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A HAMILTON-JACOBI EQUATION WITH MEASURES ARISING IN Γ-CONVERGENCE OF OPTIMAL CONTROL PROBLEMS

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Abstract. We consider a Hamilton-Jacobi equation with a measure in the Hamiltonian arising from the Γ-limit of some optimal control problems. We give a definition of viscosity solution in this case, by adapting the method of the reparametrization of Dal Maso and Rampazzo [9].

1. Introduction. In this paper we study a Hamilton-Jacobi equation with a measure in the Hamiltonian and we give a definition of viscosity solution in this case. This kind of Hamiltonian arises if one looks to the Hamilton-Jacobi equation solved by the value function of the Γ-limit of suitable sequences of optimal control problems.

Since our aim is to justify the study of this Hamiltonian and to show the ideas behind our definition, we will describe the problem in the framework of Γ-limits of optimal control problems. The study of a general Hamilton-Jacobi equation with a measure in the Hamiltonian will be the object of a future work. For every $n \in \mathbb{N}$ consider the state equation

$$\begin{cases} \dot{x}_n(t) = a_n(t, x_n) + b_n(t)u(t), & t \in (\tau, T] \\ x_n(\tau) = z \end{cases}$$

where $b_n(t) \in L^2(0,T)$ and $u \in \mathcal{U} \equiv L^2(0,T)$ with $T < \infty$. The cost functionals are defined by

$$J_n(\tau, z, u) = \int_\tau^T \big(g(x_n(t)) + \frac{1}{2}|u(t)|^2\big)dt$$

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so that the value functions, $v_n(\tau, z) = \inf_{u\in\mathcal{U}} J_n(\tau, z, u)$, are viscosity solutions of the Hamilton-Jacobi equations

$$\begin{cases} -\frac{\partial v_n}{\partial t}(\tau, z) + H_n(\tau, z, \frac{\partial}{\partial x} v_n(\tau, z)) = 0 & \text{in } [0, T] \times \mathbb{R} \\ v_n(T, z) = 0 & \text{in } \mathbb{R} \end{cases}$$

having as Hamiltonian

$$H_n(\tau, z, p) = -g(z) - pa_n(\tau, z) + \frac{1}{2} p^2 b_n(\tau)^2$$

(this fact will be proved in Section 5). Assuming that

$$a_n(\cdot, y) \to a(\cdot, y) \text{ weakly } L^1(0, T)\ \forall y \in \mathbb{R} \tag{1.1}$$

$$b_n(t) \to b(t) \text{ weakly } L^2(0, T) \tag{1.2}$$

$$b_n(t)^2 \to \mu \text{ weakly}^* \ \mathcal{M}^+(0, T) \tag{1.3}$$

(where $\mathcal{M}^+(0, T)$ denotes the class of all nonnegative finite Borel measures on $(0, T)$), Buttazzo and Freddi in [8] proved that the limit problem (in the sense of Γ-convergence) is the optimal control problem whose state equation is the following identity between measures

$$\begin{cases} \dot{x}(t) = (a(t, x) + v(t)) \cdot dt + \xi(t) \cdot \mu^s & \text{in } (\tau, T] \\ x(\tau^-) = z \end{cases}$$

where $v \in L^2(0, T)$, μ^s is the singular part of the measure μ with respect to the Lebesgue measure and $\xi \in L^2(0, T; \mu^s)$, and whose cost functional is

$$J(\tau, z, U) = \int_\tau^T f(t, x(t), U(t))dt + \int_{[\tau,T)} \frac{1}{2}\xi^2(t) d\mu^s$$

where $U(t) = (u(t), v(t), \xi(t)) \in \mathcal{U} = L^2(0, T) \times L^2(0, T) \times L^2(0, T; \mu^s)$ and $f(t, y, U) = \frac{|u|^2}{2} + g(y) + \frac{|v-bu|^2}{2(\mu^a - b^2)}$ being μ^a the derivative of the measure μ with respect to the Lebesgue measure.

In [5] we proved that if we assume, instead of (1.2)-(1.3), that $b_n(t) \to b(t)$ weakly* $L^\infty(0, T)$ and $b_n(t)^2 \to \beta(t)$ weakly* $L^\infty(0, T)$ then the Hamiltonians H_n converge to the Hamiltonian of the Γ-limit problem (given by Buttazzo and Cavazzuti in [7]). In order to extend this result to the measure case we expected as Hamiltonian of the limit problem the limit of $H_n(\tau, z, p)$ under the convergence assumptions (1.1) and (1.3). This limit is given by

$$H(\tau, z, p) = -g(z) - p\, a(\tau, z) + \frac{1}{2}p^2 \cdot \mu. \tag{1.4}$$

The problem is now to understand in what sense the value function, $v(\tau, z) = \inf_{U \in \mathcal{U}} J(\tau, z, U)$, is the solution of the Hamilton-Jacobi equation

$$\begin{cases} -\frac{\partial v}{\partial t}(\tau, z) + H(\tau, z, \frac{\partial v}{\partial x}(\tau, z)) = 0 & \text{in } [0, T] \times \mathbb{R} \\ v(T, z) = 0 & \text{in } \mathbb{R} \end{cases} \tag{1.5}$$

if μ^s is concentrated on a jump of Dv. In order to explain this problem in a precise way we consider the special case $a_n = 0$, $g(z) = |z|^2$, $b_n(t) = 1_{(1,1+1/n)}(t)\sqrt{n}$ and $T = 2$, that will be detailed in Section 3. The limit Hamilton-Jacobi equation is formally given by

$$\begin{cases} -\frac{\partial v}{\partial t}(\tau, z) - |z|^2 + \frac{1}{2}\big(\frac{\partial v}{\partial x}(\tau, z)\big)^2 \cdot \delta_1 = 0 & \text{in } [0, 2] \times \mathbb{R} \\ v(2, z) = 0 & \text{in } \mathbb{R} \end{cases} \tag{1.6}$$

and the limit value function can be explicitly computed:

$$v(\tau, z) = \begin{cases} |z|^2(\frac{4}{3} - \tau) & 0 \le \tau \le 1 \\ |z|^2(2 - \tau) & 1 < \tau \le 2. \end{cases}$$

In this case it is clear that a difficulty arises at point $\tau = 1$ where $\frac{\partial v}{\partial x}$ has a jump and δ_1 is concentrated. Since $\frac{\partial v}{\partial t}(\tau, z) = -|z|^2 + \frac{2}{3}|z|^2 \cdot \delta_1$ the question is to understand the meaning of the term $\frac{1}{2}\big(\frac{\partial v}{\partial x}(\tau, z)\big)^2 \cdot \delta_1$ in order to obtain $\frac{2}{3}|z|^2 \cdot \delta_1$ from it.

Since this example is a linear quadratic problem we can suppose $v(\tau, z) = p(\tau)|z|^2$, the Riccati equation for p is then

$$\begin{cases} \dot{p}(\tau) = -1 + 2p(\tau) \cdot \delta_1 & \text{in } [0, 2] \\ p(2) = 0. \end{cases} \tag{1.7}$$

It is clear that we still have a problem for $\tau = 1$ but now we deal with an ordinary differential equation.

This kind of problem has been widely studied in the literature. For the Russian school we can refer, for example to [13] or [16] and for the west side school one can see [3], [4], [9], [15] or [17]. In this work we refer in particular to [9]. The basic idea is to consider a graph's reparametrization of the primitive of the measure appearing in the equation and to construct a more regular equivalent equation.

Without going into details we only observe that computing the system $(E)_s$, page 741 of [9], for the (1.7) we obtain

$$\begin{cases} \dot{p}(\tau) = -1 + \frac{2}{3} \cdot \delta_1 & \text{in } [0,2] \\ p(2) = 0, \end{cases}$$

which is solved by $p(\tau)$ in the sense of equality between measures.

Since (1.5) is in general a partial differential equation we apply here the same method but starting from the state equations: hence we obtain one reparametrized optimal control problem at every fixed n and one for the limit problem. The advantage is that for this limit problem we can define the Hamiltonian and give a meaning to the Hamilton-Jacobi equation. More precisely we use the definition of viscosity solution for measurable-time dependent Hamiltonians introduced by Ishii in [11]. We prove also that the Hamiltonians of the reparametrized problems at fixed n converge in a natural way to the Hamiltonian of the reparametrized limit problem.

This approach to optimal control problems has been already introduced in the literature, see for example [14]. But all the papers in this direction deal with problems where the measure appears in the state equation as control while in our case the measures are coefficients in the state equation. This is why our problem is different from the ones previously treated.

We remark that from the point of view of viscosity solutions the assumptions that the Hamiltonian is merely measurable with respect to t, and that the set of controls is unbounded may appear as useless and artificial generalizations. However, in the framework of Γ-convergence t-measurability is a natural assumption since in this kind of problems one deals with weak types of convergence which generally preserve measurability but not continuity. Moreover, the measure case is ruled out if we suppose the control set bounded, in fact, if we consider the Γ-limit of this sequence of optimal control problems with $u \in K$ (K compact subset of $\mathbb{R}$) and we suppose (1.1) and (1.3) in the limit problem we will not have a measure but a function (we refer to [8] for all details). The relationship between the model with measures and the one with unbounded control set has been also studied in [2], where the authors study a Hamilton-Jacobi equation for optimal control problem with measures as controls (which, as we said before, is slightly different from our case).

This paper is organized as follows. In Section 2 we construct the reparametrized optimal control problems and we justify our definition. Section 3 is devoted of the description of an example in all details. In Section 4 we prove the fundamental result about the linking between the optimal control problem and the reparametrized one. To get this we prove the chain rule in dimension one for distributional derivatives of functions with bounded variation which are the composition of an absolutely continuous function with a monotone increasing function. In Section 5 we prove all results about viscosity solutions.

2. The basic framework and the definition. We start by recalling some definitions concerning functions with bounded variation.

Definition 2.1. A function $f : [0,T] \to \mathbb{R}$ is said to have bounded variation on the subinterval $[a,b] \subseteq [0,T]$ if there exists a constant $C \geq 0$ such that, for each finite set of points $t_0, t_1, ..., t_p$ satisfying $a = t_0 < t_1 < ... < t_p = b$ the inequality

$$\sum_{k=1}^{p} |f(t_k) - f(t_{k-1})| \leq C$$

holds, where $|\cdot|$ denotes the Euclidean norm. The least C which satisfies the above condition is called the variation of f on $[a,b]$, and is denoted by $V_a^b(f)$. The number $V_0^T(f)$ will be called the total variation of f.

The symbol $BV^-([0,T])$ will indicate the set of all left continuous functions with bounded total variation which are right continuous at 0. For every $u \in BV^-([0,T])$, the distributional derivatives $\dot{u}$ is a signed measure on $]0,T[$ characterized by

$$\dot{u}([t_1, t_2[) = u(t_2) - u(t_1)$$

for every subinterval $[t_1, t_2[\subseteq]0,T[$. In particular, for every $t \in]0,T[$,

$$\dot{u}(t) = [u](t) = u(t^+) - u(t^-)$$

where $u(t^+) = \lim_{s \to t^+} u(s)$ and $u(t^-) = \lim_{s \to t^-} u(s)$.

Here and subsequently we use the following notation:

$$L^p(0,T;\mu) = \Big\{ f : [0,T] \to \mathbb{R} : \int_{[0,T]} |f(t)|^p d\mu < \infty \Big\},$$

for simplicity if μ is the Lebesgue measure we write $L^p(0,T)$ instead of $L^p(0,T;\mu)$.

As we said in the Introduction our aim is to justify our definition in the framework of $\Gamma-$limit of optimal control problems. This is why instead of giving the definition directly we begin with the reparametrization of the optimal control problem at every fixed n. We recall that our state is the solution of the following equation

$$\begin{cases} \dot{x}_n(t) = a_n(t,x_n) + b_n(t)u(t) & t \in (\tau,T] \\ x_n(\tau) = z \end{cases} \tag{2.1}$$

where $\tau \in [0,T]$, $b_n \in L^2(0,T)$ and $u \in L^2(0,T)$. Our control space is $\mathcal{U} = L^2(0,T)$.

We will denote by $\|\cdot\|_r$ the $L^r(0,T)$ norm for $r \in [1,+\infty)$ and by $\dot{f}$ the t-derivative of a function f that can be in general a measure.

The cost functional is

$$J_n(\tau,z,u) = \int_\tau^T \big(g(x_n(t)) + \frac{1}{2}|u(t)|^2\big)dt \tag{2.2}$$

so that the value function is defined by

$$v_n(\tau,z) = \inf_{u\in\mathcal{U}} J_n(\tau,z,u). \tag{2.3}$$

On the Borel functions $a_n : [0,T]\times\mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to [0,+\infty]$ we always assume that:

a) there exists a sequence of functions M_n from $[0,T]$ into $\mathbb{R}$ with $\| M_n \|_1 \leq M < \infty$ such that, for every $t \in [0,T]$

$$|a_n(t,0)| \leq M_n(t) \ \forall n \in \mathbb{N}; \tag{2.4}$$

b) there exists a sequence of functions α_n from $[0,T]$ into $\mathbb{R}$ with $\| \alpha_n \|_1 \leq \alpha < \infty$ such that, for every $t \in [0,T]$ and for every $x_1, x_2 \in \mathbb{R}$

$$|a_n(t,x_1) - a_n(t,x_2)| \leq \alpha_n(t)|x_1 - x_2| \ \forall n \in \mathbb{N}; \tag{2.5}$$

c) there exists a function $C(v,w)$ from $\mathbb{R}\times\mathbb{R}$ into $\mathbb{R}$, increasing in v and w such that, for every $y, z \in \mathbb{R}$

$$|g(y) - g(z)| \leq C(|y|,|z|)|y - z|. \tag{2.6}$$

In Section 5 we will prove the following result:

Theorem 2.2. *The value function v_n is a viscosity solution of the Hamilton Jacobi equation*

$$\begin{cases} -\frac{\partial v_n}{\partial t}(\tau, z) + H_n(\tau, z, \frac{\partial}{\partial x} v_n(\tau, z)) = 0 & \text{in } [0,T] \times \mathbb{R} \\ v_n(T, z) = 0 & \text{in } \mathbb{R} \end{cases} \tag{2.7}$$

where the Hamiltonian $H_n : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is

$$H_n(\tau, z, p) = -g(z) - pa_n(\tau, z) + \frac{1}{2}p^2 b_n(\tau)^2. \tag{2.8}$$

Since the idea is to reparametrize the graph of the primitive of the measure μ that appears in the limit problem, by hypothesis (1.3), (i.e., $b_n(t)^2 \to \mu$ weakly* $\mathcal{M}^+(0,T)$), it is clear that at every fixed n we have to reparametrize the graph of a function $B_n(t)$ such that $\dot{B}_n(t) = b_n(t)^2$.

Let us define $w_n(t) : [0,T] \to [0,1]$ as $w_n(t) = \frac{t + V_0^t(B_n)}{T + V_0^T(B_n)}$ and

$$\varphi_n(s) = (\varphi_n^0(s), \varphi_n^1(s)) = (t, B_n(t)) \text{ if } s = w_n(t), \quad \forall s \in [0,1]. \tag{2.9}$$

If we change variable in the state equation (2.1) as follows

$$y_n(s) := x_n(\varphi_n^0(s)), \tag{2.10}$$

we obtain our reparametrized state equations,

$$\begin{cases} \frac{dy_n}{ds}(s) = \left\{ a_n(\varphi_n^0(s), y_n(s)) + b_n(\varphi_n^0(s)) u(\varphi_n^0(s)) \right\} \frac{d\varphi_n^0}{ds}(s), \ s \in (s_1, 1] \\ y_n(s_1) = z \end{cases} \tag{2.11}$$

where s_1 is such that $\varphi_n^0(s_1) = \tau$.

Remark 2.3. It is easy to verify that by construction $x_n(t) = y_n(w_n(t))$ for all $t \in (\tau, T]$. □

We want to apply the same change of variable in the cost functional, so if $t = \varphi_n^0(s)$ formula (2.2) becomes

$$J_n(\tau, z, u) = \int_{s_1}^1 \left(g(y_n(s)) + \frac{1}{2}|\tilde{u}(s)|^2 \right) \frac{d\varphi_n^0}{ds}(s) \, ds$$

by definition of $y_n(s)$ and where $\tilde{u}(s) = u(\varphi_n^0(s))$. Now, if we define

$$\overline{J}_n(s_1, z, \tilde{u}) = \int_{s_1}^{1} \big(g(y_n(s)) + \frac{1}{2}|\tilde{u}(s)|^2\big) \frac{d\varphi_n^0}{ds}(s)\, ds \tag{2.12}$$

by construction we get $J_n(\tau, z, u) = \overline{J}_n(s_1, z, \tilde{u})$, and, since $\mathcal{U} \equiv L^2(0,T) = L^2(0,1; \frac{d\varphi_n^0}{ds}(s)ds)$, we can conclude that

$$v_n(\tau, z) = \overline{v}_n(s_1, z) \tag{2.13}$$

where

$$\overline{v}_n(s_1, z) = \inf_{\tilde{u} \in \mathcal{U}} \overline{J}_n(s_1, z, \tilde{u}). \tag{2.14}$$

Remark 2.4. We underline that, since $\tau = \varphi_n^0(s_1)$ if and only if $s_1 = w_n(\tau)$, (2.13) can be written as the relationship between the state equations (see Remark 2.3), i.e.,

$$v_n(\tau, z) = \overline{v}_n(w_n(\tau), z). \tag{2.15}$$

Concerning the Hamilton-Jacobi equation of the problem (2.11)-(2.12) we can prove the following.

Theorem 2.5. *Let $\alpha_n(t)$ in hypothesis* (2.5) *be in $L^\infty(0,T)$. The value function $\overline{v}_n$ is the viscosity solution of the following Hamilton-Jacobi equation*

$$\begin{cases} -\frac{\partial}{\partial s}\overline{v}_n(s_1, z) + \overline{H}_n(s_1, z, \frac{\partial}{\partial y}\overline{v}_n(s_1, z)) = 0 & \text{in } (0,1) \times \mathbb{R} \\ \overline{v}_n(1, z) = 0 & \text{in } \mathbb{R} \end{cases} \tag{2.16}$$

where the Hamiltonian $\overline{H}_n : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is

$$\overline{H}_n(s_1, z, p) = \big(-p\, a_n(\varphi_n^0(s_1), z) - g(z)\big) \frac{d\varphi_n^0}{ds}(s_1) + \frac{1}{2}p^2 b_n(\varphi_n^0(s_1))^2 \frac{d\varphi_n^0}{ds}(s_1).$$

(The proof of Theorem 2.5 will be given in Section 5.) Furthermore, since by definition $b_n(\varphi_n^0(\cdot))^2 = \dot{B}_n(\varphi_n^0(\cdot))$ using (2.9) we get

$$b_n(\varphi_n^0(s))^2 (\frac{d\varphi_n^0}{ds}(s)) = \frac{d\varphi_n^1}{ds}(s)\ \forall s \in [0,1], \tag{2.17}$$

so that the Hamiltonian can be written in the simpler form

$$\overline{H}_n(s_1, z, p) = \big(-p\, a_n(\varphi_n^0(s_1), z) - g(z)\big) \frac{d\varphi_n^0}{ds}(s_1) + \frac{1}{2}p^2 \frac{d\varphi_n^1}{ds}(s_1). \tag{2.18}$$

Remark 2.6. If we observe that

$$\overline{H}_n(s_1, z, p) = H_n(\tau, z, p)\frac{d\varphi_n^0}{ds}(s_1)$$
$$-\frac{\partial \overline{v}_n}{\partial s}(s_1, z) = -\frac{\partial v_n}{\partial t}(\tau, z)\frac{d\varphi_n^0}{ds}(s_1)$$

by an easy calculation we can conclude that v_n is a viscosity solution of (2.7) if and only if $\overline{v}_n$ is a viscosity solution of (2.16). For every fixed n the two problems are then equivalent. □

We consider now the limit problem. The state is given by the solution of the following system

$$\begin{cases} \dot{x}(t) = (a(t, x) + v(t)) \cdot dt + \xi(t) \cdot \mu^s & \text{in } (\tau, T] \\ x(\tau^-) = z. \end{cases} \tag{2.19}$$

For the definition of solution we refer to [9], Definition 2.2 page 744; here we recall only that the solution $x(t) \in BV^-([0, T])$ is implicitly given by

$$x(t) = \int_\tau^t \big(a(s, x(s)) + v(s)\big)ds + \int_{[\tau,t)} \xi(s)d\mu^s, \quad \forall t > \tau, \quad x(t) = z, \quad \forall t \le \tau.$$

The cost functional and the value function are

$$J(\tau, z, U) = \int_\tau^T f(t, x(t), U(t))dt + \int_{[\tau,T)} \frac{1}{2}\xi^2(t)d\mu^s \tag{2.20}$$

$$v(\tau, z) = \inf_{U\in\mathcal{U}} \{J(\tau, z, U)\}, \tag{2.21}$$

where $U(t) = (u(t), v(t), \xi(t))$, so that the control space is now $\mathcal{U} \equiv L^2(0, T)\times L^2(0, T)\times L^2(0, T; d\mu^s)$. We recall that $f(t, y, U) = \frac{|u|^2}{2} + g(y) + \frac{|v-bu|^2}{2(\mu^a-b^2)}$ being μ^a the derivative of the measure μ with respect to the Lebesgue measure.

In order to reparametrize the limit problem we consider $B(t)$ such that $\dot{B}(t) = \mu(t)$ and we construct a *graph completion* of B following [9]. Let $\mathcal{T}$ be a countable subset of $[0, T)$ which contains 0 and all the discontinuity points of B. Furthermore, let $(\psi_t)_{t\in\mathcal{T}}$ be a family of Lipschitz continuous maps from $[0, 1]$ into $\mathbb{R}$ such that

$$\sum_{t\in\mathcal{T}} V_0^1(\psi_t) < \infty \;\text{ and }\; \psi_t(0) = B(t^-)\;\psi_t(1) = B(t^+)\;\forall t \in \mathcal{T} \tag{2.22}$$

(if $t = 0$ we require only $\psi_0(1) = B(0^+)$). Since $\mathcal{T}$ is countable we can write $\mathcal{T} = \{t_i : i \in \mathbb{N}\}$. For every $i \in \mathbb{N}$, let us set $a_i := V_0^1(\psi_{t_i})$, $a := \sum_{i=1}^{\infty} a_i$ and define $\mathcal{W} : [0, T] \to [0, 1]$ as follows

$$\mathcal{W}(t) := \frac{1}{1+a}\big(w(t) + \sum_{t_i<t} a_i\big), \tag{2.23}$$

where $w(t) = \frac{t+V_0^t(B)}{T+V_0^T(B)}$. Let us define the graph completion of B corresponding to the family $(\psi_t)_{t\in\mathcal{T}}$ as follows

$$\varphi(s) = (\varphi^0(s), \varphi^1(s)) = \begin{cases} (t, B(t)) & \text{if } \ s = \mathcal{W}(t) \ \ t \in [0, T] \setminus \mathcal{T} \\ (t_i, \psi_{t_i}\big(\frac{s-\mathcal{W}(t_i)}{[\mathcal{W}](t_i)}\big)) & \text{if } \ s \in [\mathcal{W}(t_i), \mathcal{W}(t_i^+)], \ t_i \in \mathcal{T}. \end{cases} \tag{2.24}$$

Remark 2.7. We remark that if B is a continuous function the graph completion is natural and coincides in fact with the change of variables defined in (2.10), so the reparametrizations of the problems at every fixed n and of the limit problem are exactly the same. □

We consider now the following system on the variable $y(s) : [0, 1] \to \mathbb{R}$:

$$\begin{cases} \frac{dy}{ds}(s) = \big\{a(\varphi^0(s), y(s)) + v(\varphi^0(s))\big\}\frac{d\varphi^0}{ds}(s) + \xi(\varphi^0(s))\tilde{\mu}(s), \quad s \in (s_1, 1] \\ y(s_1) = z, \end{cases} \tag{2.25}$$

where $\tilde{\mu}(s) = \big(\frac{d\varphi^1}{ds}(s) - \mu^a(\varphi^0(s))\frac{d\varphi^0}{ds}(s)\big)$ and $s_1 = \mathcal{W}(\tau)$ (this implies $\varphi^0(s_1) = \tau$ for all $\tau \in [0, T]$).

The following theorem, that will be proved is Section 4, shows us that, using this reparametrization we can get the same relationship as in Remark 2.3 between the solution of (2.19) and of (2.25).

Theorem 2.8. *Let $B \in BV^-([0, T])$ and $(\psi_t)_{t\in\mathcal{T}}$ be a family satisfying* (2.22). *If $y(s)$ is the solution of* (2.25), *the map*

$$x(t) := y(\mathcal{W}(t)) \tag{2.26}$$

is a solution of (2.19).

With the same idea we consider the following cost functional

$$\overline{J}(s_1, z, U) = \int_{s_1}^{1} \big(f(\varphi^0(s), y(s), U(\varphi^0(s))) \frac{d\varphi^0}{ds}(s) + \frac{1}{2}\xi(\varphi^0(s))^2 \tilde{\mu}(s) \big) ds$$

so that the value function will be

$$\overline{v}(s_1, z) = \inf_{U \in \mathcal{U}} \overline{J}(s_1, z, U). \tag{2.27}$$

We observe now that we can prove the same relationship as in (2.15) between the value functions:

Theorem 2.9. *If* $\overline{v}$ *and* v *are respectively defined in* (2.27) *and* (2.21)*, then* $v(\tau, z) = \overline{v}(\mathcal{W}(\tau), z)\ \forall \tau \in [0, T]$.

(The proof will be given leather in this section). The main point is now that we can show that the value function $\overline{v}$ is the viscosity solution of the Hamilton-Jacobi equation related to the reparametrized problem. In fact we can show the following theorem (for the proof see Section 5).

Theorem 2.10. *Assume that the functions* M_n *and* α_n *in* (2.4), (2.5) *are in* $L^\infty(0, T)$ *and satisfy*

$$M_n \to \overline{M} \quad \textit{weakly } L^1(0, T) \tag{2.28}$$

$$\alpha_n(t) \to \overline{\alpha} \quad \textit{weakly } L^1(0, T), \tag{2.29}$$

then $\overline{v}$ *defined in* (2.27) *is the viscosity solution of*

$$\begin{cases} -\frac{\partial}{\partial s}\overline{v}(s_1, z)) + \overline{H}(s_1, z, \frac{\partial}{\partial y}\overline{v}(s_1, z)) = 0 & \textit{in } (0,1) \times \mathbb{R} \\ \overline{v}(1, z) = 0 & \textit{in } \mathbb{R} \end{cases} \tag{2.30}$$

with Hamiltonian

$$\overline{H}(s_1, z, p) = \big(-p\ a(\varphi^0(s_1), z) - g(z) \big) \frac{d\varphi^0}{ds}(s_1) + \frac{1}{2} p^2 \frac{d\varphi^1}{ds}(s_1). \tag{2.31}$$

Theorem 2.10 and Theorem 2.9 motivate the following definition.

Definition 2.11. We say that a function $v(\tau, z) : [0, T] \times \mathbb{R} \to \mathbb{R}$ is the viscosity solution of (1.5) with Hamiltonian (1.4) if there exist a family $(\psi_t)_{t \in \mathcal{T}}$ satisfying (2.22) and a function $\overline{v}(s, z) : [0, 1] \times \mathbb{R} \to \mathbb{R}$ such that $\overline{v}(\mathcal{W}(\tau), z) = v(\tau, z)$ and $\overline{v}(s, z)$ is the viscosity solution of (2.30) with Hamiltonian (2.31) where $\mathcal{W}(\tau)$ and $\varphi(s)$ are respectively given in (2.23) and (2.24).

Remark 2.12. We observe now the term "containing the measure" in the Hamiltonians. At level n, looking at $\overline{H}_n(s_1, z, p)$ and $H_n(\tau, z, p)$ we get

$$\frac{1}{2}p^2\frac{d\varphi_n^1}{ds}(s) = \frac{1}{2}p^2 b_n(\varphi_n^0(s))^2\frac{d\varphi_n^0}{ds}(s).$$

The idea is that our construction gives, in some sense, the same relationship between $\frac{1}{2}p^2\frac{d\varphi^1}{ds}(s)$ and $\frac{1}{2}p^2 \cdot \mu\frac{d\varphi^0}{ds}(s)$. This sense is given in the following Lemma.

Lemma 2.13. *For any function $g \in L^2(0, T; \mu)$,*

$$\int_\tau^T g(t)d\mu = \int_{s_1}^1 g(\varphi^0(s))\frac{d\varphi^1}{ds}(s)ds.$$

Proof. For simplicity we give the proof for $\tau = 0$. First we isolate the contribution of the jumps of B in the first integral to obtain

$$\int_0^T g(t)d\mu(t) = \int_{[0,T]\setminus\mathcal{T}} g(t)d\mu + \sum_{t_i\in\mathcal{T}} g(t_i)\big(B(t_i^+) - B(t_i)\big). \tag{2.32}$$

So, by construction for every $t_i \in \mathcal{T}$ we get

$$\begin{aligned}\int_{\mathcal{W}(t_i)}^{\mathcal{W}(t_i^+)} g(\varphi^0(s))\frac{d\varphi^1}{ds}(s)\, ds &= \int_{\mathcal{W}(t_i)}^{\mathcal{W}(t_i^+)} g(t_i)\frac{d\psi_{t_i}}{ds}\Big(\frac{s-\mathcal{W}(t_i)}{[\mathcal{W}](t_i)}\Big)\,\frac{1}{[\mathcal{W}](t_i)}\, ds\\ &= g(t_i)\big(B(t_i^+) - B(t_i)\big),\end{aligned}$$

then

$$\sum_{t_i\in\mathcal{T}} g(t_i)\big(B(t_i^+) - B(t_i)\big) = \sum_{t_i\in\mathcal{T}}\int_{\mathcal{W}(t_i)}^{\mathcal{W}(t_i^+)} g(\varphi^0(s))\frac{d\varphi^1}{ds}(s)\, ds. \tag{2.33}$$

Now, if $t \in [0, T] \setminus \mathcal{T}$ by the general chain rule for distributional derivatives of function with bounded variation (see Theorem 3.1, page 746 in [9]) we can deduce

$$\begin{aligned}&B(t) = \varphi^1(\mathcal{W}(t)) \Longrightarrow \dot{B}(t) = \frac{d\varphi^1}{ds}(\mathcal{W}(t))\dot{\mathcal{W}}(t)\\ &t = \varphi^0(\mathcal{W}(t)) \Longrightarrow dt = \frac{d\varphi^0}{ds}(\mathcal{W}(t))\dot{\mathcal{W}}(t),\end{aligned}$$

(where the identities on the right are between measures) that imply

$$\int_{[0,T]\setminus\mathcal{T}} g(t)d\mu = \int_{[0,T]\setminus\mathcal{T}} g(t)\dot{B}(t) \tag{2.34}$$

$$= \int_{[0,T]/\mathcal{T}} g(t)\frac{d\varphi^1}{ds}(\mathcal{W}(t))\big(\frac{d\varphi^0}{ds}(\mathcal{W}(t))\big)^{-1}dt = \int_E g(\varphi^0(s))\frac{d\varphi^1}{ds}(s)\, ds,$$

where we have set $t = \varphi^0(s)$ and $E = [0,1] \setminus \cup_{t_i\in\mathcal{T}}[\mathcal{W}(t_i), \mathcal{W}(t_i^+)]$. Since

$$\sum_{t_i\in\mathcal{T}} \int_{\mathcal{W}(t_i)}^{\mathcal{W}(t_i^+)} g(\varphi^0(s))\frac{d\varphi^1}{ds}(s)ds + \int_E g(\varphi^0(s))\frac{d\varphi^1}{ds}(s)ds = \int_0^1 g(\varphi^0(s))\frac{d\varphi^1}{ds}(s)ds$$

recalling (2.32), (2.33) and (2.34) we conclude the proof.

Proof of Theorem 2.9. Fix $u \in \mathcal{U}$, by definition

$$\overline{J}(\mathcal{W}(\tau), z, U) = \int_{\mathcal{W}(\tau)}^1 \big(f(\varphi^0(s), y(s), U(\varphi^0(s)))\frac{d\varphi^0}{ds}(s) + \frac{1}{2}\xi(\varphi^0(s))^2\tilde{\mu}(s)\big)ds$$

$$= \int_\tau^T f(t, x(t), U(t))dt + \int_{[\tau,T)} \frac{1}{2}\xi(t)^2 d\mu - \int_\tau^T \frac{1}{2}\xi(t)^2\mu^a(t)dt,$$

where we set $t = \varphi^0(s)$ and we used Lemma 2.13. Thus by definition (2.20) we obtain

$$\overline{J}(\mathcal{W}(\tau), z, U) = J(\tau, z, U). \tag{2.35}$$

Again by Lemma 2.13 and setting $t = \varphi^0(s)$ we can deduce that

$$\mathcal{U} \equiv L^2(0,T) \times L^2(0,T) \times L^2(0,T;\mu^s)$$

$$= L^2(0,1;\frac{d\varphi^0}{ds}(s)\, ds) \times L^2(0,1;\frac{d\varphi^0}{ds}(s)\, ds)) \times L^2(0,1;\tilde{\mu}(s)\, ds)$$

so that taking the infimum over $U \in \mathcal{U}$ in (2.35) we obtain the desired result.

Remark 2.14. As we said in the introduction, if we suppose (1.1) and (1.3), we have the following convergence for the Hamiltonians: $H_n(\cdot, z, p) \to H(\cdot, z, p)$ weakly$^*\mathcal{M}^+(0,T)$ for every $(z,p) \in \mathbb{R}\times\mathbb{R}$. This implies, for the reparametrized problems the following convergence result.

For every $\psi \in C^0([0,T])$ and for every $(z,p) \in \mathbb{R}\times\mathbb{R}$,

$$\lim_{n\to\infty} \int_0^1 \overline{H}_n(s,z,p)\psi(\varphi_n^0(s))\, ds = \int_0^1 \overline{H}(s,z,p)\psi(\varphi^0(s))\, ds.$$

Remark 2.15. For the sake of simplicity we treated the scalar case. The case of x vector field from $[0,T]$ into $\mathbb{R}^N$ can be treated in a similar way. More precisely the functions $w_n(t)$ and $w(t)$ have to contain all the terms b_i^2 converging to μ_i. Moreover this will be a subcase of the fully nonlinear case that will be analyzed in a forthcoming paper.

3. An example. We describe now an example where all details can be calculated. Our state equation for every fixed n is

$$\begin{cases} \dot{x}_n(t) = b_n(t)\, u(t) \quad t \in (\tau, 2] \\ x_n(\tau) = z, \end{cases} \tag{3.1}$$

where $\tau \in [0,2]$, $b_n(t) = 1_{]1,1+1/n[}(t)\sqrt{n}$ and $u \in L^2(0,2) = \mathcal{U}$. As usual we denote by $1_E(t)$ the function

$$1_E(t) = \begin{cases} 1 & t \in E \\ 0 & t \notin E. \end{cases}$$

As cost functional we take

$$J_n(\tau, z, u) = \int_\tau^2 \big(|x_n(t)|^2 + \frac{1}{2}|u(t)|^2\big) dt$$

so that the value function will be a solution of the Hamilton-Jacobi equation (2.7) with Hamiltonian

$$H_n(\tau, z, p) = -|z|^2 + \frac{1}{2} p^2 b_n(\tau)^2. \tag{3.2}$$

The Γ-convergence theory gives us the following limit problem (see [8]). The state equation is the following equality between measures

$$\begin{cases} \dot{x}(t) = c \cdot \delta_1(t) \quad \text{in } (\tau, 2] \\ x(\tau^-) = z \end{cases}$$

where $c \in \mathbb{R}$ and δ_1 is the Dirac measure for $t = 1$. The cost functional is

$$\begin{aligned} J(\tau, z, c) &= \int_\tau^2 |x(t)|^2 dt + \int_{[\tau,2)} \frac{c^2}{2} d\delta_1(t) \\ &= \begin{cases} |z|^2(1-\tau) + |z+c|^2 + \frac{1}{2}c^2 & 0 \le \tau \le 1 \\ |z|^2(2-\tau) & 1 < \tau \le 2 \end{cases} \end{aligned} \tag{3.3}$$

which is discontinuous for $\tau = 1$ if $c \neq 0$. Moreover, taking the infimum over $c \in \mathbb{R}$, we get

$$v(\tau, z) = \begin{cases} |z|^2(\frac{4}{3} - \tau) & 0 \le \tau \le 1 \\ |z|^2(2 - \tau) & 1 < \tau \le 2 \end{cases}$$

which is discontinuous for $\tau = 1$!

We want to see if $v(\tau, z)$ is a viscosity solution of the Hamilton Jacobi equation (1.5) with Hamiltonian $H(\tau, z, p) = -|z|^2 + \frac{1}{2}p^2 \cdot \delta_1$ following the Definition 2.11. So we look for a function $\overline{v}(s, z)$ that will be the value function of the reparametrized limit problem. We consider then

$$B(t) = \begin{cases} 0 & 0 \le t \le 1 \\ 1 & 1 < t \le 2 \end{cases}$$

we have $\dot{B}(t) = \delta_1(t)$ and the graph completion is

$$w(t) = \frac{t + V_0^t(B)}{T + V_0^T(B)} = \begin{cases} \frac{t}{3} & 0 \le t \le 1 \\ \frac{1}{3}(t+1) & 1 < t \le 2 \end{cases}$$

which is discontinuous in $t = 1$. So for the calculation of φ we have to use the following definition

$$\varphi(s) = \begin{cases} (t, B(t)) & \text{if} \quad s = w(t) \\ (t, B(t) + \frac{s - w(t)}{[w](t)}[B](t)) & \text{if} \quad s \in (w(t); w(t^+)). \end{cases}$$

We get

$$\varphi(s) = \begin{cases} (3s, 0) & 0 \le s \le \frac{1}{3} \\ (1, 3s - 1) & \frac{1}{3} < s < \frac{2}{3} \\ (3s - 1, 1) & \frac{2}{3} \le s \le 1. \end{cases}$$

We consider now the following system on the variable $y(s)$:

$$\begin{cases} \frac{dy}{ds}(s) = c\frac{d\varphi^1}{ds}(s) & s \in (s_1, 1] \\ y(s_1) = z. \end{cases} \tag{3.4}$$

As a cost functional we take

$$\overline{J}(s_1, z, c) = \int_{s_1}^{1} \big(|y(s)|^2 \frac{d\varphi^0}{ds}(s) + \frac{1}{2}c^2 \frac{d\varphi^1}{ds}(s)\big) ds$$

so that the value function will be $\overline{v}(s_1, z) = \inf_{c\in\mathbb{R}} \overline{J}(s_1, z, c)$, which is a continuous function. In fact

$$\overline{J}(s_1, z, c) = \begin{cases} 3|z|^2(\frac{1}{3} - s_1) + \frac{1}{2}c^2 + |z + c|^2 & 0 \le s_1 \le \frac{1}{3} \\ \frac{3}{2}c^2(\frac{2}{3} - s_1) + |2c + z - 3cs_1|^2 & \frac{1}{3} < s_1 < \frac{2}{3} \\ 3|z|^2(1 - s_1) & \frac{2}{3} \le s_1 \le 1 \end{cases}$$

and taking the infimum over $c \in \mathbb{R}$ we get

$$\overline{v}(s_1, z) = \begin{cases} |z|^2(\frac{4}{3} - 3s_1) & 0 \le s_1 \le \frac{1}{3} \\ \frac{|z|^2}{(5-6s_1)} & \frac{1}{3} < s_1 < \frac{2}{3} \\ 3|z|^2(1 - s_1) & \frac{2}{3} \le s_1 \le 1. \end{cases}$$

By simple calculations one can verify $\overline{v}(w(\tau), z) = v(\tau, z)$ and that $\overline{v}$ is a solution of the Hamilton-Jacobi equation

$$\begin{cases} -\frac{\partial}{\partial s}\overline{v}(s_1, z)) + \overline{H}(s_1, z, \frac{\partial}{\partial y}\overline{v}(s_1, z)) = 0 & \text{in } (0, 1) \times \mathbb{R} \\ \overline{v}(1, z) = 0 & \text{in } \mathbb{R} \end{cases} \tag{3.5}$$

with Hamiltonian $\overline{H}(s_1, z, p) = -|z|^2 \frac{d\varphi^0}{ds}(s_1) + \frac{1}{2}p^2 \frac{d\varphi^1}{ds}(s_1)$. In fact, since

$$\frac{\partial \overline{v}}{\partial s}(s_1, z) = \begin{cases} -3|z|^2 & 0 \le s_1 \le \frac{1}{3} \\ \frac{6|z|^2}{(5-6s_1)^2} & \frac{1}{3} < s_1 < \frac{2}{3} \\ -3|z|^2 & \frac{2}{3} \le s_1 \le 1 \end{cases}$$

$$\frac{\partial \overline{v}}{\partial y}(s_1, z) = \begin{cases} 2z(\frac{4}{3} - 3s_1) & 0 \le s_1 \le \frac{1}{3} \\ \frac{2z}{(5-6s_1)^2} & \frac{1}{3} < s_1 < \frac{2}{3} \\ 6z(1 - s_1) & \frac{2}{3} \le s_1 \le 1 \end{cases}$$

setting $p = \frac{\partial \overline{v}}{\partial y}$, we obtain that (3.5) is fulfilled. Thus Definition 2.11 is verified.

In order to complete the example we calculate also the reprametrization of the problem at level n. Let $B_n(t)$ such that $\dot{B}_n(t) = b_n(t)^2$, i.e.,

$$B_n(t) = \begin{cases} 0 & 0 \le t \le 1 \\ n(t - 1) & 1 < t < 1 + \frac{1}{n} \\ 1 & 1 + \frac{1}{n} \le t \le 2 \end{cases}$$

so the function w_n is given by

$$w_n(t) = \begin{cases} \frac{t}{3} & 0 \le t \le 1 \\ \frac{t}{3}(1+n) - \frac{n}{3} & 1 < t < 1 + \frac{1}{n} \\ \frac{1}{3}(t+1) & 1 + \frac{1}{n} \le t \le 2. \end{cases}$$

Then $\varphi_n(s)$, which by definition is $\varphi_n(s) = (\varphi_n^0(s), \varphi_n^1(s)) = (t, B_n(t))$ if $s = w_n(t)$ becomes

$$\varphi_n(s) = \begin{cases} (3s, 0) & 0 \le s \le \frac{1}{3} \\ (\frac{3s+n}{1+n}, \frac{n}{1+n}(3s-1)) & \frac{1}{3} < s < \frac{2n+1}{3n} \\ (3s-1, 1) & \frac{2n+1}{3n} \le s \le 1. \end{cases}$$

The reparametrized state equation is

$$\begin{cases} \frac{dy_n}{ds}(s) = b_n(\varphi_n^0(s))u(\varphi_n^0(s))\frac{d\varphi_n^0}{ds}(s) & s \in (s_1, 1] \\ y_n(s_1) = z \end{cases}$$

where s_1 is such that $\varphi_n^0(s_1) = \tau$. The raparametrized cost functional is by definition

$$\overline{J}_n(s_1, z, \tilde{u}) = \int_{s_1}^1 \big(|y_n(s)|^2 + \frac{1}{2}|\tilde{u}(s)|^2\big)\frac{d\varphi_n^0}{ds}(s)\, ds$$

while the value function is $\overline{v}_n(s_1, z) = \inf_{\tilde{u}\in U} \overline{J}_n(s_1, z, \tilde{u})$. The Hamilton-Jacobi equation is

$$\begin{cases} -\frac{\partial}{\partial s}\overline{v}_n(s_1, z) + \overline{H}_n(s_1, z, \frac{\partial}{\partial y}\overline{v}_n(s_1, z)) = 0 & \text{in } [0,1] \times \mathbb{R} \\ \overline{v}_n(1, z) = 0 & \text{in } \mathbb{R} \end{cases} \tag{3.6}$$

with Hamiltonian $\overline{H}_n(s_1, z, p) = -|z|^2 \frac{d\varphi_n^0}{ds}(s_1) + \frac{1}{2}p^2\frac{d\varphi_n^1}{ds}(s_1)$. The convergence of the Hamiltonians is in this case stronger than the one of the general framework (see Remark 2.14). In fact, it is clear that the pointwise convergence of the Hamiltonians is a consequence of the following limits

$$\lim_{n\to\infty} \frac{d\varphi_n^0}{ds}(s) = \frac{d\varphi^0}{ds}(s), \qquad \lim_{n\to\infty} \frac{d\varphi_n^1}{ds}(s) = \frac{d\varphi^1}{ds}(s)$$

which can be easily calculated. Thus we have the following result

$$\lim_{n\to\infty} \overline{H}_n(s, z, p) = \overline{H}(s, z, p) \ \ \forall s \in [0, 1].$$

for every fixed $(z, p) \in \mathbb{R} \times \mathbb{R}$.

Remark 3.1. We remark that in this example one can verify also the convergence of the optimal controls and of the optimal trajectories. This is in fact a consequence of the Γ-convergence of the optimal control problems.

4. Proof of Theorem 2.8. This proof is a modification of the proof of Theorem 5.2 page 750 in [9]. The main difference is that since we are considering an unbounded control set our state y it is not a Lipschitz function, so the general chain rule for distributional derivatives of function with bounded variation (see [9], Theorem 3.1 page 746) does not applies in this case. Anyway in the sequel we prove this result in the one dimensional case also for y absolutely continuous.

Theorem 4.1. *Let $\psi : [0,1] \to \mathbb{R}$ be an absolutely continuous function and let $z : [0,T] \to [0,1]$ be an increasing function. If the map α is defined by $\alpha(t) := \psi(z(t))\ \forall t \in [0,T]$, then*

i) $\alpha \in BV([0,T])$*;*

ii) *the identity of measures*

$$\dot{\alpha} = \widehat{\psi_*(z)}\dot{z} \tag{4.1}$$

holds, where ψ_* denotes any Borel function coinciding with the derivatives $\frac{d\psi}{ds}$ almost everywhere with respect to the Lebesque measure and we define (Volpert's Average Superposition)

$$\widehat{\psi_*(z)}(t) = \int_0^1 \psi_*(z(t^-) + \sigma(z(t^+) - z(t^-)))d\sigma.$$

A fundamental tool in the sequel is the following Lemma.

Lemma 4.2. *Let $u \in BV(\Omega)$ (Ω a open subset of $\mathbb{R}$) and let S_u be the set of all the discontinuity points of u. If $N \subset \mathbb{R}$ is a set of null Lebesgue measure, then $|\dot{z}|(z^{-1}(N) \setminus S_u) = 0$ where $|\dot{z}|$ is the total variation of the measure $\dot{z}$.*

Proof of Theorem 4.1. First, without loss of generality, we may suppose that ψ is an increasing function. Thus, since α is the composition of two increasing functions, i) is proved.

Before proving ii) we underline that (4.1) makes sense in all [0,T]. We first consider the countable set S_z of all the discontinuity points of z. Since

ψ is continuous this set coincides with the set of the discontinuity points of α, so fix $t \in S_z$, we have $\dot\alpha(t) = \big(\alpha(t^+) - \alpha(t^-)\big)\delta_t$. Always by the continuity of ψ we can say that $\alpha(t^+) = \psi(z(t^+))$ and $\alpha(t^-) = \psi(z(t^-))$, to get

$$\dot\alpha(t) = \big(\psi(z(t^+)) - \psi(z(t^-))\big)\delta_t = \frac{\psi(z(t^+)) - \psi(z(t^-))}{z(t^+) - z(t^-)}\big(z(t^+) - z(t^-)\big)\delta_t$$
$$= \big(\int_0^1 \frac{d\psi}{ds}(z(t^-) + \sigma(z(t^+) - z(t^-)))d\sigma\big)\big(z(t^+) - z(t^-)\big)\delta_t = \widehat{\frac{d\psi}{ds}(z)}(t)\dot z(t)$$

by an easy calculation and recalling the definition of Volpert's Average Superposition. This give us the result for all points $t \in S_z$.

Let $E = \big\{s \in [0,1] : \nexists \ \frac{d\psi}{ds}\big\}$, by Lemma 4.2, we have $|\dot z|(z^{-1}(E) \setminus S_z) = 0$ and $|\dot\alpha|(\alpha^{-1}(M) \setminus S_z) = 0$, where $M = \psi(E)$.

We approximate the function ψ as follows. We set

$$g_n = \frac{d\psi}{ds} \wedge n \quad \text{and} \quad \psi_n(s) = \int_0^s g_n(s)ds + \psi(0),$$

hence,

$$\psi_n \text{ is a Lipschitz function on } [0,1], \tag{4.2}$$
$$\lim_{n\to\infty} \psi_n(s) = \psi(s) \text{ uniformly in } [0,1], \tag{4.3}$$
$$\frac{d\psi_n}{ds} = g_n \quad \text{a.e.} \quad \text{and} \quad \frac{d\psi}{ds} = \sup_{n\in\mathbb{N}} g_n. \tag{4.4}$$

Now setting $\alpha_n(t) = \psi_n(z(t))$ for every $n \in \mathbb{N}$, by (4.2) the general chain rule, we always refer to Theorem 3.1 of [9], can be applied to obtain $\dot\alpha_n = \widehat{\psi_{n,*}(z)}\dot z$ where $\psi_{n,*}$ denotes any Borel function coinciding with the derivatives $\frac{d\psi_n}{ds}$ almost everywhere with respect to the Lebesque measure. (We remark that we can give a sense at this equality in all [0,T] as we have done for the (4.1)). Thus, to obtain the result we have to show

$$\dot\alpha_n \to \dot\alpha \quad \text{weak}^* \ \mathcal{M}(0,1) \tag{4.5}$$
$$\widehat{\psi_{*,n}(z)}\dot z \to \widehat{\psi_*(z)}\dot z \quad \text{weak}^* \ \mathcal{M}(0,1), \tag{4.6}$$

then the uniqueness of the weak limit gives us (4.1) which completes the proof.

To prove (4.5), we fix $\phi \in C_c^\infty(0,1)$, by definition of distributional derivatives

$$\int_{(0,1)} \phi(t) d\dot{\alpha}_n = -\int_{(0,1)} \phi'(t)\psi_n(z(t))dt,$$

(4.3) implies then

$$\lim_{n\to\infty}\int_{(0,1)} \phi'(t)\psi_n(z(t))dt = \int_{(0,1)} \phi'(t)\psi(z(t))dt$$

and, again by definition of distributional derivatives (using also i)),

$$-\int_{(0,1)} \phi'(t)\psi(z(t))dt = \int_{(0,1)} \phi(t)d\dot{\alpha}.$$

So, for every $\phi \in C_c^\infty(0,1)$

$$\lim_{n\to\infty}\int_A \phi(t)d\dot{\alpha}_n = \int_A \phi(t)d\dot{\alpha},$$

and, since by the uniform convergence of the ψ_n the $|\dot{\alpha}_n|$ are equibounded, this convergence is equivalent to the weak$^*\mathcal{M}(0,1)$.

In order to prove (4.6) we fix $\phi \in C_0(0,1)$ (which is the closure of $C_c(0,1)$ in the sup norm), by definition

$$\begin{aligned}&\int_{(0,1)} \phi(t)\widehat{\psi_{n,*}(z)}(t)\dot{z} \\ &= \int_{(0,1)\setminus S_z} \phi(t)\psi_{n,*}(z(t))\dot{z} + \sum_{t\in S_z}\big(\psi_n(z(t^+)) - \psi_n(z(t^-))\big)\delta_t.\end{aligned} \tag{4.7}$$

Let $F_n^1 = \big\{s \in [0,1] : \not\exists \ \frac{d\psi_n}{ds}\big\}$, $F_n^2 = \big\{s \in [0,1] : \frac{d\psi_n}{ds} \neq g_n\big\}$, $F_n^3 = \big\{s \in [0,1] : \frac{d\psi_n}{ds} \neq \psi_{n,*}\big\}$, $F^1 = \big\{s \in [0,1] : \frac{d\psi}{ds}(s) \neq \psi_*\big\}$, $F^2 = \big\{s \in [0,1] : \lim_{n\to\infty} g_n(s) \neq \frac{d\psi}{ds}(s)\big\}$, $K = F^1 \cup E \cup F^2 \cup \big(\cup_{n\in\mathbb{N}}(F_n^1 \cup F_n^2 \cup F_n^3)\big)$. Since the Lebesque measure of K is zero applying Lemma 4.2, we get

$$\int_{(0,1)\setminus S_z} \phi(t)\psi_{n,*}(z(t))\dot{z} = \int_{(0,1)\setminus(S_z\cup z^{-1}(K))} \phi(t)\frac{d\psi_n}{ds}(z(t))\dot{z}$$

then by the (4.4) we have

$$\lim_{n\to\infty}\int_{(0,1)\setminus(S_z\cup z^{-1}(K))} \phi(t)\frac{d\psi_n}{ds}(z(t))\dot{z} = \int_{(0,1)\setminus(S_z\cup z^{-1}(K))} \phi(t)\frac{d\psi}{ds}(z(t))\dot{z}$$

and applying the Lemma 4.2 again we get

$$\int_{(0,1)\setminus(S_z\cup z^{-1}(K))}\phi(t)\frac{d\psi}{ds}(z(t))\dot{z} = \int_{(0,1)\setminus S_z}\phi(t)\psi_*(z(t))\dot{z}.$$

Thus, letting $n\to\infty$ in (4.7) and recalling also (4.3), we obtain

$$\lim_{n\to\infty}\int_{(0,1)}\phi(t)\widehat{\psi_{n,*}(z)}(t)\dot{z}$$
$$= \int_{(0,1)\setminus S_z}\phi(t)\psi_*(z(t))\dot{z} + \sum_{t\in S_z}\big(\psi(z(t^+))-\psi(z(t^-))\big)\delta_t = \int_{(0,1)}\phi(t)\widehat{\psi_*(z)}(t)\dot{z}$$

for every fixed $\phi\in C_0(0,1)$, which is the (4.6) and completes the proof.

Proof of Lemma 4.2. For all the notation in this proof we refer to [1]. If $u\in BV$, the co-area formula (see [1], Theorem 1.3.8) gives us

$$|\dot{u}|(u^{-1}(N)\setminus S_u) = \int_{-\infty}^{+\infty}P(\{z>t\},u^{-1}(N)\setminus S_u)dt \tag{4.8}$$

(where $P(\{z>t\},\Omega)$ is the perimeter of the set of $x\in\Omega$ such that $z(x)>t$). If we show that

$$\int_{-\infty}^{+\infty}P(\{z>t\},u^{-1}(N)\setminus S_u)dt \le \int_{-\infty}^{+\infty}\mathcal{H}^0(\{u=t\}\cap(u^{-1}(N)\setminus S_u))dt \tag{4.9}$$

where $\mathcal{H}^0$ is the null-dimensional Hausdorff measure, to conclude we have only to show that $\mathcal{H}^0(\{u=t\}\cap(u^{-1}(N)\setminus S_u))=0$ for a.e. $t\in[0,T]$. This is equivalent to show that for almost every t and for all $x\in u^{-1}(N)\setminus S_u$, $u(x)\neq t$ which is true because the Lebesgue measure of E is zero (we recall that $E=\big\{s\in[0,1]:\nexists\ \frac{d\psi}{ds}\big\}$).

Our aim is then to prove the (4.9). If t is such that $A=\{u>t\}$ has finite perimeter (which is true for almost every t), we have following inequality between measures $|D\chi_A|\le\mathcal{H}^0\llcorner\partial A$ where ∂A is the topological boundary of A. So, for every Borel set B, $P(A;B\setminus S_u)\le\mathcal{H}^0(\partial A\cap(B\setminus S_u))$. Now it is easy to see that $\partial\{u>t\}\setminus S_u=\{u=t\}\setminus S_u$ so we can conclude that $P(\{u>t\},u^{-1}(N)\setminus S_u)\le\mathcal{H}^0(\{u=t\}\cap(u^{-1}(N)\setminus S_u))$ for almost every t. This implies the (4.9) which completes the proof.

Proof of Theorem 2.8. Let y be the solution of (2.25); if we set $x(\cdot):=y(\mathcal{W}(\cdot))$ we want to show that it is a solution of (2.19). By definition y is an absolutely continuous function from $[0,1]$ into $\mathbb{R}$ and $\mathcal{W}(t)$ is an increasing function from $[0,T]$ into $[0,1]$, thus we can apply Theorem 4.1 to get the following identity between measures

$$\dot{x} = \widehat{\frac{dy}{ds}(\mathcal{W})}\dot{\mathcal{W}}. \tag{4.10}$$

Fix $A \subset [0,T] \setminus \mathcal{T}$, then (2.25) implies

$$\int_A \dot{x} = \int_A \frac{dy}{ds}(\mathcal{W})(t)\dot{\mathcal{W}}(t) = \int_A \Big(a(\varphi^0(\mathcal{W}(t)), y(\mathcal{W}(t))) \\ + v(\varphi^0(\mathcal{W}(t)))\Big)\frac{d\varphi^0}{ds}(\mathcal{W}(t))\dot{\mathcal{W}} + \int_A \xi(\varphi^0(\mathcal{W}(t)))\tilde{\mu}(\mathcal{W}(t))\dot{\mathcal{W}}.$$

Since from the general chain rule we can deduce for $t \in A$

$$B(t) = \varphi^1(\mathcal{W}(t)) \Longrightarrow \dot{B}(t) = \frac{d\varphi^1}{ds}(\mathcal{W}(t))\dot{\mathcal{W}}(t), \\ t = \varphi^0(\mathcal{W}(t)) \Longrightarrow dt = \frac{d\varphi^0}{ds}(\mathcal{W}(t))\dot{\mathcal{W}}(t),$$

we get (also using the definition of x)

$$\int_A \dot{x} = \int_A (a(t,x(t)) + v(t))\, dt + \int_A \xi(t) d\mu^s(t). \tag{4.11}$$

Let $t_j \in]0,T[\cap \mathcal{T}$, again by (4.10) we have

$$\dot{x}(t_j) = \widehat{\frac{dy}{ds}(\mathcal{W})}(t_j)\dot{\mathcal{W}}(t_j). \tag{4.12}$$

By definition

$$\widehat{\frac{dy}{ds}}(\mathcal{W})(t_j) = \int_0^1 \frac{dy}{ds}(\mathcal{W}(t_j) + \sigma\dot{\mathcal{W}}(t_j))\, d\sigma.$$

Now, if $\sigma \in [0,1]$ then $s = \mathcal{W}(t_j) + \sigma\dot{\mathcal{W}}(t_j) \in [\mathcal{W}(t_j), \mathcal{W}(t_j^+)]$ and since y is the solution of (2.25), we have

$$\widehat{\frac{dy}{ds}(\mathcal{W})}(t_j) = \int_0^1 \frac{dy}{ds}(\mathcal{W}(t_j) + \sigma\dot{\mathcal{W}}(t_j))d\sigma = \int_0^1 \xi(t_j)\frac{d\varphi^1}{ds}\big(\mathcal{W}(t_j) + \sigma\dot{\mathcal{W}}(t_j)\big)d\sigma \\ = \int_0^1 \xi(t_j)\frac{d\psi_{t_j}}{ds}(\sigma)\frac{1}{\dot{\mathcal{W}}(t_j)}\ d\sigma = \frac{\xi(t_j)}{\dot{\mathcal{W}}(t_j)}\big(B(t_j^+) - B(t_j^-)\big),$$

where we used the definition of φ^1. So, from (4.12) we obtain for every $t_j \in \mathcal{T}$,

$$\dot{x}(t_j) = \xi(t_j)\big(B(t_j^+) - B(t_j^-)\big).$$

This identity and the (4.11) tell us that x is the solution of (2.19). Observing that if $y(s_1) = z$ then $x(\tau^-) = y(\mathcal{W}(\tau^-)) = y(\mathcal{W}(\tau)) = y(s_1) = z$ the initial condition is also fulfilled.

5. The Hamilton-Jacobi equation. In this section we want to prove the results about the viscosity solutions. We start by giving the result in a general framework. We consider an optimal control problem with state equation

$$\begin{cases} y'(s) = A(s,y) + B(s)U(s) & s \in (t,T] \\ y(t) = x \end{cases} \tag{5.1}$$

where $A : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ is a Borel function, $B \in L^2(0,T;\mathbb{R}^{n\times m})$, $U \in L^2(0,T;\mathbb{R}^m) \equiv \mathcal{U}$ and cost functional

$$J(t,x,U) = \int_t^T f(s, y^{x,t}(s;U), U(s))\, ds \tag{5.2}$$

where $f : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \to [0,+\infty]$ is a Borel function, and we will denote by $y^{x,t}(s;U)$ the solution of (5.1) with initial condition x at time t and control U.

The value function of this problem is then defined as follows

$$v(t,x) = \inf_{U\in\mathcal{U}} \{J(t,x,U)\}. \tag{5.3}$$

We will make the following assumptions :

a) There exists a sequence of functions M_h from $(0,T)$ into $\mathbb{R}$ with $\| M_h \|_1 \leq M < \infty$ such that, for every $t \in (0,T)$

$$|A(t,0)| \leq M_h(t); \tag{5.4}$$

b) there exists a sequence of functions α_h from $(0,T)$ into $\mathbb{R}$ with $\| \alpha_h \|_1 \leq \alpha < \infty$ such that, for every $t \in (0,T)$ and for every $y_1, y_2 \in \mathbb{R}^n$

$$|A(t,y_1) - A(t,y_2)| \leq \alpha_h(t)|y_1 - y_2|; \tag{5.5}$$

c) there exists a function $C(t,v,w)$, from $(0,T)\times\mathbb{R}\times\mathbb{R}$ into $\mathbb{R}$, integrable with respect to t, increasing in v and w such that, for every $U\in\mathbb{R}^m$ and for every $y,z\in\mathbb{R}^n$

$$|f(t,y,U)-f(t,z,U)|\le C(t,|y|,|z|)|y-z|; \tag{5.6}$$

d) there exist two constants $a>0$ and $b\ge 0$ such that, for every $(t,y,u)\in(0,T)\times\mathbb{R}^n\times\mathbb{R}^m$

$$f(t,y,U)\ge a|U|^2-b; \tag{5.7}$$

e) there exists a control function $u_0\in\mathbb{R}^m$ such that, for every $y\in W^{1,1}(0,T;\mathbb{R}^n)$

$$f(\cdot,y(\cdot),u_0)\in L^1(0,T). \tag{5.8}$$

We recall that in this framework for every fixed control U and initial condition x at time t there exists a unique absolutely continuous function y solution of (5.1). Our aim is to prove that the value function is a viscosity solution of the following Hamilton-Jacobi equation

$$\begin{cases} -\frac{\partial}{\partial t}v(t,x)+H(t,x,Dv(t,x))=0 & \text{in } (0,T)\times\mathbb{R}^n \\ v(T,x)=0 & \text{in } \mathbb{R}^n \end{cases} \tag{5.9}$$

where $Dv(t,x)$ denotes the gradient with respect to the x-variables of $v(t,x)$, and the Hamiltonian is given by

$$H(t,x,p)=\sup_{U\in\mathbb{R}^m}\big\{-p\cdot(A(t,x)+B(t)U)-f(t,x,U)\big\}. \tag{5.10}$$

We use the following definition of viscosity solution given by Ishii in [11].

Definition 5.1. Let $v(t,x)$ be a continuous function on $(0,T)\times\mathbb{R}^n$. We say that $v(t,x)$ is a *viscosity subsolution* of (5.9) in $(0,T)\times\mathbb{R}^n$ if the following condition is satisfied. If $b\in L^1(0,T)$, $F(t,x,p)\in C([0,T]\times\mathbb{R}^n\times\mathbb{R}^n)$, and F satisfies for some $\delta>0$, $(t_0,x_0)\in(0,T)\times\mathbb{R}^n$, $\phi\in C^1([0,T]\times\mathbb{R}^n)$,

$$-b(t)+F(t,x,p)\le H(t,x,p) \tag{5.11}$$

for $|x-x_0|<\delta$, $|p-D\phi(t_0,x_0)|<\delta$, a.e. $|t-t_0|<\delta$ and if $v(t,x)+\int_0^t b(s)ds-\phi(t,x)$ has a local maximum at (t_0,x_0), then

$$-\frac{\partial}{\partial t}\phi(t_0,x_0)+F(t_0,x_0,D\phi(t_0,x_0))\le 0.$$

We say that $v(t,x)$ is a *viscosity supersolution* of (5.9) in $(0,T)\times\mathbb{R}^n$ if the following condition is satisfied. If $b \in L^1(0,T)$, $F(t,x,p) \in C([0,T]\times\mathbb{R}^n\times\mathbb{R}^n)$, and F satisfies for some $\delta>0$, $(t_0,x_0)\in(0,T)\times\mathbb{R}^n$, $\phi\in C^1([0,T]\times\mathbb{R}^n)$,

$$-b(t)+F(t,x,p)\geq H(t,x,p) \tag{5.12}$$

for $|x-x_0|<\delta$, $|p-D\phi(t_0,x_0)|<\delta$, a.e. $|t-t_0|<\delta$ and if $v(t,x)+\int_0^t b(s)ds-\phi(t,x)$ has a local minimum at (t_0,x_0), then

$$-\frac{\partial}{\partial t}\phi(t_0,x_0)+F(t_0,x_0,D\phi(t_0,x_0))\geq 0.$$

Finally, we say that $v(t,x)$ is a *viscosity solution* of (5.9) in $(0,T)\times\mathbb{R}^n$ if it is both viscosity subsolution and viscosity supersolution of (5.9) in $(0,T)\times\mathbb{R}^n$ and if $v(T,x)=0$ for every $x\in\mathbb{R}^n$.

The main result is the following.

Theorem 5.2. *If we suppose* (5.4)-(5.8), *then* $v(t,x)$ *is a viscosity solution of* (5.9) *in* $(0,T)\times\mathbb{R}^n$.

We need to prove this theorem because in the case of measurable time dependent Hamiltonians it does not fit into classical results (see for example [10], Section II.7). In the proof of Theorem 5.2 we will need some preliminary results that we state below.

Lemma 5.3. *Under the hypotheses* (5.4) *and* (5.5), *the solution* $y^{x,t}$ *of the state equation* (5.1) *satisfies the inequality*

$$|y^{x,t}(s)|\leq C_h(|x|,U)\ \forall s\in[t,T], \tag{5.13}$$

where $A_h=1+\parallel\alpha_h\parallel_1 e^{\|\alpha_h\|_1}$ *and* $C_h(|x|,U)=A_h\{|x|+M+\|B(\cdot)U(\cdot))\|_1\}$.

Moreover, if we denote by $y^i(t,U)$ *the solution of* (5.1) *with initial condition* $y(t_i)=x_i$ *for* $i=0,1$, *we have*

$$|y^0(t,U)-y^1(t,U)|\leq A_h\left\{|x_0-x_1|+\int_{t_0}^{t_1}\gamma_h(s,|x_0|,U)\,ds\right\}\ \textit{if}\ \ t_0<t_1$$
$$|y^1(t,U)-y^0(t,U)|\leq A_h\left\{|x_0-x_1|+\int_{t_1}^{t_0}\gamma_h(s,|x_1|,U)\,ds\right\}\ \textit{if}\ \ t_1<t_0,$$

where $\gamma_h(s,|x|,U)=\alpha_h(s)C_h(|x|,U)+M_h(s)+|B(s)U(s)|$.

Proof. The proof is a standard application of Gronwall's Lemma and it is left to the reader.

Remark 5.4. It is easy to see that $C_h(|x|, U)$ and $\gamma_h(s, |x|, U)$ are both increasing functions on the variable $|x|$, and $\gamma_h(\cdot, x, U) \in L^1(0, T)$ for every fixed $(x, U) \in \mathbb{R}^n \times \mathbb{R}^m$. $\square$

A basic (and standard) tool in the proof of Theorem 5.2 is the Dynamic Programming Principle. Also if our assumption does not include the continuity of $A(\cdot, x)$, $B(\cdot)U(\cdot)$, and $f(\cdot, y, U)$ the standard proof works in this framework so we will not give it here (see, for example [10], page 9).

Theorem 5.5. (Dynamic Programming Principle). *If* (5.4) *and* (5.5) *are fulfilled and the space of controls is* $\mathcal{U} = L^2(0, T; \mathbb{R}^m)$*, then for every* $\tau \in [t, T]$

$$v(t, x) = \inf_{U \in \mathcal{U}} \left\{ \int_t^\tau f(s, y^{x,t}(s; U), U(s))\, ds + v(\tau, y^{x,t}(\tau)) \right\}. \tag{5.14}$$

From now on we will denote by $\omega(\delta, x)$ a function such that $\lim_{\delta \to 0} \omega(\delta, x) = 0$ and that is monotone increasing on $|x|$ and by $\omega(\delta)$ a function such that $\lim_{\delta \to 0} \omega(\delta) = 0$. In the same way we will denote by $K(|x|)$ a function monotone increasing on $|x|$ and by K an arbitrary constant. We remark that ω and K may possibly vary from line to line.

Lemma 5.6. *Under the hypotheses* (5.4)-(5.8)*, for every* $t \in [0, T]$*, for every* $x \in \mathbb{R}^n$*, for every* $\varepsilon > 0$ *if the control* u_ε *is such that*

$$v(t, x) + \varepsilon > \int_t^T f(s, y^{x,t}(s, u_\varepsilon), u_\varepsilon(s)) ds, \tag{5.15}$$

then we have the following estimate

$$\| B(\cdot) u_\varepsilon(\cdot) \|_{L^1(t,T)} \leq K(|x|) + K\varepsilon + K, \tag{5.16}$$

where the constant K *does not depend on* u_ε *and* t.

Proof. We first verify that $v(t, x)$ is locally bounded in $(0, T) \times \mathbb{R}^n$. Let u_0 given by hypothesis (5.8), by definition and by (5.6) we can write

$$\begin{aligned} v(t, x) &\leq \int_t^T |f(s, y^{x,t}(s; u_0), u_0) - f(s, 0, u_0)|\, ds + \int_t^T f(s, 0, u_0)\, ds \\ &\leq \int_t^T C(s, |y^{x,t}(s; u_0)|, 0) |y^{x,t}(s; u_0)|\, ds + \int_t^T f(s, 0, u_0)\, ds. \end{aligned}$$

Applying (5.13) in Lemma 5.3, we have that

$$v(t,x) \le \int_t^T C(s, C_h(|x|, u_0), 0) C_h(|x|, u_0)\, ds + \int_t^T f(s, 0, u_0)\, ds$$

which, together with Remark 5.4, gives us the locally boundedness of $v(t,x)$ in $(0,T) \times \mathbb{R}^n$, i.e.,

$$v(t,x) \le K(|x|). \tag{5.17}$$

Let u_ε satisfy (5.15), by (5.7)

$$\begin{aligned}
\int_t^T |B(s)u_\varepsilon(s))| ds &\le \int_t^T \Big(\frac{1}{2}|B(s)|^2 + \frac{1}{2}|u_\varepsilon(s)|^2\Big) ds \\
&\le \frac{1}{2} \parallel B(s) \parallel_{L^2(0,T)}^2 + \frac{1}{2a} \int_t^T f(s, y^{x,t}(s, u_\varepsilon), u_\varepsilon(s)) ds + \frac{bT}{a} \\
&< \frac{1}{2} \parallel B(s) \parallel_{L^2(0,T)}^2 + \frac{1}{2a}(v(t,x) + \varepsilon) + \frac{bT}{a} \\
&\le \frac{1}{2} \parallel B(s) \parallel_{L^2(0,T)}^2 + \frac{1}{2a}(K(|x|) + \varepsilon) + \frac{bT}{a} = K + K(|x|) + K\varepsilon,
\end{aligned}$$

where we used also (5.17). □

We are now in position to prove that the value function is continuous in both variables.

Theorem 5.7. *If we suppose* (5.4)-(5.8), *then the value function* $v(t,x)$ *is a continuous function on* $[0,T] \times \mathbb{R}^n$.

Proof. We split the proof in 3 steps.

Step 1. (Continuity in x). Fix $\varepsilon > 0$ and let u_ε be a control such that

$$v(t,w) + \varepsilon > \int_t^T f(s, y_\varepsilon^{w,t}(s), u_\varepsilon(s)) ds,$$

where $y_\varepsilon^{w,t}(s)$ is the solution of (5.1) with control u_ε and initial condition w at time t. By definition, we have

$$v(t,x) \le \int_t^T f(s, y_\varepsilon^{x,t}(s), u_\varepsilon(s)) ds,$$

where similarly $y_\varepsilon^{x,t}(s)$ is the solution of (5.1) with control u_ε but initial condition x at time t. Then by (5.6) and using Lemma 5.3,

$$\begin{aligned} v(t,x) - v(t,w) &\le \int_t^T f(s, y_\varepsilon^{x,t}(s), u_\varepsilon(s))ds - \int_t^T f(s, y_\varepsilon^{w,t}(s), u_\varepsilon(s))ds + \varepsilon \\ &\le A_h|w-x| \int_t^T C(s, C_h(|w|, u_\varepsilon), C_h(|x|, u_\varepsilon))ds + \varepsilon. \end{aligned}$$

By definition

$$\begin{aligned} C_h(|w|, u_\varepsilon) &= A_h\{|w| + M + \| B_h(\cdot)u_\varepsilon(\cdot) \|_{L^1(t,T)}\} \\ &\le A_h\{|w| + M + K + K(|x|) + K\varepsilon\} \end{aligned}$$

by Lemma 5.6. Since w tends to x we can suppose that there exists a constant K such that $|x| + |x-w| \le K$, then, using this in the previous estimate we get

$$C_h(|w|, u_\varepsilon) \le K(|x|, |x-w|) + K\varepsilon.$$

With a similar argument we have also

$$C_h(|x|, u_\varepsilon) \le A_h\{|x| + M + K + K(|x|) + K\varepsilon\} = K(|x|) + K\varepsilon.$$

Our estimate becomes

$$\begin{aligned} v(t,x) - v(t,w) &\le A_h|w-x| \int_t^T C(s, K(|x|, |x-w|) + K\varepsilon, K(|x|) + K\varepsilon))ds + \varepsilon \\ &\le A_h|w-x| \int_0^T C(s, K(|x|, |x-w|) + K\varepsilon, K(|x|) + K\varepsilon))ds + \varepsilon. \end{aligned}$$

Let ε going to zero we obtain, using also hypothesis (5.6),

$$\begin{aligned} v(t,x) - v(t,w) &\le A_h|w-x| \int_0^T C(s, K(|x|, |x-w|), K(|x|))ds \\ &= K(|x|, |x-w|)|w-x|. \end{aligned}$$

With a similar argument we can easily get the reverse inequality, to obtain the following estimate

$$|v(t,x) - v(t,w)| \le K(|x|, |w-x|)|w-x| \tag{5.18}$$

which gives us the continuity in x of the value function.

Step 2. (Continuity in t). We begin proving the following inequality:

$$v(t,x) - v(t+\delta,x) \le K(|x|)\omega(\delta,x). \tag{5.19}$$

We consider first the case $\delta > 0$. By Dynamic Programming Principle we have the following inequality:

$$v(t,x) \le \int_t^{t+\delta} f(s,y^{x,t}(s;u_0),u_0)ds + v(t+\delta, y^{x,t}(t+\delta;u_0))$$

where the control u_0 satisfies hypothesis (5.8). Then

$$\begin{aligned} &v(t,x) - v(t+\delta,x) \\ &\quad\le \int_t^{t+\delta} f(s,y^{x,t}(s;u_0),u_0)ds + v(t+\delta, y^{x,t}(t+\delta;u_0)) - v(t+\delta,x). \end{aligned} \tag{5.20}$$

Using hypothesis (5.6) we can estimate

$$\begin{aligned} &\int_t^{t+\delta} f(s,y^{x,t}(s;u_0),u_0)ds \\ &\le \int_t^{t+\delta} \big(f(s,y^{x,t}(s;u_0),u_0) - f(s,0,u_0)\big)\,ds + \int_t^{t+\delta} f(s,0,u_0)ds \\ &\le \int_t^{t+\delta} C(s,|y^{x,t}(x;u_0)|,0)|y^{x,t}(x;u_0)|ds + \int_t^{t+\delta} f(s,0,u_0)ds \\ &\le C_h(|x|,u_0)\int_t^{t+\delta} C(s,C_h(|x|,u_0),0)ds + \int_t^{t+\delta} f(s,0,u_0)ds, \end{aligned}$$

where we used (5.13). Thus, by (5.6) and (5.8) we obtain:

$$\int_t^{t+\delta} f(s,y^{x,t}(s;u_0),u_0)ds \le K(|x|)\omega(\delta,x). \tag{5.21}$$

In order to estimate the second part of (5.20) we need the following inequalities which can be easily deduced from Gronwall's Lemma, and hypotheses (5.4), (5.5):

$$|y^{x,t}(t+\delta;u_0) - x| \le \omega(\delta,x) \quad , \quad |y^{x,t}(t+\delta;u_0) - x| \le K(|x|). \tag{5.22}$$

Using the continuity in x of the value function, in particular (5.18), we can write

$$v(t+\delta, y^{x,t}(t+\delta; u_0)) - v(t+\delta, x) \le |y^{x,t}(t+\delta; u_0) - x| K(|x|, |y^{x,t}(t+\delta; u_0) - x|)$$

that, by (5.22), becomes

$$v(t+\delta, y^{x,t}(t+\delta; u_0)) - v(t+\delta, x) \le K(|x|)\omega(\delta, x). \tag{5.23}$$

Finally we can write the estimate for $\delta > 0$ inserting (5.21) and (5.23) in (5.20)

$$v(t,x) - v(t+\delta, x) \le K(|x|)\omega(\delta, x). \tag{5.24}$$

We consider now the case $\delta < 0$. Fix $\varepsilon > 0$, let u_ε^δ be the control such that

$$v(t+\delta, x) + \varepsilon > \int_{t+\delta}^T f(s, y^{x,t+\delta}(s; u_\varepsilon^\delta), u_\varepsilon^\delta) ds \tag{5.25}$$

by definition of value function we get

$$v(t,x) \le \int_t^T f(s, y^{x,t}(s; u_\varepsilon^\delta), u_\varepsilon^\delta) ds.$$

Then

$$\begin{aligned} & v(t,x) - v(t+\delta, x) \\ & \le \int_t^T f(s, y^{x,t}(s; u_\varepsilon^\delta), u_\varepsilon^\delta) ds - \int_{t+\delta}^T f(s, y^{x,t+\delta}(s; u_\varepsilon^\delta), u_\varepsilon^\delta) ds + \varepsilon \qquad (5.26) \\ & \le \int_t^T \Big(C(s, |y^{x,t}(s; u_\varepsilon^\delta)|, |y^{x,t+\delta}(s; u_\varepsilon^\delta)|) |y^{x,t}(s; u_\varepsilon^\delta) - y^{x,t+\delta}(s; u_\varepsilon^\delta)| \Big) ds + \varepsilon, \end{aligned}$$

where we used (5.6) and the fact that f is a positive function. By Lemma 5.3 and Lemma 5.6 we estimate

$$\begin{aligned} |y^{x,t}(s; u_\varepsilon^\delta)| & \le A_h\big(|x| + M + \| B_h(\cdot) u_\varepsilon^\delta(\cdot) \|_{L^1(t,T)} \big) \\ & \le A_h\big(|x| + M + \| B_h(\cdot) u_\varepsilon^\delta(\cdot) \|_{L^1(t+\delta,T)} \big) \\ & \le A_h(|x| + M + K + K(|x|) + K\varepsilon) = K(|x|) + K\varepsilon \qquad (5.27) \end{aligned}$$

and

$$|y^{x,t+\delta}(s;u_\varepsilon^\delta)| \leq A_h(|x| + M + K + K(|x|) + K\varepsilon) = K(|x|) + K\varepsilon, \quad (5.28)$$

so, using again Lemma 5.3 we have

$$|y^{x,t}(s;u_\varepsilon^\delta) - y^{x,t+\delta}(s;u_\varepsilon^\delta)| \quad (5.29)$$
$$\leq A_h \left\{ \int_{t+\delta}^t \big[\alpha_h(s)(K(|x|) + K\varepsilon) + M_h(s) + |B(s)u_\varepsilon^\delta(s)|\big] ds \right\}.$$

In order to estimate the last integral we calculate using (5.7)

$$\int_{t+\delta}^t |u_\varepsilon^\delta(s)|^2 ds \leq \frac{1}{a} \int_{t+\delta}^t f(s, y^{x,t+\delta}(s;u_\varepsilon^\delta), u_\varepsilon^\delta) ds + \frac{b}{a}(-\delta)$$
$$\leq \frac{1}{a}(v(t+\delta,x) + \varepsilon) + \frac{bT}{a} \leq \frac{1}{a}(K(|x|) + \varepsilon) + \frac{bT}{a},$$

where we used (5.17) and (5.25). Using this we obtain

$$\int_{t+\delta}^t |B(s)u_\varepsilon^\delta(s)| ds \leq \omega(\delta)\big[K + K(|x|) + \frac{\varepsilon}{a}\big]^{1/2}$$

that plugged in (5.29) gives

$$|y^{x,t}(s;u_\varepsilon^\delta) - y^{x,t+\delta}(s;u_\varepsilon^\delta)| \quad (5.30)$$
$$\leq A_h \left\{ \int_{t+\delta}^t \alpha_h(s)(K(|x|) + K\varepsilon) + M_h(s) ds + \omega(\delta)\big[K + K(|x|) + \frac{\varepsilon}{a}\big]^{1/2} \right\}.$$

Inserting (5.27), (5.28), and (5.30) in (5.26), we have

$$v(t,x) - v(t+\delta,x) \leq \int_t^T C(s, K(|x|) + K\varepsilon, K(|x|) + K\varepsilon) ds$$
$$\times \Big(A_h \Big\{ \int_{t+\delta}^t (\alpha_h(s)(K(|x|) + K\varepsilon) + M_h(s)) ds$$
$$+ \omega(\delta)\big[K(|x|) + K\varepsilon + \frac{\varepsilon}{a}\big]^{1/2} \Big\} + \varepsilon \Big)$$

and letting ε go to zero, we obtain

$$v(t,x) - v(t+\delta,x) \leq \int_t^T C(s, K(|x|), K(|x|)) ds$$
$$\times \Big(A_h \Big\{ \int_{t+\delta}^t (\alpha_h(s)K(|x|) + M_h(s)) ds + \omega(\delta)K(|x|) \Big\} \Big).$$

By hypotheses (5.4), (5.5) and (5.6) we can conclude

$$v(t,x) - v(t+\delta,x) \le K(|x|)(\omega(\delta,x) + \omega(\delta)) = K(|x|)\omega(\delta,x)$$

which, with (5.24) gives us the required inequality. With the same arguments we obtain the reverse inequality, i.e.

$$v(t+\delta,x) - v(t,x) \le K(|x|)\omega(\delta,x). \tag{5.31}$$

Step 3. (Final estimate). Let $t \in [0,T]$, $\delta > 0$ and $x, w \in \mathbb{R}^n$, by (5.18), (5.19) and (5.31) we obtain

$$|v(t,x) - v(t+\delta,w)| \le K(|x|)\omega(\delta,x) + K(|x|,|x-w|)|x-w|.$$

This implies

$$\lim_{\delta\to 0, w\to x} |v(t,x) - v(t+\delta,w)| = 0$$

which completes the proof. □

Proof of Theorem 5.2. We observe that by definition of $v(t,x)$, we have $v(T,x) = 0 \ \ \forall x \in \mathbb{R}^n$. In order to show that $v(t,x)$ is a viscosity solution of (5.9) in $(0,T)\times\mathbb{R}^n$ we have to prove that it is a viscosity subsolution and a viscosity supersolution of (5.9) in $(0,T)\times\mathbb{R}^n$.

We start proving that $v(t,x)$ is a viscosity subsolution. Following the definition we take a test function $\phi \in C^1([0,T]\times\mathbb{R}^n)$, $b \in L^1(0,T)$ and a function $F(t,x,p) \in C([0,T]\times\mathbb{R}^n\times\mathbb{R}^n)$ such that (5.11) is fulfilled. Let (t_0,x_0) be a local maximum for $v(t,x) + \int_0^t b(s)ds - \phi(t,x)$, so there exists a $\delta > 0$ such that

$$\begin{aligned} &|t-t_0| \le \delta \quad , \quad |x-x_0| \le \delta \\ &\Rightarrow \int_{t_0}^t b(s)ds + \phi(t_0,x_0) - \phi(t,x) \le v(t_0,x_0) - v(t,x). \end{aligned} \tag{5.32}$$

Let $u_0 \in \mathbb{R}^m$ be fixed satisfying hypothesis (5.8), and we denote by $y^0(t)$ the solution of our state equation (5.1) with initial condition $y(t_0) = x_0$ and control u_0. The Dynamic Programming Principle gives for every $\tau \in [t_0,T]$

$$v(t_0,x_0) \le \int_{t_0}^{\tau} f(t,y^0(t),u_0)\, dt + v(\tau,y^0(\tau)). \tag{5.33}$$

By Gronwall's Lemma we can easily deduce that there exists a $\tau \in]t_0, t_0 + \delta]$ such that $|y^0(t) - x_0| \leq \delta$ for all $t \in]t_0, \tau]$. It follows that in (5.32) we can choose $x = y^0(\tau)$ and $t = \tau$ to get (using also (5.33))

$$\int_{t_0}^{\tau} b(s)ds + \phi(t_0, x_0) - \phi(\tau, y^0(\tau)) \leq \int_{t_0}^{\tau} f(t, y^0(t), u_0)\, dt. \tag{5.34}$$

Since

$$\begin{aligned}\phi(t_0, x_0) - \phi(\tau, y^0(\tau)) &= \phi(t_0, x_0) - \phi(\tau, x_0) + \int_{t_0}^{\tau} -\frac{d}{dt}\phi(\tau, y^0(t))\, dt \\ &= \phi(t_0, x_0) - \phi(\tau, x_0) + \int_{t_0}^{\tau} -D\phi(\tau, y^0(t)) \cdot (A(t, y^0(t)) + B(t)u_0)dt\end{aligned}$$

(5.34) becomes

$$\begin{aligned}\int_{t_0}^{\tau} b(s)ds + \phi(t_0, x_0) - \phi(\tau, x_0) + \int_{t_0}^{\tau} -D\phi(\tau, y^0(t)) \cdot (A(t, y^0(t)) + B(t)u_0)dt \\ \leq \int_{t_0}^{\tau} f(t, y^0(t), u_0)\, dt.\end{aligned}$$

Dividing by $\tau - t_0$ and passing to the limit as $\tau \to t_0$ we have for a.e. t_0

$$b(t_0) - \frac{\partial}{\partial t}\phi(t_0, x_0) + \{-D\phi(t_0, x_0) \cdot (A(t_0, x_0) + B(t_0)u_0 - f(t_0, x_0, u_0)\} \leq 0.$$

Taking the supremum over $U \in \mathbb{R}^m$ and recalling the definition of the Hamiltonian (5.10) we can write

$$b(t_0) - \frac{\partial}{\partial t}\phi(t_0, x_0) + H(t_0, x_0, D\phi(t_0, x_0)) \leq 0 \quad \text{a.e. with respect to } \ t_0$$

which implies

$$-\frac{\partial}{\partial t}\phi(t_0, x_0) + F(t_0, x_0, D\phi(t_0, x_0)) \leq 0$$

and this completes the subsolution proof.

To prove that $v(t, x)$ is a viscosity supersolution of (5.9) in $(0, T) \times \mathbb{R}^n$ we take $\phi \in C^1([0, T] \times \mathbb{R}^n)$, $b \in L^1(0, T)$ and a function $F(t, x, p) \in C([0, T] \times$

$\mathbb{R}^n \times \mathbb{R}^n$) such that (5.12) is fulfilled. Let (t_0, x_0) be a local minimum for $v(t,x) + \int_0^t b(s)ds - \phi(t,x)$, so there exists a $\delta > 0$ such that

$$|t - t_0| \leq \delta \quad , \quad |x - x_0| \leq \delta \tag{5.35}$$
$$\Rightarrow \int_{t_0}^{t} b(s)ds + \phi(t_0, x_0) - \phi(t,x) \geq v(t_0, x_0) - v(t,x).$$

Applying the Dynamic Programming Principle for every $\varepsilon > 0$ there exists a u_ε such that, denoting by $y_\varepsilon^0(t)$ the solution of (5.1) with initial condition $y(t_0) = x_0$ and control u_ε,

$$v(t_0, x_0) + \varepsilon > \int_{t_0}^{\tau} f(t, y_\varepsilon^0(t), u_\varepsilon(t))\, dt + v(\tau, y_\varepsilon^0(\tau)) \tag{5.36}$$

for every $\tau \in [t_0, T]$. Then in (5.35) we have for $|x - x_0| \leq \delta$ and $|t - t_0| \leq \delta$

$$\int_{t_0}^{t} b(s)ds + \phi(t_0, x_0) - \phi(t,x) \tag{5.37}$$
$$\geq \int_{t_0}^{\tau} f(t, y_\varepsilon^0(t), u_\varepsilon(t))\, ds - \varepsilon + v(\tau, y_\varepsilon^0(\tau)) - v(t,x).$$

For any $\tau \in [t_0, T]$, by the Hölder's inequality we have

$$\int_{t_0}^{\tau} |B(t)u_\varepsilon(t))|\, dt \leq \| B(t) \|_{L^2(t_0,\tau)} \| u_\varepsilon(t) \|_{L^2(t_0,\tau)}$$

and from hypotheses (5.6) we deduce then

$$\int_{t_0}^{\tau} |u_\varepsilon(t)|^2\, dt \leq \frac{1}{a} \int_{t_0}^{\tau} f(t, y_\varepsilon^0(t), u_\varepsilon(t))\, dt + \frac{bT}{a} \leq \frac{1}{a}(v(t_0, x_0) + \varepsilon) + \frac{bT}{a},$$

where we used the (5.36). Moreover, since $v(t,x)$ is locally bounded,

$$\int_{t_0}^{\tau} |B(t)u_\varepsilon(t))|\, dt \leq \omega(t_0 - \tau)\big[K(|x_0|) + \frac{\varepsilon}{a} + K\big]^{1/2}. \tag{5.38}$$

Applying Gronwall's Lemma and hypotheses (5.4) and (5.5), by (5.38) it follows that

$$|y_\varepsilon^0(\tau) - x_0| \leq A_h \int_{t_0}^{\tau} (\alpha_h(s)|x_0| + M_h(s))\, ds + \omega(\tau - t_0)\ \big[K(x_0) + \frac{\varepsilon}{a} + K\big]^{1/2}.$$

Since it is not restrictive to suppose $\varepsilon \leq 1$, we can fix a $\tau \in]t_0, t_0+\delta]$ such that $|y^0_\varepsilon(t) - x_0| \leq \delta$ for all $t \in]t_0, \tau]$. So, in the inequality (5.37) we can take $x = y^0_\varepsilon(\tau)$ and $t = \tau$ to obtain

$$\int_{t_0}^{\tau} b(s)ds + \phi(t_0, x_0) - \phi(\tau, y^0_\varepsilon(\tau)) \geq \int_{t_0}^{\tau} f(t, y^0_\varepsilon(t), u_\varepsilon(t))\, dt - \varepsilon. \tag{5.39}$$

Since

$$\begin{aligned}\phi(t_0, x_0) - \phi(\tau, y^0_\varepsilon(\tau)) &= \phi(t_0, x_0) - \phi(\tau, x_0) \\ &\quad - \int_{t_0}^{\tau} D\phi(\tau, y^0_\varepsilon(t)) \cdot (A(t, y^0_\varepsilon(t)) + B(t)u_\varepsilon(t))dt,\end{aligned}$$

(5.39) becomes

$$\begin{aligned}&\int_{t_0}^{\tau} b(s)ds + \int_{t_0}^{\tau} \big[-D\phi(\tau, y^0_\varepsilon(t)) \cdot (A(t, y^0_\varepsilon(t)) + B(t)u_\varepsilon(t)) - f(t, y^0_\varepsilon(t), u_\varepsilon(t))\big]\, dt \\ &\qquad + \phi(t_0, x_0) - \phi(\tau, x_0) \geq -\varepsilon.\end{aligned} \tag{5.40}$$

Before passing to the limit as $\varepsilon \to 0$ we observe that in this case y^0_ε depends on ε, so we first estimate its distance from x_0 in all the terms. By hypothesis (5.6), inequality (5.40) becomes

$$\begin{aligned}&\int_{t_0}^{\tau} b(s)ds + \phi(t_0, x_0) - \phi(\tau, x_0) + \int_{t_0}^{\tau} (-f(t, x_0, u_\varepsilon(t)) + C(t, |x_0| + \delta, |x_0|)\delta)\, dt \\ &+ \int_{t_0}^{\tau} \Phi |A(t, x_0) - A(t, y^0_\varepsilon(t))|\, dt + \int_{t_0}^{\tau} \omega(\delta) |A(t, x_0) + B(t)u_\varepsilon(t)| dt \\ &- \int_{t_0}^{\tau} D\phi(\tau, x_0) \cdot (A(t, x_0) + B(t)u_\varepsilon(t))\, dt \geq -\varepsilon,\end{aligned}$$

where Φ is such that $|D\phi(y^0_\varepsilon(t), \tau)| \leq \Phi$. Using the estimate (5.38) and recalling the definition of the Hamiltonian (5.10) we obtain

$$\begin{aligned}&\int_{t_0}^{\tau} b(s)ds + \phi(t_0, x_0) - \phi(\tau, x_0) + \int_{t_0}^{\tau} H(t, x_0, D\phi(\tau, x_0))dt + \varepsilon + \delta \int_{t_0}^{\tau} \eta(t)dt \\ &+ \int_{t_0}^{\tau} \omega(\delta) |A(t, x_0)|\, dt + \omega(\delta) \big[K(|x_0|) + K + \frac{\varepsilon}{a}\big]^{1/2} \omega(\tau - t_0) \geq 0\end{aligned}$$

where $\eta(t) = C(t, |x_0| + \delta, |x_0|) + \Phi\alpha_h(t) \in L^1(0, T)$ and $|A(t, x_0)| \in L^1(0, T)$.

It is easy to check that $H(t,x_0,D\phi(\tau,x_0)) \in L^1(0,T)$, then passing to the limit as $\varepsilon \to 0$, we obtain

$$\int_{t_0}^{\tau} b(s)ds + \phi(t_0,x_0) - \phi(\tau,x_0) + \int_{t_0}^{\tau} H(t,x_0,D\phi(\tau,x_0))dt \tag{5.41}$$
$$+\delta\int_{t_0}^{\tau}\eta(t)dt + \int_{t_0}^{\tau}\omega(\delta)|A(t,x_0)|dt + \omega(\delta)\big[K(|x_0|)+K\big]^{1/2}\omega(\tau-t_0) \geq 0.$$

Finally, dividing by $\tau - t_0$ and passing to the limit as $\delta \to 0$ (this implies $\tau \to t_0$), the inequality (5.41) becomes

$$b(t_0) - \frac{\partial}{\partial t}\phi(t_0,x_0) + H(t_0,x_0,D\phi(t_0,x_0)) \geq 0 \quad \text{a.e. with respect to } \; t_0.$$

This implies

$$-\frac{\partial}{\partial t}\phi(t_0,x_0) + F(t_0,x_0,D\phi(t_0,x_0)) \geq 0$$

which proves that $v(t,x)$ is a viscosity supersolution of (5.9) and completes the proof. □

We are finally ready to give the proofs of the results given in Section 2.

Proof of Theorem 2.2. We set $A(t,y) = a_n(t,x_n)$, $B(t) = b_n(t)$, $U(t) = u(t)$ and $f(t,x,U) = g(x_n(t)) + \frac{1}{2}|u(t)|^2$. Thanks to assumptions (2.4), (2.5) and (2.6) we can verify (5.4)-(5.8) at every fixed n. Applying Theorem 5.2 we obtain the desired result.□

Proof of Theorem 2.5. In order to apply Theorem 5.2 we set $A(s,y) = a_n(\varphi_n^0(s),y_n^1(s))\frac{d\varphi_n^0}{ds}(s)$. Since

$$\int_0^1 a_n(\varphi_n^0(s),y)\frac{d\varphi_n^0}{ds}(s)ds = \int_0^T a_n(t,y)dt$$

for every $y \in \mathbb{R}$, hypotheses (2.4) and (2.5) imply assumptions (5.4) and (5.5). If $B(s) = b_n(\varphi_n^0(s))\big(\frac{d\varphi_n^0}{ds}(s)\big)^{1/2}$ and $U(s) = u(\varphi_n^0(s))\big(\frac{d\varphi_n^0}{ds}(s)\big)^{1/2}$ setting $t = \varphi_n^0(s)$ we can easily deduce that $B \in L^2(0,1)$ and $U \in L^2(0,1)$. In order to verify assumptions (5.6), (5.7) and (5.8) we set

$$f(t,y,U) = \big(g(y_n(s)) + \frac{1}{2}|u(\varphi_n^0(s))|^2\big)\frac{d\varphi_n^0}{ds}(s),$$

then, (5.6) follows from assumption (2.6), (5.8) is fulfilled with $u_0 \equiv 0$ and by

$$f(t,y,U) > \frac{1}{2}|u(\varphi_n^0(s))|^2\frac{d\varphi_n^0}{ds}(s) = \frac{1}{2}|U|^2$$

we obtain (5.7). For the proof of uniqueness we refer to Theorem 3.1.4 in [6]. □

Proof of Theorem 2.10. In this case we set $A(s,y) = a(\varphi^0(s), y^1(s))\frac{d\varphi^0}{ds}(s)$ then (2.28) and (2.29) implies (5.4) and (5.5) (using the change of variable $t = \varphi^0(s)$ and hypotheses (2.4), (2.5)). We consider

$$B(s) = \begin{pmatrix} 0 & \left(\frac{d\varphi^0}{ds}(s)\right)^{1/2} & (\tilde{\mu}(s))^{1/2} \end{pmatrix}$$

and

$$U(s) = \begin{pmatrix} u(\varphi^0(s))\left(\frac{d\varphi^0}{ds}(s)\right)^{1/2} \\ v(\varphi^0(s))\left(\frac{d\varphi^0}{ds}(s)\right)^{1/2} \\ \xi(\varphi^0(s))\left(\tilde{\mu}(s)\right)^{1/2} \end{pmatrix},$$

then $B \in L^2(0,1;\mathbb{R}^{1\times 3})$ and $U \in L^2(0,1;\mathbb{R}^3)$ by Lemma 2.13. With an abuse of notation we set

$$f(s,y,U) = f(y^0(s), y(s), U(y^0(s)))\frac{d\varphi^0}{ds}(s) + \frac{1}{2}\xi(\varphi^0(s))^2\tilde{\mu}(s)$$

then if $u_0 \equiv 0$ we obtain (5.8), the (5.6) follows from (2.6) and from

$$f(s,y,U) > (a(|u|^2 + |v|^2) - b)\frac{d\varphi^0}{ds}(s) + \frac{1}{2}|\xi|^2\tilde{\mu} \geq \min(a, 1/2)|U|^2 - b\frac{d\varphi^0}{ds}(s)$$

we get the (5.7). Theorem 5.2 applies then to this case too. The proof of uniqueness is given in Theorem 3.1.9 of [6]. □

Remark 5.8. As we observed for the example in Section 3, in the limit problem the value function v it is not continuous in the t-variable, but it is left continuous and only right lower semi-continuous. This regularity can be verified also in the general case with arguments similar to Theorem 5.7. Thus the reparametrized limit problem has also the advantage of having a continuous value function.

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REFERENCES

[1] L. Ambrosio, *Corso introduttivo alla Teoria Geometrica della Misura ed alle Superfici Minime*, Lecture notes, Scuola Normale Superiore, Pisa 1997.

[2] E.N. Barron, R. Jensen, and J.L. Menaldi, *Optimal control and differential games with measures*, Nonlinear Analysis, Theory, Methods & Applications, **21** (4) (1993), 241-268.

[3] A. Bressan, *On differential systems with impulsive controls*, Rend. Sem. Mat. Univ. Padova, 78 (1987), 227-236.

[4] A. Bressan and F. Rampazzo, *On differential systems with vector-valued impulsive controls*, Boll. Un. Mat. Ital., 2-B (7) (1988), 641-656.

[5] A. Briani, *Convergence of Hamilton-Jacobi equations for sequences of optimal control problems*, Communications in Applied Analysis, to appear.

[6] A. Briani, *Hamilton-Jacobi-Bellman equations and* Γ*-convergence for optimal control problems*, Phd Thesis, Departement of Mathematics, University of Pisa (1999).

[7] G. Buttazzo and E. Cavazzuti, *Limit problems in optimal control theory*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 6 Suppl. (1989), 151-160.

[8] G. Buttazzo and L. Freddi, *Sequences of optimal control problems with measures as controls*, Adv. Math. Sci. Appl., 2 (1993), 215-230.

[9] G. Dal Maso and F. Rampazzo, *On systems of ordinary differential equations with measures as controls*, Differential and Integral equations, 4 (1991), 739-765.

[10] W.H. Fleming and M. Soner, "Controlled Markov Processes and Viscosity Solutions," Springer-Verlag, New York (1992).

[11] H. Ishii, *Hamilton-Jacobi equations with discontinuous Hamiltonians on arbitrary open sets*, Bull. Fac. Sci. Engng Chou Univ., 28 (1985), 33-77.

[12] P.L. Lions and B. Perthame, *Remarks on Hamilton-Jacobi equations with time-dependent Hamiltonians* Nonlinear Analysis, Theory, Methods & Applications, 11 (1987), 613-621.

[13] B.M. Miller, *Optimization of dynamic systems with a generalized control*, Automation and Remote Control, 50 (1989).

[14] M. Motta and F. Rampazzo, *Dynamic programming for nonlinear systems driven by ordinary and impulsive controls*, SIAM Journal of Control and Optimization, 34 (1996), 199-225.

[15] R.W. Rishel, *An extended Pontriagin principle for control systems whose controls laws containing measures*, SIAM J. Control, 3 (1965), 191-205.

[16] A.V. Sarychev, *Nonlinear systems with impulsive and generalized functions controls*, Proc. Conf. on Nonlinear Synthesis, Sopron, Hungary, (1989).

[17] H.J. Sussmann, *On the gap between deterministic and stochastic ordinary differential equations*, Ann. of Probability, 6 (1978), 17-41.