot)\|_\infty)^{s-1}}{(4\pi(t-\tau))^{\frac{n}{2}}} d\tau + \int_{\frac{t}{2}}^t (\|u(\tau,\cdot)\|_\infty)^s d\tau\Big) \\ &\leq A\|u\|_{k/2,T}(\|u\|_{(n+k)/2,\infty,T})^{s-1}\Big(\int_0^{t/2} \frac{\tau^{n/2-s(n+k)/2}}{(2\pi t)^{n/2}} d\tau + \int_{t/2}^t \tau^{-s(n+k)/2} d\tau\Big) \\ &= AB' t^{-\theta_k(\infty)}\|u\|_{k/2,T}(\|u\|_{(n+k)/2,\infty,T})^{s-1}.\end{aligned}$$

We note that we have used the condition (3.2) in the inequalities above since the integral $\int_0^{t/2} \tau^{n/2-s(n+k)/2} d\tau$ is convergent iff (3.2) is true. □

The estimates above combined with arguments similar to those used for the FPT in $B_R(T)$ lead to $u \in X_{k/2}(T)$. The interpolation between L^1 and L^∞ spaces yields (3.4) for $\alpha = 0$.

Now we take $\alpha \in \mathbb{Z}_+^n$ such that $|\alpha| = 1$. We get from (3.11)

$$\partial_x^\alpha v(t,\cdot) = \partial_x^\alpha K_o(u)(t,\cdot) = \int_0^t \partial_x^\alpha E_n(t-\tau,\cdot) * g(u)(\tau,\cdot) d\tau. \tag{3.27}$$

According to Proposition 2.2 we can write

$$\begin{aligned}\|\partial_x^\alpha v(t,\cdot)\|_1 = \|\partial_x^\alpha K_o(u)(t,\cdot)\|_1 &\leq A\chi(1-\frac{1}{s}) \int_0^t \frac{1}{\sqrt{t-\tau}} \|u(\tau,\cdot)\|_s d\tau \\ &\leq A t^{1/2+n/2-s(n+k)/2}(\|u\|_{k/2+n/2(1-1/s),s,t})^{s-1},\end{aligned} \tag{3.28}$$

for all $0 < t \leq T$. For the L^∞ estimate of $\partial_x^\alpha v$ we repeat the arguments of Lemma 3.1 but multiplying with $t^{1/2+\theta_k(\infty)}$ instead of $t^{\theta_k(\infty)}$ and using the estimates for the operator $\partial_x^\alpha e^{t\triangle}$ instead of the ones for $e^{t\triangle}$.

Now we suppose that (3.1) holds. In that case we claim:

Lemma 3.2. *Under the assumption* (3.1) *the operator* K_o *satisfies*

$$\|K_o(u)\|_{k/2,T} \le A_1 T^{\rho_k(s)}(\|u\|_{k/2,T})^s + A_2 T\|u\|_{k/2,T}, \tag{3.29}$$

where $A_j = A_j(s,k,n) > 0$, $j = 1,2$.

Proof. In view of (3.1) we have

$$\begin{aligned} t^{\frac{k}{2}}\|K_o u(t,\cdot)\|_1 &\le A t^{\frac{k}{2}} \int_0^t [\|u(\tau,\cdot)\|_1 + (\|u(\tau,\cdot)\|_s)^s] d\tau \\ &\le A\|u\|_{\frac{k}{2},1,T}\Big[\frac{2t}{2-k} + \frac{t^{1-\frac{(k+n)(s-1)}{2}}}{1-\frac{k}{2}-\frac{(k+n)(s-1)}{2}}(\|u\|_{\frac{k+n}{2},\infty,T})^{s-1}\Big] \end{aligned} \tag{3.30}$$

and

$$t^{\frac{n+k}{2}}\|K_o u(t,\cdot)\|_\infty \le t^{\frac{n+k}{2}}\Big[\int_0^{\frac{t}{2}} \frac{\|g(u(\tau,\cdot))\|_1}{(4\pi(t-\tau))^{\frac{n}{2}}} d\tau + \int_{\frac{t}{2}}^t \|g(u(\tau,\cdot))\|_\infty d\tau\Big]$$

$$\begin{aligned} \le& A t^{\frac{k+n}{2}}\Big[\int_0^{t/2} \frac{\tau^{-k/2}}{(4\pi(t-\tau))^{n/2}} d\tau \|u\|_{\frac{k}{2},T} + \int_0^{t/2} \frac{\tau^{-\frac{(n+k)(s-1)}{2}-\frac{k}{2}}}{(4\pi(t-\tau))^{n/2}} d\tau (\|u\|_{\frac{k}{2},T})^s\Big] \\ &+ A t^{\frac{k+n}{2}}\Big[\int_{t/2}^t \tau^{-\frac{n+k}{2}} d\tau \|u\|_{\frac{k}{2},T} + \int_{t/2}^t \tau^{-\frac{(n+k)s}{2}} d\tau (\|u\|_{\frac{k}{2},T})^s\Big]. \end{aligned} \tag{3.31}$$

Inequality (3.29) is obtained summing (3.30) and (3.31). □

Continuing the proof of Theorem 3.1, from (3.27) and (2.15) we get

$$\|Ku\|_{\frac{k}{2},T} \le \sum_{j=0}^k (C'_j) T^{\frac{j}{2}} |\mu|_k + A_1 T^{\rho_k(s)}(\|u\|_{\frac{k}{2},T})^s + A_2 T\|u\|_{\frac{k}{2},T}, \tag{3.32}$$

where $C'_j = C_j(0,0,1,n) + C_j(0,0,\infty,n)$. Analogously, we get

$$\begin{aligned} \|Ku_1 - Ku_2\|_{\frac{k}{2},T} \le& 2^{s-1} A_1 \|u_1-u_2\|_{\frac{k}{2},T} T^{\rho_k(s)} \max_{i=1,2}(\|u_i\|_{\frac{k}{2},T})^{s-1} \\ &+ A_2 T\|u_1-u_2\|_{\frac{k}{2},T}. \end{aligned} \tag{3.33}$$

Now, in order to apply FPT we have to solve the inequalities:

$$\sum_{j=0}^k C'_j T^{\frac{j}{2}} |\mu|_k + (A_1 T^{\rho_k(s)} + A_2 T) R^s \le R \tag{3.34}$$

$$2^{s-1} A_1 T^{\rho_k(s)} R^{s-1} + A_2 T := \delta < 1, \tag{3.35}$$

which may be handled analogously to (3.15)–(3.16).

For the life-span, in the particular case of g satisfying (3.7), it suffices to set $u = e^{-ct}w$ and w will satisfy (1.1) with $g(u)$ being replaced by $h(t,w) = e^{ct}G(e^{-ct}w)$ and

$$|h(t,w_1) - h(t,w_2)| \le Ae^{-c(s-1)t}|w_1 - w_2|(|w_1| + |w_2|)^{s-1}. \tag{3.36}$$

Then with the obvious changes in the proof of (3.6) we get (3.9).

As to the the initial data, we have that for each $\psi \in C_b^k(\mathbb{R}^n)$ the following inequalities hold

$$\begin{aligned}|(u(t,\cdot) - \mu, \psi)| &\le |(e^{t\triangle}\mu(\cdot) - \mu, \psi)| + |(K_0(u)(t,\cdot), \psi)| \\ &\le |(e^{t\triangle}\mu - \mu, \psi)| + t^{\rho_k(s)}\|v\|_{k/2,1,T}\|\psi\|_\infty,\end{aligned}$$

which clearly implies that $u(t,\cdot)$ tends to μ for $t \searrow 0$ in $C([0,T) : \mathcal{M}^k(\mathbb{R}^n))$, $k = 0, 1$. Finally, we note that the $C^{2,1}$ regularity of u in $(0,T) \times \mathbb{R}^n$ follows from the standard theory when the Cauchy data are L^∞. $\square$

Let now $\varphi(x) \in \mathcal{S}(\mathbb{R}^n)$ be a Friedrichs mollifier. For every $\varepsilon > 0$ we consider the Cauchy problem for (1.1) with initial data

$$u(0,x) = \mu_\varphi^\varepsilon(x) := \varphi^\varepsilon * \mu(x), \quad \varphi^\varepsilon(x) = \varepsilon^{-n}\varphi(\varepsilon^{-1}x). \tag{3.37}$$

Theorem 3.2. *Let the assumptions of Theorem* 3.1 *hold and let*

$$\mu \in \mathcal{M}^k(\mathbb{R}^n).$$

Then if $(3.1)'$ (*respectively,* (3.1)) *holds, we claim that one can find* $T > 0$ *such that for every mollifier* $\varphi(x)$ *and every* $\varepsilon > 0$ *there exists a unique* $C^{2,1}([0,T) \times \mathbb{R}^n) \bigcap C_{k/2+n/2(1-1/s)}(L^s(\mathbb{R}^n) : T)$ (*respectively,* $C^{2,1}([0,T) \times \mathbb{R}^n) \bigcap X_{k/2}(T)$) *solution* $u_\varphi^\varepsilon(t,x)$ *of the Cauchy problem* (1.1), (3.37) *such that*

$$\lim_{\varepsilon \searrow 0} u_\varphi^\varepsilon = u \text{ in } C_{k/2+n/2(1-1/s)}(L^s(\mathbb{R}^n) : T)(\text{ respectively, } X_{k/2}(T)),$$

$$\lim_{\varepsilon \searrow 0} \partial_x^\alpha v_\varphi^\varepsilon = \partial_x^\alpha v \text{ in } \bigcap_{p=1}^{\infty} C_{\theta_k(p)+|\alpha|/2}(L^p(\mathbb{R}^n) : T), \alpha \in \mathbb{Z}_+^n, |\alpha| \le 1,$$

where $v_\varphi^\varepsilon(t,x) := u_\varphi^\varepsilon(t,x) - U^0_{\mu_\varphi^\varepsilon}(t,x)$.

Proof. First, we observe that $\|e^{t\triangle}\mu_\varepsilon\|_p \le \|e^{t\triangle}\mu\|_p$, $\varepsilon > 0$. Then we follow the arguments of the proof of Theorem 3.1 substituting u (respectively, v) by

u^ε (respectively, by v^ε) and noting that the difference $u^\varepsilon - u$ (respectively, $v^\varepsilon - v$) can be estimated uniformly with respect to $\varepsilon \searrow 0$ by the difference $e^{t\triangle}(\mu^\varepsilon - \mu)$ in the corresponding weighted spaces. $\square$

Now we present the main result for singular initial data in the superlinear case using anisotropic spaces. For the sake of simplicity we deal with condition $(3.1)'$ only. We fix an integer d, $0 \le d \le n$, with the convention $x' = x$ (respectively, $x'' = x$) if $d = n$ (respectively, $d = 0$).

Theorem 3.3. *Let $g(u)$ satisfy $(3.1)'$ with some $s \ge 1$, and let $q_1,\ q_2 \ge s$ and $r \ge 0$ satisfy the conditions*

$$(\frac{d}{2q_1} + \frac{n-d}{2q_2})(s-1) < 1, \tag{3.38}$$

$$rs < 1, \tag{3.39}$$

$$1 - r(s-1) - (\frac{d}{2q_1} + \frac{n-d}{2q_2})(s-1) > 0. \tag{3.40}$$

Suppose that $\mu(x) \in \mathcal{S}'(\mathbb{R}^n)$ and

$$U^0_\mu \in C_r(L^{q_1,q_2}_d(\mathbb{R}^n) : T), \quad T > 0. \tag{3.41}$$

Then there exists $T_\mu > 0$ such that (1.1) admits a unique solution

$$u \in C_r(L^{q_1,q_2}_d(\mathbb{R}^n) : T_\mu) \cap C([0, T_\mu) : \mathcal{S}'(\mathbb{R}^n))$$

with initial value $u(0,\cdot) = \mu$.

If, instead of (3.40), we have

$$1 - r(s-1) - (\frac{d}{2q_1} + \frac{n-d}{2q_2})(s-1) = 0 \quad \text{and}$$
$$\lim_{T\to 0} \mu(T) < M_h \frac{C_0^{1/(1-s)}}{2}, \tag{3.42}$$

where $\mu(T) = \|U^0_\mu\|_{r,(q_1,q_2),T}$, $C_0 = A\Theta(\gamma)B^d_k$, $B^d_k = B(1 - (\frac{d}{2q_1} + \frac{n-d}{2q_2})(s - 1), 1 - rs)$, with $\Theta(\gamma)$ defined by (2.12), taking $r_1 = q_1/s$, $r_2 = q_2/s$, and letting M_h be the supremum over $]0,1[$ of the function $h(\delta)$ given by (3.20), then the same conclusions are true. Moreover, if (3.42) holds and

$$\mu(+\infty) := \lim_{T\to\infty} \mu(T) < (s-1)s^{\frac{s}{1-s}} C_o^{\frac{1}{1-s}}, \tag{3.43}$$

then the solution is global in $t > 0$*. Finally we note that* (3.38), (3.39), (3.42) *lead to* $s > 1 + \frac{2}{n}$ *and therefore our conditions for the global solvability of the Cauchy problem imply that* s *must be supercritical.*

Proof. Propositions 2.2, 2.3, $(3.1)'$ and the hypotheses (3.38), (3.39) imply

$$\begin{aligned}\|K_o(u)(t,\cdot)\|_{q_1,q_2} &\le \Theta(\gamma)\int_0^t (t-\tau)^{-(\frac{d}{2q_1}+\frac{n-d}{2q_2})(s-1)}\|g(u(\tau,\cdot))\|_{\frac{q_1}{s},\frac{q_2}{s}}\,d\tau \\ &\le A\Theta(\gamma)\int_0^t \frac{(\|u\|_{r,(q_1,q_2),T})^s}{(t-\tau)^{(\frac{d}{2q_1}+\frac{n-d}{2q_2})(s-1)}\tau^{rs}}d\tau \\ &= C_0T^{1-rs-(\frac{d}{2q_1}+\frac{n-d}{2q_2})(s-1)}(\|u\|_{r,(q_1,q_2),T})^s\end{aligned}$$

for all $u \in C_r(L_d^{q_1,q_2}(\mathbb{R}^n):T)$. Using exactly the same arguments as in the proof of Theorem 3.1 we reduce the proof to the application of the FPT in

$$B_R^d(T) = \{u \in C_r(L_d^{q_1,q_2}(\mathbb{R}^n):T) : \|u\|_{r,(q_1,q_2),T} \le R\}$$

for some $R > 0$, $T > 0$, namely, it will be possible if we find $R > 0$ and $T > 0$ satisfying the following estimates

$$\mu(T) + C_0T^{1-r(s-1)-(\frac{d}{2q_1}+\frac{n-d}{2q_2})(s-1)}R^s \le R, \tag{3.44}$$

$$2^{s-1}C_0T^{1-r(s-1)-(\frac{d}{2q_1}+\frac{n-d}{2q_2})(s-1)}R^{s-1} \le \delta, \quad 0 < \delta < 1. \tag{3.45}$$

Now, taking into account that $\mu(T)$ is a nondecreasing function of T, if the case (3.40) holds we can satisfy (3.44), (3.45) by choosing first some $T_0 > 0$, then $R = \delta^{-1}\mu(T_0)$ and at the end we choose T small enough. If (3.42) holds we substitute $R = 2^{-1}(\delta C_0^{-1})^{1/(s-1)}$ in (3.44) and find $T_0 > 0$ small enough such that

$$\mu(T_0) - \frac{1}{2}C_0^{1/(1-s)}M_h < 0$$

and the continuity of $h(\delta)$ yields the desired conclusion. Concerning the global existence, put $z(t) = \|u\|_{r,(q_1,q_2),t}$ for $0 < t < T_\mu$, T_μ being the life span. From (3.11) and (3.44) we have

$$z(t) \le \mu(T) + C_o(z(t))^s \le \mu(+\infty) + C_o(z(t))^s.$$

Hence standard arguments (see [23, p.38]) imply that $z(t)$ cannot become infinity in finite time provided (3.43) is satisfied. □

If $d = 0$ or $d = n$, Theorem 3.3 generalizes in anisotropic spaces results on the local existence for (1.1) with $L^p(\mathbb{R}^n)$ initial data in [22], [15] (see also the discussion in [17], pp. 991–992) and with distributional initial data in [17], [20], [21]. More precisely:

Corollary 3.1. *Let* $p \geq \max\{1, \frac{n(s-1)}{2}\}$ *and* $p > 1$ *when* $s = \frac{n+2}{n}$. *Given a function* $\psi \in L^p(\mathbb{R}^n)$ *and a real number* k *satisfying* $0 \leq k < k(s,n,p)$ *where*

$$k(s,n,p) = \begin{cases} \text{a) } \frac{2}{s-1} - \frac{n}{p} & \text{if } s > \frac{n+2}{n}, \\ & \quad \max\{s, \frac{n(s-1)}{2}\} \leq p < \frac{ns(s-1)}{2} \\ \text{b) } \frac{2}{s} & \text{if } s > \frac{n+2}{n},\ p \geq \frac{ns(s-1)}{2} \\ \text{c) } \frac{2}{s-1} - \frac{n}{p} & \text{if } s > \frac{n+2}{n},\ \frac{n(s-1)}{2} \leq p < s \\ \text{d) } \frac{2}{s} & \text{if } s \leq \frac{n+2}{n},\ p \geq s \\ \text{e) } \frac{2}{s} - \frac{n(s-p)}{ps} & \text{if } s \leq \frac{n+2}{n},\ 1 < p < s; \end{cases} \tag{3.46}$$

k may be equal to $k(s,n,p)$ in the cases a) *and* c), *we set* $\mu = |D|^k\psi$. *Then we can find* $T = T_\mu > 0$ *such that there exists a unique solution* u *of the equation* (1.1) *with initial data* μ *in*

$$C([0,T) : \mathcal{S}'(\mathbb{R}^n)) \cap C_{\frac{k}{2}+\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}(L^q(\mathbb{R}^n) : T),$$

where $q = p + \varepsilon$ *if* $p \geq s$ *and* $q = s + \varepsilon$ *otherwise, with* $\varepsilon > 0$ *small enough, with* $\varepsilon = 0$ *when* $k < k(s,n,p)$. *The life-span* T_μ, *when* $k < \frac{2}{s-1} - \frac{n}{p}$ *is bounded from below by* $\overline{T}_\mu = C\|\psi\|_p^\ell$, *with* $\ell = -\frac{s-1}{1-\frac{s-1}{2}(k+\frac{n}{p})}$. *Furthermore,* $u \in C([0,T) : \dot{L}_{-k,p}(\mathbb{R}^n))$, $\dot{L}_{-k,p}(\mathbb{R}^n)$ *being the homogeneous Lebesgue space* $|D|^k L^p(\mathbb{R}^n)$ *and the solution* u *is global provided* $\|\psi\|_p$ *is small enough.*

Proof. As in the proof of Theorem 3.3 we choose $r = \frac{k}{2} + \frac{n}{2}(\frac{1}{p} - \frac{1}{q})$. Straightforward calculations show that (3.46) and the choice of q imply that the assumptions of the theorem are verified, modulo the smallness condition in (3.42) when $k = k(s,n,p)$, which follows from the fact that, if $q > p$,

$$\lim_{t\to 0} t^{\frac{k}{2}+\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|e^{t\triangle}|D|^k\psi\|_q = 0$$

for all $\psi \in L^p(\mathbb{R}^n)$ (see Proposition 2.3). Finally, we can show the life-span estimate by a suitable change of the arguments used in showing (3.6).

Remark 3.1. We observe that $k(s,n,p) = 0$ iff $p = \frac{n(s-1)}{2}$ and either $n \geq 3$, $s \geq \frac{n}{n-2}$ or $\frac{n+2}{n} < s$ and $s < \frac{n}{n-2}$ provided $n \geq 3$ and therefore the result of the corollary coincides with Theorem 1 in [22] when $k(s,n,p) = 0$.

Corollary 3.2.. *Let $n \geq 4,\ s > 1,\ 1 \leq d < n-2,\ q_2 > 1$ and $k \in \mathbb{N}$ satisfy*

$$s < \frac{d+2}{d+k}, \tag{3.47}$$

$$q_2 \geq \max\{s, \frac{n-d}{\frac{2}{s-1} - d}\}, \tag{3.48}$$

$$k < \frac{2}{s} - d(1 - \frac{1}{s}) =: k_o \tag{3.49}$$

$$k \leq \frac{2}{s-1} - d - n - dq_2 =: k_1. \tag{3.50}$$

If $\mu = |D'|^k \mu_1 \otimes f,\ \mu_1 \in \mathcal{M}(\mathbb{R}^d),\ f \in L^{q_2}(\mathbb{R}^{n-d})$, then the hypotheses of Theorem 3.3 hold, with $q_1 = s,\ r = \frac{k}{2} + \frac{d}{2}(1 - \frac{1}{s})$ and, as to (3.42), it is satisfied provided $k = k_1$ and

$$\lim_{t \to 0}(\sup_{0<\tau<t} \tau^{\frac{k}{2} + \frac{d}{2}(1-\frac{1}{s})} \| |D'|^k e^{\tau\triangle'} \mu_1 \|_s) \|f\|_{q_2} < \frac{M_h}{2} C_o^{\frac{1}{1-s}}. \tag{3.51}$$

The global solution exists if $k = k_1$ and

$$|\!|\mu_1|\!| \, \|f\|_{q_2} \leq \frac{M_h}{2\tilde{C}} C_o^{\frac{1}{1-s}} \tag{3.52}$$

where $\tilde{C} = \sup_{t>0} t^{\frac{k}{2} + \frac{d}{2}(1-\frac{1}{s})} \| |D'|^k e^{t\triangle'} \|_{\mathcal{M} \to L^s}$.

Proof. In fact, inequalities (3.47) and (3.48) imply (3.38); (3.47) and (3.49) imply (3.39) and (3.48), (3.50) imply (3.40). Concerning the global solution, we recall that

$$t^{\frac{k}{2} + \frac{d}{2}(1-\frac{1}{s})} \|e^{t\triangle'} |D'|^k \mu_1\|_s \|e^{t\triangle''} f\|_{q_2} \leq \tilde{C} |\!|\mu|\!| \|f\|_{q_2}$$

and therefore (3.52) yields (3.43).

Remark 3.2. Note that $k_o > k_1$ iff $q_2 < \frac{(s-1)(n-d)}{d+2-sd}$. From (3.48) one gets that $k_o > k_1$ might occur iff $s > \frac{n+2}{n}$. So, in this case we may have equality in (3.50) and subsequently global solutions for small data. If $s \leq \frac{n+2}{n}$, then the condition (3.49) implies (3.50) with strict inequality.

4. The sublinear case. Throughout this section we suppose that for some positive numbers s, $s < 1$, A_0 and A_1, the following hypotheses hold

$$|g(u)| \leq A_0 |u|^s, \quad u \in \mathbb{R}, \tag{4.1}$$

$$|g(u) - g(v)| \leq A_1 |u - v|, \quad u, v \in \mathbb{R}. \tag{4.2}$$

Theorem 4.1. *Let $k \in \mathbb{Z}_+$ and let the number s in (4.1) and (4.2) satisfy*

$$0 < s < \frac{n+2}{n+k}\ . \tag{4.3}$$

We define p^* as $+\infty$ if $\theta_k(+\infty) \le 0$ and in the case $\theta_k(+\infty) > 0$ we set p^* to be the unique zero of $\theta_k(\cdot)$, where $\theta_k(p)$ is defined in (3.5). Then for any given $\mu \in \mathcal{M}^{k,s}_{delta}(\mathbb{R}^n)$ the Cauchy problem (1.1), (1.2) has a unique global solution $u \in C([0,+\infty) : \mathcal{M}^k(\mathbb{R}^n))$ such that for every $T > 0$

$$v := u - U^0_\mu \in C_0(L^p(\mathbb{R}^n) : T), \quad 1 \le p < p^*, \tag{4.4}$$

$$\partial^\alpha_x v \in \bigcap_{p=1}^{\infty} C_{\theta_k(p)}(L^p(\mathbb{R}^n) : T), \quad \alpha \in \mathbb{Z}^n_+,\ |\alpha| \le 1. \tag{4.5}$$

Proof. Set $v(t,\cdot) = u(t,\cdot) - U^0_\mu(t,\cdot)$. Then v satisfies

$$\partial_t v(t,\cdot) - \triangle v(t,\cdot) = F_\mu(v(t,\cdot)) + g(U^0_\mu(t,\cdot)), \quad v(0,\cdot) = 0, \tag{4.6}$$

in a weak sense where

$$F_\mu(v)(t,\cdot) = g(v(t,\cdot) + U^0_\mu(t,\cdot)) - g(U^0_\mu(t,\cdot)). \tag{4.7}$$

We point out that $g(U^0_\mu)$ is well defined in view of Proposition 2.3, while if $v(t,\cdot) \in L^p(\mathbb{R}^n)$, $F_\mu(v)(t,\cdot) \in L^p(\mathbb{R}^n)$ as well, in view of (4.2).

Now we write (4.6) as an integral equation

$$v(t,\cdot) = K_\mu(v)(t,\cdot) = K^0_\mu(v)(t,\cdot) + v^0(t,\cdot),\ 0 < t < T,\ x \in \mathbb{R}^n, \tag{4.8}$$

where

$$K^0_\mu(v)(t,\cdot) = \int_0^t e^{(t-\tau)\triangle} F_\mu(v)(\tau,\cdot)d\tau,\ v^0(t,\cdot) = \int_0^t e^{(t-\tau)\triangle} g(U^0_\mu(\tau,\cdot))d\tau. \tag{4.9}$$

Next we show that for each $1 \le p \le \infty$, $T > 0$

$$\|v^0(t,\cdot)\|_{\theta_k(p),p,T} < \infty. \tag{4.10}$$

Indeed, let us write $v^0(t,\cdot) = v^0_1(t,\cdot) + v^0_2(t,\cdot)$ with

$$v^0_1(t,\cdot) = \int_0^{t/2} e^{(t-\tau)\triangle} g(U^0_\mu(t,\cdot))d\tau,\ v^0_2(t,\cdot) = \int_{t/2}^t e^{(t-\tau)\triangle} g(U^0_\mu(\tau,\cdot))d\tau.$$

Taking into account (2.20) we estimate the L^p norms as follows

$$\|v_1^0(t,\cdot)\|_p \le \Theta(n/2(1-1/p)) \int_0^{t/2} \frac{\|g(U_\mu^0(\tau,\cdot)\|_1}{(t-\tau)^{n/2(1-1/p)}} d\tau$$
$$\le t^{-\theta_k(p)} \Theta(n/2(1-1/p)) 2^{-n/2(1-1/p)} C_{k,s} \sum_{|\alpha|\le k} \|(\{b_{j\alpha}\}_{j=1}^\infty\|_{\ell^s})^s t^{\frac{(k-|\alpha|)s}{2}},$$
$$\|v_2^0(t,\cdot)\|_p \le \int_{t/2}^t \|g(U_\mu^0(\tau,\cdot)\|_p d\tau$$
$$\le t^{-\theta_k(p)} \Theta(n/2(1-1/p)) C_{k,s} \sum_{|\alpha|\le k} \|(\{b_{j\alpha}\}_{j=1}^\infty\|_{\ell^s})^s \, t^{\frac{(k-|\alpha|)s}{2}}$$

for all $t > 0$. Clearly, the two estimates above yield (4.10). In particular we have $v^0(t,\cdot) \in \dot{C}_0(L^p(\mathbb{R}^n))$, $1 \le p < p^*$.

The global Lipschitz property of $g(u)$ and (4.10) imply that for $T > 0$, $p \in [1, p^*[$ one can find a constant $C > 0$ such that

$$\|K_\mu(v)(t,\cdot)\|_p \le C \int_0^t \|v(\tau,\cdot)\|_p d\tau + \|v^0(t,\cdot)\|_p,$$
$$\|K_\mu(v)(t,\cdot) - K_\mu(w)(t,\cdot)\|_p \le C \int_0^t \|v(\tau,\cdot) - w(\tau,\cdot)\|_p d\tau,$$

for all $v, w \in C([0,+\infty) : L^p(\mathbb{R}^n))$ and $0 < t \le T$. Then a standard iterative scheme yields (4.4). As to (4.5), it follows from the a priori estimate

$$\|K_\mu^0(v)\|_{\theta_k(1),T} \le CT\|v\|_{\theta_k(1),T}, \ v \in X_{\theta_k(1)}(T), \ T > 0, \, C = C(T) > 0,$$

which is obtained by similar arguments to those used above. □

Evidently Theorem 4.1 leads to

Corollary 4.1. *Let $g(u)$ be a real-valued globally Lipschitz function and assume that for some $r > 0$ and $C > 0$ it satisfies*

$$|g(u)| \le C(\ln(1+|u|))^r, \quad u \in \mathbb{R}. \tag{4.11}$$

Then for each $k \in \mathbb{N}$ and each $\mu \in \mathcal{M}_{delta}^{k,s}(\mathbb{R}^n)$ the conclusions of Theorem 4.1 *hold.*

Now we study the weak limits of the solutions of the corresponding family of Cauchy problems (1.1), (3.37).

Theorem 4.2. *Let the assumptions of Theorem* 4.1 *hold and let*

$$\mu \in \mathcal{M}^{k,s}_{delta}(\mathbb{R}^n).$$

Then for every mollifier $\varphi(x)$ *and every* $\varepsilon > 0$ *there exists a unique solution* $u^\varepsilon_\varphi(t,x)$ *of the Cauchy problem* (1.1), (3.37) *belonging to* $C^{2,1}([0,+\infty) \times \mathbb{R}^n) \cap C_0(L^p(\mathbb{R}^n))$, $1 \le p \le \infty$. *Furthermore,*

$$\lim_{\varepsilon \searrow 0} u^\varepsilon_\varphi = u \ \textit{in} \ \bigcap_{p=1/s}^{\infty} C_{k/2+n/2(1-1/p)}(L^p(\mathbb{R}^n) : T), \quad \forall T > 0, \tag{4.12}$$

$$\lim_{\varepsilon \searrow 0} \partial^\alpha_x v^\varepsilon_\varphi = \partial^\alpha_x v \ \textit{in} \ \bigcap_{p=1/s}^{\infty} C_{\theta_k(p)}(L^p(\mathbb{R}^n) : T),\ |\alpha| \le 1, \quad \forall T > 0, \tag{4.13}$$

where $v^\varepsilon_\varphi := u^\varepsilon_\varphi - U^0_{\varphi^\varepsilon * \mu}$ *and* u *and* v *are as in Theorem* 4.1.

Proof. We apply the same arguments as in the proof of Theorem 3.2 with the difference that if $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is a Friedrichs mollifier we can estimate the $L^p(\mathbb{R}^n)$ norm of $g(U^0_{\varphi_\varepsilon * \mu}(t,\cdot))$ uniformly in $0 < \varepsilon \le 1$ only for $p \ge 1/s$, namely,

$$\|g(U^0_{\varphi_\varepsilon * \mu}(t,\cdot))\|_p \le A(\|U^0_{\varphi_\varepsilon * \mu}(t,\cdot)\|_{ps})^s \le A(\|U^0_\mu(t,\cdot)\|_{ps})^s, \tag{4.14}$$

for all $\varepsilon > 0$.

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