

STABLE RAPIDLY OSCILLATING SOLUTIONS IN DELAY DIFFERENTIAL EQUATIONS WITH NEGATIVE FEEDBACK

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Abstract. In this paper we show that first order differential delay equations with negative feedback can possess asymptotically stable rapidly oscillating solutions. We construct an analytically tractable example in which the feedback is piecewise constant. In this case, the continuous-time dynamics on a proper subset of the phase space can be reduced exactly to a three-dimensional discrete-time map. The existence and stability properties of the delay equation's rapidly oscillating periodic solutions are given by the existence and stability of one of the fixed points of the corresponding map. When the feedback is smoothed appropriately, the stable rapidly oscillating periodic solution is shown to persist.

1. Introduction. Delay differential equations play an important role in attempts to model the dynamics of many control mechanisms encountered in fields of research as diverse as physiology, nonlinear optics and economy. To mention a few applications, the oscillations of neutrophil populations in certain cases of chronic myelogenous leukemia [7] and the dynamics of the pupil light reflex [5] were investigated using scalar first order delay equations. Similar models were used to explain the origin of the four-year cycle in the price of pork-bellies [6], as well as the transmission properties of various nonlinear resonant cavities [3]. Frequently, though by no means exclusively, those systems are governed by negative feedback mechanisms, meaning that a deviation in the state variable from a steady state is followed by the movement

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of the system in the opposite direction (mathematically this type of feedback is described by formula (2) below). In first order scalar delay equations with a single delay, the presence of negative feedback implies that solutions with consecutive zeros spaced apart by distances larger than the delay tend to be “typical”, or frequently encountered. Such solutions are referred to as being slowly oscillating (see the exact definition below).

Recently scalar delay equations with negative feedback have been the focus of intense scrutiny (cf. the review [4] and references therein). Their dynamics is relatively well understood, and various phenomena characteristic of nonlinear (semi)dynamical systems, such as period doubling bifurcations, and chaotic behavior have been observed and rigorously proved. In spite of this theoretical activity, many mathematically challenging problems remain unsolved. One of them is the existence of asymptotically stable rapidly oscillating periodic solutions. In this paper we propose to prove the existence of such solutions in a specific class of scalar delay equations.

The paper is organized as follows. Section 2 presents an overview of some important concepts relevant to our description of delayed dynamics. The Morse decomposition of attractors is briefly discussed. Section 3 introduces a first order delay equation with a piecewise constant nonlinearity which is then reduced to a three-dimensional discrete-time map, whose fixed points represent the rapidly oscillating solution of the original delay equation. The stability of these fixed points is then investigated in Section 4. Section 5 extends our main results to smooth feedback situations. Section 6 briefly reflects on the numerical aspects of the paper.

2. Preliminaries and description of main result. In this paper, we consider equations of the form

$$\dot{x}(t) = -\alpha x(t) + F(x(t-1)), \quad \alpha > 0 \quad (1)$$

where the nonlinearity F satisfies the negative feedback condition

$$xF(x) < 0, \quad \forall x \neq 0. \quad (2)$$

As usual, a solution of equation (1) is a continuous function on $[-1, \infty)$ satisfying (1) for $t > 0$. For every initial function φ from $C := C([-1, 0], \mathbb{R})$ the corresponding solution $x_\varphi(t)$ is readily obtained by integrating (1) for $t > 0$. A solution of equation (1) is called slowly oscillating if the time interval between subsequent zeros is greater than the delay 1. Such solutions play an important role in many differential delay systems because they are frequently encountered (a notion taken up below more precisely when we discuss the

Morse decomposition of the attractors for delay equations). Equation (1) can be viewed as generating a dynamical system on C with the semiflow F^t defined by $(F^t)\varphi(s) = x_\varphi(t+s)$, $s \in [-1, 0]$. A more detailed presentation can be found e.g. in [2].

The set \mathbb{S} of all slowly oscillating solutions of (1) is open: any sufficiently small perturbation of an initial function for a slowly oscillating solution gives rise to another slowly oscillating solution. The set \mathbb{S} is also large in the sense that any initial function $\varphi \in \mathbb{K}$ gives rise to a slowly oscillating solution, where \mathbb{K} is defined by $\mathbb{K} = \{\varphi \in C([-1, 0]) \mid \varphi(t) \text{ has at most one zero in } [-1, 0]\}$ (one has to assume that $F'(0) < -1/e$ to guarantee the oscillation of all solutions of (1)).

It is also of interest to investigate the structure of the set $\mathbb{S}^\varphi \subset C$ of all initial functions whose elements give rise to eventually slowly oscillating solutions ($x_\varphi(t)$ is called eventually slowly oscillating if there exists time $T = T(\varphi)$ such that $x_\varphi(t)$ is slowly oscillating for all $t \geq T$). Clearly, \mathbb{S}^φ is open in C . A solution which is not eventually slowly oscillating is called rapidly oscillating.

Recently it was proved in [10] that the set \mathbb{S}^φ is dense in C provided $F(x)$ is decreasing and either $\sup F(x) < \infty$ or $\inf F(x) > -\infty$. This shows that rapid oscillations are rare in delay equations with monotone negative feedback. A natural question here is whether this result holds for nonmonotonic negative feedback nonlinearities. Our example shows that in general it does not, and that rapid oscillations can be typical. More specifically, a consequence of our work is that the complementary set $C \setminus \mathbb{S}^\varphi$ may contain an open subset when F is not monotone.

Our example also provides a deeper insight into possible structure of the Morse decomposition for differential delay equations with the negative feedback, which was introduced and studied in [8,9]. Under certain assumptions equation (1) has a global attractor Ω consisting of bounded solutions defined for all $t \in \mathbb{R}$. Those assumptions require, for example, that $F \in C^\infty$ is bounded from below or above, and 0 is an unstable hyperbolic equilibrium of equation (1) (for more details see [8]). The attractor Ω has the Morse decomposition $\Omega = \mathbb{S}_0 \cup \mathbb{S}_1 \cup \mathbb{S}_3 \cup \mathbb{S}_5 \cup \dots \cup \mathbb{S}_{2N+1}$, $\mathbb{S}_0 := \{0\}$, where the Morse sets $\mathbb{S}_k, k = 0, 1, 3, 5, \dots, 2N + 1$, are invariant under the semiflow defined by equation (1). Each set \mathbb{S}_k contains solutions which appropriately normalized projections ($x_\varphi(-1) = 0$) onto C have exactly k zeros; in particular, every \mathbb{S}_k contains a periodic solution. Every solution $x_\varphi(t) \in C$ is attracted by one of the sets \mathbb{S}_k at both positive and negative (if extendable) infinity: $\omega(x_\varphi(t)) \subset \mathbb{S}_m, \alpha(x_\varphi(t)) \subset \mathbb{S}_n$ with $m \leq n$, where $\omega(\cdot)$ and $\alpha(\cdot)$ stand for ω - and α -limit sets respectively. The set \mathbb{S}_1 consists of slowly oscillating

solutions while every $\mathbb{S}_k, k > 1$, contains only rapidly oscillating solutions. The Morse decomposition shows the gradient-like structure of the dynamics of equation (1) with negative feedback: as t increases the motion either stays on one of the invariant sets or else can only go from Morse sets with larger indices to those with smaller ones. Determining the detailed structure of Morse sets is generally a challenging problem. In fact, at this point, little is known besides the nonemptiness of each set and the existence of connecting solutions (cf. [1, 11]).

In the case of a monotone nonlinearity F , as the above mentioned result from [10] shows, the generic motion is the one which eventually rests on the set \mathbb{S}_1 . All other motions are exceptional. Our example shows that in the case of nonmonotone nonlinearities sets $\mathbb{S}_k, k > 1$, can be large enough. In particular, they may contain attracting periodic solutions.

In the present paper we investigate an equation of the form (1) which possesses an asymptotically stable periodic solution which is not slowly oscillating: in fact, it belongs to the set \mathbb{S}_3 of the Morse decomposition.

As a first step in our construction, we consider a piecewise constant nonlinearity F characterized by 6 real parameters $a, b, c, d, \theta_1, \theta_2$ (the exact definition is given below in Section 3 and illustrated in Figure 1). This simple functional form allows us a direct integration of the delay equation (1), and the solutions are found to be piecewise exponential continuous functions for all $t > 0$. For a specific choice of the parameters, namely $a = 6, b = 14, c = -13.5, d = -3.85, \theta_1 = -\theta_2 = 1/3, \alpha = 7.75$, it is possible to define a set \mathbb{D} of initial functions, characterized by three real parameters, such that corresponding solutions belong to \mathbb{D} again for some $t > 0$.

The dynamics of the delay equation on the set \mathbb{D} can be explicitly reduced to a three-dimensional discrete map Φ . To be more specific, let $\varphi \in \mathcal{C}$ be such that also $\varphi \in \mathbb{D}$. Then φ is uniquely characterized by three real parameters u, v , and h , and there exists a uniquely determined solution $x_\varphi(t)$ of the delay equation (1) through φ for all $t > 0$. For every $\varphi \in \mathbb{D}$ there exists first time $T = T(\varphi) > 0$ such that $(F^T)\varphi(s) = x_\varphi(T + s), s \in [-1, 0]$, is an element of \mathbb{D} , thus characterized by 3 real parameters, say u', v', h' . We derive an exact form for the map $\Phi : (u, v, h) \rightarrow (u', v', h')$ and prove that it possesses an attracting fixed point. The above map is shown to be conjugate to a somewhat simpler map $\Psi : (h, w, z) \rightarrow (h', w', z')$. Its fixed points are found from the three dimensional system: $\Psi(w, z, h) = (w, z, h)$, which reduces to a single nonlinear equation $G(w) = 0$, where $G(w)$ is a polynomial of degree 4. To prove the existence of a real solution of the latter the intermediate value theorem is applied to G on the interval $[0.05, 0.2]$.

The fixed point of the map Ψ corresponds to a periodic solution $x := p(t)$

of the original delay differential equation (1). Due to the piecewise constant form of the nonlinearity F it follows that there is a neighborhood $U_\varepsilon \subset C$ of the periodic solution $p(t)$ such that for every $\varphi \in U_\varepsilon$ there exists first time $\tau_0 = \tau_0(\varphi) \geq 0$ such that $(F^{\tau_0})\varphi(s) = x_\varphi(\tau_0 + s) \in \mathbb{D}$, $s \in [-1, 0]$. This implies that the local behavior of solutions close to the periodic solution is entirely determined by the three dimensional map Ψ .

The stability of the periodic solution in the space C follows then from the stability of the fixed point of the discrete map. The latter is determined by the location of the eigenvalues of the matrix $\Psi' := \partial\Psi/\partial(h, w, z)$. In general the entries for Ψ' can be calculated explicitly in terms of the parameters $a, b, c, d, \theta_1, \theta_2, \alpha$. Straightforward but lengthy estimates of the eigenvalues λ_i of Ψ' show that $|\lambda_i| < 1, i = 1, 2, 3$ uniformly for $(w, z, h) \in \Pi$, where $\Pi \in \mathbb{R}^3$ is a box containing the fixed point. The stability of the fixed point follows.

An obvious drawback of the piecewise constant nonlinearity F is that the solutions of the delay differential equation (1) are not differentiable at a discrete set of points. This creates a formal difficulty with the variational equation along the periodic solution and its stability. This difficulty is overcome by the standard by now procedure of the “smoothing” of the nonlinearity F . In a small neighborhood V_ε of its discontinuity set function F is replaced by a “close” function \tilde{F} which is continuous (in fact can be made C^∞) and coincides with F everywhere outside the neighborhood. It is shown then that the above three dimensional map Φ_ε , now dependent on ε , remains well defined and that it is uniformly C^1 -convergent to $\Phi : \Phi_\varepsilon \rightarrow \Phi$, $\partial\Phi_\varepsilon/\partial(w, z, h) \rightarrow \partial\Phi/\partial(w, z, h)$ as $\varepsilon \rightarrow 0+$.

3. A piecewise constant nonlinearity: reduction to a three-dimensional map. In this section we consider the delay differential equation

$$\frac{dx}{dt} = -\alpha x(t) + F(x(t - 1)) \tag{3}$$

with a piecewise constant nonlinearity F defined by

$$F(x) = \begin{cases} a & \text{if } x < \theta_1 \\ b & \text{if } \theta_1 \leq x < 0 \\ c & \text{if } 0 \leq x < \theta_2 \\ d & \text{if } x \geq \theta_2 \end{cases} \tag{4}$$

where $b > a > 0 > d > c$, $\theta_1 < 0 < \theta_2$, and $\alpha > 0$. A typical F is shown in Figure 1.

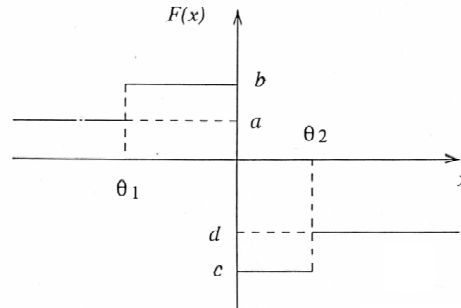


Figure 1. The piecewise constant nonlinearity defined in (4) and used to prove the existence of stable rapidly oscillating solutions of equation (1)

For any continuous initial function $\varphi(s) \in C$, it is customary to denote the corresponding solution of (3) by $x_\varphi(t)$. This solution is a continuous piecewise exponential function which can be obtained from (3) by direct integration. We note that generally, such a solution is not differentiable at a discrete set of points in $\mathbb{R}_+ = \{t : t \geq 0\}$.

We shall now restrict ourselves to a specially chosen set of initial functions $\mathbb{D} \subset C$ which is characterized by three real parameters, u , v and h . It will be shown that for any particular $\varphi \in \mathbb{D}$ there exists a time $T = T[\varphi]$ such that the segment of solution $x_\varphi(t)$ defined on the time interval $[T - 1, T]$ belongs to the set \mathbb{D} . Thus it is again characterized by the three real variables, say u' , v' , h' , and the dynamics of equation (3) is reducible to a three dimensional map $\Phi : (u, v, h) \rightarrow (u', v', h')$. The exact functional form of the transformation Φ is derived in this Section. The existence of a fixed point and its stability are also discussed.

3.1. The set \mathbb{D} of initial functions. The solution of the initial value problem

$$\frac{dx}{dt} = -\alpha x(t) + A, \quad x(t_0) \equiv x_0, \quad A - \text{a constant}$$

is

$$x(t) = \gamma_A + (x_0 - \gamma_A)e^{-\alpha(t-t_0)}, \quad \gamma_A \equiv \frac{A}{\alpha}. \quad (5)$$

A solution of the form (5) will be referred to as the “ γ_A -exponential.”

Since F given by (4) is piecewise constant, the solutions to (3) will in general be piecewise exponential continuous functions made up of pieces of

Typical Initial Function in \mathbb{D}

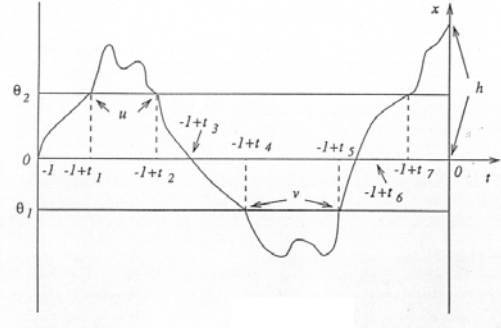


Figure 2. Schematic illustration of the form of an initial function belonging to the set \mathbb{D} defined in Section 2.1

γ_k -exponents where $k = a, b, c$ or d . Let

$$t_1 = \frac{1}{\alpha} \ln \left(\frac{\gamma_b}{\gamma_b - \theta_2} \right), \quad t_3 - t_2 = \frac{1}{\alpha} \ln \left(\frac{\gamma_d - \theta_2}{\gamma_d} \right), \quad t_4 - t_3 = \frac{1}{\alpha} \ln \left(\frac{\gamma_d}{\gamma_d - \theta_1} \right),$$

$$t_6 - t_5 = \frac{1}{\alpha} \ln \left(\frac{\gamma_b - \theta_1}{\gamma_b} \right), \quad t_7 - t_6 = \frac{1}{\alpha} \ln \left(\frac{\gamma_b}{\gamma_b - \theta_2} \right).$$

Choose positive numbers u, v and h such that $0 < u, v < 1$ and $h > \theta_2$. We now define the set \mathbb{D} of initial functions φ as follows:

$$\varphi(s) = \begin{cases} \gamma_b \text{-exponent for } s \in [-1, -1 + t_1] \text{ and } \varphi(-1) = 0 \\ > \theta_2 \text{ for } s \in (-1 + t_1, -1 + t_2) \text{ where } t_2 - t_1 \equiv u \\ \gamma_d \text{-exponent for } s \in [-1 + t_2, -1 + t_4] \\ < \theta_1 \text{ for } s \in (-1 + t_4, -1 + t_5), \text{ where } t_5 - t_4 = v \\ \gamma_b \text{-exponent for } s \in [-1 + t_5, -1 + t_7] \\ > \theta_2 \text{ for } s \in (-1 + t_7, 0) \text{ and } \varphi(0) = h. \end{cases}$$

In order for this definition to be meaningful for the resolution of the problem, we further require that

$$0 < t_1 + u + (t_4 - t_2) + v + (t_7 - t_5) < 1. \tag{6}$$

A typical geometric shape of an initial function $\varphi \in \mathbb{D}$ and the meaning of the $t_i, i = 1, 2, \dots, 7$ can be seen from Figure 2.

3.2. Solutions for $t \in [0, 1]$. For any $\varphi \in \mathbb{D}$ the corresponding solution $x_\varphi(t)$, $0 \leq t \leq 1$, is a piecewise exponential function defined as follows

$$x_\varphi(t) = \begin{cases} \gamma_c\text{-exponent in } [0, t_1], & \gamma_d\text{-exponent in } [t_1, t_2] \\ \gamma_c\text{-exponent in } [t_2, t_3], & \gamma_b\text{-exponent in } [t_3, t_4] \\ \gamma_a\text{-exponent in } [t_4, t_5], & \gamma_b\text{-exponent in } [t_5, t_6] \\ \gamma_c\text{-exponent in } [t_6, t_7], & \gamma_d\text{-exponent in } [t_7, 1]. \end{cases}$$

If we now define $x_\varphi(t_k) \equiv x_{h_{k-1}}$, we get the following set of equations for the x_{h_k} in terms of h, a, b, c, d , and α :

$$x_{h_0} = \gamma_c + \frac{(h - \gamma_c)(\gamma_b - \theta_2)}{\gamma_b} \quad (7)$$

$$x_{h_1} = \gamma_d + (x_{h_0} - \gamma_d)e^{-\alpha u} \quad (8)$$

$$x_{h_2} = \gamma_c + \frac{(x_{h_1} - \gamma_c)\gamma_d}{\gamma_d - \theta_2} \quad (9)$$

$$x_{h_3} = \gamma_b + \frac{(x_{h_2} - \gamma_b)(\gamma_d - \theta_1)}{\gamma_d} \quad (10)$$

$$x_{h_4} = \gamma_a + (x_{h_3} - \gamma_a)e^{-\alpha v} \quad (11)$$

$$x_{h_5} = \gamma_b + \frac{(x_{h_4} - \gamma_b)\gamma_b}{\gamma_b - \theta_1} \quad (12)$$

$$x_{h_6} = \gamma_c + \frac{(x_{h_5} - \gamma_c)(\gamma_b - \theta_2)}{\gamma_b}$$

$$x_{h_7} := x_\varphi(1) = \gamma_d + \frac{(x_{h_6} - \gamma_d)\gamma_b(\gamma_d - \theta_2)(\gamma_b - \theta_1)}{(\gamma_b - \theta_2)(\gamma_d - \theta_1)(\gamma_b - \theta_2)} \cdot e^{\alpha(u+v-1)}.$$

A typical solution is sketched in Figure 3. As explained in Section 3 the reduction of the original delay equation to a map rests on the fact that given an ‘‘appropriate’’ initial function φ (*i.e.* one which belongs to \mathbb{D}), the restriction of the corresponding solution $x_\varphi(t)$ to a time interval of length 1 at a later time also belongs to \mathbb{D} . This results in constraints both on the parameters of the original equation and on those describing the initial function φ (*i.e.* $\{u, v, h\}$). To simplify our discussion, we will fix the parameters of the equation and then examine the constraints on $\{u, v, h\}$.

Lemma 1. *Let $a = 6$, $b = 14$, $c = -13.5$, $d = -3.85$, $\alpha = 7.75$ and $\theta_1 = -\theta_2 = -1/3$. Then if the parameters u, v, h are chosen to satisfy the inequalities*

$$x_{h_0} > \theta_2, \quad x_{h_1} < \theta_1, \quad x_{h_3} > \theta_2, \quad (13)$$

Typical Initial Function in \mathbb{D} Sketch of corresponding solution

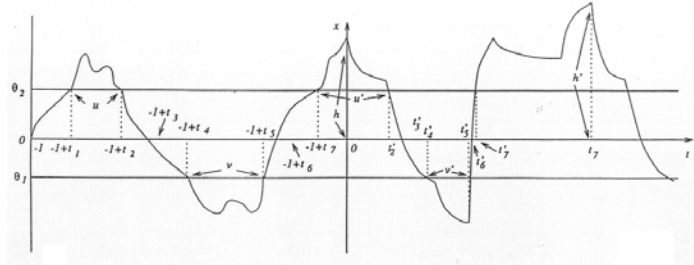


Figure 3. Schematic illustration of the form of a rapidly oscillating solution of equation (3) with nonlinearity (4)

the corresponding solution $x_\varphi(t)$, $t \in [-1 + t_6, t_6]$ is an element of \mathbb{D} .

Proof. Since $x_\varphi(t)$ is piecewise monotone for $t \geq 0$ the required condition $x_\varphi \in \mathbb{D}$ follows if the inequalities

$$x_{h_0} > \theta_2, x_{h_1} < \theta_1, x_{h_2} < \theta_1, x_{h_3} > \theta_2, x_{h_4} > \theta_2, x_{h_5} > \theta_2 \quad (14)$$

are satisfied, see Figure 3 for the geometric meaning. From (8)-(9), $x_{h_2} < x_{h_1}$ and so $x_{h_1} < \theta_1$ implies $x_{h_2} < \theta_1$; furthermore, from (11)-(12) $x_{h_5} > x_{h_4}$ and so $x_{h_4} > \theta_2$ implies $x_{h_5} > \theta_2$. Note also that since $\gamma_a > \theta_2$, $x_{h_4} > \theta_2$ automatically if $x_{h_3} > \theta_2$. As a result, the inequalities in (14) all follow from $x_{h_0} > \theta_2$, $x_{h_1} < \theta_1$, and $x_{h_3} > \theta_2$. Solving inequalities (13) gives us the following set of conditions

$$\begin{aligned} h &> \gamma_c + \frac{\gamma_b(\theta_2 - \gamma_c)}{\gamma_b - \theta_2}, \quad h < \gamma_c + \frac{\gamma_b(\gamma_d - \gamma_c)}{\gamma_b - \theta_2} + \frac{\gamma_b(\theta_1 - \gamma_d)}{\gamma_b - \theta_2} \cdot e^{\alpha u} \\ h &> \frac{\gamma_b\{(\gamma_d - \theta_2)[\gamma_d(\theta_2 - \gamma_b + (\gamma_b - \gamma_c)(\gamma_d - \theta_1))] + \gamma_d(\gamma_d - \theta_1)(\gamma_c - \gamma_d)\}}{\gamma_d(\gamma_b - \theta_2)(\gamma_d - \theta_1)} \\ &\times e^{\alpha u} + \frac{\gamma_b\gamma_d - \gamma_c\theta_2}{\gamma_b - \theta_2}. \end{aligned}$$

For the specified values of the parameters $a, b, c, d, \theta_1, \theta_2, \alpha$ it is straightforward to check that the latter inequalities define a nonempty subset in the set $h > \theta_2, 0 < u < 1, 0 < v < 1$. Note that Lemma 1 does not include any “nontrivial” lower bound on v . The bound is determined by (6). \square

With the assumptions as in Lemma 1, there exists a sequence t'_2, t'_4, t'_5, t'_6 , all in $[0, 1]$, such that the solution $x_\varphi(t)$, $\varphi \in \mathbb{D}$, has the following property:

$x_\varphi(t'_2) = \theta_2$, $x_\varphi(t'_4) = \theta_1$, $x_\varphi(t)$ is a γ_d -exponent in $[t'_2, t'_4]$, and $x_\varphi(t'_5) = \theta_1$, $x_\varphi(t'_7) = \theta_2$, $x_\varphi(t)$ is a γ_b -exponent in $[t'_5, t'_7]$ (see Figure 3). It follows that $x_\varphi(t) > \theta_2$ in $(t'_7, t'_5]$ and $x_\varphi(t) < \theta_1$ in (t'_4, t'_5) . Also, $x_\varphi(t) > \theta_2$ in $(-1 + t_7, t_1)$ (which includes a segment of the initial function φ).

Note that the solution $x_\varphi(t)$, considered within the interval $[-1 + t_6, t_6]$ as an element of C , is of the same form as the initial function φ itself. From Figure 3, it is straightforward to note that new values u' , v' , and h' can be defined for the solution $x_\varphi(t)$ ($t \in [-1 + t_6, t_6]$) in analogy to the definition of u , v and h for the initial condition. These new values are given by

$$\begin{aligned} u' &= 1 - [u + v + t_1 + (t_4 - t_2) + (t_7 - t_5)] + t'_2, \\ v' &= t'_5 - t'_4, \quad h' = x_{h_5}. \end{aligned} \quad (15)$$

Thus (15) defines a three dimensional map $\Phi : (u, v, h) \rightarrow (u', v', h')$ on a subset of $[0, 1] \times [0, 1] \times [c/\alpha, d/\alpha]$.

3.3. Exact form of the map. Since $x_\varphi(t)$ is a γ_d -exponent in $[t_1, t_2]$, t'_2 is obtained from the condition $x_\varphi(t'_2) = \theta_2$:

$$t'_2 - t_1 = \frac{1}{\alpha} \ln \frac{\gamma_d - x_{h_0}}{\gamma_d - \theta_2}. \quad (16)$$

Hence the first equation of system (15) becomes

$$u' = 1 - \frac{1}{\alpha} \ln \frac{(\gamma_d - \theta_2)^2 (\gamma_b - \theta_1)}{(\gamma_d - \theta_1)(\gamma_b - \theta_2)(\gamma_d - x_{h_0})} - u - v$$

where x_{h_0} is given by (7). To compute v' in (15) we make use of the relation

$$(t_7 - t_6) + u' + (t'_4 - t'_2) + v' + (t'_7 - t'_5) + (t_4 - t'_7) + v + (t_6 - t_5) = 1.$$

But note that $t'_4 - t'_2 = t_4 - t_2$ and $t'_7 - t'_5 = t_7 - t_5$, and therefore

$$(t_7 - t_6) + u' + (t_4 - t_2) + v' + (t_7 - t_5) + (t_4 - t'_7) + v + (t_6 - t_5) = 1$$

or $(t_4 - t'_7) + u' + v' + (t_4 - t_2) + 2(t_7 - t_5) + v = 1$. Making use of the first relation of system (15) we obtain

$$(t_4 - t'_7) + [1 - (t_4 - t_2) - (t_7 - t_5) - u - v - (t'_2 - t_1)] + v' + (t_4 - t_2) + 2(t_7 - t_5) + v = 1$$

or $(t_4 - t'_7) + (t_7 - t_5) + (t'_2 - t_1) - u + v' = 0$, which we rewrite for convenience as

$$v' = u - (t_4 - t'_7) - (t_7 - t_5) - (t'_2 - t_1). \quad (17)$$

To compute $(t_4 - t'_7)$, we exploit the fact that the solution is a γ_b -exponent in $[t'_7, t_4]$ and $x_\varphi(t_4) = x_{h_3}$. It follows that $t_4 - t'_7 = \frac{1}{\alpha} \ln \frac{\gamma_b - \theta_2}{\gamma_b - x_{h_3}}$. Combining the above with (16), equation (17) becomes

$$v' = u - \frac{1}{\alpha} \ln \frac{(\gamma_b - \theta_1)(\gamma_d - x_{h_0})}{\gamma_d(\gamma_b - x_{h_3})},$$

where x_{h_0} and x_{h_3} are given by (7) and (10) respectively. From the form of $x_\varphi(t)$ in subsection 3.2 we know that it is a γ_b -exponent on the interval $[t_5, t_6]$, and so x_{h_5} in (15) is easily computed in terms of x_{h_4} . Hence the map Φ defined by system (15) assumes the form

$$\begin{aligned} u' &= 1 - u - v - \frac{1}{\alpha} \ln \frac{(\gamma_d - \theta_2)^2(\gamma_b - \theta_1)}{(\gamma_d - \theta_1)(\gamma_b - \theta_2)(\gamma_d - x_{h_0})} \\ v' &= u - \frac{1}{\alpha} \ln \frac{(\gamma_b - \theta_1)(\gamma_d - x_{h_0})}{\gamma_d(\gamma_b - x_{h_3})}, \quad h' = \gamma_b + \frac{\gamma_b(x_{h_4} - \gamma_b)}{\gamma_b - \theta_1} \end{aligned} \tag{18}$$

where x_{h_4} , x_{h_3} , and x_{h_0} are given by (11), (10) and (7) respectively.

3.4. Reduction to a conjugate map. It is possible to rewrite the transformation (18) in a somewhat simpler form by introducing the variables

$$w \equiv e^{-\alpha u}, \quad z \equiv e^{-\alpha v}, \quad h \equiv h. \tag{19}$$

Using these (18) can be written

$$\begin{aligned} w' &= \frac{e^{-\alpha(\gamma_d - \theta_2)^2(\gamma_b - \theta_1)}}{(\gamma_d - \theta_1)(\gamma_b - \theta_2) \left[(\gamma_d - \gamma_c) - \frac{(h - \gamma_c)(\gamma_b - \theta_2)}{\gamma_b} \right] w z}, \\ z' &= \frac{(\gamma_b - \theta_1) \left[(\gamma_d - \gamma_c) - \frac{(h - \gamma_c)(\gamma_b - \theta_2)}{\gamma_b} \right] w}{(\gamma_d - \theta_1)(\gamma_b - x_{h_2})}, \quad h' = \gamma_b + \frac{\gamma_b(x_{h_4} - \gamma_b)}{\gamma_b - \theta_1} \end{aligned} \tag{20}$$

where x_{h_2} is given by (9). In deriving (20) from (18) we have made use of the expressions for h_0 and h_3 given in (7) and (10) respectively. In terms of the new variables h , w and z , the formulas (7)–(11) can be written

$$x_{h_4} = \gamma_a + (x_{h_3} - \gamma_a)z \tag{21}$$

$$x_{h_3} = \gamma_b + \frac{(x_{h_2} - \gamma_b)(\gamma_d - \theta_1)}{\gamma_d} \tag{22}$$

$$x_{h_2} = \gamma_c + \frac{(x_{h_1} - \gamma_c)\gamma_d}{\gamma_d - \theta_2} \tag{23}$$

$$x_{h_1} = \gamma_d + (x_{h_0} - \gamma_d)w \tag{24}$$

$$x_{h_0} = \gamma_c + \frac{(h - \gamma_c)(\gamma_b - \theta_2)}{\gamma_b}. \tag{25}$$

Now using (21)-(25) the map (20) can be expressed directly in terms of the parameters of the original delay equation (3) (i.e., $\alpha, a, b, c, d, \theta_1$, and θ_2). The calculations are elementary but rather lengthy, and they yield the following expression for the map (20)

$$h' = A + Bz + Czw + Dzwh, \quad w' = \frac{E}{Fwz + Gzwh}, \quad z' = \frac{Hw + Iwh}{J + Kw + Lwh}, \quad (26)$$

where

$$\begin{aligned} A &= \frac{b(a - \theta_1 \alpha)}{\alpha(b - \theta_1 \alpha)}, \\ B &= -\frac{b(ad^2 - ad\theta_2\alpha + dc\alpha\theta_2 - \theta_1\alpha^2c\theta_2 - \theta_1\alpha bd + \theta_1\alpha^2b\theta_2 - d^3 + \theta_1\alpha d^2)}{\alpha(b - \theta_1\alpha)d(d - \theta_2\alpha)}, \\ C &= \frac{(d - \theta_1\alpha)(c\alpha\theta_2 - bd)}{\alpha(b - \theta_1\alpha)(d - \theta_2\alpha)}, & D &= \frac{(b - \theta_2\alpha)(d - \theta_1\alpha)}{(b - \theta_1\alpha)(d - \theta_2\alpha)}, \\ E &= -\frac{(d - \theta_2\alpha)^2(b - \theta_1\alpha)be^{-\alpha}}{\alpha^4}, & F &= \frac{(d - \theta_1\alpha)(b - \theta_2\alpha)(c\alpha\theta_2 - bd)}{\alpha^4}, \\ G &= \frac{(d - \theta_1\alpha)(b - \theta_2\alpha)^2}{\alpha^3}, & H &= \frac{(b - \theta_1\alpha)(c\alpha\theta_2 - bd)}{\alpha^3}, \\ I &= \frac{(b - \theta_1\alpha)(b - \theta_2\alpha)}{\alpha^2}, \\ J &= \frac{(d^2 - c\alpha\theta_2 - bd + b\theta_2\alpha)(d - \theta_1\alpha)b}{\alpha^3(d - \alpha\theta_2)}, \\ K &= \frac{(d - \theta_1\alpha)(c\alpha\theta_2 - bd)d}{\alpha^3(d - \alpha\theta_2)}, & L &= \frac{(b - \theta_2\alpha)(d - \theta_1\alpha)d}{\alpha^2(d - \alpha\theta_2)}. \end{aligned}$$

The map Ψ given by system (26) is topologically conjugate, by means of the change of variables (19), to the map Φ , hence both exhibiting the same dynamical behavior.

Note that the second iteration of the map Ψ is in fact a two-dimensional map (it does not depend on z). Considering this second iteration rather than map Ψ , however, does not simplify our principal calculations. Therefore, in the sequel we will address the problem on the existence and stability of fixed points of the formally three-dimensional map Ψ .

3.5. Existence of a fixed point. The fixed points of the map Ψ are found from the system:

$$h = A + Bz + Czw + Dzwh, \quad w = \frac{E}{Fwz + Gzwh}, \quad z = \frac{Hw + Iwh}{J + Kw + Lwh}. \quad (27)$$

By using its third equation system (27) is reducible to a two-dimensional system

$$\begin{aligned} w^2(h - A)(F + Gh) &= E[B + (C + Dh)w], \\ w^3(H + Ih)(F + Gh) &= E[J + (K + Lh)w], \end{aligned} \tag{28}$$

which allows us to solve h in terms of w :

$$h = \frac{F(AI + H)w^3 + ECIw^2 + E(BI - K)w - EJ}{-G(AI + H)w^3 - DEIw^2 + ELw}.$$

Substituting this into the first equation of system (28) and using the fact that $CG - DF \equiv 0$ we obtain the following polynomial equation of degree 4:

$$(\alpha_3w^3 + \alpha_2w^2 + \alpha_1w + \alpha_0)(\beta_1w + \beta_0) + (\gamma_2w^2 + \gamma_1w + \gamma_0)(\delta_2w^2 + \delta_1w + \delta_0) = 0,$$

where $\alpha_3 = (AI + H)(AG + F)$, $\alpha_2 = EI(AD + C)$, $\alpha_1 = E(BI - AL - K)$, $\alpha_0 = -EJ$, $\beta_1 = G(BI - K) + FL$, $\beta_0 = -GJ$, $\gamma_2 = ECI(1 - D) - BG(AI + H)$, $\gamma_1 = E(BI + CL - K - BDI)$, $\gamma_0 = E(BL - DJ)$, $\delta_2 = G(AI + H)$, $\delta_1 = DEI$, $\delta_0 = -EL$. The polynomial equation can be rewritten as

$$P_4(w) := b_4w^4 + b_3w^3 + b_2w^2 + b_1w + b_0 = 0,$$

where the coefficients $b_i, i = 0, 1, 2, 3, 4$ are given by the following formulas: $b_4 = \alpha_3\beta_1 + \gamma_2\delta_2$, $b_3 = \alpha_3\beta_0 + \alpha_2\beta_1 + \gamma_2\delta_1 + \gamma_1\delta_2$, $b_2 = \alpha_2\beta_0 + \alpha_1\beta_1 + \gamma_2\delta_0 + \gamma_1\delta_1 + \gamma_0\delta_2$, $b_1 = \alpha_1\beta_0 + \alpha_0\beta_1 + \gamma_1\delta_0 + \gamma_0\delta_1$, $b_0 = \alpha_0\beta_0 + \gamma_0\delta_0$, and therefore expressible in terms of the coefficients $A, B, C, D, E, F, G, H, I, J, K, L$ and eventually in terms of the original coefficients $a, b, c, d, \theta_1, \theta_2, \alpha$ of the delay differential equation. We do not include the explicit form of the expressions here due to their length. Note that the above reduction formally leads to a polynomial of degree 6, however, the first two coefficients before w^6 and w^5 appear to be 0.

It is straightforward to calculate that $P_4(0.05) > 0$ and that $P_4(0.2) < 0$, therefore, the equation $P_4(w) = 0$ has a solution w^* inside the interval $[0.05, 0.2]$. The numerical value of this solution with the accuracy of 10^{-10} can be found as $w^* \simeq 0.0999515864$. This gives the corresponding numerical values for the other two coordinates $h^* \simeq 0.9648653295$, $z^* \simeq 0.2744271004$. The values of the original variables at this fixed point are found then as

$$u^* \simeq 0.2971702382, \quad v^* \simeq 0.1668476936, \quad h^* \simeq 0.9648653295. \tag{29}$$

4. Stability of the fixed point. Clearly, in order to prove the stability of the fixed point (h^*, w^*, z^*) given here, it is necessary to determine the eigenvalues of the linearization of the map (26) about the fixed point, in order to check that all three lie inside the unit disc. The matrix whose eigenvalues we are interested in is:

$$\Psi' := \partial\Psi/\partial(h, w, z)|_{(h^*, w^*, z^*)} := (a_{ij}), \quad i, j = 1, 2, 3.$$

It is straightforward to evaluate the derivatives of the three functions in (26) and to obtain the entries of matrix Ψ' :

$$\begin{aligned} a_{11} &= Dwz, & a_{12} &= Cz + Dhz, \\ a_{13} &= B + Cw + Dhwr, & a_{21} &= -\frac{GE}{wz(F + Gh)^2}, \\ a_{22} &= -\frac{E}{(Fz + Ghz)w^2}, & a_{23} &= -\frac{E}{Fw + Ghw}z^2, \\ a_{31} &= \frac{IJw + (IK - LH)w^2}{J + Kw + Lwh}, & a_{32} &= \frac{J(H + Ih)}{[J + (K + Lh)w]^2}, \quad a_{33} = 0. \end{aligned}$$

The numerical value at the parameters for which the delay equation (3) yields the solution of Figure 3 is:

$$\Psi' = \begin{pmatrix} 0.0037180093 & 0.0438895824 & 0.1088367774 \\ -0.0847127965 & -1 & -0.3642190814 \\ 0.2374628028 & 2.8031515040 & 0 \end{pmatrix}.$$

Since second iteration of Ψ is a two-dimensional map, at least one eigenvalue of matrix Ψ' , say λ_1 , is zero (for all values of (h, w, z)). Additionally, the determinant of matrix Ψ' can be expressed explicitly in terms of the coefficients $A, B, C, D, E, F, G, H, I, J, K, L$ and then eventually in terms of the original parameters $a, b, c, d, \theta_1, \theta_2, \alpha$. The formulas are too lengthy to be included here, however, the direct evaluation of the determinant at the parameter values $a = 6, b = 14, c = -13.5, d = -3.85, \alpha = 7.75$ and $\theta_1 = -\theta_2 = -1/3$ shows that $\det \Psi' = 0$. Two other eigenvalues can be found by solving the corresponding quadratic equation obtained from the characteristic equation of matrix Ψ' by dividing by λ . With accuracy of 10^{-10} one finds

$$\lambda_{2,3} \simeq -0.4981409953 \pm 0.8642754940I, \quad \text{and} \quad |\lambda_{2,3}| \simeq 0.9951165807.$$

We shall show next that there exists a box $\Pi \in \mathbb{R}^3$ containing the fixed point (h^*, w^*, z^*) such that for every $(h, w, z) \in \Pi$ all eigenvalues of matrix

Ψ' lie strictly inside the unit disc. Before giving more specific detail on the proof we note that the statement is in fact obvious due to the continuous dependence of the eigenvalues on the coefficients a_{ij} which in turn depends continuously on (h, w, z) .

Since $\lambda_1 = 0$ it follows that $\lambda_2 \cdot \lambda_3 = a_{11}a_{22} - a_{12}a_{21} - a_{13}a_{31} - a_{23}a_{32}$. In view of $|\lambda_2| = |\lambda_3|$ and using the explicit form of the coefficients a_{ij} one easily finds:

$$\begin{aligned}
 |\lambda_{2,3}|^2 = & -\frac{DE}{(F + Gh)w} + \frac{GE(C + Dh)}{(F + Gh)^2w} \\
 & - \frac{(B + Cw + Dhw)[IJw + (IK - LH)w^2]}{(J + Kw + Lhw)^2} \\
 & + \frac{EJ(H + Ih)}{z^2w(F + Gh)[J + (K + Lh)w]^2}.
 \end{aligned} \tag{30}$$

One can formally differentiate the expression on the right of (30) to show that it is monotone with respect to each of the variables h, w , and z . More specifically

- it is monotonically increasing for $h \in [h^* - 2 \cdot 10^{-4}, h^* + 2 \cdot 10^{-4}]$
- it is monotonically decreasing for $w \in [w^* - 2 \cdot 10^{-4}, w^* + 2 \cdot 10^{-4}]$
- it is monotonically decreasing for $z \in [z^* - 2 \cdot 10^{-4}, z^* + 2 \cdot 10^{-4}]$.

We do not include the exact expressions for the partial derivatives and their estimates here due to their length.

Since $\lambda_1 = 0$ for all $(h, w, z) \in \Pi := [h^* - 2 \cdot 10^{-4}, h^* + 2 \cdot 10^{-4}] \times [w^* - 2 \cdot 10^{-4}, w^* + 2 \cdot 10^{-4}] \times [z^* - 2 \cdot 10^{-4}, z^* + 2 \cdot 10^{-4}]$ it follows that the maximum of the absolute value of the eigenvalues $\lambda_{2,3}$ is attained when $h_c = h^* + 2 \cdot 10^{-4}, w_c = w^* - 2 \cdot 10^{-4}, z_c = z^* - 2 \cdot 10^{-4}$. It is straightforward to evaluate the numerical value of the above expression for $\lambda_{2,3}$ at (h_c, w_c, z_c) in order to get $|\lambda_{2,3}|^2 = 0.998638301$.

Since all the eigenvalues evaluated inside the box Π , which obviously includes the fixed point, lie inside the the unit disc, we have proved that at the parameter values $a = 6, b = 14, c = -13.5, d = -3.85, \alpha = 7.75, \theta_1 = -\theta_2 = -1/3$, the map Ψ given by (25) has an attracting fixed point.

4.1. Stability in the delay equation. Fixed point (u^*, v^*, h^*) of the map (15) corresponds to a rapidly oscillating periodic solution $x_* := p(t)$ of the original delay differential equation. Since the corresponding initial function $\varphi_* \in \mathbb{D}$ has three zeros, this periodic solution belongs to the set \mathbb{S}_3 of the Morse decomposition (see Section 2 and [8]). To show that the periodic solution $p(t)$ is asymptotically stable in the phase space $C = C[-1, 0]$ we need

to prove that there exists a neighborhood $U_\delta \subset C$ of the periodic solution such that for every $\varphi \in U_\delta$ the corresponding solution $x_\varphi := (F^t\varphi)(s)$ gets attracted by x_* . This fact follows from the following statement.

Lemma 2. *There exists a neighborhood $U_\delta \subset C$, $\delta > 0$ of the initial function $\varphi_* \in \mathbb{D}$ such that for every $\varphi \in U_\delta$ the corresponding solution $x_\varphi(t)$ has the property:*

- a) *there exists first time $t_6 = t_6(\varphi) > 0$ such that $x_\varphi(t_6 + s) \in \mathbb{D}$, $s \in [-1, 0]$, and*
- b) *$t_6(\varphi) \rightarrow t_6(\varphi_*)$ as $\delta \rightarrow 0$.*

The proof of this statement is trivial and it follows immediately, due to the continuous dependence of solutions of the delay differential equation (3) on the initial conditions and the fact that the values of $\dot{x}^*(t)$ at the discontinuity set (7 points on the period, see Fig. 3) lie outside the discontinuity set of the nonlinearity F . Small perturbations in C of the initial function $\varphi_* \in \mathbb{D}$ result in piecewise exponential solutions which are close to $x_*(t)$ and made up of the same type exponents.

Lemma 2 implies that for $\varphi \in U_\delta$ the local dynamics of the semiflow $(F^t\varphi)(s)$, $s \in [-1, 0]$ as $t \rightarrow \infty$ is governed by the three-dimensional map Ψ . This in turn implies that the periodic solution $x_*(t)$ is asymptotically stable.

5. Smoothed piecewise constant nonlinearity: Persistence of dynamics. In this section, we consider the equation

$$\frac{dx}{dt} = -\alpha x(t) + \tilde{F}(x(t-1)), \quad (31)$$

where the nonlinearity \tilde{F} is a continuous function which is close to the nonlinearity F of the previous sections. More precisely, \tilde{F} is defined as follows:

$$\tilde{F} = \begin{cases} a & \text{if } x < \theta \\ a + [(b-a)/2\varepsilon]x & \text{if } x \in [\theta_1 - \varepsilon, \theta_1 + \varepsilon] \\ b & \text{if } x \in (\theta_1 + \varepsilon, -\varepsilon) \\ b + [(c-b)/2\varepsilon]x & \text{if } x \in [-\varepsilon, \varepsilon] \\ c & \text{if } x \in (\varepsilon, \theta_2 - \varepsilon) \\ c + [(d-c)/2\varepsilon]x & \text{if } x \in [\theta_2 - \varepsilon, \theta_2 + \varepsilon] \\ d & \text{if } x > \theta_2 + \varepsilon. \end{cases}$$

Hence, \tilde{F} is chosen so as to be piecewise constant and coinciding with F everywhere except in ε -neighborhoods of θ_1 , θ_2 and 0, in which it is piecewise linear.

If ε is small enough, we say that \tilde{F} is close to F . Note also that in general \tilde{F} can be made arbitrarily smooth by appropriately defining the “matching” in the ε -neighborhoods of the discontinuity points of F .

5.1. A set of initial functions. Fix small $\varepsilon > 0$, and define the set \mathbb{D}_ε of initial functions φ as follows

$$\varphi(s) = \begin{cases} \gamma_b - \text{exponent for } s \in [-1, -1 + s_1] \text{ and } \varphi(-1) = 0 \\ > \theta_2 + \varepsilon \text{ for } s \in (-1 + s_1, -1 + s_2) \text{ where } s_2 - s_1 \equiv u \\ \gamma_d - \text{exponent for } s \in [-1 + s_2, -1 + s_4] \\ < \theta_1 - \varepsilon \text{ for } s \in (-1 + s_4, -1 + s_5), \text{ where } s_5 - s_4 = v \\ \gamma_b - \text{exponent for } s \in [-1 + s_5, -1 + s_7] \\ > \theta_2 + \varepsilon \text{ for } s \in (-1 + s_7, 0) \text{ and } \varphi(0) = h. \end{cases}$$

Note that when u, v , and h are fixed, all the s_i are determined uniquely, since the initial function is piecewise exponential between the levels $\theta_1 - \varepsilon$ and $\theta_2 + \varepsilon$. It is therefore straightforward to obtain

$$\begin{aligned} s_1 &= \frac{1}{\alpha} \ln \frac{\gamma_b}{\gamma_b - (\theta_2 + \varepsilon)}, & s_3 - s_2 &= \frac{1}{\alpha} \ln \frac{\gamma_d - (\theta_2 + \varepsilon)}{\gamma_d}, \\ s_4 - s_3 &= \frac{1}{\alpha} \ln \frac{\gamma_d}{\gamma_d - (\theta_1 - \varepsilon)}, & s_6 - s_5 &= \frac{1}{\alpha} \ln \frac{\gamma_b - (\theta_1 - \varepsilon)}{\gamma_b}, \\ s_7 - s_6 &= \frac{1}{\alpha} \ln \frac{\gamma_b}{\gamma_b - (\theta_2 + \varepsilon)}. \end{aligned}$$

Given these preliminaries, we are now ready to proceed as in Section 3, by reducing (31) to a discrete time map.

5.2. Reduction to a three dimensional map. In this section, it is shown that if $\varepsilon > 0$ is small enough, choosing u, v and h appropriately allows us the reduction of the continuous time dynamics to a discrete-time map in a manner analogous to the way in which we obtained (15) in the previous section. This follows from the fact that as $\varepsilon \rightarrow 0$, the solutions of (31) converge to those of (3) (*cf.* Lemma 3 below).

Let $\varphi \in \mathbb{D}_\varepsilon$, and denote by $x_\varepsilon(t, \varphi)$ and $x(t, \varphi)$ the corresponding solutions of (31) and (3) respectively.

Lemma 3 (continuous dependence on ε). $x_\varepsilon(t, \varphi) \rightarrow x(t, \varphi)$ as $\varepsilon \rightarrow 0$ uniformly for $t \in [0, 1]$.

Proof. Let $r_1 \equiv \frac{1}{\alpha} \ln \gamma_b / (\gamma_b - \varepsilon)$. r_1 is the first point larger than -1 at which $\varphi = \varepsilon$. Since F and \tilde{F} are different in a small neighborhood of the

discontinuity points of F and coincide elsewhere, it is sufficient to show that solutions $x_\varepsilon(t, \varphi)$ and $x(t, \varphi)$ are uniformly close in $[0, r_1]$. The uniform closeness in $[0, 1]$ then follows by induction arguments.

The solution $x(t, \varphi)$ on $[0, r_1]$ is a γ_b -exponent and is given by

$$x(t, \varphi) = \gamma_b + (h - \gamma_b)e^{-\alpha t}.$$

The solution $x_\varepsilon(t, \varphi)$ is given explicitly by

$$x_\varepsilon(t, \varphi) = \left[h + \int_0^t \frac{\gamma_b(c-b)}{2\varepsilon} (e^{\alpha s} - 1) : ds \right] e^{-\alpha t}.$$

Clearly, $x(t, \varphi) \leq x_\varepsilon(t, \varphi) \leq h$. Since $x(0, \varphi) = x_\varepsilon(0, \varphi) = h$ and $r_1 \rightarrow 0$ as $\varepsilon \rightarrow 0$, the uniform closeness in $[0, r_1]$ follows. \square

Lemma 3 implies that if h , u , and v are chosen close to the fixed point (h_*, u_*, v_*) of the map Φ , then if ε is small enough, one can define h' , u' and v' in a manner which is analogous to their definition in Section 2. We now introduce the following notations:

- s'_2 : first time in $[0, 1]$ at which $x_\varepsilon(t, \varphi) = \theta_2 + \varepsilon$
- s'_3 : first time in $[0, 1]$ at which $x_\varepsilon(t, \varphi) = 0$
- s'_4 : first time in $[0, 1]$ at which $x_\varepsilon(t, \varphi) = \theta_1 - \varepsilon$
- s'_5 : second time in $[0, 1]$ at which $x_\varepsilon(t, \varphi) = \theta_1 - \varepsilon$
- s'_6 : second time in $[0, 1]$ at which $x_\varepsilon(t, \varphi) = 0$
- s'_7 : second time in $[0, 1]$ at which $x_\varepsilon(t, \varphi) = \theta_2 + \varepsilon$.

The existence of the s_i 's is a consequence of the continuous dependence of the solution of (31) (with nonlinearity \tilde{F}) on ε , and the explicit solution of equation (3) (with nonlinearity F given by (4) (cf. Figure 3)). Moreover, the solution $x_\varepsilon(t, \varphi)$ is a γ_b -exponent in $[s'_2, s'_3]$, a γ_b -exponent in $[s'_2, s'_4]$, and a γ_d -exponent in $[s'_5, s'_7]$. Thus one can define the discrete time map $\Phi_\varepsilon : (u, v, h) \rightarrow (u', v', h')$ as follows:

$$\begin{aligned} u' &= 1 - [u + v + s_1 + (s_4 - s_2) + (s_7 - s_5)] + s'_2 \\ v' &= s'_5 - s'_4, \quad h' = x_{h_5}, \end{aligned} \tag{32}$$

where $x_{h_5} \equiv x_\varepsilon(s_6, \varphi)$.

5.3. C^1 -closeness between Φ_ε and Φ . In this section, we shall show that the map Φ_ε is C^1 -close to the map Φ which was derived in Section 2. From this property, one deduces that the map Φ_ε possesses a fixed point $(u_\varepsilon^*, v_\varepsilon^*, h_\varepsilon^*)$ close to the fixed point (u^*, v^*, h^*) of the map Φ . Thus, the delay equation (31) with the continuous nonlinearity \tilde{F} possesses an asymptotically stable rapidly oscillating periodic solution.

Theorem 1. $\Phi_\varepsilon \rightarrow \Phi$ and $\Phi'_\varepsilon \rightarrow \Phi'$ as $\varepsilon \rightarrow 0^+$.

Proof. We shall prove the convergence for the first component of the map Φ_ε . For other components, the proof is analogous. To obtain the expression for this first component of the map, we note that

$$\begin{aligned} s'_4 - s'_2 &= s_4 - s_2 = \frac{1}{\alpha} \ln \frac{\gamma_d - (\theta_2 + \varepsilon)}{\gamma_d - (\theta_1 - \varepsilon)}, \\ s'_7 - s'_5 &= s_7 - s_5 = \frac{1}{\alpha} \ln \frac{\gamma_b - (\theta_1 - \varepsilon)}{\gamma_b - (\theta_2 + \varepsilon)}, \end{aligned}$$

and, as calculated earlier, $s_1 = \frac{1}{\alpha} \ln \frac{\gamma_b}{\gamma_b - (\theta_2 + \varepsilon)}$. To find s'_2 , determine $x_\varepsilon(t, \varphi)$ explicitly. For $t \in [0, r_1]$, $x_\varepsilon(t, \varphi)$ is the solution of the following initial value problem:

$$\frac{dx(t)}{dt} = -\alpha x(t) + \frac{c-b}{2\varepsilon} x(t-1), \quad x(0) = h$$

which can be written

$$\frac{dx(t)}{dt} = -\alpha x(t) + \frac{c-b}{2\varepsilon} (\gamma_b - \gamma_b e^{-\alpha t}), \quad x(0) = h$$

since $\varphi(s) = \gamma_b - \gamma_b e^{-\alpha(s+1)}$, for $s \in [-1, -1 + r_1]$. Therefore,

$$\tilde{h} \equiv x_\varepsilon(r_1, \varphi) = \left[h + \int_0^{r_1} \frac{\gamma_b(c-b)}{2\varepsilon} (e^{\alpha s} - 1) : ds \right] \frac{\gamma_b - \varepsilon}{\gamma_b}. \quad (33)$$

Let $x_\varepsilon(s_1, \varphi) \equiv x_{h_0}$. Since $x_\varepsilon(t, \varphi)$ is a γ_c -exponent for $t \in [r_1, s_1]$ and $s_1 - r_1 = \frac{1}{\alpha} \ln \frac{\gamma_b - \varepsilon}{\gamma_b - \varepsilon - \theta_2}$, it is straightforward to obtain

$$x_{h_0} = \gamma_c + (\tilde{h} - \gamma_c) \frac{\gamma_b - \theta_2 - \varepsilon}{\gamma_b - \varepsilon}. \quad (34)$$

Finally, $s'_2 - s_1$ is found from the fact that $x_\varepsilon(t, \varphi)$ is a γ_d -exponent in $[s_1, s'_2]$, where $s'_2 - s_1 = \frac{1}{\alpha} \ln \frac{x_{h_0} - \gamma_d}{\theta_2 + \varepsilon - \gamma_d}$. Substitution into the first equation of (32) yields

$$u' = 1 - \frac{1}{\alpha} \ln \frac{(\gamma_d - \theta_2 - \varepsilon)(\gamma_b - \theta_1 + \varepsilon)(\theta_2 + \varepsilon - \gamma_d)}{(\gamma_d - \theta_1 + \varepsilon)(\gamma_b - \theta_2 - \varepsilon)(x_{h_0} - \gamma_d)} - u - v \quad (35)$$

where x_{h_0} is given by (34) and \tilde{h} by (33). Direct evaluation of (35) and its partial derivatives when $\varepsilon = 0$ and comparing with the first equation in (15) concludes the proof of the convergence for the first component of Ψ_ε .

Typical Initial Function in \mathbb{D} Sketch of corresponding solution

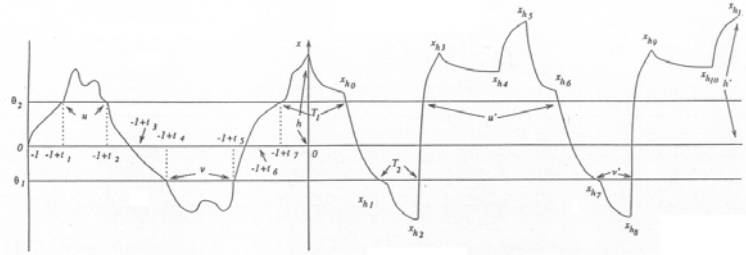


Figure 4. Illustration of the position of the x_{h_i} 's which are used to derive the map (36)

6. Numerical aspects. In the above proof of the existence and stability of a fixed point several numerical calculations were employed. Specifically, we have determined the location of the fixed point (h^*, w^*, z^*) with accuracy 10^{-10} . Likewise, we gave an upper estimate on the location of the eigenvalues of the linearization of the map Ψ about the fixed point (h^*, w^*, z^*) . The choice of this accuracy is arbitrary, and we could have illustrated our analysis with numerical results of precision 10^{-5} or 10^{-200} (the latter was actually used to do some additional numerical check). Clearly, in principal these computations can be carried out manually, but the help of a symbolic manipulator greatly accelerates the process.

It should be noted that the map (26) above is not the only possible choice for our description of the rapidly oscillating solution of Figure 3. As an alternative, one can choose the following system, which is much more cumbersome to treat analytically but which provides us with an independent “check” for the algebra above (it is obtained by considering Figure 4 rather than Figure 3):

$$\begin{aligned}
 u_{t+1} &= v_t + \frac{1}{\alpha} \ln \left[\frac{(xh_{6t} - \gamma_d)(\gamma_b - \theta_1)}{(\gamma_b - xh_{3t})(\theta_2 - \gamma_d)} \right], \\
 v_{t+1} &= \frac{1}{\alpha} \ln \left[\frac{(xh_{7t} - \gamma_d)(\theta_1 - \gamma_b)\gamma_d}{(\theta_1 - \gamma_d)(xh_{8t} - \gamma_b)(\gamma_d - \theta_2)} \right], \quad h_{t+1} = \gamma_b - \frac{\gamma_b(xh_{10t} - \gamma_b)}{(\gamma_b - \theta_1)},
 \end{aligned}
 \tag{36}$$

where the following relations, obtained from a careful consideration of Figure 4, implicitly define the dependents of the variables $h_{t+1}, u_{t+1}, v_{t+1}$ on their preimages:

$$xh_{10t} = \gamma_a + (xh_{9t} - \gamma_a)e^{-\alpha T_{2t}}, \quad xh_{9t} = \gamma_b + \frac{(xh_{8t} - \gamma_b)(\gamma_d - \theta_1)}{\gamma_d}$$

$$\begin{aligned}
 xh_{8t} &= \gamma_c + \frac{(xh_{7t} - \gamma_c)\gamma_d}{(\gamma_d - \theta_2)}, & xh_{7t} &= \gamma_d + (xh_{6t} - \gamma_d)e^{-\alpha T_{1t}} \\
 xh_{6t} &= \gamma_c + \frac{(xh_{5t} - \gamma_c)(\gamma_b - \theta_2)}{\gamma_b}, & xh_{5t} &= \gamma_b + \frac{\gamma_b(xh_{4t} - \gamma_b)}{(\gamma_b - \theta_1)} \\
 xh_{4t} &= \gamma_a + (xh_{3t} - \gamma_a)e^{-\alpha v_t}, & xh_{3t} &= \gamma_b + \frac{(xh_{2t} - \gamma_b)(\gamma_d - \theta_2)}{\gamma_d} \\
 xh_{2t} &= \gamma_c + \frac{\gamma_d(xh_{1t} - \gamma_c)}{(\gamma_d - \theta_1)}, & xh_{1t} &= \gamma_d + (xh_{0t} - \gamma_d)e^{-\alpha u_t}, \\
 xh_{0t} &= \gamma_c + \frac{(h_t - \gamma_c)(\gamma_b - \theta_1)}{\gamma_b}, & T_{2t} &= \frac{-1}{\alpha} \ln \left[\frac{(\gamma_d - xh_{1t})\gamma_b}{(\theta_1 - \gamma_d)(xh_{2t} - \gamma_b)} \right] \\
 T_{1t} &= 1 - u_t - v_t - \frac{1}{\alpha} \ln \left[\frac{(\gamma_d - \theta_2)^2(\gamma_b - \theta_1)}{(\gamma_d - xh_{0t})(\gamma_b - \theta_2)(\gamma_d - \theta_1)} \right].
 \end{aligned}$$

Once again, it is possible, using a symbolic manipulator, to determine the fixed points of the three dimensional map (36), but the closed form expression of this fixed point is too lengthy to be printed out in the scope of a paper. It is possible however to evaluate this expression at specific parameter values to an arbitrary degree of accuracy. For the parameters given above, the fixed point $(\tilde{h}, \tilde{u}, \tilde{v})$ of map (36) is

$$\tilde{h} \simeq 0.9648653295, \quad \tilde{u} \simeq 0.2971702382, \quad \tilde{v} \simeq 0.1668476936,$$

and as expected it coincides with that given in (29).

As an additional check, it is possible to integrate the original delay differential equation using a simple integration scheme, which is very inefficient, but is known to converge to the solution of the delay equation as the integration step is going to 0. This gives a third way to check the results presented here, which yields a numerical check of accuracy 10^{-5} which agrees with (29). It is not feasible to increase the accuracy of this third check because the computing times required become prohibitively large.

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