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ALMOST–PERIODIC AND BOUNDED SOLUTIONS OF CARATHÉODORY DIFFERENTIAL INCLUSIONS

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Abstract. The main purpose of this paper is two-fold: to find sufficient conditions for the existence of entirely bounded solutions of Carathéodory quasi-linear differential inclusions and to show that, if the coefficients are specially constant and the right-hand sides are additionally Lipschitz continuous (with a sufficiently small Lipschitz constant) and almost-periodic in time, then these solutions become almost-periodic as well. The almost-periodicity is understood in the sense of H. Weyl and, because of set-valued analysis, we introduce for the first time the appropriately generalized concept. The related methods, including the fixed-point theorem for a class of $\mathcal{J}$-maps in locally convex topological vector spaces, are developed here too. In the single-valued case, the obtained criteria generalize those of the other authors.

1. Introduction. In difference to the classical H. Bohr theory of uniform almost-periodic (a.p.) functions, the generalized concepts of measurable a.p. functions due to V.V. Stepanov, H. Weyl, A.S. Besicovitch, ... (see e.g. [6], [20], [26]) have found so far only rare applications to differential systems (see e.g. [19], [26], [30] and the references therein). Moreover, for multivalued maps, as far as we know, the appropriate generalization does not exist at all.

Hence, it is our aim here to do the first step in this direction and try to get some application for Carathéodory differential inclusions. For the latter, we need to solve the asymptotic boundary value problems (BVPs), leading to the existence of an entirely bounded solution on the real line, at first. Although we have developed recently in [1] such a technique, because of convenience, we decided to present here this method once again, but this time based on a slightly different approach.

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Thus, our paper is organized as follows. In the next section, the topological terminology and needed results, concerning the locally convex (in particular, Fréchet) topological vector spaces and the class of so called $\mathcal{J}$-maps operating in these spaces, are recalled in order to obtain the fixed-point theorem (Theorem 1). We suppose that the reader is familiar with the elements of algebraic topology and topological vector spaces (for example, with the notions of retracts and the Fréchet spaces); otherwise, these elements can be found in all related text-books like [16], [27].

Then, we develop the method (Theorem 3) allowing us to solve the large family of multivalued BVPs on noncompact intervals. There we presume besides another the elementary knowledge of set-valued analysis (for example, semi-continuous maps, measurable maps and differential inclusions); these elements can be found e.g. in [4], [14].

In Part 4, Theorem 3 is improved for special (quasi-linear) systems, after investigating the topological structure of solution sets to the associated linearized system, and so an effective criterium for the existence of an entirely bounded solution is given in Theorem 4.

Part 5 is devoted to the new definitions of almost-periodicity in the sense of H. Weyl for multivalued measurable functions which are used in Part 6, where the existence result (Theorem 5) is obtained for a.p. solutions of special Carathéodory-Lipschitz differential inclusions. In concluding remarks (Part 6), some related open problems are posed.

2. Topological preliminaries. The main purpose of this preliminary part consists, besides giving the necessary definitions, in developing the fixed-point theorem (Theorem 1) for the class of so called $\mathcal{J}$-maps in locally convex topological vector spaces. The approach here as well as in the next section is similar to that in [1] and, in fact, the results of these two parts can be regarded as a joint work with the coauthors in [1].

Let I be a real interval (not necessarily compact) and $n \in \mathbb{N}$. By $C(I, \mathbb{R}^n)$ we mean the space of all continuous functions $X(t) : I \to \mathbb{R}^n$ with the topology of the uniform convergence on compact subintervals of I. The set of nonnegative functions $\{p_K : K$ is a compact subinterval of $I\}$, where $p_K(X) := \sup_{t\in K}|X(t)|$, is the family of seminorms generating this topology. It is well-known that it can be, equivalently, generated by the metric

$$\mathrm{d}(X,Y) = \sum_{n=1}^{\infty} 2^{-n} \frac{p_{K_n}(X-Y)}{1 + p_{K_n}(X-Y)},$$

where $\{K_n\}$ is the class of compact subsets of I such that $\cup_{n=1}^{\infty} K_n = I$ and $K_n \subset K_{n+1}$. With this metric the space $C(I, \mathbb{R}^n)$ is the *Fréchet space* (i.e., completely metrizable).

We say that a subset $Q \subset C(I, \mathbb{R}^n)$ is *bounded* if there exists a positive function $\varphi : I \to \mathbb{R}$ such that $|X(t)| \leq \Phi(t)$ for all $t \in I$ and $X \in Q$.

Recall that, by the well-known Ascoli theorem, Q is relatively compact if and only if it is bounded and the functions of Q are equicontinuous at each $t \in I$.

Let S_1 and S_2 be two subsets of the topological vector spaces E_1 and E_2, respectively, and $\varphi : S_1 \to S_2$ be a map. The notation $\varphi : S_1 \to S_2$ means, as usual, a set-valued map from S_1 to nonempty subsets of S_2. Let U_1 and U_2 be two open neighbourhoods of the origins in E_1 and E_2, respectively. A continuous function $f : S_1 \to S_2$ is said to be a (U_1, U_2)-*approximation* of φ (we write $f \in a(\varphi, U_1, U_2)$) if $f(X) \in (\varphi[(X+U_1) \cap S_1] + U_2) \cap S_2$ for each $X \in S_1$. Recall that if E_1, E_2 are locally convex, then there exist fundamental bases $\mathcal{N}_{E_1}(0), \mathcal{N}_{E_2}(0)$ of open, symmetric, convex neighbourhoods of the origins. We will consider only locally convex topological vector spaces. Thus, the definition of approximable maps will be related just to this case.

We say that φ is *approximable* if for every pair $(U_1, U_2) \in \mathcal{N}_{E_1}(0) \times \mathcal{N}_{E_2}(0)$ there exists a (U_1, U_2)-approximation of φ. It can be easily seen that, if φ has compact values, then it is approximable if and only if for any open neighbourhood $W \subset E_1 \times E_2$ of the graph Γ_φ of φ there exists $f : S_1 \to S_2$ such that $\Gamma_f \subset \Gamma_\varphi + W$.

Let M_1 and M_2 be two metric spaces. A multivalued map $\varphi : M_1 \to M_2$ is called *upper-semi-continuous* (u.s.c) if for any open subset $B \subset M_2$ the set $\{X \in M_1 : \varphi(X) \subset B\}$ is an open subset of M_1; φ is called *lower-semi-continuous* (l.s.c.) if for any open subset $B \subset M_2$, the set $\{X \in M_1 : \varphi(X) \cap B \neq \emptyset\}$ is an open subset of M_1. Finally, φ is called *continuous* if it is both u.s.c. and l.s.c.

Lemma 1 ([8]). *Let S_1 be a convex, compact subset of a locally convex topological vector space E_1 and $\varphi : S_1 \to S_1$ be an approximable u.s.c. map with closed values. Then φ has a fixed point $\widehat{X} \in S_1$, i.e., $\widehat{X} \in \varphi(\widehat{X})$.*

The following $\mathcal{J}$-*class* of maps (for details see e.g. [7], [19] and the references therein), namely those being u.s.c. with compact proximally ∞-connected values, will play an essential role. Let E be a metric space and K be a compact subset of E.

We say that K is ∞-*proximally connected* if, for any $\varepsilon > 0$, there exists

$\eta, 0 < \eta < \varepsilon$, such that

(i) any two points of $N_\eta(K)$, an η-neighbourhood of K in E, can be joined by a path in $N_\varepsilon(K)$, and
(ii) for each positive integer n and a base point $Y_o \in N_\eta(K)$, the inclusion $N_\eta(K) \subset N_\varepsilon(K)$ induces the trivial homomorphism:

$$\pi_n(N_\eta(K), Y_o) \to \pi_n(N_\varepsilon(K), Y_o),$$

where π_n stands for the n-th homotopy groups.

Lemma 2. *Let S be a convex, compact subset of a locally convex topological vector space E and $\varphi \in \mathcal{J}(S)$ (i.e., φ is u.s.c. with compact proximally ∞-connected values). Then φ is approximable.*

Proof. Let $(U, V) \in \mathcal{N}(0) \times \mathcal{N}(0)$, where $\mathcal{N}(0) := \mathcal{N}_E(0)$ (see above). We show that there exists a (U, V)-approximation of φ.

Put $U' = \frac{1}{3}U$. There exists a finite dimensional polyhedron $P = \operatorname{conv}(N)$, where $N = \{X_1, \ldots, X_n\} \subset S$ and $\pi : S \to P \subset S$ such that π and id_S are U'-near. Above, "conv" and "id_S" denote the convex hull and identity on S, respectively.

By the continuity of π, for every $Z \in S$, there is $U_Z \in \mathcal{N}(0)$ such that

$$\pi((Z + 2U_Z) \cap S) \subset (\pi(Z) + U') \cap P.$$

By the compactness of S, we have $S = \bigcup_{i=1}^m (Z_i + U_{Z_i})$. Let $\widehat{U} = \cap_{i=1}^m U_{Z_i} \cap U'$. There are $U'', V'' \in \mathcal{N}(0)$ such that $f\pi \in a(\varphi_P \pi, \widehat{U}, V)$ whenever $f \in a(\varphi_P, U'', V'')$, where $\varphi_P := \varphi|_P : P \to X$. However, in view of Theorem 4.2 in [8], φ_P is approximable. Therefore, if $f \in a(\varphi_P, U'', V'')$, then $g := f\pi \in a(\varphi_P \pi, \widehat{U}, V)$.

Now, if $X \in S$, then $X \in Z_i + U_{Z_i}$ for some Z_i. Thus, $X + \widehat{U} \subset Z_i + \widehat{U} + U_{Z_i} \subset Z_i + 2U_{Z_i}$, and so $\pi((X + \widehat{U}) \cap S) \subset (\pi(Z_i) + U') \cap P$ and $\varphi\pi(X + \widehat{U}) \subset \varphi((\pi(Z_i + U') \cap P) \subset \varphi((Z_i + U' + U') \cap S) = \varphi((Z_i + 2U') \cap S$. Since $Z_i \in X + U_{Z_i} \subset X + U'$, we get $\varphi\pi(X + \widehat{U}) \subset \varphi((X + 3U') \cap S)$.

Hence, for $g \in a(\varphi_P \pi, \widehat{U}, V)$, we have $g(X) \in (\varphi(\pi((X + \widehat{U}) \cap P)) + V) \cap S$, which implies that $g(X) \in (\varphi((X + U) \cap S) + V) \cap S$. So we arrived at $g \in a(\varphi, U, V)$. $\square$

From Lemma 1 and Lemma 2 we obtain immediately

Lemma 3. *Every $\varphi \in \mathcal{J}(S)$, where S is a compact, convex subset of a locally convex topological vector space, has a fixed point.*

Before the next result, let us note that in the Fréchet spaces each convex closure of a compact subset is also compact.

Lemma 4. *Let S be a closed, convex subset of a locally convex topological vector space [Fréchet space] and $\varphi \in \mathcal{J}(S)$ be such that $\overline{\text{conv}\ \varphi(S)} = C[\overline{\varphi(S)} = C]$ is compact. Then φ has a fixed point.*

Proof. It is sufficient to take $\varphi|_C : C \to C$ and apply Lemma 3.

Theorem 1. *Let S be a retract of a locally convex topological vector space [Fréchet space] E and $\varphi \in \mathcal{J}(S)$ be such that $\overline{\text{conv}\ \varphi(S)} = C \subset E[\overline{\varphi(X)} = C \subset E]$ is compact. Then φ has a fixed point.*

Proof. Let $r : E \to S$ be a retraction. Define $\psi := \varphi r : E \to E$. Obviously, $\overline{\text{conv}\ \psi(E)}$ is compact and so, according to Lemma 4, there exists a fixed point $X \in \psi(X) = \varphi r(X)$. But $X = r(X)$ because of $X \in S$, and so X is a fixed point of φ as well.

Remark 1. Since every convex subset of a locally convex topological vector space is an absolute extensive space (AES) for a class of metric spaces (see [17] and [8]), every convex subset of the Fréchet space (which is metric) must be an absolute retract (AR). It follows from here that in the Fréchet spaces all contractible subsets and R_δ-subsets are proximally ∞-connected (see [18]).

Let us recall that a nonempty metric space A is called an *R_δ-set* if there exists a decreasing sequence $\{A_n\}$ of compact contractible spaces A_n such that $A = \cap_{n\geq 1} A_n$. By contractibility we mean that there exists a (continuous) homotopy $h : A_n \times [0,1] \to A_n$ such that $h(X,0) = X$ and $h(X,1) = X_0$ for every $X \in A_n$. Note that any R_δ-set is acyclic w.r.t. any continuous homology theory (e.g. the Čech theory), i.e. only the zero-th homology group of such a set is nontrivial, or equivalently, all the relative homology groups are trivial. In particular, it is compact nonempty and connected. The notion of the ∞-proximal connectedness above is more general in the sense that every R_δ-subset of an absolute neighbourhood retract (ANR) is proximally ∞-connected. For more details and appropriate definitions see e.g. [7], [18].

3. Boundary value problems on noncompact intervals. In this part we develop a technique allowing us to solve the large family of boundary value problems (BVPs) on possibly infinite intervals. The main result, Theorem 3, represents a slight generalization of Corollary 2.37 in [1], consisting in replacing the assumed R_δ-solution set of the linearized problem by a proximally ∞-connected set. Even in the single-valued case, Theorem 3 below improves its analogies in [9] and [10]. Some convexity restrictions imposed there (those related to the set Q in condition (i) below) are namely shown superfluous, because Q can be a bounded retract of $C(I,\mathbb{R}^n)$.

Hence, it is time to apply the (fixed-point) Theorem 1 to the multivalued boundary value problems on not necessarily compact intervals. For this, the technique similar to Schauder's (quasi) linearization method will be employed.

We say that a map $X(t): I \to \mathbb{R}^n$ is *locally absolutely continuous* if it is absolutely continuous on every compact subinterval of $I \subset \mathbb{R}$. The set of all locally absolutely continuous maps from I to $\mathbb{R}^n$ is denoted by $AC_{loc}(I,\mathbb{R}^n)$.

Consider the inclusion

$$X' \in F(t,X), \tag{1}$$

where $F: I \times \mathbb{R}^n \to \mathbb{R}^n$ is a set-valued *Carathéodory map*, i.e.

(i) the set of values of $F(t,X)$ is nonempty, compact and convex for all $(t,X) \in I \times \mathbb{R}^n$,
(ii) the map $F(t,.)$ is u.s.c. (see the definition in Part 2) for a.a. $t \in I$,
(iii) the map $F(.,X)$ is *measurable* for all $X \in \mathbb{R}^n$, i. e. for any open $U \subset \mathbb{R}^n$ and every $X \in \mathbb{R}^n$ the set $\{t \in I : F(.,X) \cap U \neq \emptyset\}$ is measurable.

By a *solution* of (1) we mean always the one *in the sense of Carathéodory*, i.e. a locally absolutely continuous map $X(t)$ (i.e. $X(t) \in AC_{loc}(I,\mathbb{R}^n)$) satisfying (1) a.e.

Lemma 5. (*cf. Theorem* 0.3.4 *in* [4]). *Assume that the sequence of absolutely continuous maps $X_k(t): K \to \mathbb{R}^n$ (K is a closed interval) satisfies the following conditions:*

(i) *the set $\{X_k(t) : k \in \mathbb{N}\}$ is bounded for every $t \in K$,*
(ii) *there is an integrable function (in the sense of Lebesgue) $\alpha : K \to \mathbb{R}$ such that*

$$|X_k'(t)| \leq \alpha(t) \text{ for a.a. } t \in K.$$

Then there exists a subsequence $\{X_l\}$ convergent to an absolutely continuous map $X: K \to \mathbb{R}^n$ in the following sense:

(iii) *$\{X_l\}$ uniformly converges to X;*
(iv) *$\{X_l'\}$ weakly converges in $L^1(K,\mathbb{R}^n)$ to X'.*

Lemma 6. (*due to S. Mazur, cf. e.g. Theorem* 2.1.4 *in* [22]). *If E is a normed space and the sequence $\{X_k\} \subset E$ is weakly convergent to $X \in E$, then there exists a sequence of linear combinations $Y_m = \sum_{k=1}^m a_{mk} X_k$, where $a_{mk} \geq 0$ for $k = 1, 2, \ldots, m$, and $\sum_{k=1}^m a_{mk} = 1$, which is strongly convergent to X.*

Now, we can state the multivalued improved analogies of the single-valued results in [10, Theorems 1.1 and 1.2] (cf. also Corollary 2.37 in [1]).

Theorem 2. *Let $G : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory map and let S be a nonempty subset of $AC_{loc}(I, \mathbb{R}^n)$. Assume that*

(i) *there exists a bounded subset Q of $C(I, \mathbb{R}^n)$ such that, for any $q \in Q$, the set $T(q)$ of all solutions of the boundary value problem*

$$X' \in G(t, X, q(t)), \qquad X \in S$$

on $I \subset \mathbb{R}$ is nonempty;

(ii) *$T(Q)$ is bounded in $C(I, \mathbb{R}^n)$;*

(iii) *there exists a locally* (*Lebesgue*) *integrable function $\alpha : I \to \mathbb{R}$ such that*

$$|G(t, X(t), q(t))| = \sup\{|y| : y \in G(t, X(t), q(t))\} \leq \alpha(t) \ \textit{a.e. in } I,$$

for any pair $(q, X) \in \Gamma_T$, i.e. in the graph of T.

Then $C = \operatorname{conv} T(Q)$ is a relatively compact subset of $C(I, \mathbb{R}^n)$. Moreover, under the assumptions (i)–(iii), *the multivalued operator $T : Q \to S$ is u.s.c. with compact values if and only if the following condition is satisfied:*

(iv) *given a sequence $\{(q_k, X_k)\} \subset \Gamma_T$, if $\{(q_k, X_k)\}$ converges to (q, X) with $q \in Q$, then $X \in S$.*

Proof. Using the standard arguments, we get that the set C is bounded. Moreover, in view of (iii), we have $|X'(t)| \leq \alpha(t)$ for every $X \in T(Q)$ and all $t \in I$, by which

$$|X(t_1) - X(t_2)| \leq |\int_{t_1}^{t_2} \alpha(t) dt|.$$

It follows that the elements of C are equicontinuous.

We will show that the set Γ_T, denoting the graph of T, is closed. Let K be an arbitrary compact interval such that α is integrable on K. By conditions (ii) and (iii), the sequence $\{X_k\}$ satisfies the assumptions of Lemma 5.

Thus, there exists a subsequence $\{X_l\}$, uniformly convergent to X on K (since the limit is unique), and such that $\{X_l'\}$ weakly converges to X' in L^1. Therefore, X' belongs to the weak closure of the set $\operatorname{conv}\{X_m' : m \geq l\}$, for every $l \geq 1$. According to Lemma 6, X' belongs also to the strong closure of this set. Hence, for every $l \geq 1$, there is $Z_l \in \operatorname{conv}\{X_m' : m \geq l\}$ such that $||Z_k - X'||_{L^1} \leq \frac{1}{k}$. This implies the existence of a subsequence $Z_{l_p} \to X'$ a.e. in K.

Let $s \in K$ be such that:

$$G(s,.,.) \text{ is u.s.c.}, \quad \lim_{p\to\infty} Z_{l_p}(s) = X'(s), \quad X_l'(s) \in G(s, X_l(s), q_l(s)),$$

and fix ε.

There is $\delta > 0$ such that $G(s,Z,r) \subset N_\varepsilon(G(s,X(s),q(s)))$ whenever $|X(s) - Z| < \delta$ and $|q(s) - r| < \delta$. We know, however, that there exists $N \geq 1$ such that $|X(s) - X_m(s)| < \delta$ and $|q(s) - q_m(s)| < \delta$ for every $m \geq N$. Hence,

$$X_l'(s) \in G(s, X_l(s), q_l(s)) \subset N_\varepsilon(G(s, X(s), q(s))).$$

Because of convexity of $G(s, X(s), q(s))$, we have

$$Z_{l_p}(s) \in N_\varepsilon(G(s, X(s), q(s)))$$

for $l_p \geq N$. Thus, $X'(s) \in N_\varepsilon(G(s,X(s),q(s)))$ for every $\varepsilon > 0$, which implies that $X'(s) \in G(s,X(s),q(s))$. Since K was arbitrary, $X'(t) \in G(t,X(t),q(t))$ a.e. in I. □

As an immediate consequence of the above Theorem 2 and Theorem 1, we get the main statement of this section, having for us the character of the method in the sequel.

Theorem 3. *Consider the boundary value problem*

$$X' \in F(t, X), \qquad X \in S, \tag{2}$$

on a given interval $I \subset \mathbb{R}$, where $F : I \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory map and S is a subset of $AC_{loc}(I, \mathbb{R}^n)$. Let $G : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory map such that $G(t,c,c) \subset F(t,c)$ for all $(t,c) \in I \times \mathbb{R}^n$.

Assume that:

(i) *there exists a bounded retract $Q \subset C(I,\mathbb{R}^n)$ of $C(I,\mathbb{R}^n)$ such that the associated problem*

$$X' \in G(t, X, q(t)), \qquad X \in S \cap Q, \tag{3}$$

is solvable on I with a proximally ∞-connected set of solutions for each $q \in Q$;

(ii) *there exists a locally (Lebesgue) integrable function $\alpha : I \to \mathbb{R}$ such that*

$$|G(t, X(t), q(t))| = \sup\{|y| : y \in G(t, X(t), q(t))\} \leq \alpha(t) \text{ a.e. in } I,$$

for any pair $(q, X) \in \Gamma_T$;

(iii) $\overline{T(Q)} \subset S$ *(for the meaning of the symbol T see condition* (i) *in Theorem* 2*).*

Then problem (2) *has a solution.*

In condition (i), we can particularly assume (in view of Remark 1) the solvability with an R_δ-set of solutions or that the set of solutions is, according to Theorem 2, nonempty and convex. On the other hand, if the set Q in condition (i) is additionally convex and closed (and so retract of $C(I, \mathbb{R}^n)$), then we have proved in [1, Corollary 2.38] that the same is true, when the set of solutions of (3), is only acyclic for each $q \in Q$. The notion of acyclicity is more general that the one of the proximal ∞-connectedness in the sense that any proximally ∞-connected subset of an ANR is acyclic (in the Alexander-Spanier homology theory with integer coefficients). For more details and the appropriate definitions see [1] and [7], [18].

So, we can conclude this section by the statement generalizing and (in single-valued case) improving the single-valued analogies in [9], [10].

Corollary 1. *The assertion of Theorem* 3 *remains valid, provided the same but* (i) *replaced by the following condition:*

(i)′ *there exists a nonempty, closed, convex, bounded subset Q of $C(I, \mathbb{R}^n)$ such that the associated problem* (3) *has an acyclic, or specially proximally ∞-connected, or more specially R_δ, or extra specially convex set of solutions for each $q \in Q$.*

4. Bounded solutions of quasi-linear differential inclusions. Since the main obstruction in the application of Theorem 3 or Corollary 1 is related to the knowledge of the topological structure of the solution set for (3), let us consider problem (2) with a special structure of the right-hand side of the inclusion, namely

$$X' + A(t, X)X \in F(t, X), \qquad X \in S, \tag{4}$$

where

$$F(t,X) : I \times \mathbb{R}^n \to \mathbb{R}^n$$

is a Carathéodory function and $S \subset AC_{loc}(I,\mathbb{R}^n)$ is as above, but $A(t,X) : I \times \mathbb{R}^n \to \mathbb{R}^{n^2}$ is a single-valued continuous $(n \times n)$-matrix. Thus, as we shall see, the study of the structure of the solution set related to the associated linearized problem, namely

$$X' + A(t,q(t))X \in F(t,q(t)), \qquad X \in S \cap Q, \tag{5}$$

for each $q(t) \in Q \subset C(I,\mathbb{R}^n)$, becomes significantly easier, provided Q as well as S are nonempty, convex, and Q is bounded and closed in the given topology of the uniform convergence on compact subintervals of I.

Lemma 7. *Under the above assumptions, the solution set of problem (5) is convex.*

Proof. It is well-known (see e.g [3], p. 35]) that, under the above assumptions, $F(t,q(t))$ is measurably selectionable. Let $\mu(t), \nu(t) \subset F(t,q(t))$ be two such selectors. Thus, $\lambda\mu(t), (1-\lambda)\nu(t)$ and $\lambda\mu(t)+(1-\lambda)\nu(t)$ are obviously measurable functions for each $\lambda \in [0,1]$ as well and, because of convex valued F, we have $\lambda\mu(t) + (1-\lambda)\nu(t) \subset F(t,q(t))$ for a.a. $t \in I$. Hence, $\lambda\mu(t)+(1-\lambda)\nu(t)$ is for every $\lambda \in [0,1]$ a measurable selector of $F(t,q(t))$, by which the set of all measurable selectors is convex.

Now, we will prove that $T(q(t))$ is convex, when following the argument in [11, Proposition 2.1]. Assume that $X_1(t), X_2(t) \in T(q(t))$, i.e.,

$$X_1'(t) = A(t,q(t))X_1(t) + f_q^1(t) \text{ a.e.}, \qquad X_1(t) \in S \cap Q,$$

and

$$X_2'(t) = A(t,q(t))X_2(t) + f_q^2(t) \text{ a.e.}, \qquad X_2(t) \in S \cap Q,$$

where f_q^1, f_q^2 are the related measurable selectors of $F(t,q(t))$.

Since the set of such selectors has been proved to be convex and since $S \cap Q$ is, by the hypothesis, convex as well, we obtain for every $\lambda \in [0,1]$ immediately that

$$\lambda X_1(t) + (1-\lambda)X_2(t) \in T(q(t)),$$

which completes the proof. □

Since condition (ii) in Theorem 3, related to (5), follows for a bounded Q directly from the inequality

$$\sup_{|X|\le D}|F(t,X)-A(t,X)X|\le \alpha(t) \text{ for a.a. } t\in I, \tag{6}$$

where D is a sufficiently big number and $\alpha(t): I\to\mathbb{R}$ is a suitable locally (Lebesgue) integrable function, we can reformulate Corollary 1 for (4) in an extremely simple way.

Corollary 2. *Consider the boundary value problem* (4) *on a given interval* $I\subset\mathbb{R}$*, where* $F: I\times\mathbb{R}^n\to\mathbb{R}^n$ *is a Carathéodory map,* $A: I\times\mathbb{R}^n\to\mathbb{R}^{n^2}$ *is a single-valued continuous* $(n\times n)$*-matrix and* $S\subset AC_{loc}(I,\mathbb{R}^n)$ *is nonempty and convex.*

Let there exist a nonempty, convex, closed and bounded set $Q\subset C(I,\mathbb{R}^n)$ *such that the associated linearized problem* (5) *admits for each* $q(t)\in Q$ *a solution and let* $T(q(t))$ *be the set of such solutions. Then the original problem* (4) *is solvable, provided only* (6) *and* (iii) $\overline{T(Q)}\subset S$.

Proof follows immediately from Corollary 1 and Lemma 7.

Remark 2. If S is closed in the given topology of the uniform convergence on compact subintervals of I, then (iii) is satisfied. Therefore, if we are interested only in the sole existence of (entirely) bounded solutions with no additionally prescribed properties, then this can be achieved, when $\sup\operatorname{ess}_{t\in I}|s'(t)|\le D'$ for all $s(t)\in S$ with a sufficiently big number D'. In order to show that all bounded solutions of the linearized system belong to S, we will assume that A is bounded in t, F is essentially bounded in t and can be absolutely estimated in X by a single-valued continuous function.

Moreover, (6) is trivially fulfilled in this way as well.

As already pointed out, $F(t,q(t))$ from above is measurably selectionable for every $q(t)\in Q\subset C(I,\mathbb{R}^n)$. Hence, let $f_q(t)$ be a measurable selector for a fixed $q(t)\in Q$ and consider the problem

$$\begin{cases} X'+A(t,q(t))X=f_q(t), \text{ where } f_q(t)\subset F(t,q(t)) \text{ for a.a. } t\in I,\\ X\in Q. \end{cases} \tag{7}$$

In order to prove the solvability of (7), we need to assume that an exponential dichotomy (shortly e.d.) holds for the homogeneous equation in (7), namely

$$X'+A(t,q(t))X=0. \tag{8}$$

Let $X(t)$ be a fundamental matrix of system (8) satisfying $X(0) = I$, where I is the unit matrix.

We say that (8) possesses an *exponential dichotomy* (the standard references are [12] and [21]) on the interval I, if there exists a projection matrix P $(P = P^2)$ and constants $k > 0, \lambda > 0$ such that

$$\begin{cases} |X(t)PX^{-1}(s)| \leq k \exp(-\lambda(t-s)) \text{ for } s \leq t, \\ |X(t)(I-P)X^{-1}(s)| \leq k \exp(-\lambda(s-t)) \text{ for } t \leq s. \end{cases} \tag{9}$$

It is well-known that (see e.g. [12], [28] and the references therein) if in particular A in (8) is constant, then (8) has an e.d. on an infinite interval I if and only if all eigenvalues of A have nonzero real parts. If A in (8), considered as a composed function, is periodic, then (8) has an e.d. on an infinite interval I if only if all associated Floquet multipliers lie off the unit cycle. If (8), where $A \equiv A(t)$ is uniformly almost-periodic, has an e.d. on $[a, \infty)$, then the same is true on $(-\infty, \infty)$ (see [12, p. 70]). This result has been still improved in [25] by showing hat for an e.d. on $(-\infty, \infty)$ it is enough the one on a sufficiently long finite interval I.

In general, the situation becomes more delicate, but we have to our disposal several sufficient conditions for an e.d. on $(-\infty, \infty)$ like a diagonal dominance criterion (see [23] and the references therein) or some more advanced criteria like those in [24] and [28], [29]. *The diagonal dominance criterion* in [23], implying an e.d. on $(-\infty, \infty)$, consists in satisfying the following three conditions, namely

(i) $A = (a_{ij})$ is bounded,
(ii) $\inf_{t \in \mathbb{R}} |\det A(.)| > 0$,
(iii) $|a_{ii}| \geq \sum_{j=1 (j \neq i)}^{n} |a_{ji}|$ or $|a_{ii}| \geq \sum_{j=1 (j \neq i)}^{n} |a_{ij}|$.

Furthermore, it is well-known (see e.g. [21] or [28]) that (8) with an e.d. on $(-\infty, \infty)$ has no nontrivial entirely bounded solutions and subsequently (because of the valid Fredholm alternative) the inhomogeneous system in (7) admits a unique bounded solution on $(-\infty, \infty)$ in the form

$$\widehat{X}(t) = \int_{-\infty}^{\infty} G(t,s) f_q(s) ds, \tag{10}$$

where (cf. (9))

$$G(t,s) = \begin{cases} X(t)PX^{-1}(s) & \text{for } t \geq s, \\ X(t)(I-P)X^{-1}(s) & \text{for } t \leq s \end{cases}$$

is the associated piece-wise continuous Green function.

Because of (9), we can get successively (when using the compatible vector and matrix norms)

$$
\begin{aligned}
|\widehat{X}(t)| &= |\int_{-\infty}^{\infty} G(t,s) f_q(s) ds| \le ||f_q(t)|| \int_{-\infty}^{\infty} |G(t,s)| ds \\
&= ||f_q(t)|| \int_{-\infty}^{t} |G(t,s)| ds + ||f_q(t)|| \int_{t}^{\infty} |G(t,s)| ds \\
&\le ||f_q(t)|| \int_{-\infty}^{t} |X(t) P X^{-1}(s)| ds + ||f_q(t)|| \int_{t}^{\infty} ||X(t)(I-P) X^{-1}(s)| ds \\
&\le ||f_q(t)|| \int_{-\infty}^{t} k \exp(-\lambda(t-s)) ds + ||f_q(t)|| \int_{t}^{\infty} k \exp(-\lambda(s-t)) ds \\
&\le 2\frac{k}{\lambda} ||f_q(t)||,
\end{aligned}
$$

i.e. after all,

$$
\sup_{t\in\mathbb{R}} |\widehat{X}(t)| \le 2\frac{k}{\lambda} ||f_q(t)||, \tag{11}
$$

where $\| \, . \, \| := \sup \operatorname{ess}_{t\in\mathbb{R}} | \, . \, |$.

Now, it is time to give the main statement of this part.

Theorem 4. *Let a continuous single-valued matrix function $A(t,X) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n^2}$ be bounded in t and a Carathéodory* (*multivalued*) *map $F(t,X) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be essentially bounded in t and can be for a.a. $t \in \mathbb{R}$ globally absolutely estimated by a single-valued continuous function $\Phi(X)$, i.e. $|F(t,X)| \le \Phi(X)$ for all $X \in \mathbb{R}^n$ and a.a. $t \in \mathbb{R}$.*

Let, furthermore, there exist a sufficiently big number D such that system (8) *possesses on $I = (-\infty, \infty)$ an exponential dichotomy* (9) *uniformly w.r.t. $q : \sup_{t\in\mathbb{R}} |q(t)| \le D$ [for this is sufficient to assume, for example,* (i)–(iii) *above].*

At last, let there exist a sufficiently small constant C such that

$$
\limsup_{|X|\to\infty} \frac{\Phi(X)}{|X|} \le C \tag{12}
$$

is satisfied with Φ defined above.

Then the inclusion

$$
X' + A(t,X)X \in F(t,X) \tag{13}
$$

admits an entirely bounded solution $X(t)$ such that

$$\sup_{t\in\mathbb{R}} |X(t)| \leq D.$$

Proof. In view of Remark 2, it is sufficient to show the solvability of (7), where $q \in Q = \{r(t) \in C(\mathbb{R}) : \sup_{t\in\mathbb{R}}|r(t)| \leq D\}$, when applying Corollary 2.

Because of an exponential dichotomy of (8) on $(-\infty, \infty)$, we could see that the inhomogeneous system in (7) admits a unique entirely bounded solution $\widehat{X}(t)$ in the form (9), satisfying (11).

Since, for given positive constants $C < C_o < \frac{\lambda}{2k}$, condition (12) implies obviously the existence of sufficiently big number M such that

$$\Phi(X) \leq C_0|X| + M$$

holds for all $X \in \mathbb{R}^n$, we arrive at the inequality (see (11))

$$\begin{aligned} 2\frac{k}{\lambda}||f_q(t)|| &\leq 2\frac{k}{\lambda}||\Phi(q(t))|| \\ &\leq 2\frac{k}{\lambda}[C_0||q(t)|| + M] \leq (1-\varepsilon)||q(t)|| + 2C_0M\frac{k}{\lambda}, \end{aligned}$$

where $0 < \varepsilon < 1$ is a suitable positive number implied by the magnitude of $C_0 < \frac{\lambda}{2k}$. Thus, one can readily check that, if we take D in the above definition of Q as $D \geq 2C_0Mk/\lambda\varepsilon$, then we come to

$$\sup_{t\in\mathbb{R}} ||\widehat{X}(t)|| \leq D \text{ for all } q \in Q,$$

by which $T(Q) \subset Q$, as required.

Remark 3. Theorem 4 generalizes its single-valued analogy in [2] and significantly extends and improves the well-known (single-valued) boundedness theorem of P. Bohl (see e.g. [15, pp. 360–361]). Changing slightly the approach, our technique is available to the boundedness results on the half-line in [11] as well.

Remark 4. Assuming additionally that $F(t+T,X) \equiv F(t,X)$ and $A(t+T,X) \equiv A(t,X)$, we can obtain the existence of a T-periodic solution of the inclusion $X' + A(t,X) \in F(t,X)$. For an almost-periodicity, it is however (as we shall see) much more delicate problem.

5. Weyl-like almost-periodic multifunctions and selectors. Since we consider the Carathéodory differential inclusions, it seems natural to use one of the concepts of nonuniform almost-periodicity, when investigating the related almost-periodic problems. For our purposes, the generalized concept of almost-periodicity in the sense of H. Weyl will be of a particular importance because of its effective applications.

At first, let us recall (see e.g. [20]) the classical definition of Weyl-like a.p. functions.

Definition 1. A locally Lebesgue integrable single-valued function $p(t)$ with nonempty values is called a.p. in the sense of Weyl (shortly, W-a.p.) if for every $\varepsilon > 0$ there exists a positive number $k = k(\varepsilon)$ such that in each interval of the length k there is at least one number τ satisfying

$$\lim_{l\to\infty} \big[\sup_{a\in\mathbb{R}} \frac{1}{l}\big\{ \int_a^{a+l} \big|p(t+\tau) - p(t)\big| dt\big\}\big] < \varepsilon. \tag{14}$$

Remark 5. Since the Lebesgue integral is absolutely convergent, Definition 1 has a meaning, provided the (finite or infinite) limit $\lim_{l\to\infty}[\,.\,]$, exists. But this is always true (see [20, p. 221]). On the other hand, we must understand that the space of all W-a.p. functions is not complete with the above metric (see [20]).

If a (multivalued) essentially bounded vector map $P(t) \in \mathbb{R}^n$ with nonempty closed values is *(Lebesgue) measurable*, i.e., if for any open $U \subset \mathbb{R}^n$ the set

$$\{t \in (-\infty, \infty) : \; P(t) \cap U \neq \emptyset\}$$

is measurable, then we can replace (14) by

$$\lim_{l\to\infty} \big[\sup_{a\in\mathbb{R}} \frac{1}{l}\big\{ \int_a^{a+l} \rho\big(P(t+\tau), P(t)\big) dt\big\}\big] < \varepsilon \tag{15}$$

for the same goal, provided $\rho(P(\,.\,+\tau), P(\,.\,))$ is measurable, where

$$\rho(M, N) := \inf\{d(x,y) : \, x \in M, \, y \in N\}$$

is the standard distance between the sets $M, N \subset \mathbb{R}^n$ and

$$d(x,y) = |x - y| = \sum_{i=1}^{n} |x_i - y_i|,$$

$x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$.

Indeed. This is true, because $P(t)$ is known (see e.g. [14, p. 23], [3, p. 18] and the references therein) to be measurable if and only if there exists a sequence $\{p_n(t)\}$ of measurable selectors of $P(t)$ such that

$$P(t) = \overline{\cup_{n\in\mathbb{N}}\, p_n(t)},$$

by which $\rho(P(t+\tau), P(t)) = \inf_{m,n\in\mathbb{N}} |p_m(t+\tau) - p_n(t)|$. From this so called Castaing representation, the desired measurability follows immediately by the standard (single-valued) arguments.

The local convergence of the Lebesgue integral in (15) follows then by means of the well-known Lebesgue dominated theorem. The (finite or infinite) limit, $\lim_{l\to\infty}[\,.\,]$, exists by the same reasons as in the single-valued case.

Hence we can give

Definition 2. An essentially bounded measurable (multivalued) function $P(t) \in \mathbb{R}^n$ with nonempty closed values is called weakly a.p. in the sense of Weyl (shortly, W_w-a.p.) if for every $\varepsilon > 0$ there exists a positive number $k = k(\varepsilon)$ such that in each interval of the length k there is at least one number τ satisfying (15).

In the single-valued case, Definition 2 reduces to Definition 1. For the continuous (multivalued) functions, condition (15) can be simply replaced by

$$\rho(P(t+\tau), P(t)) < \varepsilon \text{ for all } t \in \mathbb{R}.$$

For a T-periodic $P(t)$, the last inequality leads to the identity $\rho(P(t + T), P(t)) \equiv 0$, which looks rather unnatural.

Therefore, generalizing the concept of periodicity, one needs to use the well-known Hausdorff distance $h(\,.\,,\,.\,)$, namely

$$h(M,N) := \max\{h^+(M,N), h^-(M,N)\},$$

where

$$h^+ := \sup\{\rho(z,N) :\, z \in M\}, \quad h^-(M,N) = h^+(N,M).$$

Thus, $P(t+T) \equiv P(t)$ is equivalent with $h(P(t+T), P(t)) \equiv 0$, and so condition (15) changes to

$$\lim_{l\to\infty} \Big[\sup_{a\in\mathbb{R}} \frac{1}{l}\Big\{ \int_a^{a+l} h\big(P(t+\tau), P(t)\big)dt\Big\}\Big] < \varepsilon, \tag{16}$$

provided $h(P(\,.\,+\tau), P(\,.\,))$ is measurable.

If

$$\sup_{t\in\mathbb{R}} \operatorname{ess}|P(t)| < \infty, \tag{17}$$

then in view of

$$h(P(t+T), P(t)) \le 2\sup\{d(y,z) : z \in P(t+\tau)\} + 2\sup\{d(y,z) : z \in P(t)\},$$

where $y \in \operatorname{Im} P(t)$, and the Lebesgue dominated theorem, relation (16) has a meaning again, because, by the same reason as in Definition 2, $h(P(\,.\,+\tau), P(\,.\,))$ can be proved measurable again (see also [14, Problem 6a, p. 45]).

Definition 3. A measurable (multivalued) function $P(t) \in \mathbb{R}^n$ with nonempty closed values and satisfying (17) is called strongly a.p. in the sense of Weyl (shortly, W_s-a.p.) if for every $\varepsilon > 0$ there exists a positive number $k = k(\varepsilon)$ such that in each interval of the length k there is at least one number τ satisfying (16).

Although in the periodic case the stronger concept, related to the usage of the Hausdorff distance, is (under the assumptions of Definition 3) sufficient to possess a periodic measurable selector., it is not clear whether or not it is so for W_s-a.p. multifunctions. Therefore, we introduce still another definition allowing us to deal with a.p. measurable selectors.

Definition 4. A (multivalued) function $P(t) \in \mathbb{R}^n$ is called selectionally a.p. in the sense of Weyl (shortly, W_S-a.p.) if it can be written as the sum $P = P_1 + P_2$, where P_1 is a finite linear combination of essentially bounded measurable (multivalued) periodic functions with nonempty closed values and P_2 is a measurable essentially bounded function with nonempty closed values having the following property:

"for every $\varepsilon > 0$ there exists a positive number $k = k(\varepsilon)$ such that in each interval of the length k there is at least one number τ satisfying

$$\lim_{l\to\infty} \Big[\sup_{a\in\mathbb{R}} \frac{1}{l}\Big\{ \int_a^{a+l} \big|P_2(t+\tau) - P_2(t)\big| dt\Big\}\Big] < \varepsilon, \tag{18}$$

where $\int_a^{a+l} |P_2(t+\tau) - P_2(t)| dt = \{\int_a^{a+l} |f(t+\tau) - f(t)| dt : f$ is a measurable selector of $P_2\}$, i.e., the integrals in (18) are considered in the sense of Aumann (see [5] and cf. [3, p. 72]).

Example 1. As a concrete example of a W_S-a.p. multifunction, we can take e.g. $P(t) = P_1 + P_2$, where the 1-periodic P_1 defined as

$$P_1(t+1) \equiv P_1 = \begin{cases} [0, \frac{1}{2}] & \text{for } x \neq \frac{1}{2}, \\ [0, 1] & \text{for } x = \frac{1}{2}, \end{cases}$$

is well-known to be u.s.c. with convex compact values (and subsequently, it is measurable with nonempty, bounded, closed values) and

$$P_2(t) = \sin \frac{1}{2 + \cos\, t + \cos\, \sqrt{2}\, t}$$

is a.p. in the sense of Stepanov, but not uniformly a.p. (see [20, pp. 212–213]), (and subsequently, it is W-a.p.).

Remark 6. Since a measurable multifunction with nonempty closed values is always measurably selectionable (see e.g. [3, p. 18]), the integral set in (18) is nonempty (eventually a trivial singleton). Thus, every W_S-a.p. function $P(t)$ has (under the assumptions of Definition 4) a W-a.p. selector in the form of the sum of W-a.p. selectors of P_1 and P_2. Moreover, one can readily check that Definition 4 generalizes the one for (multivalued) periodic functions as well as Definition 1.

In the sequel, we need still to consider (because of applications) W-a.p. multifunctions containing a parameter, where the following kind of uniformity is necessary to take place. Since for measurable and essentially bounded (see (17)) multifunctions with nonempty closed values any of the concepts has a meaning, namely

$$\text{W-a.p. or } \mathrm{W}_S\text{-a.p. or } \mathrm{W}_s\text{-a.p. or } \mathrm{W}_w\text{-a.p.,} \tag{19}$$

we can give at the same time the appropriate definition for all of them together.

Definition 5. An essentially bounded in t (multivalued) Carathéodory function $F(t,X) \in \mathbb{R}^n$ (i.e. measurable in t, upper semi-continuous for a.a. X and with nonempty, compact and convex values) is called a.p.* in the sense of Weyl (shortly, W_*-a.p.), where the asterisk means one of the concepts in (19), uniformly w.r.t. $X \in \mathbb{R}^n$, if for every $\varepsilon > 0$ and every $D > 0$ there exists a positive number $k = k(\varepsilon, D)$ such that in each interval of the length k there is at least one number τ satisfying

$$\lim_{l \to \infty} \big[\sup_{a \in \mathbb{R}} \frac{1}{l} \big\{ \int_a^{a+l} d^*\big(F(t+\tau, X), F(t,X)\big) dt \big\}\big] < \varepsilon, \tag{20}$$

for $|X| \le D$, where $d^*(\,.\,,\,.\,)$ in (20) denotes the respective distance in (15) or (16) or (18) [in (18), only for $F_2 = F - F_1$, where F_1 is a possible finite linear combination of (multivalued) Carathéodory functions which are essentially bounded and periodic in t and (Lipschitz-) continuous in X for a.a. $t \in \mathbb{R}$, while F_2 is a Carathéodory multifunction which is essentially bounded in t and (Lipschitz-) continuous in X for a.a. $t \in \mathbb{R}$ (cf. Proposition 1 below)].

Remark 7. Under the assumptions of Definition 5 (which are for $F(t, X) \equiv P(t)$ more restrictive than those in previous Definitions 2, 3 and 4), a Carathéodory function $F(t, X)$ is known (see e.g. [3, p. 35]) to be weakly superpositionally measurable. It means that the Nemytskii operator $F(t, X(t))$, where $X(t)$ is a continuous single-valued function, possesses a (Lebesgue) measurable selector. If $F(t, .)$ is additionally continuous (i.e. also lower-semi-continuous; see the definition in Part 2) for a.a. $t \in \mathbb{R}$, then $F(t, X)$ is even product-measurable, and consequently superpositionally measurable (see e.g. [3, p. 34]) and having a measurable selector (see e.g. [3, p. 18]). It means that the Nemytskii operator $F(t, X(t))$, where $X(t)$ is again a continuous single-valued function, becomes measurable.

We conclude this section by the following proposition which is crucial (jointly with Definition 5) for the applications to differential inclusions. For more details see [3, pp. 24–25] and the references therein.

Proposition 1. *Let $F(t, X) : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be a multivalued function such that:*

(i) $F(t,\,.\,)$ *with nonempty closed convex values is* (*Lipschitz-*) *continuous* (*with a sufficiently small Lipschitz constant L*), *namely*

$$h(F(t, X), F(t, Y)) \le L|X - Y| \text{ for a.a } t \in \mathbb{R},$$

where $h(\,.\,,\,.\,)$ denotes the Hausdorff distance,

(ii) $F(\,.\,, X)$ *is measurable with nonempty closed convex values.*

Let, furthermore, $X(t) : \mathbb{R} \to \mathbb{R}^n$ be a single-valued continuous map and $Y(t)$ be a measurable selector of $F(\,.\,, X(\,.\,))$ (*which exists according to Remark* 7), *i.e. $Y(t) \subset F(t, X(t))$ for a.a. $t \in \mathbb{R}$.*

Then there exists a Carathéodory selector f of F such that $f(t,\,.\,)$ is (*Lipschitz-*) *continuous* (*with not necessarily the same, but sufficiently small Lipschitz constant* (cf. [4, p. 77])) *and satisfies $Y(t) = f(t, X(t))$ for a.a. $t \in \mathbb{R}$.*

6. Existence criteria for almost-periodic solutions. Let us start with summarizing the conclusions of Theorem 4 in Part 4 (see also (10)) and Proposition 1 in Part 5 for the following special form of inclusion (13), namely

$$X' + AX \in F(t, X), \tag{21}$$

where the matrix $A(t, X) \equiv A$ is constant.

Proposition 2. *Under the assumptions of Theorem* 4 *and Proposition* 1*, inclusion* (21) *admits a bounded solution* $X(t)$ *such that*

$$X(t) = \int_{-\infty}^{\infty} G(t, s) f(s, X(s)) ds, \tag{22}$$

where $f(t, X(t)) \subset F(t, X(t))$ *for a.a.* $t \in \mathbb{R}$ *and* $G(t, s)$ *is same as in* (10)(*i.e. satisfying* (9))*. Moreover, the Carathéodory selector* $f(t, X)$ *of* F *is Lipschitz-continuous in* $X \in \mathbb{R}^n$ *with a sufficiently small Lipschitz constant, say* L_0.

Our last but not least goal is to prove that $X(t)$ in (22) is a.p., provided (besides another) (20).

Theorem 5. *Let the following assumptions be satisfied:*

(i) *a* (*single-valued*) *constant* $(n \times n)$*-matrix* A *is hyperbolic, i.e., all the associated eigenvalues have nonzero real parts;*
(ii) *a (multivalued) Carathéodory map* $F(t, X)$: $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ *is essentially bounded in* t *and Lipschitz-continuous in* X *for a.a.* $t \in \mathbb{R}$ (*see Proposition* 1) *with a sufficiently small constant* L;
(iii) $F(t, X)$ *is* W_S *- a.p. in* t*, uniformly w.r.t.* $X \in \mathbb{R}^n$, (*see Definitions* 4, 5).[1]

Then inclusion (21) *admits an almost-periodic solution.*

Proof. Conditions (i),(ii) represent the assumptions of Proposition 2 implying the existence of an entirely bounded solution $X(t)$ of (21) in the form (22), where the Green function $G(t, s)$ satisfies (9) and $f(t, X) \subset F$ is a (single-valued) Lipschitz-continuous function with a sufficiently small Lipschitz constant L_0.

By the hypothesis, for every $\varepsilon > 0$ and every $D > 0$ there exists a positive number $k = k(\varepsilon, D)$ such that in each interval of the length k there is at

[1]or $F(t, X)$ is W_*-a.p., uniformly w.r.t. $X \in \mathbb{R}^n$, (see (20)) and the existing (according to Proposition 1) Carathéodory selector $f(t, X) \subset F$ is W-a.p. in t, uniformly w.r.t. $X \in \mathbb{R}^n$.

least one number τ satisfying

$$\lim_{l\to\infty}\big[\sup_{a\in\mathbb{R}}\frac{1}{l}\big\{\int_a^{a+l}|f(t+\tau,X)-f(t,X)|dt\big\}\big]<\varepsilon \tag{23}$$

for $|X|\le D$, where the constants ε, D, k, τ are not necessarily the same as those imposed in (20) on the multivalued function $F\supset f$ (see Definition 5).

Since for $X(t)$ in (22) we have also (cf. [15, p. 426])

$$X(t+\tau)=\int_{-\infty}^{\infty}G(t+\tau,s)f(s,X(s))ds=\int_{-\infty}^{\infty}G(t,s)f(s+\tau,X(s+\tau))ds,$$

one can deal with

$$\int_a^{a+l}|X(t+\tau)-X(t)|\ dt,$$

where $l>0$ is an arbitrary constant, to prove the almost-periodicity of X(t) in the Weyl metric used in (23). Hence, applying (after several steps) the well-known Fubini theorem and using the norm $\|\,.\,\|=\sup_{t\in\mathbb{R}}|\,.\,|$ and compatible vector and matrix norms, we obtain successively

$$\begin{aligned}
&\int_a^{a+l}|X(t+\tau)-X(t)|dt\\
&=\int_a^{a+l}\left|\int_{-\infty}^{\infty}G(t,s)[f(s+\tau,x(s+\tau))-f(s,X(s))]ds\right|\ dt\\
&\le\int_a^{a+l}dt\int_{-\infty}^{\infty}|G(t,s)||f(s+\tau,X(s+\tau))-f(s,X(s))|\,ds\\
&\le\int_a^{a+l}dt\big[\int_{-\infty}^{t}k\ e^{-\lambda(t-s)}|f(s+\tau,X(s+\tau))-f(s,X(s))|ds\\
&+\int_t^{\infty}k\ e^{\lambda(t-s)}|f(s+\tau,X(s+\tau))-f(s,X(s))|ds\big]\\
&=k\int_{-\infty}^{a}|f(s+\tau,X(s+\tau))-f(s,X(s))|ds\int_a^{a+l}e^{-\lambda(t-s)}dt\\
&+k\int_a^{a+l}|f(s+\tau,X(s+\tau))-f(s,X(s))|ds\int_s^{a+l}e^{-\lambda(t-s)}dt
\end{aligned}$$

$$
\begin{aligned}
&+ k\int_a^{a+l} |f(s+\tau, X(s+\tau)) - f(s, X(s))| ds \int_a^s e^{\lambda(t-s)} dt \\
&+ k\int_{a+l}^{\infty} |f(s+\tau, X(s+\tau)) - f(s, X(s))| ds \int_a^{a+l} e^{\lambda(t-s)} dt \\
&= \frac{k}{\lambda} e^{-\lambda a}(1 - e^{-\lambda l}) \int_{-\infty}^{a} e^{\lambda s} |f(s+\tau, X(s+\tau)) - f(s, X(s))| ds \\
&+ \frac{k}{\lambda} \int_a^{a+l} (1 - e^{-\lambda(a+l-s)}) |f(s+\tau, X(s+\tau)) - f(s, X(s))| ds \\
&+ \frac{k}{\lambda} \int_a^{a+l} (1 - e^{\lambda(a-s)}) |f(s+\tau, X(s+\tau)) - f(s, X(s))| ds \\
&+ \frac{k}{\lambda} e^{\lambda a}(e^{\lambda l} - 1) \int_{a+l}^{\infty} e^{-\lambda s} |f(s+\tau, X(s+\tau)) - f(s, X(s))| ds \\
&\le 2\frac{k}{\lambda} \int_a^{a+l} |f(s+\tau, X(s+\tau)) - f(s, X(s))| ds \\
&+ 2\frac{k}{\lambda^2} \|f(t+\tau, X(t+\tau)) - f(t, X(t))\|.
\end{aligned}
$$

Since $X(t)$ is bounded, and subsequently $\|f(t, X(t))\| \le \infty$, the last term vanishes in the Weyl metric used in (23). Thus, we get furthermore, by means of (23) and the Lipschitz property, that

$$
\begin{aligned}
&\lim_{l\to\infty} \big[\sup_{a\in\mathbb{R}} \frac{1}{l} \big\{ \int_a^{a+l} |X(t+\tau) - X(t)| dt \big\}\big] \\
&\le 2\frac{k}{\lambda} \lim_{l\to\infty} \big[\sup_{a\in\mathbb{R}} \frac{1}{l} \big\{ \int_a^{a+l} |f(t+\tau, X(t+\tau)) - f(t, X(t))| dt \big\}\big] \\
&\le 2\frac{k}{\lambda} \lim_{l\to\infty} \big[\sup_{a\in\mathbb{R}} \frac{1}{l} \big\{ \int_a^{a+l} |f(t+\tau, X(t+\tau)) - f(t+\tau, X(t))| \\
&\qquad + |f(t+\tau, X(t)) - f(t, X(t))| dt \big\}\big] \\
&< 2\varepsilon\frac{k}{\lambda} + 2L_0\frac{k}{\lambda} \lim_{l\to\infty} \big[\sup_{a\in\mathbb{R}} \frac{1}{l} \big\{ \int_a^{a+l} |X(t+\tau) - X(t)| dt \big\}\big],
\end{aligned}
$$

where L_0 is sufficiently small.

After all, we arrive at

$$
\lim_{l\to\infty} \big[\sup_{a\in\mathbb{R}} \frac{1}{l} \big\{ \int_a^{a+l} |X(t+\tau) - X(t)| dt \big\}\big] < \frac{2\varepsilon k}{\lambda - 2L_0 k},
$$

as far as $L_0 < \frac{\lambda}{2k}$, which already verifies, according to Definition 1, the desired almost-periodicity of $X(t)$. So, the proof is complete. □

For the equation

$$X' + AX = f(t, X) \tag{24}$$

we can also easily obtain the uniqueness result.

Corollary 3. *Let A fulfil condition* (i) *in Theorem* 5 *and $f(t, X)$ be a single-valued Carathéodory function which is essentially bounded in t. If $f(t, X)$ is still W-a.p. in t (see* (23)*), uniformly w.r.t $X \in \mathbb{R}^n$, and*

$$|f(t, X) - f(t, Y)| \leq L_0 |X - Y| \tag{25}$$

holds for a.a. $t \in \mathbb{R}$ and all $X, Y \in \mathbb{R}^n$ with a sufficiently small constant L_0, then equation (24) *admits a unique a.p. solution.*

Proof. Since the existence part represents only a single-valued case of Theorem 5, it is enough to show the uniqueness of $X(t)$ in (22).

Hence, assume that there exists another bounded (not necessarily a.p.) solution of (24), namely

$$Y(t) = \int_{-\infty}^{\infty} G(t, s) f(s, Y(s)) ds.$$

By virtue of (25), we have

$$\begin{aligned}
&\|X(t) - Y(t)\| = \| \int_{-\infty}^{\infty} G(t, s)[f(s, X(s)) - f(s, Y(s))] ds \| \\
&\leq \| \int_{-\infty}^{\infty} |G(t, s)| |f(s, X(s)) - f(s, Y(s))| ds \| \\
&\leq L_0 \| \int_{-\infty}^{\infty} |G(t, s)| |X(s) - Y(s)| ds \| \\
&\leq L_0 \|X(t) - Y(t)\| \int_{-\infty}^{\infty} |G(t, s)| ds \\
&\leq L_0 K \ \|X(t) - Y(t)\|,
\end{aligned}$$

where K is a sufficiently big constant implied by (9) (see above).

This already leads to

$$\|X(t) - Y(t)\|(1 - L_0 K) \leq 0, \text{ whenever } L_0 < K^{-1},$$

by which $X(t) \equiv Y(t)$ as claimed.

Remark 8. Corollary 3 generalizes the analogical theorem due to G.I. Birjuk (see [15, p. 426]) who studied equation (24) with "only" a uniformly a.p. $f(t,X)$ in t. For a stable constant matrix A, the magnitude of L_0 in (25) can be detected explicitly (see [2]).

Let us conclude this section by the following more explicit

Example 2. Consider the inclusion

$$X' + AX \in F(X) + P(t), \tag{26}$$

where A is a constant hyperbolic $(n \times n)$-matrix, $[F(x)+P(t)]$ is a Carathéodory map, $F(x)$ is Lipschitz-continuous with a Lipschitz constant L (see Proposition 1), $P(t)$ is essentially bounded and takes the form of the sum of a W-a.p. (single-valued) function plus a finite linear combination of essentially bounded periodic multifunctions with nonempty closed values (e.g. as in Example 1).

Then inclusion (26) admits an a.p. solution, provided (see [2])

$$LC < \frac{|\lambda|}{2},$$

whenever A is symmetric, or (see [2])

$$LC < 1 / \sum_{k=0}^{n-1} \frac{2^k |A|^k}{|Re\lambda|^{k+1}},$$

whenever A is stable (Hurwitzean), where $|\lambda|$ or $|Re\lambda|$ denote the respective minima of absolute values or absolute values of real parts of the eigenvalues associated to A. The factor C comes from the possibility of a barycentric selection and its (rather cumbersome) computation can be found e.g. in [4, pp. 77–80]).

In the single-valued case, we can take $C = 1$, and then the a.p. solution is unique.

7. Concluding remarks. We could see that an important question arises whether W_s-almost-periodicity (see Definition 3) implies the existence of an W-a.p. selector (see Definition 1) or what are the additional restrictions for this implication. After its answering, condition (iii) in Theorem 5 could take a more effective form.

Another question is related to the absence of the Lipschitz-continuity in the right-hand sides of the studied inclusions (and subsequently in the Carathéodory selectors), because otherwise we would be able to solve by the same manner important technical (a.p.) problems like those related to the dry (Coulomb) friction (see e.g. [13]). One might expect that it could be the case, when defining Q in Theorem 3 directly as a nonempty, convex, bounded set of a.p. functions, but then as we already pointed out in [2] for uniformly a.p. functions, Q is unfortunately not closed in the given topology of the uniform convergence on compact subintervals of the real line.

At the moment, we are also trying to replace the notion of almost-periodicity in the sense of H. Weyl (used here) by the one in the sense of A.S. Besicovitch, by means of the results for the Besicovitch spaces of a.p. functions developed in [6], [26].

REFERENCES

[1] J. Andres, G. Gabor, and L. Górniewicz, *Boundary value problems on infinite intervals,* Trans. Amer. Math. Soc., to appear.

[2] J. Andres and B. Krajc, *Unified approach to bounded, periodic and almost periodic solutions of differential systems,* Annal. Math. Silesianae., 11 (1997), 39–53.

[3] J. Appel, E. DePascale, N. H. Thái, and P.P. Zabreiko, *Multivalued superpositions,* Dissertationes Mathematicae 345 (1995), 1–97.

[4] J.-P. Aubin and A. Cellina, "Differential Inclusions," Springer, Berlin, 1982.

[5] J. Aumann, *Integrals of set-valued functions,* J. Math. Anal. Appl., 12 (1965), 1–12.

[6] A. Avantaggiati, G. Bruno, and R. Iannacci, *Classical and new results on Besicovitch spaces of almost-periodic functions and their duals,* Quaderni del Dipart. di ME-MO-MAT per le Scienze Applicate, La Sapienza-Roma (1993), 1–50.

[7] R. Bader and W. Kryszewski, *Fixed-point index for compositions of set-valued maps with proximally ∞-connected values on arbitrary ANR's,* Set-Valued Analysis, 2 (1994), 459–480.

[8] H. Ben-El-Mechaiekh and P. Deguire, *Approachability and fixed points for non-convex set-valued maps,* J. Math. Anal. Appl., 170 (1992), 472–500.

[9] M. Cecchi, M. Furi, and M. Marini, *On continuity and compactness of some nonlinear operators associated with differential equations in noncompact intervals,* Nonlin. Anal., 9 (1985), 171–180.

[10] M. Cecchi, M. Furi, and M. Marini, *About the solvability of ordinary differential equations with asymptotic boundary conditions,* Boll. U.M.I.. (6) 4-C, 1 (1985), 329–345.

[11] M. Cecchi, M. Marini, and P. Zecca, *Existence of bounded solutions for multivalued differential systems,* Nonlin. Anal., 9, 8 (1985), 775–786.

[12] W.A. Coppel, "Dichotomies in Stability Theory," Springer, Berlin, 1978.

[13] K. Deimling, G. Hetzer, and Wenxian Shen, *Almost periodicity enforced by Coulomb friction,* Advances in Diff. Eqns., 1, 2 (1996), 265–281.

[14] K. Deimling, "Multivalued Differential Equations," De Gruyter, Berlin, 1992.

[15] B.P. Demidovitch, "Lectures on the Mathematical Stability Theory," Nauka, Moscow, 1967, (Russian).
[16] A. Dold, "Lectures on Algebraic Topology," Springer, Berlin, 1972.
[17] J. Dugundji, *An extension of Tietze's theorem,* Pacific J. Math., 1 (1951), 353–367.
[18] L. Górniewicz, A. Granas, and W. Kryszewski, *On the homotopy method in the fixed point index theory of multi-valued mappings of compact absolute neighbourhood retracts,* J. Math. Anal. Appl., 161(2) (1991), 457–473.
[19] A. Haraux, *Asymptotic behaviour for two-dimensional, quasi-autonomous, almost-periodic evolution equations,* J. Diff. Eqns., 66(1) (1987), 62–70.
[20] B.M. Levitan, "Almost-Periodic Functions," GIT-TL, Moscow, 1953. (Russian).
[21] J.L. Massera and J.J. Schaeffer, "Linear Differential Equations and Function Spaces," Academic Press, New York, 1966.
[22] J. Musielak, "Introduction to Functional Analysis," PWN, Warsaw, 1976, (Polish).
[23] K.J. Palmer, *A diagonal dominance criterion for exponential dichotomy,* Bull. Austral. Math. Soc., 17 (1977), 363–374.
[24] K.J. Palmer, *Two linear systems criteria for exponential dichotomy,* Ann. Mat. Pura Appl., 124(4) (1980), 199–216.
[25] K.J. Palmer, *Exponential dichotomies for almost-periodic equations,* Proceed. Amer. Math. Soc., 101(2) (1987), 293–298.
[26] A.N. Pankov, "Bounded and Almost Periodic Solutions of Nonlinear Operator Differential Equations," Kluwer Acad. Publ., Dordrecht, 1990.
[27] A.P. Robertson and W. Robertson, "Topological Vector Spaces," Cambridge Univ. Press, Cambridge, 1964.
[28] R.J. Sacker and G.R. Sell, *Existence of dichotomies and invariant splittings for linear differential systems I,* J. Diff. Eqns., 15(3) (1974), 429–458.
[29] R.J. Sacker and G.R. Sell, *Existence of dichotomies and invariant splittings for linear differential systems II,* J. Diff. Eqns., 22(2) (1976), 478–496.
[30] V.V. Zhikov and B.M. Levitan, *The Favard theory,* Uspekhi Matem. Nauk, 32(2) (1977), 123–171, (Russian).