Differential and Integral Equations Volume 12, Number 6, November 1999, pp. 833– 848

ON THE UNIQUENESS OF THE COEXISTENCE STATE OF PREDATOR-PREY SYSTEMS ON $\mathbb{R}^1$

Sandro Merino
Universität Zürich, Mathematisches Institut
Winterthurerstrasse 190, CH-8057 Zürich, Switzerland

Rosa Pardo San Gil
Departamento de Matemática Aplicada, Universidad Complutense de Madrid
Madrid 28040, Spain

(Submitted by: Herbert Amann)

Abstract. It is shown that predator-prey type reaction-diffusion systems on the whole real line have a unique coexistence state.

1. Introduction. We study the following semilinear reaction-diffusion system of predator-prey type on the whole of $\mathbb{R}^1$:

$$\begin{aligned} \partial_t u - u_{xx} &= \lambda(x)u - a(x)u^2 - b(x)uv \\ \partial_t v - v_{xx} &= \mu(x)v + c(x)uv - d(x)v^2 \end{aligned} \quad \text{in } \mathbb{R} \times (0,\infty) . \tag{1}$$

Here u and v model the respective densities of the prey and predator populations. The assumptions on the coefficients λ, μ, a, b, c, d are specified in $(A1), \dots, (A4)$. For initial data in the positive cone X^+ of the Banach space $X := BUC(\mathbb{R}) \times BUC(\mathbb{R})$ the Cauchy problem associated with (1) has been studied in [14], where it is shown that (1) generates an asymptotically compact semiflow (ϕ, X^+). That semiflow possesses a compact global B-attractor $\mathcal{A}$ of finite Hausdorff dimension. In this paper we are interested in the uniqueness of coexistence solutions. A coexistence solution of (1) is a stationary state $(u_0, v_0) \in X^+$ with both components u_0 and v_0 nonzero. Under the assumptions (A1)-(A4) necessary and sufficient conditions for the

Accepted for publication March 1998.
AMS Subject Classifications: 35K55, 35K57, 35K22, 92B05.

existence of coexistence solutions are given in [13, Theorem 11.1] as a special case of more general results. Under weak additional conditions ((8), (9)), that we believe to be only of a technical nature, we will prove that if (1) admits a coexistence solution, then it is unique. The proof of this result is based on earlier results on the analogous problem on a bounded interval (cf. [11] for Dirichlet boundary conditions and [12] for general boundary conditions).

We note that a functional analytic framework for parabolic problems such as (1) was discussed in several recent papers (cf. [2, 5, 8]). Related questions regarding principal eigenfunctions have been addressed in (cf. [3, 4]). Therefore we will be rather brief when we encounter these topics in our exposition, merely referring to the relevant references for details.

An important tool in our approach is a result on the invertibility of certain non-cooperative linear elliptic systems (cf. [12, Theorem 3.1]). That paper contains an assumption, namely (**A2**) , that is not satisfied in the situation that is of interest for our purposes. In the Appendix we show how the arguments in [12] have to be modified in order to avoid the assumption (**A2**) .

2. The coexistence result. In this section we briefly review some of the known results on the predator prey system (1). Before we specify the assumptions on the coefficients appearing in (1) we introduce a convenient terminology: we say that a weight function $m \in L_\infty(\mathbb{R})$ is *strongly absorbing* (cf. [1]) if its positive and negative parts m^+ and m^- have the following properties

- $m^+ \in C_0(\mathbb{R})$.
- There exist positive constants C and R such that

$$\int_x^{x+R} m^-(s)ds \geq C \quad \text{for each } x \in \mathbb{R}\,.$$

Throughout this paper we make the following assumptions on the coefficients λ, μ, a, b, c, d .

(A1) λ, μ, a, b, c, $d \in BUC^\alpha(\mathbb{R})$ for some $\alpha \in (0,1)$.

(A2) λ and μ are strongly absorbing.

(A3) The functions a, b, c, d are nonnegative.

(A4) $\mathrm{supp}(\lambda^+) \subset \mathrm{supp}(a)$, $\mathrm{supp}(\mu^+) \subset \mathrm{supp}(d)$.

For $m \in BUC(\mathbb{R})$ we consider the eigenvalue problem

$$-\varphi_{xx} - m(x)\varphi = \sigma\varphi \quad \text{in } \mathbb{R}\,. \tag{2}$$

By $\sigma_1(m)$ we denote the infimum of the spectrum of the $BUC(\mathbb{R})$-realization (cf. [5]) of the operator

$$-\frac{d^2}{dx^2} - m(x)\,.$$

If m is a strongly absorbing weight function, then $\sigma_1(m) < 0$ implies that $\sigma_1(m)$ is the unique principal eigenvalue for (2) (cf. [1], [3], [4], [5] and [13, Proposition 4.1]).

Necessary and sufficient conditions for the existence of coexistence solutions to the system (1) have been obtained in [13, Theorem 11.1]. The system (1) admits a coexistence solution if and only if either one of the following sets of conditions is satisfied.

$$\sigma_1(\lambda) < 0\,,\ \sigma_1(\mu) < 0\,,\ \sigma_1(\lambda - bv^*) < 0\,. \tag{3}$$

$$\sigma_1(\lambda) < 0\,,\ \sigma_1(\mu) \geq 0\,,\ \sigma_1(\mu + cu^*) < 0\,. \tag{4}$$

Here, u^* and v^* are the unique positive solutions of the semilinear elliptic equations

$$-\,u_{xx} = \lambda(x)u - a(x)u^2 \quad \text{in } \mathbb{R}\,, \tag{5}$$

$$-\,v_{xx} = \mu(x)v - d(x)v^2 \quad \text{in } \mathbb{R}\,, \tag{6}$$

respectively. We refer to [13, Section 5] for more details on equations with "logistic-type" nonlinearities. Observe that a coexistence solution of (1) is necessarily contained in the closed subspace $C_0(\mathbb{R}) \times C_0(\mathbb{R})$ of $BUC(\mathbb{R}) \times BUC(\mathbb{R})$. In fact, as proved in [14], the parabolic initial value problem associated with (1) generates a global, asymptotically compact semiflow on the positive cone of the Banach space $BUC(\mathbb{R}) \times BUC(\mathbb{R})$. That semiflow possesses a compact B-attractor $\mathcal{A}$ of finite Hausdorff dimension and

$$\mathcal{A} \subset C_0(\mathbb{R})^+ \times C_0(\mathbb{R})^+ \ . \tag{7}$$

In particular, the B-attractor $\mathcal{A}$ contains all bounded (nonnegative) stationary solutions of (1) and by (7) their components vanish at infinity.

3. Uniqueness of the coexistence solution. In this section we will prove that the system (1) admits at most one coexistence solution. We require two additional technical conditions, namely:

$$\text{Either } b \text{ or } c \text{ has compact support.} \tag{8}$$

There exist positive constants R_0 and C_0 such that

$$\lambda(x) \leq -C_0 \quad \text{and} \quad \mu(x) \leq -C_0 \quad \text{for} \quad |x| \geq R_0 \,. \tag{9}$$

The following theorem is our main result. We conjecture that the assumptions (8) and (9) can be omitted and only arise as a consequence of the techniques that we use.

Theorem 3.1. *Assume that* (8) *and* (9) *hold. Then the predator-prey system* (1) *admits at most one coexistence solution.*

The proof of the above theorem will be accomplished in a series of lemmas. The main step consists in proving that coexistence solutions of (1) cannot be degenerate. Let (u_0, v_0) be a coexistence solution. For notational convenience we introduce the elliptic operators

$$\mathcal{L}_1 := -\frac{d^2}{dx^2} - \Big[\, \lambda(x) - 2a(x)u_0(x) - b(x)v_0(x) \Big] \tag{10}$$

and

$$\mathcal{L}_2 := -\frac{d^2}{dx^2} - \Big[\, \mu(x) + c(x)u_0(x) - 2d(x)v_0(x) \Big] \;. \tag{11}$$

We will show that the linearization at (u_0, v_0) of the steady state equation associated with (1)

$$\begin{array}{rcl} \mathcal{L}_1\varphi & = & -b(x)u_0(x)\psi \\ \mathcal{L}_2\psi & = & c(x)v_0(x)\varphi \end{array} \quad \text{in } \mathbb{R}\,, \tag{12}$$

admits only the trivial solution. The main tool for showing this nondegeneracy is a result on the invertibility of linear noncooperative elliptic systems on bounded intervals under general boundary conditions (cf. [12, Theorem 3.1]). However, in [12] there is a hypothesis, namely (**A2**), that we do not require. The hypothesis is the following

(A2) : $b \geq 0$, $c \geq 0$, and the two sets $\{x \in \mathbb{R} \mid b(x) = 0\}$ and $\{x \in \mathbb{R} \mid c(x) = 0\}$ have empty interior.

In the Appendix we will prove that this hypothesis is superfluous.

Lemma 3.2. *Assume* (9) *and let* (u_0, v_0) *be any coexistence solution of* (1). *Then for* $L \in \mathbb{R}$ *sufficiently large the principal eigenvalues of both of the two eigenvalue problems with Robin boundary conditions*

$$\begin{gathered} \mathcal{L}_1 \varphi = \sigma \varphi , \qquad x \in (-L, L) , \\ \varphi(-L) - \frac{u_0(-L)}{u_0'(-L)} \varphi'(-L) = 0 , \qquad \varphi(L) - \frac{u_0(L)}{u_0'(L)} \varphi'(L) = 0 , \end{gathered} \tag{13}$$

and

$$\begin{gathered} \mathcal{L}_2 \psi = \sigma \psi , \qquad x \in (-L, L) , \\ \psi(-L) - \frac{v_0(-L)}{v_0'(-L)} \psi'(-L) = 0 , \qquad \psi(L) - \frac{v_0(L)}{v_0'(L)} \psi'(L) = 0 , \end{gathered} \tag{14}$$

are positive.

Proof. We only prove the positivity of the principal eigenvalue of (13). The arguments are analogous for the problem (14). We prove that for L sufficiently large the problem (13) admits a strict supersolution. This implies the positivity of the principal eigenvalue (cf. [12, Theorem 2.1.]). First note that u_0 satisfies

$$-u_0'' = (\lambda - a u_0 - b v_0) u_0 \quad , \qquad \text{in } \mathbb{R} .$$

By (9) and since $u_0,\ v_0$ vanish at infinity, we obtain that $u_{0,xx}(x) > 0$ for $|x|$ large enough. Hence $u_0(x)$ has no local maximum for large $|x|$. Furthermore since u_0 vanishes at infinity it is clear that - for large $|x|$ - u_0 can not have a local minimum either. We conclude that $u_0'(x) < 0$ for $x >> 0$ and $u_0'(x) > 0$ for $x << 0$. This implies that the coefficients in the boundary conditions of (13) satisfy

$$\frac{u_0(-L)}{u_0'(-L)} > 0 \quad , \qquad \frac{u_0(L)}{u_0'(L)} < 0 \tag{15}$$

for any sufficiently large L. It is immediately verified that u_0 satisfies the equation

$$\mathcal{L}_1(u_0) = a u_0^2\,,\ \text{in } \mathbb{R}\,.$$

We note that without loss of generality we can exclude that a is identically zero on $\mathbb{R}$. In fact, an inspection of the coexistence conditions (3), (4) and of assumption $(A4)$ yields that no coexistence solutions can exist for $a \equiv 0$. Since u_0 satisfies the boundary conditions in (13), where by (15) the coefficients have the appropriate sign, we conclude that, for L large, u_0 is a supersolution for (13). Since $a \not\equiv 0$ this supersolution is strict. □

In the next lemmas we prove a-priori decay and non-oscillation properties of nontrivial solutions of (12).

Lemma 3.3. *Let* (u_0, v_0) *be a coexistence solution of* (1) *and let* $(\varphi, \psi) \in BUC(\mathbb{R}) \times BUC(\mathbb{R})$ *be a solution of* (12)*. Then* $(\varphi, \psi) \in C_0(\mathbb{R}) \times C_0(\mathbb{R})$.

Proof. The assertion follows along the lines of the proof of [14, Lemma 3.2.]. Similar ideas were used earlier in [4]. □

Lemma 3.4. *Assume* (9) *and that* b *has compact support. Let* (u_0, v_0) *be a coexistence solution of* (1) *and let* $(\varphi, \psi) \in BUC(\mathbb{R}) \times BUC(\mathbb{R})$ *be a non-trivial solution of* (12)*. Then there exists a constant* $R_1 > 0$ *such that either*

$$\varphi(x) = 0 \quad \textit{for} \;\; x \in [R_1, \infty)\,, \quad \textit{or} \tag{16}$$

$$\varphi(x) = \varphi'(x) = 0 \quad \textit{never holds for } x \in [R_1, \infty)\,, \tag{17}$$

is true. On the interval $(-\infty, -R_1]$ *the function* φ *is subject to the analogous alternative. In the case that* (17) *holds* φ *does not change sign on* $[R_1, \infty)$ *(on* $(-\infty, -R_1]$*, respectively) and*

$$\varphi(x)\,\varphi'(x) \,<\, 0 \quad , \quad (\varphi(-x)\,\varphi'(-x) \,>\, 0) \tag{18}$$

for $x \in [R_1, \infty)$. *Moreover,*

$$\lim_{x\to\infty} \varphi'(x) \,=\, 0 \quad , \quad (\lim_{x\to-\infty} \varphi'(x) \,=\, 0)\ . \tag{19}$$

On the assumption that c *has compact support analogous assertions hold for* ψ.

Proof. We only consider the component φ. The arguments are analogous for ψ. By (8) and (9) we find a constant $R_0 > 0$ such that $\mathrm{supp}(b) \subset [-R_0, R_0]$ and $\lambda(x) \leq -C_0 < 0$ for $|x| \geq R_0$. Assume that $\varphi(x_0) = \varphi'(x_0) = 0$ at some $x_0 \in [R_0, \infty)$. Then φ solves the second order initial value problem

$$\varphi'' = \big[-\lambda + 2au_0 + bv_0\big]\varphi \;, \tag{20}$$

$$\varphi(x_0) = 0 \;\;,\;\; \varphi'(x_0) = 0 \;\;. \tag{21}$$

By the unique solvability of (20), (21) we obtain that $\varphi(x) = 0$ for $x \in [x_0, \infty)$. This shows that either (16) or (17) holds.

Next assume that φ satisfies (17). Observe that φ satisfies (20). Since u_0 and v_0 vanish at infinity and by (9) we find that for x sufficiently large φ'' and φ have the same sign. We show that this implies that $\varphi'(x) \neq 0$ for x large enough. In fact, if $\varphi'(x) = 0$, then $\varphi(x) \neq 0$. As a consequence of (20), (21) we obtain that x is a strict local maximum if $\varphi(x) < 0$ and x is a strict local minimum if $\varphi(x) > 0$. Since $\varphi(x) \to 0$ as $x \to \infty$ this implies the existence of either a positive maximum or a negative minimum of φ, which is impossible. Thus there exists a constant $R_1 > 0$ such that $\varphi'(x) \neq 0$ and $\varphi(x)\,\varphi'(x) < 0$ for $x \in [R_1, \infty)$.

Finally, to show that $\lim_{x\to\infty} \varphi'(x) = 0$ we shall argue by contradiction. Assume without loss of generality that $\varphi > 0$ in $[R_1, \infty)$. Then we know that $\varphi' < 0$ and $\varphi'' > 0$ for x sufficiently large. Therefore φ' is increasing and bounded above by zero. Hence there exists $\lim\limits_{x\to\infty} \varphi'(x) = M \leq 0$. Assume that $M < 0$. Then $\varphi'(x) < M/2 < 0$ for x large enough. Integrating this inequality in $L < x < T$ and letting $T \to \infty$ yields that φ is not bounded, which is a contradiction. □

Remark 3.5. The argument used in the above proof also shows that any coexistence solution (u_0, v_0) satisfies

$$\lim_{x\to\pm\infty} u_0'(x) = 0 \;, \qquad \lim_{x\to\pm\infty} v_0'(x) = 0 \;\;. \tag{22}$$

In the following lemma it is shown that the asymptotic behaviour for $x \to \pm\infty$ of eigenfunctions of (12) and coexistence solutions is the same.

Lemma 3.6. *Let* (u_0, v_0) *be a coexistence solution of* (1) *and let* $(\varphi, \psi) \in BUC(\mathbb{R}) \times BUC(\mathbb{R})$ *be a nontrivial solution of* (12). *Assume that* b *has compact support. Whenever* φ *is not vanishing identically on a neighborhood*

$[R,\infty)$ of ∞ (or on a neighborhood $(-\infty,-R]$ of $-\infty$, respectively), the following relations hold

$$\lim_{x\to\infty}\left\{\frac{u_0(x)}{u_0'(x)}-\frac{\varphi(x)}{\varphi'(x)}\right\}=0,\quad \left(\lim_{x\to-\infty}\left\{\frac{u_0(x)}{u_0'(x)}-\frac{\varphi(x)}{\varphi'(x)}\right\}=0,\ \textit{respectively}\right).$$

Analogously if c has compact support and ψ is not vanishing identically on a neighborhood $[R,\infty)$ of ∞ (or on a neighborhood $(-\infty,-R]$ of $-\infty$, respectively), then

$$\lim_{x\to\infty}\left\{\frac{v_0(x)}{v_0'(x)}-\frac{\psi(x)}{\psi'(x)}\right\}=0,\quad \left(\lim_{x\to-\infty}\left\{\frac{v_0(x)}{v_0'(x)}-\frac{\psi(x)}{\psi'(x)}\right\}=0,\ \textit{respectively}\right).$$

Proof. We only proof the assertions for $x\to+\infty$. The other cases are similar. Multiply (12) by u_0 and integrate by parts twice in an interval $[L,T]$ with $L<T$ sufficiently large. Taking into account that u_0 satisfies the equation $\mathcal{L}_1(u_0)=au_0^2$, in $\mathbb{R}$, we obtain

$$(\varphi' u_0-\varphi u_0')\big|_L^T=\int_L^T au_0^2\varphi\,dx\,.$$

Letting $T\to\infty$ yields

$$\varphi(L)u_0'(L)-\varphi'(L)u_0(L)=\int_L^\infty au_0^2\varphi\,dx\,,$$

and therefore

$$\frac{u_0'(L)}{u_0(L)}-\frac{\varphi'(L)}{\varphi(L)}=\int_L^\infty a\frac{u_0^2(x)}{u_0(L)}\frac{\varphi(x)}{\varphi(L)}\,dx. \tag{23}$$

By comparison principles for boundary value problems and using (9) it is seen that for any $L>R_0$

$$0\le u_0(x)\le u_0(L)e^{-\sqrt{C_0}(x-L)},\quad \forall x>L. \tag{24}$$

Consequently, $u_0\in L^1([0,\infty))$ and

$$\lim_{L\to\infty}\int_L^\infty u_0\,dx=0\,.$$

These equalities and using the relation (18) in (23) lead to

$$\lim_{L\to\infty} \left\{\frac{u_0'(L)}{u_0(L)} - \frac{\varphi'(L)}{\varphi(L)}\right\} = 0.$$

It is clear that to conclude the proof it only remains to show that there exist two constants L_0 and C such that

$$0 \le -\frac{u_0(x)}{u_0'(x)} \le C, \quad 0 \le -\frac{\varphi(x)}{\varphi'(x)} \le C, \quad \forall x > L_0 \ .$$

We only prove this assertion for u_0 and $x > L$, the proof for φ is analogous. We apply monotone methods for initial value problems (cf. [15, Theorem 13, §1.6]) : it is known that if there exists an L_1 sufficiently large such that

$$u_0'(L_1) \ge -\sqrt{C_0}u_0(L_1) \ , \tag{25}$$

then

$$u_0(x) \ge u_0(L_1)e^{-\sqrt{C_0}(x-L_1)}, \ \ u_0'(x) \ge -\sqrt{C_0}u_0(L_1)e^{-\sqrt{C_0}(x-L_1)}, \ \forall x \ge L_1. \tag{26}$$

Here, R_0 and C_0 are the constants appearing in (9) . Now, integrating $-u_0'' = \lambda(x)u_0 - a(x)u_0^2 - b(x)u_0v_0$ in $[x,T]$ for any $T > x > L_2 := \max\{L, L_1\}$ and taking into account (26) and (9) we have

$$-u_0'(T) + u_0'(x) \le \sqrt{C_0}u_0(L_1)[e^{-\sqrt{C_0}(T-L_1)} - e^{-\sqrt{C_0}(x-L_1)}] \ .$$

Letting $T \to \infty$ and taking into account (22), we find

$$u_0'(x) \le -\sqrt{C_0}u_0(L_1)e^{-\sqrt{C_0}(x-L_1)}, \quad \forall x \ge L_2$$

and due to (26)

$$u_0'(x) = -\sqrt{C_0}u_0(L_1)e^{-\sqrt{C_0}(x-L_1)}, \quad \forall x \ge L_2 \ .$$

In that case

$$0 \le -\frac{u_0(x)}{u_0'(x)} = \frac{1}{\sqrt{C_0}}, \quad \forall x \ge L_2 \ .$$

The only other possibility, refusing (25), is that $u_0'(x) < -\sqrt{C_0}u_0(x)$ for x sufficiently large. In either case we obtain the desired inequality and the proof is accomplished. □

Remark 3.7. Assume that (16) holds. Then ψ solves the differential equation

$$-\psi'' = \big[\mu + cu_0 - 2dv_0\big]\psi, \quad x \in [R_1, \infty). \tag{27}$$

and all the assertions in the Lemmas 3.4 and 3.6 hold for ψ, without making the assumption that c has compact support. Analogous assertions hold for φ.

Remark 3.7 will be used in the proof of Theorem 3.1. Under the hypothesis that b has compact support, we have characterized the non-oscillation behavior of φ at infinity. The following proposition will prove that these non-oscillation properties imply $\varphi \equiv 0$, and from Remark 3.7 we are able to apply the Lemmas 3.4 and 3.6 to ψ, again ψ is non-oscillating at infinity and the following proposition imply $\psi \equiv 0$.

Proposition 3.8. *Assume* (8) *and* (9) *and let* (u_0, v_0) *be a coexistence solution of* (1) *. Then the only solution of* (12) *in* $BUC(\mathbb{R}) \times BUC(\mathbb{R})$ *is the trivial one.*

Proof. Assume without loss of generality that supp(b) is compact. Suppose on the contrary that (φ, ψ) is a nontrivial solution of (12) . Then for any $L > 0$ the restriction to $[-L, L]$ of (φ, ψ) is a nontrivial solution of the boundary value problem

$$\begin{aligned}
&\left.\begin{aligned} \mathcal{L}_1\varphi &= -b\, u_0\, \psi \\ \mathcal{L}_2\psi &= c\, v_0\, \varphi \end{aligned}\right. \qquad x \in (-L, L)\ , \\
&\varphi(-L) - \frac{\varphi(-L)}{\varphi'(-L)}\,\varphi'(-L) = 0\ , \qquad \varphi(L) - \frac{\varphi(L)}{\varphi'(L)}\,\varphi'(L) = 0\ , \\
&\psi(-L) - \frac{\psi(-L)}{\psi'(-L)}\,\psi'(-L) = 0\ , \qquad \psi(L) - \frac{\psi_(L)}{\psi'(L)}\,\psi'(L) = 0\ .
\end{aligned} \tag{28}$$

Observe that we can restrict ourselves to the difficult case where φ and ψ do not have compact support. In fact, in the case of φ or ψ having compact support we simply consider the corresponding eigenvalue problem on a large bounded interval with Dirichlet boundary conditions. Then we can apply the same arguments that are given below.

We verify that the Assumption **(A1)** of [12, Theorem 3.1] is satisfied if we choose L large enough. In the Appendix we show how to avoid assumption **(A2)**. From [12, Theorem 2.1], what needs to be shown is that for L

sufficiently large the principal eigenvalues of the Sturm Liouville problems

$$\begin{aligned} &\mathcal{L}_1 w = \sigma\, w && x \in\ (-L, L)\ , \\ &w(-L) - \frac{\varphi(-L)}{\varphi'(-L)}\, w'(-L) = 0\ , && w(L) - \frac{\varphi(L)}{\varphi'(L)}\, w'(L) = 0\ , \end{aligned} \tag{29}$$

and

$$\begin{aligned} &\mathcal{L}_2 w = \sigma\, w && x \in\ (-L, L)\ , \\ &w(-L) - \frac{\psi(-L)}{\psi'(-L)}\, w'(-L) = 0\ , && w(L) - \frac{\psi(L)}{\psi'(L)}\, w'(L) = 0\ , \end{aligned} \tag{30}$$

are positive. This is indeed true as a consequence of Lemma 3.6 and the continuous dependence of the eigenvalues on the coefficient in the Robin boundary condition (cf. e.g. [9]), and Lemma 3.2. By [12, Theorem 3.1] we conclude that $\varphi \equiv 0$. Now, we apply Lemma 3.6 and Remark 3.7 to ψ and finally obtain that the boundary value problem (28) has no nontrivial solutions. □

We are now in possession of all the tools needed in the proof of Theorem 3.1.

Proof of Theorem 3.1. Consider the parameter dependent problem that is obtained from (1) by introducing the parameter $\delta \in [0, 1]$

$$\begin{aligned} \partial_t u - u_{xx} &= \lambda(x)u - a(x)u^2 - (1-\delta)b(x)uv \\ \partial_t v - v_{xx} &= \mu(x)v + c(x)uv - d(x)v^2 \end{aligned} \qquad \text{in } \mathbb{R} \times (0, \infty)\ . \tag{31}$$

It is clear that all the results that were obtained above for (1) still hold for (31) since the parameter $(1-\delta)$ can be absorbed into the coefficient b without violating the assumptions (A1)-(A4). We introduce this parameter for the following reason : for $\delta = 1$ the system (31) is decoupled and it is easily seen that in that case coexistence solutions are unique (cf. [13, Lemma 9.5]). Note that since

$$\sigma_1\big(\lambda - (1-\delta)b\, v^*\big) \leq \sigma_1\big(\lambda - bv^*\big)$$

for $\delta \in [0, 1]$, the coexistence conditions (3) and (4) for (31) are satisfied for each $\delta \in [0, 1]$ if they are satisfied for $\delta = 0$. Without loss of generality we can thus assume that either (3) or (4) is satisfied for (1). Consequently (31) has a coexistence solution for each $\delta \in [0,1]$. Assume that (1) admits two distinct coexistence solutions $w_1 = (u_1, v_1)$ and $w_2 = (u_2, v_2)$. In the sequel we will prove that there exist two curves $w_1(s)$ and $w_2(s)$ of coexistence solutions of (31) with $\delta = s$ such that $w_1(1) = w_2(1)$. In other words we show that by the uniqueness of the coexistence state of the decoupled system ($\delta = 1$) the two curves are forced to meet for $s = 1$. This will turn out to be a contradiction to the nondegeneracy of the coexistence states that was proved in Proposition 3.8.

We now present the technical aspects of the above argument in more detail: consider the map $\phi : [0,1] \times X^+ \times [0,\infty) \to X^+$, where $\phi(s,\cdot,\cdot)$ is the global differentiable semiflow on $X^+ := BUC^+(\mathbb{R}) \times BUC^+(\mathbb{R})$ that is generated by (31) for $\delta = s$. Fix any $T > 0$ and define the map

$$F \ : \ [0,1] \times X^+ \to X^+ \ , \quad F(\delta, w) := w - \phi(\delta, w, T) \ .$$

The Fréchet derivative of F with respect to the second variable w at a coexistence solution w_0 of (31) is given by

$$D_w F(\delta, w_0) \ = \ 1 \ - \ D_w \phi(\delta, w_0, T) \ .$$

We now show that the linear operator $D_w F(\delta, w_0) \in \mathcal{L}(X)$ has a bounded inverse whenever w_0 is a coexistence solution of (31). First note that by a perturbation argument as used e.g. in [14, Proof of Theorem 3.5] and which is based on the results in [4, Section 5], the radius of the essential spectrum of $D_w\phi(\delta, w_0, T)$ is smaller than 1. Thus zero is not in the essential spectrum of $D_w F(\delta, w_0)$ and if zero is in the spectrum of $D_w F(\delta, w_0)$, then it is a pole of the resolvent (cf. [4, Remarks 5.1]). In particular, in that case zero is an eigenvalue. As a consequence of Proposition 3.8 the linear operator $D_w F(\delta, w_0)$ has a bounded inverse at any coexistence solution w_0 of (31). If (1) possesses two coexistence solutions w_1, w_2, then we can apply the implicit function theorem and obtain two curves $w_i : [0,1] \to X^+$, $i = 1, 2$, where $w_i(s)$ is a coexistence solution of (31) for $\delta = s$. Note that as a consequence of the maximum principle the curves can not leave the positive cone X^+. Furthermore, as is shown in [13, Proposition 9.3] they can not connect the coexistence solutions $w_1(0)$ and $w_2(0)$ with any of the trivial or semitrivial solutions of (1). Since for $\delta = 1$ the system (31) possesses a unique coexistence solution we conclude that $w_1(1) = w_2(1)$, which is in contradiction with the nondegeneracy of the coexistence states of (31). □

4. Appendix. We explain the modifications that are needed in [12] in order to avoid making the hypothesis **(A2)**. We first prove in [12, Lemma 3.3] that the set of zeros of φ, ψ is discrete in a compact set, say $[-L, L]$, and consequently the sets $\{x \in [-L, L] \ : \ \varphi(x) = 0\}$ and $\{x \in [-L, L] \ : \ \psi(x) = 0\}$ have empty interior. Next we prove in [12, Lemma 3.4] that if there exists $(z_1, z_2) \subset [-L, L]$ such that $\varphi(z_1) = \varphi(z_2) = 0$, $\varphi < 0$ in (z_1, z_2) and $\psi(z_1) \leq 0$ then $\psi(z_2) > 0$. The oscillation argument contained in [12, Theorem 3.1], consists in choosing a finite partition $\mathcal{P}$ of zeros of φ and applying the Maximum Principle in each sub-interval where $\varphi \neq 0$ has constant sign.

Actually we have only assumed that b or c has compact support, refusing the hypothesis **(A2)**. We have to consider the possibility of φ or ψ vanishing in some intervals. What we really need is to be able to choose, for any compact set, a finite partition $\mathcal{P}$ of points, boundary points of intervals where $\varphi \neq 0$ has constant sign. We do not exclude the possibility that φ vanishes on a finite number of intervals. We observe that we only need to look at the zeros of φ where it changes sign. We start with the modifications introduced in [12, Lemma 3.3], by considering the simple zeros and the accumulation of zeros.

Lemma 4.1. *Assume that* (12) *has a solution with* $\varphi \not\equiv 0$ *and* $\psi \not\equiv 0$. *Then the following assertions are true*

(i) *If there exists a sequence* $x_m \downarrow x_0$ *such that* $\varphi(x_m) = 0$ *for all* $m \geq 1$, *then there exists* $N \in \mathbb{N}$ *such that* $\varphi(x) \equiv 0$ *and* $b(x)\psi(x) \equiv 0$ *for all* $x \in [x_0, x_N]$. *Analogous assertions hold if* $x_m \uparrow x_0$.

(ii) *The set of simple zeros of* φ *and* ψ *is finite in any compact set.*

Proof. (i) Assume without loss of generality that there exists a convergent sequence $x_m \downarrow x_0$ of zeros of φ. Then $\varphi(x_0) = \varphi'(x_0) = 0$ and because of the uniqueness of the Cauchy problem at x_0 associated with (12), either $\psi(x_0) \neq 0$ or $\psi'(x_0) \neq 0$. In both cases $\psi(x)$ has constant sign in some interval $(x_0, x_0 + \varepsilon)$. If $b(x)\psi(x) \neq 0$ for $x \in [x_0, x_0 + \varepsilon]$ we can argue as in [12, Lemma 3.3] choosing N sufficiently large so that $x_N < x_0 + \varepsilon$ and the first eigenvalue of the operator $\mathcal{L}_1$ in the interval (x_0, x_N), with Dirichlet boundary conditions is positive. The Maximum Principle applied to this BVP implies that φ does not vanish on (x_0, x_N), which is a contradiction. However, if $b(x)\psi(x) \equiv 0$ for all $x \in [x_0, x_N]$, then $\varphi(x) \equiv 0$ for $x \in [x_0, x_N]$.

(ii) Assume without loss of generality that there exists a convergent sequence $x_m \downarrow x_0$ of simple zeros of φ. Then, by part (i), $\varphi(x) \equiv 0$ for $x \in [x_0, x_N]$. Therefore $x_m,\ m \geq N$ are not simple zeros. □

The next lemma is a modified version of [12, Lemma 3.4].

Lemma 4.2. *Let (φ, ψ) be a nontrivial solution of (12) such that $\varphi(z_1) = \varphi(z_2) = 0$, $\varphi \leq 0$ and $\varphi \neq 0$ in (z_1, z_2). Assume that $\psi(z_1) \leq 0$. Then $\psi(z_2) > 0$.*

Proof. If $\psi(z_2) \leq 0$, then $\mathcal{L}_2\psi = c(x)v_0(x)\varphi \leq 0$ in (z_1, z_2), $\psi(z_1) \leq 0$, $\psi(z_2) \leq 0$. Due to $\mathcal{L}_2 v_0 = d(x)v_0^2 \geq 0$, $v_0(z_1) > 0$, $v_0(z_2) > 0$, v_0 is a positive strict supersolution of the associated eigenvalue problem and the corresponding first eigenvalue will be strictly positive. Now, the Maximum Principle implies that $\psi(z) \leq 0$ in $[z_1, z_2]$, therefore $\mathcal{L}_1\varphi = -b(x)u_0(x)\psi \geq 0$ in (z_1, z_2), $\varphi(z_1) = \varphi(z_2) = 0$. Again the Maximum Principle implies that $\varphi \geq 0$ in (z_1, z_2) which is a contradiction. □

We next discuss how the proof of Theorem 3.1 in [12] needs to be changed in order to cover the present situation. We do not exclude that φ or ψ vanishes in some intervals. We will choose, for any compact set, a partition $\mathcal{P}$ of boundary points of intervals where $\varphi \neq 0$ is either $\varphi \geq 0$ or $\varphi \leq 0$. If $\varphi \equiv 0$ on a finite number of intervals, we join this vanishing interval with the next one non-vanishing, letting a finite number of intervals where φ changes sign; in that case, this partition $\mathcal{P}$ is not unique, but we will prove that $\mathcal{P}$ is finite.

Theorem 4.3. *Assume there exists $L > 0$ such that the principal eigenvalues of the Sturm Liouville problems* (29) *and* (30) *are positive. Then $(\varphi, \psi) = (0, 0)$ is the unique solution of* (28) *in $BUC(\mathbb{R}) \times BUC(\mathbb{R})$.*

Proof. What needs to be shown is that the set of zeros of φ where a change of sign occurs is discrete in $[-L, L]$. Assume by contradiction $x_m \downarrow x_0$, $\varphi(x_m) = 0$. Then by Lemma 4.1 $\varphi(x) \equiv 0$, $x \in [x_0, x_N]$, for some $N \in \mathbb{N}$. Assume that there is an infinite sequence $[x_m, y_m]$ of sub-intervals of $[-L, L]$ such that $\varphi(x) \equiv 0$ in $[x_m, y_m]$, $y_m < x_{m+1}$ and $|\varphi(x)| > 0$ for a.e. $x \in (y_m, x_{m+1})$. Then y_m is an increasing bounded sequence, therefore converging to some y_0 and again by Lemma 4.1, there exists $N \in \mathbb{N}$, $\varphi(x) \equiv 0$, for all $x \in [y_N, y_0]$, which is a contradiction. Consequently, we can choose a finite set

$$\mathcal{P} = \{-L = z_0 < \cdots < z_{N+1} = L\}$$

such that $\varphi \geq 0$, $\varphi \neq 0$, in (z_{2j}, z_{2j+1}), $\varphi \leq 0$, $\varphi \neq 0$, in (z_{2j-1}, z_{2j}). The oscillation argument used in the proof of [12, Theorem 3.1] remains essentially unchanged for the present situation. □

Final remarks. a) As in the case of bounded domains the following problems are open.

- Is the coexistence solution locally or even globally stable?
- Is the uniqueness result still valid in dimension greater than one?

b) More general nonlinearities can be discussed (cf. [13]). In the case of a bounded interval results in this direction were obtained in [7].

REFERENCES

[1] W. Arendt and C.J.K. Batty, *Exponential stability of a diffusion equation with absorption,* Differential Integral Equations, 6 (1993), 1009-1024.

[2] D. Daners and P. Koch Medina, "Abstract Evolution Equations, Periodic Problems and Applications," Longman Scientific & Technical, Pitman Research Notes in Mathematics Series 279, Harlow, Essex, 1992.

[3] D. Daners and P. Koch Medina, *Superconvexity of the evolution operator and parabolic eigenvalue problems on* $\mathbb{R}^N$, Differential Integral Equations, 7 (1994), 235-255.

[4] D. Daners and P. Koch Medina, *Exponential stability, change of stability and eigenvalue problems for linear time periodic parabolic equations on* $\mathbb{R}^N$, Differential Integral Equations, 7 (1994), 1265-1284.

[5] M. Hieber, P. Koch Medina and S. Merino, *Linear and semilinear parabolic equations on* $BUC(\mathbb{R}^N)$, Mathematische Nachrichten, 179 (1996), 107-118.

[6] M. Hieber, P. Koch Medina and S. Merino, *Diffusive logistic growth on* $\mathbb{R}^N$, Nonlinear Analysis TMA, 27 (1996), 879-894.

[7] N.I.A. Karahalios, *Uniqueness of positive steady states for a nonlinear reaction-diffusion system,* MSc Thesis, Heriot-Watt University, 1995.

[8] P. Koch Medina and G. Schätti, *Longtime behaviour of reaction-diffusion equations on* $\mathbb{R}^N$, Nonlinear Analysis TMA, 25 (1995), 831-870.

[9] Q. Kong and A. Zettl, *Eigenvalues of regular Sturm-Liouville problems,* J. of Differential Equations, 131 (1996), 1-19.

[10] O. Ladyzhenskaya, "Attractors for Semigroups and Evolution Equations," Lezioni Lincee, Cambridge University Press, 1991.

[11] J. López-Gómez and R.M. Pardo, *Existence and uniqueness of coexistence states for the predator-prey model with diffusion: the scalar case,* Differential and Integral Equations, 6 (1993), 1025-1032.

[12] J. López-Gómez and R.M. Pardo, *Invertibility of linear non-cooperative elliptic systems,* Nonl. Anal. T.M.A., 31 (1998), 687-699.

[13] S. Merino, *Positive periodic solutions for semilinear reaction diffusion systems on* $\mathbb{R}^N$*,* Advances in Differential Equations, 1 (1996), 579-609.

[14] S. Merino, *On the existence of the compact global attractor for semilinear reaction diffusion systems on* $\mathbb{R}^N$*,* J. of Differential Equations, 132 (1996), 87-106.

[15] M.H. Protter and H.F. Weinberger, "Maximum Principles in Differential Equations," Springer-Verlag (1984), New York.