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SOME CHARACTERIZATIONS OF THE TAYLOR–COUETTE ATTRACTOR

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Abstract. The Taylor–Couette problem in infinite cylinders is considered for weakly unstable Couette flow in case of fixed outer cylinder. The following results about the attractor are established in this paper: 1) The upper–semicontinuity of the rescaled Taylor–Couette attractor towards the associated Ginzburg–Landau attractor holds. 2) Every solution in the Taylor–Couette attractor can be shadowed by some pseudo–orbit in this Ginzburg–Landau attractor. 3) The Taylor–Couette attractor only contains rotational symmetric solutions. Similar results hold for corresponding hydrodynamical stability problems, like Bénard's problem.

1. Introduction. In this paper we prove some results about the Taylor-Couette attractor in infinite cylinders in case of weakly unstable Couette flow. Similar results hold for corresponding hydrodynamical stability problems, like Bénard's problem, or reaction–diffusion systems, as the Brusselator, and many other systems.

The Taylor–Couette problem is a classical hydrodynamical stability problem, where a fluid is contained between two rotating concentric cylinders. Originally designed for the study of turbulence, it is nowadays a well studied problem in pattern formation. This problem is modeled with infinite cylinders in order to neglect the boundaries at the top and at the bottom of the cylinders, and to obtain the main issues of the pattern forming processes, i.e., to obtain a dynamics independent of the special length of the cylinders.

The Taylor–Couette problem with periodic boundary conditions can be studied with classical bifurcation analysis. An almost complete overview about this case can be found in the textbook [3] to which we refer for additional information. Here we consider the situation where the trivial ground

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state, the Couette flow, loses stability and bifurcates under periodic boundary conditions into a family of spatially periodic equilibria, the Taylor vortices. We are interested in the collection of all bifurcating solutions for the same situation, but now for the problem on the infinite line, i.e., we are interested in the attractor of the problem.

Since we are dealing with a 3D Navier–Stokes problem the existence of a global attractor is still unknown, even for bounded domains (cf. [2]). Under the assumption that such an attractor exists for bounded domains, there are characterizations, for instance in terms of its fractal dimension (cf. [9]). In some limit situations a local attractor can be defined and described in more detail (cf. [19]). In [16] it was pointed out that a useful definition of a local attractor on the infinite line is as follows. (See also [1] or [8].) It is an invariant, non–empty set which is bounded in a global (L^∞)–topology and which attracts sets contained in a positively invariant set in a local topology, for instance convergence on every compact interval. Moreover, for the linear diffusion equation it was observed that stronger convergence cannot be expected. This lack is due to the fact that the nonlinear semiflow is no longer compact on infinite domains, in contrast to bounded ones (cf. [10], [27]).

It is the aim of this paper to describe this local Taylor–Couette attractor in more detail. One of the tools which we use is the Ginzburg–Landau formalism. This theory, which is explained in section 3, allows us to describe the solutions of the more complicated Taylor–Couette problem with the solutions of the Ginzburg–Landau equation, a complex–valued nonlinear diffusion equation. In [24] with the help of this tool the existence of a positively invariant set for the weakly unstable Taylor–Couette problem has been established. In Theorem 2.2 a local attractor in the above sense is constructed for this positively invariant set. This attractor can be described by the associated Ginzburg–Landau attractor. In Theorem 4.2 the upper–semicontinuity of the rescaled Taylor–Couette attractor is shown, i.e., it converges into the Ginzburg–Landau attractor, as the bifurcation parameter goes to zero. Such a result has already been established in [16] for the Swift–Hohenberg equation. In Theorem 5.3 we prove that every solution in the Taylor–Couette attractor can be shadowed (approximated) for all times by a pseudo–orbit solution in the Ginzburg–Landau attractor. A (T_1, κ)–pseudo–orbit solution solves the Ginzburg–Landau equation on the time–intervals $[nT_1, (n+1)T_1)$ and makes jumps of size less than κ at the times $T = nT_1$ with $n \in \mathbb{N}$. This main result goes beyond [21] and [16].

Finally, in Theorem 6.1 we prove that the Taylor–Couette attractor only contains rotationally symmetric solutions.

The Ginzburg–Landau equation as an amplitude equation was first formally derived in [17] for Bénard's problem. In the following years the Ginzburg–Landau equation has been derived as an amplitude equation in many different situations in nonlinear physics (cf. [25], [15], [5]). The mathematical results obtained in [4], [28], [7], [20], [21], and [16] established the method of amplitude– or modulation equations as a new mathematical tool on unbounded domains. The results in this paper are one of the final points in the mathematical justification of the Ginzburg–Landau equation as an amplitude– or modulation equation, as they showed a deep connection between the attractors of the systems.

In [23] and [24] we explained in which sense the theory of modulation equations can be seen as generalization of the center manifold theorem in case of continuous spectrum. See also section 3.

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2. The functional set–up for the Taylor–Couette problem and existence of a local attractor. The Taylor–Couette problem consists in finding the motion a viscous incompressible fluid filling the domain $\Omega = \{(x,y) \in \mathbb{R} \times \Sigma\} = \{(x,y) \in \mathbb{R} \times \mathbb{R}^2 \mid |y|_2 \in (R_1, R_2)\}$ between two concentric rotating infinite cylinders with radii R_1 and R_2. Here, $|\cdot|_2$ denotes the euclidean norm in $\mathbb{R}^2$. The problem is governed by the Navier–Stokes equations on Ω with no–slip boundary conditions. For simplicity we set the rotational velocity of the outer cylinder to zero, and so the Reynolds number $\mathcal{R}$ is proportional to the rotational velocity of the inner cylinder.

The system possesses an exact solution, called the Couette flow $U_{\mathrm{Cou}} = U_{\mathrm{Cou}}(y)$. The deviation (U,q) from Couette–flow satisfies

$$\begin{aligned} &\partial_t U = \Delta U - \mathcal{R}[(U_{\mathrm{Cou}} \cdot \nabla)U + (U \cdot \nabla)U_{\mathrm{Cou}} + (U \cdot \nabla)U] - \nabla q, \\ &\nabla \cdot U = 0, \qquad [U_{(x)}]_\Sigma = \frac{1}{|\Sigma|}\int_\Sigma U_{(x)} = 0, \qquad U|_{\mathbb{R}\times\partial\Sigma} = 0, \end{aligned} \tag{2.1}$$

where $U_{(x)}$ denotes the component of U in x–direction.

In order to formulate our results we introduce some function spaces. We let $\rho(x) = (1+|x|^2)^{-1}$ and define

$$H^m_\rho(\Omega,\mathbb{R}) = \{u : \Omega \to \mathbb{R} \mid \ \|u\|_{H^m_\rho} = \|u\rho\|_{H^m} < \infty\}.$$

Then we define the translation operator T_z as $(T_z u)(x,y) = u(x+z,y)$ and

$$H^m_{l,u}(\Omega,\mathbb{R}) = \{u \in H^m_\rho \mid \|u\|_{H^m_{l,u}} = \sup_{z\in\mathbb{R}} \|T_z u\|_{H^m_\rho} < \infty, \|T_z u - u\|_{H^m_{l,u}} \to 0 \text{ as } z \to 0\}.$$

We write (2.1) as a dynamical system and eliminate $\nabla \cdot U = 0$ and the pressure term by introducing the projection $\Pi : L^2_\rho(\Omega)^3 \to \Pi L^2_\rho(\Omega)^3 = \mathrm{cl}_{L^2_\rho(\Omega)^3}\{U \in H^2_\rho(\Omega)^3 \mid \nabla \cdot U = 0,\ U \cdot n|_{\mathbb{R}\times\partial\Sigma} = 0, [U_{(x)}]_\Sigma = 0\}$ which is also continuous in H^m_ρ, and consequently in each $H^m_{l,u}$. See [24, Lemma 9.1] which is slight generalization of the usual L^2–theorem (cf. [27]). We define the linear and nonlinear terms, $\Lambda_\mathcal{R}$ and $N(\mathcal{R},\cdot)$, as

$$\begin{aligned} \Lambda_\mathcal{R} U &= \Pi[\Delta U - \mathcal{R}[(U_{\mathrm{Cou}} \cdot \nabla)U + (U \cdot \nabla)U_{\mathrm{Cou}}]] \\ N(\mathcal{R},U) &= -\mathcal{R}\Pi(U \cdot \nabla)U \end{aligned}$$

and write (2.1) as

$$\partial_t U = \Lambda_\mathcal{R} U + N(\mathcal{R},U). \tag{2.2}$$

We define the domain of definition for $\Lambda_\mathcal{R}$ by

$$\begin{aligned} Z &= \{U \in H^2_{l,u}(\Omega)^3 \mid U|_{\partial\Sigma\times\mathbb{R}} = 0, \Pi U = U\} \\ Z_\rho &= \{U \in H^2_\rho(\Omega)^3 \mid U|_{\partial\Sigma\times\mathbb{R}} = 0, \Pi U = U\} \end{aligned}$$

and the spaces $Z^* = \Pi(H^1_{l,u})^3$ and $Z^*_\rho = \Pi(H^1_\rho)^3$. For low Reynolds number $\mathcal{R} < \mathcal{R}_c$ Couette flow is exponentially stable. For $\mathcal{R} \geq \mathcal{R}_c$ we define the small bifurcation parameter $\varepsilon^2 = \mathcal{R} - \mathcal{R}_c$. In [24, Theorem 1.4] it has been shown:

Theorem 2.1. *There exist C_1, C_2, C_3 and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ the following holds. Let U_0 be an initial condition of (2.2) with $\|U_0\|_Z \leq C_1$. Then we have a unique solution U with $U|_{t=0} = U_0$ and*

$$\sup_{t\in[0,\infty)} \|U(t)\|_Z \leq C_2 \quad \text{and} \quad \limsup_{t\to\infty} \|U(t)\|_Z \leq C_3\varepsilon.$$

From this theorem we have that $\mathcal{B} = \bigcup_{t\geq 0} \mathcal{S}^\varepsilon_t \{U_0 \mid \|U_0\|_Z \leq C_1\}$ is a positively invariant set, where $\mathcal{S}^\varepsilon_t$ denotes the nonlinear evolution operator of (2.2).

Theorem 2.2. *There exists an* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$ *the Taylor–Couette problem* (2.1) *possesses a local* (Z, Z_ρ)*–attractor* $\mathcal{A}^\varepsilon_{\rm TC}$*. In detail: There exists a non–empty invariant set* $\mathcal{A}^\varepsilon_{\rm TC}$ *with*
a) $\mathcal{A}^\varepsilon_{\rm TC}$ *is bounded in the norm of* Z*.*
b) *For all* $\delta > 0$ *there exists a* $t_0 > 0$ *such that for all* $t \geq t_0$*:*

$$\sup_{b\in\mathcal{B}} \inf_{a\in\mathcal{A}^\varepsilon_{\rm TC}} \|a - \mathcal{S}^\varepsilon_t b\|_{Z_\rho} < \delta.$$

Proof. In order to show the existence we apply [16, Lemma 2.1]. The assumptions (A1) and (A3) of the lemma cited above are obviously satisfied. So, it remains to show the continuity of $\mathcal{S}^\varepsilon_t$ with respect to the norm of Z_ρ. This follows exactly as in [18]. □

3. The weakly unstable Taylor–Couette problem. In order to describe the Taylor–Couette attractor $\mathcal{A}^\varepsilon_{\rm TC}$ in more detail we cite here some results which are needed later on. In case $\varepsilon^2 = \mathcal{R} - \mathcal{R}_c > 0$ we have a curve of positive eigenvalues $\lambda(k)$ with associated eigenfunctions $\hat{U}_k e^{ikx}$ of $\Lambda_\mathcal{R}$, i.e., $\Lambda_\mathcal{R}(\hat{U}_k e^{ikx}) = \lambda(k)\hat{U}_k e^{ikx}$. The eigenfunctions $\hat{U}_k$ only depend on $|y|$ in cylindrical coordinates. In [24] it is shown that in this situation the solutions of (2.1) can be approximated by

$$U \sim \tilde{\psi} = \varepsilon A(\varepsilon x, \varepsilon^2 t)\hat{U}_{k_c} e^{ik_c x} + \text{c.c.}, \tag{3.1}$$

on time scales $\mathcal{O}(1/\varepsilon^2)$, where the critical wavenumber $k_c > 0$ is defined in Figure 1. Herein, A is a solution of the Ginzburg–Landau equation

$$\partial_T A = \alpha A + \beta \partial_X^2 A - \gamma A|A|^2, \tag{3.2}$$

where $A(X,T) \in \mathbb{C}$, $T = \varepsilon^2 t$, $X = \varepsilon x$, and real–valued constants $\alpha, \beta, \gamma > 0$. This is now explained in more detail, where the following lines are very close to the associated section in [16].

In order to separate the critical ($\lambda(k)$ positive or weakly negative) from the non–critical (Fourier–) modes we introduce the Fourier transform

$$(\mathcal{F}u)(k,y) = \hat{u}(k,y) = \frac{1}{2\pi}\int_\mathbb{R} u(x,y)e^{-ikx}dx$$

and some mode filters by the multiplication operators

$$\hat{E}_c(k) = \int_\Gamma (\hat{\Lambda}(k) - s)^{-1} ds \chi(||k| - k_c|/\rho_0),$$
$$\hat{E}_s(k) = (1 - \int_\Gamma (\hat{\Lambda}(k) - s)^{-1} ds)(1 - \chi(||k| - k_c|/\rho_0)).$$

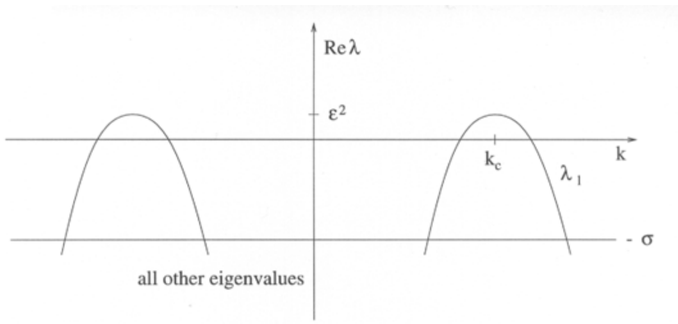

Figure 1: The spectrum of $\Lambda_{\mathcal{R}}$ for weakly unstable Couette flow drawn as function over the wavenumbers k.

The complex–valued operator $\hat{E}_1(k)$ is defined implicitly by $(\hat{E}_1(k)u)\hat{U}_k = \hat{E}_c(k)(u)$ for $|k-k_c| < 2\rho_0$. Here, for each fixed wavenumber k, Γ is a curve in the complex plane surrounding 0; $\chi \in C_0^\infty$ is a cut–off function with $\chi(k) = 1$ for $|k| \leq 1$, $\chi(k) \in [0,1]$ and $\chi(k) = 0$ for $|k| \geq 2$. We choose ρ_0 independent of ε, so small, that the curve of largest eigenvalues $k \mapsto \lambda(k)$ is separated from the rest of the spectrum for $|k| \leq 4\rho$. We use [20, Lemma 5] to define (in $H^m_{l,u}$ for $m = 0,1,2$) continuous operators in physical space by $E_c = \mathcal{F}^{-1}\hat{E}_c\mathcal{F}$ as an example. We define a scaling operator S_ε by $(S_\varepsilon A)(x) = A(\varepsilon x)$, and the multiplication operator $\mathbf{E}$ by $(u\mathbf{E})(x) = u(x)e^{ikx}$. Finally, the operator $\mathcal{U}$ is defined by the multiplier $\hat{U}_k\chi(|k-k_c|/(2\rho_0))$ using again [20, Lemma 5]. We introduce

$$\Phi_\varepsilon : u \mapsto \frac{1}{\varepsilon}S_{1/\varepsilon}[(E_1 u)\mathbf{E}^{-1}] \tag{3.3}$$

which maps initial conditions of the Taylor–Couette problem to initial conditions of the Ginzburg–Landau equation and $\psi_\varepsilon(A) = \varepsilon(\mathcal{U}S_\varepsilon A + \overline{\mathcal{U}S_\varepsilon A})$ which maps solutions of the Ginzburg–Landau equation to approximations of the Taylor–Couette problem.

For the Ginzburg–Landau equation (3.2) we introduce the spaces

$$Y = H^2_{l,u}(\mathbb{R},\mathbb{C}),\ Y_\rho = H^2_\rho(\mathbb{R},\mathbb{C}),\ Y^* = H^1_{l,u}(\mathbb{R},\mathbb{C}),\ \text{and}\ Y^*_\rho = H^1_\rho(\mathbb{R},\mathbb{C}).$$

Lemma 3.1. *There exist $C, \varepsilon_0 > 0$ such that the linear operator $\Phi_\varepsilon : Z \to Y$ satisfies for all $\varepsilon \in (0,\varepsilon_0]$ the estimate $\|\Phi_\varepsilon u\|_Y \leq C\varepsilon^{-1}\|u\|_Z$, and the operator $\psi_\varepsilon : Y \to Z$ satisfies*

$$\|E_s\psi_\varepsilon(A)\|_Z \leq C\varepsilon^2\|A\|_Y \text{ and } \|\psi_\varepsilon(A)\|_Z < C\varepsilon\|A\|_Y \quad \textit{for all } A \in Y. \tag{3.4}$$

Moreover, we have

$$\|E_s\psi_\varepsilon(A)\|_{Z^*} \le C\varepsilon^{3/2}\|A\|_{Y^*} \text{ and } \|\psi_\varepsilon(A)\|_{Z^*} < C\varepsilon\|A\|_{Y^*} \quad \textit{for all } A \in Y^*.$$

Proof. Similar to [16, Lemma 4.1], where H^1 is replaced by H^2 which gives ε^2 instead of $\varepsilon^{3/2}$. □

There are two principles, attractivity and approximation property, which are the mathematical connection between the dynamics of the Taylor–Couette problem and the Ginzburg–Landau equation.

The first principle is the *attractivity* of the set of functions which are in the Ginzburg–Landau form (3.1). This principle was stated in [6], first proved in [7], and improved in [22]. It is very similar to the attractivity of the center manifold in case of discrete spectrum (cf. [12]).

Theorem 3.2. *For each $r_0 > 0$ there exist constants $C, T_0, R_1, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the following estimates hold:*

$$\operatorname{dist}_Z(\mathcal{S}^\varepsilon_{T_0/\varepsilon^2}(B_Z(\varepsilon r_0)), \psi_\varepsilon(B_Y(R_1))) \le C\varepsilon^{5/4}, \tag{3.5}$$

$$\operatorname{dist}_Y(\Phi_\varepsilon[\mathcal{S}^\varepsilon_{T_0/\varepsilon^2}(B_Z(\varepsilon r_0))], B_Y(R_1)) \le C\varepsilon^{1/4}, \tag{3.6}$$

where $B_Z(r) = \{u \in Z \mid \|u\|_Z \le r\}$ and

$$\operatorname{dist}_Z(A, B) = \sup_{a\in A}\inf_{b\in B}\|a - b\|_Z.$$

Proof. Our version is a direct consequence of [24, Theorem 1.5]. □

The second principle is that every solution of the Taylor–Couette problem (2.2) which is close to the Ginzburg–Landau form (3.1) can be *approximated* by the solutions of the Ginzburg–Landau equation. A special case was first proved by [4]. The general scalar case was first proved in [28]. A simplified version of the proofs can be found in [14] and [20], respectively. The Ginzburg–Landau equation describes the dynamics in the attractive set established in Theorem 3.2 similar to the amplitude equations on the center manifold in case of discrete spectrum. The last reduction is exact although in general only approximations of the exact reduction are known. In this sense Theorem 3.2 and Theorem 3.3 are optimal results in case of continuous spectrum.

Theorem 3.3. *For all $R_1, T_1, d > 0$ there exists $C, \varepsilon_0 > 0$ such that for all $0 < \varepsilon \le \varepsilon_0$ the following holds: Let $A_0 \in B_Y(R_1)$ and $u_0 \in Z$ with*

$\|u_0 - \psi_\varepsilon(A_0)\|_Z \le d\varepsilon^{5/4}$, *then*

$$\sup_{0\le t\le T_1/\varepsilon^2} \|\mathcal{S}_t^\varepsilon(u_0) - \psi_\varepsilon(\mathcal{G}_{\varepsilon^2 t}(A_0))\|_Z \le C\varepsilon^{5/4}, \tag{3.7}$$

$$\sup_{0\le t\le T_1/\varepsilon^2} \|\Phi_\varepsilon(\mathcal{S}_t^\varepsilon(u_0)) - \mathcal{G}_{\varepsilon^2 t}(A_0)\|_{Y^*} \le C\varepsilon^{1/4}, \tag{3.8}$$

$$\|\Phi_\varepsilon(\mathcal{S}_{T_1/\varepsilon^2}^\varepsilon(u_0)) - \mathcal{G}_{T_1}(A_0)\|_Y \le C\varepsilon^{1/4}, \tag{3.9}$$

where $\mathcal{G}_T$ denotes the nonlinear evolution operator for the Ginzburg–Landau equation.

Proof. Our version is a direct consequence of [24, Theorem 8.2]. □
See [23] for a different kind of instability, where the Ginzburg–Landau equation occurs as an amplitude equation.

4. Upper–semicontinuity of the rescaled Taylor–Couette attractor. As in [16] these two principles, attractivity and approximation property, can be used to establish the upper-semicontinuity of the rescaled Taylor-Couette attractor $\Phi_\varepsilon \mathcal{A}_{\mathrm{TC}}^\varepsilon$, i.e., it converges into the Ginzburg–Landau attractor $\mathcal{A}_\mathcal{G}$, as the bifurcation parameter goes to zero. The existence of $\mathcal{A}_\mathcal{G}$ is guaranteed by the following theorem (cf. [16]).

Theorem 4.1. *The Ginzburg-Landau equation has a global (Y, Y_ρ)-attractor $\mathcal{A}_\mathcal{G}$ which is translationally invariant and invariant under the rotations R_ϕ : $A \mapsto e^{i\phi}A$.*

Then, we have the upper–semicontinuity (cf. [1]). Lower semicontinuity cannot be expected (cf. [11]).

Theorem 4.2. *For every $\sigma > 0$ there exist $C, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ the estimates*

$$\mathrm{dist}_{Y_\rho}(\Phi_\varepsilon \mathcal{A}_{\mathrm{TC}}^\varepsilon, \mathcal{A}_\mathcal{G}) \le \sigma \quad \textit{and} \quad \mathrm{dist}_Z(E_s \mathcal{A}_{\mathrm{TC}}^\varepsilon, \{0\}) \le C\varepsilon^{5/4}$$

hold.

Proof. From Theorem 2.1 we know that $\mathcal{A}_{\mathrm{TC}}^\varepsilon$ is contained in $B_Z(\varepsilon r_0)$ for some $r_0 > 0$. Let $v \in \mathcal{A}_{\mathrm{TC}}^\varepsilon$. Since $\mathcal{A}_{\mathrm{TC}}^\varepsilon$ is invariant under the flow $\mathcal{S}_t^\varepsilon$, there is a $u_0 \in \mathcal{A}_{\mathrm{TC}}^\varepsilon$ such that $v = \mathcal{S}_{T_0/\varepsilon^2}^\varepsilon(u_0)$, where T_0 is chosen according to Theorem 3.2. Hence,

$$\begin{aligned}\|E_s v\|_Z &= \|E_s \mathcal{S}_{T_0/\varepsilon^2}^\varepsilon(u_0)\|_Z \\ &\le \|E_s(\mathcal{S}_{T_0/\varepsilon^2}^\varepsilon(u_0) - \psi_\varepsilon(A_0))\|_Z + \|E_s \psi_\varepsilon(A_0)\|_Z \le C\varepsilon^{5/4},\end{aligned}$$

where (3.5) and (3.4) was used. This shows the second result as $v \in \mathcal{A}^{\varepsilon}_{\mathrm{TC}}$ was arbitrary.

From the attractivity in (3.6) we find $R_1 > 0$ such that

$$\mathrm{dist}_Y(\Phi_\varepsilon \mathcal{A}^{\varepsilon}_{\mathrm{TC}}, B_Y(R_1)) \leq C\varepsilon^{1/4}.$$

Since $\mathcal{A}_{\mathcal{G}}$ is an attractor, there exists, for given σ, a time $T_2 > 0$ such that $\mathrm{dist}_{Y_\rho}(\mathcal{G}_{T_2}(B_Y(R_1)), \mathcal{A}_{\mathcal{G}}) \leq \sigma/2$. Now let $v \in \mathcal{A}^{\varepsilon}_{\mathrm{TC}}$ be arbitrary. By invariance there is a $u_0 \in \mathcal{A}^{\varepsilon}_{\mathrm{TC}}$ with $v = \mathcal{S}^{\varepsilon}_{T_2/\varepsilon^2}(u_0)$. Applying the approximation result (3.9) with $T_1 = T_2$ we find

$$\mathrm{dist}_Y(\Phi_\varepsilon v, \mathcal{G}_{T_2}(B_Y(R_1))) = \mathrm{dist}_Y(\Phi_\varepsilon \mathcal{S}^{\varepsilon}_{T_2/\varepsilon^2}(u_0), \mathcal{G}_{T_2}(B_Y(R_1))) \leq C\varepsilon^{1/4}.$$

We complete the proof by

$$\begin{aligned}\mathrm{dist}_{Y_\rho}(\Phi_\varepsilon \mathcal{A}^{\varepsilon}_{\mathrm{TC}}, \mathcal{A}_{\mathcal{G}}) &\leq \mathrm{dist}_Y(\Phi_\varepsilon \mathcal{A}^{\varepsilon}_{\mathrm{TC}}, \mathcal{G}_{T_2}(B_Y(R_1))) \\ &\quad + \mathrm{dist}_{Y_\rho}(\mathcal{G}_{T_2}(B_Y(R_1)), \mathcal{A}_{\mathcal{G}}) \leq \sigma,\end{aligned}$$

choosing $\varepsilon_0 > 0$ so small that $C\varepsilon^{1/4} < \sigma/2$. □

5. Pseudo–orbit approximation in the Ginzburg–Landau attractor. In [24] for the Taylor–Couette problem and in [21] for the Kuramoto–Shivashinsky equation it has been shown that every solution of (2.2) can be shadowed by a so called pseudo–orbit solution for the Ginzburg–Landau equation.

Definition 5.1. Let $T_1 > 0$ and $\kappa > 0$. We call a function $A = A(T)$ a (T_1, κ)–*pseudo–orbit* in the Banach space Y for the Ginzburg–Landau equation (3.2) if the relations

$$\begin{aligned}&A((n-1)T_1 + \tau) = \mathcal{G}_\tau(A((n-1)T_1)) \text{ for all } \tau \in [0, T_1), \text{ and} \\ &\|A(nT_1{+}0) - \mathcal{G}_{T_1}(A((n-1)T_1))\|_Y \leq \kappa\end{aligned}$$

hold for all $n \in \mathbb{N}$, where $\mathcal{G}_T$ is the nonlinear semigroup associated with (3.2) and $A(T{+}0) = \lim_{\tau\to T, \tau>T} A(\tau)$.

The jumps of size κ do not destroy the existence of an absorbing ball for the Ginzburg–Landau equation. It is possible to iterate the approximation process for the Taylor–Couette problem and to control the size of the solutions by associated pseudo–orbits in the Ginzburg–Landau equation alone.

Theorem 5.2. *For all $T_1, \tilde{C}_0 > 0$ there exist positive constants $\varepsilon_0, \tilde{C}_1, \tilde{C}_2$ and T_0 such that for all $\varepsilon \in (0, \varepsilon_0]$ the following is true:*

For all initial conditions U_0 with $\|U_0\|_Z \leq \tilde{C}_0\varepsilon$ the solution $U(t) = \mathcal{S}_t^\varepsilon(U_0)$ of (2.2) exists for all time, and there is a $(T_1, \tilde{C}_1\varepsilon^{1/4})$–pseudo–orbit A for (3.2) which satisfies $\sup_{T\geq 0} \|A(T)\|_Y \leq \tilde{C}_2$ and approximates $U(t)$ for all $t \geq T_0/\varepsilon^2$ as follows:

$$\|U(t) - \psi_\varepsilon(A(\varepsilon^2 t - T_0))\|_Z \leq \tilde{C}_1\varepsilon^{5/4},$$
$$\|\Phi_\varepsilon U(t) - A(\varepsilon^2 t - T_0)\|_{Y^*} \leq \tilde{C}_1\varepsilon^{1/4}.$$

Proof. See [24, Theorem 8.2]. □

There exists an associated pseudo–orbit in the Ginzburg–Landau attractor $\mathcal{A}_\mathcal{G}$. Since attractivity to $\mathcal{A}_\mathcal{G}$ only holds in Y_ρ the pseudo–orbit is in Y_ρ instead of Y.

Theorem 5.3. *For all $C_1 > 0$ there exists a $\tilde{T}_0 > 0$ such that for all $T_1 > \tilde{T}_0$, we have $\varepsilon_0 > 0$, and $n \in \mathbb{N}$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the following holds. To every solution U with values in $\mathcal{A}_{\mathrm{TC}}^\varepsilon$ there exists a (T_1, C_1)–pseudo–orbit A in Y_ρ with values in $\mathcal{A}_\mathcal{G}$ such that*

$$\sup_{t\geq 0} \|\Phi_\varepsilon U(t) - A(\varepsilon^2 t)\|_{Y_\rho^*} \leq CC_1,$$

where C is a constant independent of C_1 and ε.

Proof. From [24, Lemma 1.10] it follows that there exists a $C_2 > 0$ such that each (T_1, C_1)–pseudo–orbit A satisfies $\sup_{T\geq 0} \|A(T)\|_Y < C_2$ for each $T_1 > 1$ and $C_1 \in [0, 1]$. From the attractivity of $\mathcal{A}_\mathcal{G}$ we have: For these $C_1, C_2 > 0$ there exists a $\tilde{T}_0 \geq 1$ such that

$$\mathrm{dist}_Y(A_0, \mathcal{A}_\mathcal{G}) < C_2 \text{ yields } \mathrm{dist}_{Y_\rho}(\mathcal{G}_{T_1} A_0, \mathcal{A}_\mathcal{G}) < C_1/2$$

for all $T_1 \geq \tilde{T}_0$. From the Lipschitz–continuity of $\mathcal{G}_T$ in $B_Y(C_2)$ with respect to the Y_ρ–norm it follows that there exists a $C_3 > 0$ such that

$$\|A_0 - \tilde{A}_0\|_{Y_\rho} < C_3 \quad \text{yields} \quad \|\mathcal{G}_T A_0 - \mathcal{G}_T \tilde{A}_0\|_{Y_\rho} < C_1/2. \tag{5.1}$$

Again, from the attractivity of $\mathcal{A}_\mathcal{G}$ it follows that there exists a $n \in \mathbb{N}$ such that

$$\mathcal{G}_{nT_1} B_Y(C_2) \subset \{u \in Y_\rho \mid \inf_{a\in\mathcal{A}_\mathcal{G}} \|u - a\|_{Y_\rho} \leq C_3/4\}.$$

Moreover, let $\tilde{A}$ be the $(T_1, \tilde{C}_1\varepsilon^{1/4})$–pseudo–orbit in Y (and so also in Y_ρ), whose existence is guaranteed by Theorem 5.2. Since $\mathcal{G}_{T_1}$ is Lipschitz–continuous in $B_Y(C_2)$ with respect to the Y_ρ–norm, and since $\tilde{A}$ makes finitely many jumps there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have

$$\mathrm{dist}_{Y_\rho}(\tilde{A}(T), \mathcal{A}_\mathcal{G}) < C_3/2 \quad \text{for all } T \geq nT_1.$$

Therefore, there exist $A_m \in \mathcal{A}_\mathcal{G}$ with $\|\tilde{A}(mT_1) - A_n\|_{Y_\rho} < C_3/2$ for $m \geq n$.

We define the pseudo–orbit A by

$$A(T) = \mathcal{G}_{T-mT_1}A_m \in \mathcal{A}_\mathcal{G} \quad \text{for } T \in [mT_1, (m+1)T_1).$$

The jumps made by A at the times $T = mT_1$ are of the size $C_1/2 + \tilde{C}_1\varepsilon^{1/4}$ in Y_ρ (due to (5.1)) which is less than C_1 for $\varepsilon > 0$ sufficiently small.

Finally, for some constant $C \geq 1$ independent of ε we have

$$\begin{aligned}
&\|\Phi_\varepsilon U(t) - A(\varepsilon^2 t - T_0 - nT_1)\|_{Y_\rho^*} \leq \|\Phi_\varepsilon U(t) - \tilde{A}(\varepsilon^2 t - T_0 - nT_1)\|_{Y^*} \\
&+ \|\tilde{A}(\varepsilon^2 t - T_0 - nT_1) - A(\varepsilon^2 t - T_0 - nT_1)\|_{Y_\rho} \\
&\leq \tilde{C}_1\varepsilon^{1/4} + CC_1/2 \leq CC_1,
\end{aligned}$$

provided $\varepsilon > 0$ is sufficiently small. Since $\mathcal{A}_\mathcal{G}$ and $\mathcal{A}^\varepsilon_{\mathrm{TC}}$ are invariant under the flows, we can shift time. Thus, we are done. □

Remark 5.4. We introduce the norm $\|u\|_{Z^*_{\rho,\varepsilon}} = \|u(S_\varepsilon\rho)\|_{H^1}$. It is easy to compute that we have

$$\begin{aligned}
&\|U(t) - \psi_\varepsilon(A(\varepsilon^2 t - T_0 - nT_1))\|_{Z^*_{\rho,\varepsilon}} \\
&\leq C\varepsilon^{-1/2}\|U(t) - \psi_\varepsilon(\tilde{A}(\varepsilon^2 t - T_0 - nT_1))\|_Z \\
&\qquad + \|\psi_\varepsilon(\tilde{A}(\varepsilon^2 t - T_0 - nT_1)) - \psi_\varepsilon(A(\varepsilon^2 t - T_0 - nT_1))\|_{Z^*_{\rho,\varepsilon}} \\
&\leq C\varepsilon^{-1/2}(\tilde{C}_1\varepsilon^{5/4} + CC_1\varepsilon/2) \leq C^2C_1\varepsilon^{1/2},
\end{aligned}$$

again provided $\varepsilon > 0$ is sufficiently small. Herein, we have used $\|u\|_{Z^*_{\rho,\varepsilon}} \leq C\varepsilon^{-1/2}\|u\|_Z$ and $\|\psi_\varepsilon A\|_{Z^*_{\rho,\varepsilon}} \leq C\varepsilon^{-1/2}\|A\|_{Y^*_\rho}$.

As a consequence, in physical space this approximation result only makes sense for solutions which are already in Ginzburg–Landau form. They are typically of order $\mathcal{O}(\varepsilon^{1/2})$ in $Z^*_{\rho,\varepsilon}$. Thus, for these solutions the approximation error is smaller than the magnitude of the solutions.

It is a typical situation that the right limit, here (3.3), has to be considered (cf. [26]).

Remark 5.5. In Theorem 2.2 and in the sections 4 and 5 the results can be improved by replacing convergence with respect to the Y_ρ–distance by convergence with respect to the distance $\text{dist}(A, B) = \sup_{y\in\mathbb{R}} \text{dist}_{Y_\rho}(T_y A, T_y B)$ due to the translation invariance of the problem.

6. Reduction to the two–dimensional Taylor–Couette problem. Finally, we show that the Taylor–Couette attractor $\mathcal{A}^{\varepsilon}_{\text{TC}}$ only contains rotationally symmetric solutions, i.e., it is sufficient to look at the so called two–dimensional Taylor–Couette problem. We introduce cylindrical coordinates $U = U_{(x)}e_x + U_{(r)}e_r + U_{(\theta)}e_\theta$, where $(x, r, \theta) \in \mathbb{R} \times \Sigma$ with $r = |y|$ and $\theta = y/|y| \in S^1$.

Theorem 6.1. *There exists an* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$ *the following holds: From* $U \in \mathcal{A}^{\varepsilon}_{\text{TC}}$ *follows that* U *possesses a representation* $U = U_{(x)}(x, r)e_x + U_{(r)}(x, r)e_r$.

Proof. We consider again

$$\partial_t U = \Lambda_{\mathcal{R}} U + \mathcal{N}(\mathcal{R}, U) = \Lambda_{\mathcal{R}} U + B(U, U), \tag{6.1}$$

where we have written the nonlinear terms as symmetric bilinear mapping $B(U, U)$. Then we define the projections

$$Q^+ U = \frac{1}{2\pi} \int_{S^1} U d\theta \text{ and } \ Q^- U = U - Q^+ U \tag{6.2}$$

for U in cylindrical coordinates. Note that $Q^{\pm}\Pi = \Pi Q^{\pm}$. Since we know already that the solutions in the attractor are of order $\mathcal{O}(\varepsilon)$ we make the ansatz $U = \varepsilon U^+ + \varepsilon U^-$ with $U^+ = Q^+ U^+$, $U^- = Q^- U^-$, and $\varepsilon > 0$ sufficiently small. Inserting this into (6.1) gives

$$\begin{aligned} \partial_t U^+ &= \Lambda_{\mathcal{R}} U^+ + \varepsilon Q^+ (B(U^+, U^+) + 2B(U^-, U^+) + B(U^-, U^-)), \\ \partial_t U^- &= \Lambda_{\mathcal{R}} U^- + \varepsilon Q^- (B(U^+, U^+) + 2B(U^-, U^+) + B(U^-, U^-)). \end{aligned} \tag{6.3}$$

Note that $Q^- B(U^+, U^+) = 0$. Moreover, we know

$$\sup_{t\in[0,\infty)} (\|U^+\|_Z + \|U^-\|_Z) \le C.$$

We have

$$\|B(U^+, U^-)\|_{Z^*} \le C\|U^-\|_Z, \quad \|B(U^-, U^-)\|_{Z^*} \le C\|U^-\|_Z,$$

and

$$\|e^{\Lambda_{\mathcal{R}} t} Q^- \|_{\mathcal{L}(Z^*,Z)} \le \max(1, t^{-3/4}) e^{-\sigma_0 t}$$

for $C, \sigma_0 > 0$ independent of $\varepsilon > 0$ sufficiently small. Thus

$$\begin{aligned}
&\|U^-(t) e^{\sigma_0 t/2}\|_Z \le \|e^{\Lambda_{\mathcal{R}} t} e^{\sigma_0 t/2} U^-|_{t=0}\|_Z \\
&+ \|\varepsilon \int_0^t e^{\Lambda_{\mathcal{R}}(t-\tau)} e^{\sigma_0 t/2} (2B(U^-, U^+) + B(U^-, U^-)) d\tau \|_Z \\
&\le C e^{-\sigma_0 t/2} + \varepsilon \int_0^t \|e^{\Lambda_{\mathcal{R}}(t-\tau)}\|_{\mathcal{L}(Z^*,Z)} e^{\sigma_0 (t-\tau)/2} d\tau \sup_{\tau \in [0,t]} \|U^-(\tau) e^{\sigma_0 \tau/2}\|_Z \\
&\le C e^{-\sigma_0 t/2} + C\varepsilon \sup_{\tau \in [0,t]} \|U^-(\tau) e^{\sigma_0 \tau/2}\|_Z
\end{aligned}$$

and so

$$\sup_{t \in [0,\infty)} \|U^-(t) e^{\sigma_0 t/2}\|_Z \le (1 - C\varepsilon)^{-1} C < \infty$$

for $\varepsilon > 0$ sufficiently small. Thus $\lim_{t\to\infty} U^-(t) = 0$, and so $Q^+ \mathcal{A}^{\varepsilon}_{\mathrm{TC}} = \mathcal{A}^{\varepsilon}_{\mathrm{TC}}$ and $Q^- \mathcal{A}^{\varepsilon}_{\mathrm{TC}} = 0$. $\square$

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