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WELL-POSEDNESS FOR THE ZAKHAROV SYSTEM WITH THE PERIODIC BOUNDARY CONDITION

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Abstract. Under a non resonance condition, we establish the unique local existence results in weak function spaces for the initial value problem of the Zakharov system with the periodic boundary condition. The proof is based on the Fourier restriction norm method, which was developed by J. Bourgain and C.E. Kenig, G. Ponce and L. Vega.

1. Introduction and Main result. In this paper we consider the initial value problem with the periodic boundary condition for the one dimensional Zakharov system:

$$\begin{cases} i\partial_t u + \alpha\partial_x^2 u = un, \ (x,t) \in \mathbb{T} \times (-T,T), \\ \beta^{-2}\partial_t^2 n - \partial_x^2 n = \partial_x^2(|u|^2), \ (x,t) \in \mathbb{T} \times (-T,T), \\ u(x,0) = u_0(x), \ n(x,0) = n_0(x), \ \partial_t n(x,0) = n_1(x), \ x \in \mathbb{T}, \end{cases} \tag{1.1}$$

where α and β are real constants with $\alpha \neq 0, \beta > 0$, u and n are a complex valued and a real valued function, respectively, $\mathbb{T}$ is a one dimensional torus which implies the periodic boundary condition and T is a positive constant to be determined later.

The system (1.1) describes the propagation of Langmuir turbulence waves in an unmagnetized completely ionized hydrogen plasma (see [26]). The function $u(x,t)$ denotes the slowly varying envelope of the electric field E with the frequency ω such that $E(x,t) = \mathrm{Re}(u(x,t)\exp(-it\omega))$. The function $n(x,t)$ denotes the deviation of the ion density from the equilibrium. The constant α is a dispersion coefficient and the constant β is the speed of ion acoustic wave in a plasma.

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Our interest is to prove the well-posedness results in a weak class for the initial value problem (1.1). The difficulty to deal with the initial value problem (1.1) is that the second equation of (1.1) has second derivatives in the nonlinear term. One derivative can be regained because of the second order hyperbolicity of the second equation of (1.1). But it is a hard task to regain one more derivative. Then we meet with the difficulty of the so-called "loss of derivative", which comes from the second derivatives of the second equation of (1.1).

There are many papers concerning the Zakharov system for the $\mathbb{R}^d$ case (see, e.g., [1, 2, 11, 12, 16, 20, 21, 22, 23]). When $d = 2, 3$, J. Bourgain and J. Colliander [11] proved the well-posedness in the energy space. In [11] the Fourier restriction norm method was used, which was developed by Bourgain and Kenig-Ponce-Vega. In [12], J. Ginibre, Y. Tsutsumi and G. Velo showed that when $d = 1$, $-1/2 < s_0 - s_1 \leq 1$ and $2s_0 \geq s_1 + 1/2 \geq 0$, the one dimensional Zakharov system is well-posed for $(u_0, n_0, n_1) \in H^{s_0} \times H^{s_1} \times H^{s_1-1}$, by using a variant of the method developed by Bourgain. Furthermore, they studied the well-posedness in higher space dimensions. Thus, the Cauchy problem of the Zakharov system for the $\mathbb{R}^d$ case has extensively been studied. However, to our knowledge, there are only a few papers concerning the problem with the periodic boundary condition (see, e.g., [10]). In this paper, we consider the initial value problem with the periodic boundary condition for the one dimensional Zakharov system. In [10] Bourgain showed that when $\alpha = \beta = 1$, the initial value problem (1.1) is locally well-posed in the following sense. Assume that the initial data (u_0, n_0, n_1) satisfy $u_0 \in H^s$ and

$$\begin{cases} \sup_{k\in\mathbb{Z}} \langle k\rangle^{s_1} |\mathcal{F}_x u_0(k)| < \infty, \\ \sup_{k\in\mathbb{Z}} \langle k\rangle^{\sigma} |\mathcal{F}_x n_0(k)| < \infty, \\ \sup_{k\in\mathbb{Z}} \langle k\rangle^{\sigma-1} |\mathcal{F}_x n_1(k)| < \infty, \end{cases} \qquad \textbf{[B]}$$

with $\sigma < 0 < s < 1/2 < s_1 < 1$, where σ and s are sufficiently close to 0 and 1/2, respectively. Then the initial value problem (1.1) is locally well-posed. In [10] the Fourier restriction norm method was used, which was developed for the Schrödinger and the KdV equation in [6] and [7], respectively.

The aim in this paper is to investigate the time local well-posedness in a weak space for (1.1), by using the method due to Bourgain [6,7,10] and Kenig-Ponce-Vega [14,15]. In this paper, under the non resonance condition that β/α is not an integer, we show the local well-posedness in $H^{s_0} \times H^{s_1} \times$

H^{s_1-1}, when $0 \le s_0 - s_1 \le 1$ and $0 \le s_1 + 1/2 \le 2s_0$. The weakest Sobolev class of solutions that our result in this paper can cover is $L^2 \times H^{-1/2} \times H^{-3/2}$. We do not have to assume the weighted L^∞ condition on the initial data in the Fourier space such as [B], while we assume the non resonance condition that β/α is not an integer. As long as we consider the case that β/α is not an integer, our result is an improvement of a previous result in [10].

Before precisely stating our results, we prepare the following notations.

Definition 1.1. Let $\mathcal{V}$ be the space of functions f such that

$$\begin{aligned} &\text{(i)} \qquad f : \mathbb{T} \times \mathbb{R} \to \mathbb{C}, \\ &\text{(ii)} \qquad f(x, \cdot) \in \mathcal{S}(\mathbb{R}) \text{ for each } x \in \mathbb{T}, \\ &\text{(iii)} \qquad f(\cdot, t) \in C^\infty(\mathbb{T}) \text{ for each } t \in \mathbb{R}. \end{aligned}$$

For $s \in \mathbb{R}$, we define the spaces X_s and Y_s to be the completion of $\mathcal{V}$ with respect to the norms: $\|f\|_{X_s} = \|f\|_{(1,s,1/2)}$, $\|f\|_{Y_s} = \|f\|_{(2,s,1/2)}$, respectively, where

$$\|f\|_{(1,s,b)} = \|\langle n\rangle^s \langle \tau + \alpha n^2\rangle^b \widehat{f}(n,\tau)\|_{l^2_n L^2_\tau},$$

$$\|f\|_{(2,s,b)} = \|\langle n\rangle^s \langle |\tau| - \beta|n|\rangle^b \widehat{f}(n,\tau)\|_{l^2_n L^2_\tau}.$$

Throughout this paper, we shall let $\psi \in C_0^\infty(\mathbb{R})$ denote a smooth cut off function such that $\psi = 1$ on $[-1,1]$ and support$\psi \subseteq (-2,2)$. For $\delta > 0$, we put $\psi_\delta(t) = \psi(t/\delta)$.

Our main result is the following theorem.

Theorem 1.1. *Assume that β/α is not an integer. Let s_0 and s_1 satisfy $0 \le s_0 - s_1 \le 1$ and $0 \le s_1 + 1/2 \le 2s_0$. For any $(u_0, n_0, n_1) \in H^{s_0} \times H^{s_1} \times H^{s_1-1}$, there exist $T = T(\|u_0\|_{L^2}, \|n_0\|_{H^{-1/2}}, \|n_1\|_{H^{-3/2}}) > 0$ and a unique solution $(u(t), n(t), \partial_t n(t))$ of the initial value problem* (1.1) *in the time interval $[-T,T]$ such that*

$$u \in C([-T,T] : H^{s_0}(\mathbb{T})), \ \psi_T u \in X_{s_0}, \tag{1.2}$$

$$n \in C([-T,T] : H^{s_1}(\mathbb{T})), \ \psi_T n \in Y_{s_1}, \tag{1.3}$$

$$\partial_t n \in C([-T,T] : H^{s_1-1}(\mathbb{T})), \ \psi_T \partial_t n \in Y_{s_1-1}. \tag{1.4}$$

For any $T' \in (0,T)$, there exists a neighborhood $\mathfrak{V}$ of $(u_0, n_0, n_1) \in H^{s_0} \times H^{s_1} \times H^{s_1-1}$ such that the map $(\widetilde{u}_0, \widetilde{n}_0, \widetilde{n}_1) \mapsto (\widetilde{u}(t), \widetilde{n}(t), \partial_t \widetilde{n}(t))$ from $\mathfrak{V}$ into the class defined in (1.2)–(1.4) *with T replaced by T' is Lipschitz.*

Remark 1.1. By Theorem 1.1, we have the local well-posedness in $H^{s_0} \times H^{s_1} \times H^{s_1-1}$, when $s_0 = s_1 + 1/2$. Of course, the result in Theorem 1.1 covers the Hamiltonian class, that is, the energy space.

Remark 1.2. The lowest admissible values $(s_0, s_1) = (0, -1/2)$ that can be reached by Theorem 1.1 are the same as in [12] for the case of $\mathbb{R}$.

In the case of $\mathbb{R}^d$, the scaling argument suggests that the critical values for the Zakharov system may be $s_0 = s_1 + 1/2 = d/2 - 3/2$ (see [12]), from which it seems natural that the difference of s_0 and s_1 be $1/2$. Notice that the difference of the lowest admissible values (s_0, s_1) reached by Theorem 1.1 is $1/2$

Remark 1.3. Notice that in the previous result [10], some weighted L^∞ condition on the initial data in the Fourier space is needed (see [B]). In Theorem 1.1, this kind of assumption is removed, under a non resonance condition.

Remark 1.4. The estimates crucial to the proof of Theorem 1.1 are the following:

$$\|un\|_{(1,s_0,1-b)} \leq c\|u\|_{(1,s_0,b)}\|n\|_{(2,s_1,b)}, \tag{1.5}$$

$$\|\partial_x(u\overline{v})\|_{(2,s_1,1-b)} \leq c\|u\|_{(1,s_0,b)}\|v\|_{(1,s_0,b)}. \tag{1.6}$$

Under the non resonance condition that β/α is not an integer, if s_0 and s_1 do not satisfy $0 \leq s_0 - s_1 \leq 1$ and $2s_0 \geq s_1 + 1/2 \geq 0$, then the above estimates (1.5) and (1.6) fail for any constants $c > 0$ and $b \in \mathbb{R}$ (see Proposition 2.6 below). Therefore, Theorem 1.1 is the best possible result given by our method.

Remark 1.5. Even if $\beta/\alpha \in \mathbb{Z}$, Theorem 1.1 holds for $0 \leq s_0 - s_1 \leq 1$ and $2s_0 \geq s_1 + 1 \geq 1$ (see Remark 2.1 below).

Remark 1.6. In [24], a non resonance condition is used for the initial value problem with the periodic boundary condition for the mKdV type equation. The non resonance condition on the coefficients α and β in this paper is similar to [24].

The proof of Theorem 1.1 is based on the argument similar to [6, 7, 10, 14, 15]. To prove Theorem 1.1, we use the two different Fourier restriction norms respectively for the first and the second equation of (1.1). When we try to evaluate each equation of (1.1), Lemma 2.3 in §2 plays an important role in our proof. The proof of Lemma 2.3 is roughly stated as follows. We consider the following two inequalities:

$$|\tau+\alpha n^2|+||\tau-\tau_1|-\beta|n-n_1||+|\tau_1+\alpha n_1^2| \geq |\alpha||n-n_1||n+n_1+\frac{\beta S_1}{\alpha}|, \quad (1.7)$$

$$||\tau|-\beta|n||+|\tau-\tau_1-\alpha(n-n_1)^2|+|\tau_1+\alpha n_1^2| \geq |\alpha||n||n-2n_1-\frac{\beta S_2}{\alpha}|, \quad (1.8)$$

where S_1 =sign$\{(\tau-\tau_1)(n-n_1)\}$ and S_2 =sign$\{\tau n\}$. When β/α is not an integer, the third factors in the right hand side of (1.7) and (1.8) never vanish. This fact enables us to overcome the difficulty of the derivative loss in the same way as the case of the KdV equation. So we are able to show the local well-posedness in $H^{s_0} \times H^{s_1} \times H^{s_1-1}$ for $0 \leq s_0 - s_1 \leq 1$ and $2s_0 \geq s_1 + 1/2 \geq 0$. Unless β/α is not an integer, the weakest Sobolev class of the solution that our proof in this paper can reach seems $H^{1/2} \times L^2 \times H^{-1}$ (see Remark 1.5).

We conclude this section by giving several notations. Let $l_n^q L_\tau^p$ denote Banach space $l_n^q(\mathbb{Z} : L_\tau^p(\mathbb{R}))$. Throughout the paper, we denote by $\widehat{\cdot}$ and $\mathcal{F}_x$ the Fourier transform with respect to the space-time and the space variables, respectively, i.e.,

$$\widehat{f}(k,\tau) = \int_{-\infty}^{\infty}\int_{\mathbb{T}} \exp(-ikx-it\tau)f(x,t)dxdt,$$

$$\mathcal{F}_x g(k) = \int_{\mathbb{T}} \exp(-ikx)g(x)dx.$$

Let $\langle\cdot\rangle = (1+|\cdot|^2)^{1/2}$.

2. Preliminaries for the proof of Theorem 1.1. First, by using Duhamel's principle, we consider the following integral equations associated with the initial value problem (1.1):

$$u(t) = S(t)u_0 - i\int_0^t S(t-\lambda)(un)(\lambda)d\lambda,$$

$$n(t) = \partial_t V(t)n_0 + V(t)n_1 + \beta^2\int_0^t V(t-\lambda)\partial_x^2(|u|^2)(\lambda)d\lambda,$$

where $S(t) = \exp(i\alpha t\partial_x^2)$ and $V(t) = \sin(\beta t(-\partial_x^2)^{1/2})/(\beta(-\partial_x^2)^{1/2})$.

We first state the lemma concerning the estimates of the linear and the nonlinear part of the Schrödinger and wave equations in the function spaces we consider.

Lemma 2.1. *For any $s \in \mathbb{R}$, we have*

$$\|\psi(t)S(t)u_0\|_{X_s} \le c\|u_0\|_{H^s},$$

$$\|\psi(t)\partial_t V(t)n_0\|_{Y_s} \le c\|n_0\|_{H^s},$$

$$\|\psi(t)V(t)n_1\|_{Y_s} \le c(1+1/\beta)\|n_1\|_{H^{s-1}},$$

$$\|\psi(t)\int_0^t S(t-\tau)F(\tau)d\tau\|_{X_s} \le c\|F\|_{(1,s,-1/2)} + c\|\langle k\rangle^s \frac{\widehat{F}(k,\tau)}{\langle \tau+\alpha k^2\rangle}\|_{l^2_k L^1_\tau},$$

$$\|\psi(t)\int_0^t V(t-\tau)\partial_x F(\tau)d\tau\|_{Y_s} \le \frac{c}{\beta}\|F\|_{(2,s,-1/2)} + \frac{c}{\beta}\|\langle k\rangle^s \frac{\widehat{F}(k,\tau)}{\langle |\tau|-\beta|k|\rangle}\|_{l^2_k L^1_\tau}.$$

See [6,7,10,14,15], for the proof of Lemma 2.1.

Lemma 2.2. *Let $\alpha, \beta > 0$ such that $\alpha+\beta > 1$. Put $d = \min\{\alpha, \beta, \alpha+\beta-1\}$. Then we have*

$$\int_{-\infty}^{\infty} \frac{d\tau}{\langle \tau\rangle^\alpha \langle \tau-\theta\rangle^\beta} \le \frac{c}{\langle \theta\rangle^d}.$$

The proof of Lemma 2.2 follows from a simple argument.

Lemma 2.3. *Assume that β/α is not an integer. Let $0 < \theta < 1/12$. For any s_0 and s_1 satisfying $0 \le s_0 - s_1 \le 1$ and $0 \le s_1 + 1/2 \le 2s_0$, there exist $a \in (1/2, 3/4)$ and $c > 0$ such that the following estimates hold.*

$$\begin{aligned} I_1 =& \frac{\langle n\rangle^{2s_0}}{\langle \tau+\alpha n^2\rangle^{2(1-a)}} \sum_{n_1} \int_{A_{n,\tau}} \frac{1}{\langle \tau_1+\alpha n_1^2\rangle^{1-2\theta} \langle n_1\rangle^{2s_0} \langle n-n_1\rangle^{2s_1}} \\ & \times \frac{1}{\langle |\tau-\tau_1| - \beta|n-n_1|\rangle^{1-2\theta}} d\tau_1 \left(\int_{-\infty}^{\infty} \frac{d\tau'}{\langle n^2 + |\tau'+\alpha n^2|\rangle^{2a}} \right) \\ \le& c, \end{aligned}$$

$$\begin{aligned} I_1' =& \frac{\langle n\rangle^{2s_0}}{\langle \tau+\alpha n^2\rangle^{2(1-a)}} \sum_{n_1} \int_{A'_{n,\tau}} \frac{1}{\langle \tau_1+\alpha n_1^2\rangle^{1-2\theta} \langle n_1\rangle^{2s_0} \langle n-n_1\rangle^{2s_1}} \\ & \times \frac{1}{\langle |\tau-\tau_1| - \beta|n-n_1|\rangle^{1-2\theta}} d\tau_1 \left(\int_{-\infty}^{\infty} \frac{d\tau'}{\langle |n| + |\tau'+\alpha n^2|\rangle^{2a}} \right) \\ \le& c, \end{aligned}$$

$$
\begin{aligned}
I_2 =& \frac{1}{\langle \tau_1 + \alpha n_1^2 \rangle \langle n_1 \rangle^{2s_0}} \sum_n \int_{B_{n_1,\tau_1}} \frac{\langle n \rangle^{2s_0}}{\langle \tau + \alpha n^2 \rangle^{2(1-a)} \langle n - n_1 \rangle^{2s_1}} \\
&\times \frac{1}{\langle |\tau - \tau_1| - \beta |n - n_1| \rangle^{1-2\theta}} d\tau \\
\le& c,
\end{aligned}
$$

$$
\begin{aligned}
I_3 =& \frac{1}{\langle |\tau_1| - \beta |n_1| \rangle \langle n_1 \rangle^{2s_1}} \sum_n \int_{C_{n_1,\tau_1}} \frac{\langle n \rangle^{2s_0}}{\langle \tau + \alpha n^2 \rangle^{2(1-a)} \langle n - n_1 \rangle^{2s_0}} \\
&\times \frac{1}{\langle \tau - \tau_1 + \alpha (n - n_1)^2 \rangle^{1-2\theta}} d\tau \\
\le& c,
\end{aligned}
$$

$$
\begin{aligned}
I_4 =& \frac{\langle n \rangle^{2s_1} |n|^2}{\langle |\tau| - \beta |n| \rangle^{2(1-a)}} \sum_{n_1} \int_{D_{n,\tau}} \frac{1}{\langle \tau_1 + \alpha n_1^2 \rangle^{1-2\theta} \langle n_1 \rangle^{2s_0} \langle n - n_1 \rangle^{2s_0}} \\
&\times \frac{1}{\langle \tau - \tau_1 - \alpha (n - n_1)^2 \rangle^{1-2\theta}} d\tau_1 \left(\int_{-\infty}^{\infty} \frac{d\tau'}{\langle n^2 + ||\tau'| - \beta |n|| \rangle^{2a}} \right) \\
\le& c,
\end{aligned}
$$

$$
\begin{aligned}
I_4' =& \frac{\langle n \rangle^{2s_1} |n|^2}{\langle |\tau| - \beta |n| \rangle^{2(1-a)}} \sum_{n_1} \int_{D'_{n,\tau}} \frac{1}{\langle \tau_1 + \alpha n_1^2 \rangle^{1-2\theta} \langle n_1 \rangle^{2s_0} \langle n - n_1 \rangle^{2s_0}} \\
&\times \frac{1}{\langle \tau - \tau_1 - \alpha (n - n_1)^2 \rangle^{1-2\theta}} d\tau_1 \left(\int_{-\infty}^{\infty} \frac{d\tau'}{\langle |n| + ||\tau'| - \beta |n|| \rangle^{2a}} \right) \\
\le& c,
\end{aligned}
$$

$$
\begin{aligned}
I_5 =& \frac{1}{\langle \tau_1 + \alpha n_1^2 \rangle \langle n_1 \rangle^{2s_0}} \sum_n \int_{E_{n_1,\tau_1}} \frac{\langle n \rangle^{2s_1} |n|^2}{\langle |\tau| - \beta |n| \rangle^{2(1-a)} \langle n - n_1 \rangle^{2s_0}} \\
&\times \frac{1}{\langle \tau - \tau_1 - \alpha (n - n_1)^2 \rangle^{1-2\theta}} d\tau \\
\le& c,
\end{aligned}
$$

where $A_{n,\tau}$, $A'_{n,\tau}$, B_{n_1,τ_1}, C_{n_1,τ_1}, $D_{n,\tau}$, $D'_{n,\tau}$ *and* E_{n_1,τ_1} *are defined as follows:*

$$
\begin{aligned}
A_{n,\tau} = \{(n_1,\tau_1) :& |n_1| < |n|/2 \text{ or } 3|n|/2 \le |n_1| \ne 0, \\
& |\tau + \alpha n^2| \ge \max\{|\tau_1 + \alpha n_1^2|, ||\tau - \tau_1| - \beta |n - n_1||\}\},
\end{aligned}
$$

$$A'_{n,\tau} = \{(n_1,\tau_1) : |n|/2 \le |n_1| < 3|n|/2 \ \textit{and}\ n \ne n_1,\\ |\tau+\alpha n^2| \ge \max\{|\tau_1+\alpha n_1^2|, ||\tau-\tau_1|-\beta|n-n_1||\}\},$$

$$B_{n_1,\tau_1} = \{(n,\tau) : n \ne n_1,\ |\tau_1+\alpha n_1^2| \ge \max\{|\tau+\alpha n^2|, ||\tau-\tau_1|-\beta|n-n_1||\}\},$$

$$C_{n_1,\tau_1} = \{(n,\tau) : n_1 \ne 0, ||\tau_1|-\beta|n_1|| \ge \max\{|\tau+\alpha n^2|, |\tau-\tau_1+\alpha(n-n_1)^2|\}\},$$

$$D_{n,\tau} = \{(n_1,\tau_1) : |n_1| < 2|n|/5 \ \textit{or}\ 2|n|/3 \le |n_1|,\\ ||\tau|-\beta|n|| \ge \max\{|\tau_1+\alpha n_1^2|, |\tau-\tau_1-\alpha(n-n_1)^2|\}\},$$

$$D'_{n,\tau} = \{(n_1,\tau_1) : 2|n|/5 \le |n_1| < 2|n|/3,\\ ||\tau|-\beta|n|| \ge \max\{|\tau_1+\alpha n_1^2|, |\tau-\tau_1-\alpha(n-n_1)^2|\}\},$$

$$E_{n_1,\tau_1} = \{(n,\tau) : |\tau_1+\alpha n_1^2| \ge \max\{||\tau|-\beta|n||, |\tau-\tau_1-\alpha(n-n_1)^2|\}\}.$$

Proof of Lemma 2.3. In the proof of this lemma, we regard S as a suitable signature.

(I_1). First, we consider I_1. According to the definition of $A_{n,\tau}$, it follows that $n \ne n_1$. Then we first note that the third factor in the right hand side of (1.7) never vanish, since β/α is not an integer. Therefore integrating with respect to τ', using (1.7) and Lemma 2.2, we have that I_1 is bounded by

$$c \sum_{\substack{S=\pm 1 \\ |n_1|<|n|/2 \text{ or } 3|n|/2 \le |n_1|}} \frac{\langle n\rangle^{2s_0-4a+2}}{\langle n_1\rangle^{2s_0}\langle n-n_1\rangle^{2s_1+2-2a}\langle n+n_1+\beta S/\alpha\rangle^{2-2a}} \times \frac{1}{\langle \tau+\alpha n_1^2-\beta S(n-n_1)\rangle^{1-\epsilon}} \tag{2.1}$$

for $\epsilon \in (4\theta, 1/2)$. Then, in the region of $|n_1| < |n|/2$ or $3|n|/2 \le |n_1|$, we have that $|n-n_1|, |n+n_1+\beta S/\alpha| \sim \max\{|n|, |n_1|\}$ for sufficiently large $|n|$.

Then (2.1) is bounded by

$$c \sum_{\substack{S=\pm 1\\ |n_1|<|n|/2 \text{ or } 3|n|/2\le |n_1|}} \frac{1}{\langle \max\{|n|,|n_1|\}\rangle^{\min\{2s_1+2-2a,2s_1-2s_0+2a\}}}$$
$$\times \frac{1}{\langle n+n_1+\beta S/\alpha\rangle^{2-2a}\langle \tau+\alpha n_1^2-\beta S(n-n_1)\rangle^{1-\epsilon}}$$
$$\le c\sum_{S=\pm1,n_1}\frac{1}{\langle \max\{|n|,|n_1|\}\rangle^{\min\{2s_1+4-4a,2s_1-2s_0+2\}}\langle \tau+\alpha n_1^2-\beta S(n-n_1)\rangle^{1-\epsilon}}$$
$$\le c\sum_{S=\pm1,n_1}\frac{1}{\langle \tau+\alpha n_1^2-\beta S(n-n_1)\rangle^{1-\epsilon}},$$

for $s_0-s_1\le 1,\ s_0\ge 0$ and $s_1\ge 2a-2$. Hence, by the argument similar to the proof in [13,14], we have that I_1 is uniformly bounded with respect to n and τ.

(I_1'). Secondly, we consider I_1'. In the region of $|n|/2\le |n_1|<3|n|/2$, we have that $|n-n_1|<3|n_1|$. In a similar way to the proof of (I_1), we have that I_1' is bounded by

$$c\sum_{\substack{S=\pm1\\ |n|/2\le|n_1|<3|n|/2}}\frac{1}{\langle n_1\rangle^{2a-1}\langle n-n_1\rangle^{2s_1+2-2a}\langle n+n_1+\beta S/\alpha\rangle^{2-2a}}$$
$$\times\frac{1}{\langle \tau+\alpha n_1^2-\beta S(n-n_1)\rangle^{1-\epsilon}}$$
$$\le c\sum_{S=\pm1,n_1}\frac{1}{\langle n-n_1\rangle^{2s_1+1}\langle n+n_1+\beta S/\alpha\rangle^{2-2a}\langle \tau+\alpha n_1^2-\beta S(n-n_1)\rangle^{1-\epsilon}}$$
$$\le c, \tag{2.2}$$

for $\epsilon\in(4\theta,1/2)$ and $s_1\ge -1/2$.

(I_2). Next, we consider I_2. In a similar way to the proof of (I_1), by using (1.7) we have that I_2 is bounded by

$$c\sum_{S=\pm1,n}\frac{\langle n\rangle^{2s_0}}{\langle n_1\rangle^{2s_0}\langle n-n_1\rangle^{2s_1+1}\langle n+n_1+\beta S/\alpha\rangle\langle \tau_1+\alpha n^2+\beta S(n-n_1)\rangle^{1-\epsilon}}, \tag{2.3}$$

for $\epsilon\in(2a-1+2\theta,1/2)$. We divide the sum into the following two regions:

$$|n|<3|n_1|/2 \text{ and } 3|n_1|/2\le|n|.$$

In the region of $|n| < 3|n_1|/2$, (2.3) restricted to the above region is bounded by

$$c\sum_{S=\pm1,n}\frac{1}{\langle n-n_1\rangle^{2s_1+1}\langle n+n_1+\beta S/\alpha\rangle\langle\tau_1+\alpha n^2+\beta S(n-n_1)\rangle^{1-\epsilon}}\leq c,$$

for $s_0\geq 0$ and $s_1\geq -1/2$. In the region of $3|n_1|/2\leq |n|$, we have that $|n-n_1|\geq |n|/3$ and $|n+n_1+\beta S/\alpha|\geq c|n|$ for sufficiently large $|n|$. In a similar way to above, (2.3) restricted to the above region is bounded by

$$\begin{aligned}&c\sum_{\substack{S=\pm1\\ 3|n_1|/2\leq|n|}}\frac{1}{\langle n_1\rangle^{2s_0}\langle n\rangle^{2s_1-2s_0+1}\langle n+n_1+\beta S/\alpha\rangle\langle\tau_1+\alpha n^2+\beta S(n-n_1)\rangle^{1-\epsilon}}\\ &\leq c\sum_{S=\pm1,n}\frac{1}{\langle n_1\rangle^{2s_0}\langle n\rangle^{2s_1-2s_0+2}\langle\tau_1+\alpha n^2+\beta S(n-n_1)\rangle^{1-\epsilon}}\\ &\leq c,\end{aligned}$$

for $s_0\geq 0$ and $s_0-s_1\leq 1$.

(I_3). We consider I_3. In a similar way to the proof of (I_1), using Lemma 2.2 and (1.7), we have that I_3 is bounded by

$$c\sum_{S=\pm1,n}\frac{\langle n\rangle^{2s_0}}{\langle n_1\rangle^{2s_1+1}\langle n-n_1\rangle^{2s_0}\langle 2n-n_1+\beta S/\alpha\rangle\langle\langle n_1\rangle|n-\gamma_1(n_1,\tau_1)|\rangle^{1-\epsilon}},\tag{2.4}$$

for $\epsilon\in(2a-1+2\theta,1)$, where $\gamma_1=\gamma_1(n_1,\tau_1)$ is the solution of the following linear equation with respect to n,

$$\tau_1-\alpha n_1^2+2\alpha nn_1=0,\ \text{i.e.,}\ \gamma_1=\frac{n_1}{2}+\frac{\tau_1}{2\alpha n_1}.$$

We divide the sum into the following three regions:

$$|n_1|\leq|n|/2,\ |n|/2<|n_1|\leq 3|n|/2 \text{ and } 3|n|/2<|n_1|.$$

In the region of $|n_1|\leq |n|/2$ or $3|n|/2<|n_1|$, we have that $|n-n_1|\geq c|n|$, then (2.4) restricted to the above region is bounded by

$$c\sum_{S=\pm1,n}\frac{1}{\langle n_1\rangle^{2s_1+1}\langle 2n-n_1+\beta S/\alpha\rangle\langle n-\gamma_1\rangle^{1-\epsilon}}\leq c,$$

for $s_0 \geq 0$ and $s_1 \geq -1/2$. In the case of $|n|/2 < |n_1| \leq 3|n|/2$, we have that (2.4) is bounded by

$$c \sum_{\substack{S=\pm 1 \\ |n|/2<|n_1|\leq 3|n|/2}} \frac{1}{\langle n\rangle^{2s_1-2s_0+1}\langle n-n_1\rangle^{2s_0}\langle 2n-n_1+\beta S/\alpha\rangle\langle\langle n_1\rangle|n-\gamma_1(n_1,\tau_1)|\rangle^{1-\epsilon}}.$$

Then, by dividing the sum into two cases that $n \sim \gamma_1$ and that $n \nsim \gamma_1$, we have that (2.4) restricted to the above region is bounded by

$$c + c \sum_{\substack{S=\pm 1 \\ |n|/2<|n_1|\leq 3|n|/2}} \frac{1}{\langle n\rangle^{2s_1-2s_0+2-\epsilon}\langle n-n_1\rangle^{2s_0}\langle 2n-n_1+\beta S/\alpha\rangle\langle n-\gamma_1\rangle^{1-\epsilon}}, \tag{2.5}$$

for $s_0 \geq 0,\ s_1 \geq -1/2,\ s_0 - s_1 \leq 1$ and $\epsilon \in (2a-1+2\theta, 1/2)$. In the region of $|n|/2 < |n_1| \leq 3|n|/2$ we have $|2n-n_1+\beta S/\alpha| \geq c|n|$ for sufficiently large $|n|$, then the second term of (2.5) is bounded by

$$c\sum_n \frac{1}{\langle n\rangle^{2s_1-2s_0+3-\epsilon}\langle n-n_1\rangle^{2s_0}\langle n-\gamma_1\rangle^{1-\epsilon}} \leq c,$$

for $s_0 \geq 0,\ s_0 - s_1 \leq 3/2 - \epsilon/2$ and $s_1 > -3/2 + \epsilon$.

(I_4). Next, we consider I_4. We can assume $n \neq 0$ in I_4. Integrating with respect to τ' and τ_1, using Lemma 2.2 and (1.8), we have that I_4 is bounded by

$$c \sum_{\substack{S=\pm 1 \\ |n_1|\leq 2|n|/5 \text{ or } 2|n|/3<|n_1|}} \frac{\langle n\rangle^{2s_1+2-2a}}{\langle n_1\rangle^{2s_0}\langle n-n_1\rangle^{2s_0}\langle n-2n_1-\beta S/\alpha\rangle^{2-2a}} \times \frac{1}{\langle\langle n\rangle|n_1-\gamma_2(n,\tau)|\rangle^{1-\epsilon}}, \tag{2.6}$$

for $\epsilon \in (4\theta, 1/2)$ where $\gamma_2 = \gamma_2(n,\tau)$ is the solution of the following linear equation with respect to n_1,

$$\tau - \alpha n^2 + 2\alpha n n_1 = 0, \text{ i.e., } \gamma_2 = \frac{n}{2} - \frac{\tau}{2\alpha n}.$$

We divide the sum with respect to n_1 into the following three regions:

$$|n_1| \leq 2|n|/5,\ 2|n|/3 < |n_1| \leq 3|n|/2 \text{ and } 3|n|/2 < |n_1|.$$

In the region of $|n_1| \le 2|n|/5$ or $3|n|/2 < |n_1|$, we have

$$|n - n_1| \sim \max\{|n|, |n_1|\},$$

then (2.6) restricted to the above region is bounded by

$$c \sum_{\substack{S=\pm 1 \\ |n_1|\le 2|n|/5 \text{ or } 3|n|/2<|n_1|}} \frac{1}{\langle n_1\rangle^{2s_0}\langle n\rangle^{2s_0-2s_1+2a-2}\langle n-2n_1-\beta S/\alpha\rangle^{2-2a}} \times \frac{1}{\langle\langle n\rangle|n_1-\gamma_2(n,\tau)|\rangle^{1-\epsilon}}. \tag{2.7}$$

In the region of $|n_1| \le 2|n|/5$ or $2|n|/3 < |n_1|$, we have $|n-2n_1-\beta S/\alpha| \sim \max\{|n|, |n_1|\}$. Then in a similar way to the proof of (I_3) by dividing the sum into two cases that $n \sim \gamma_1$ and that $n \nsim \gamma_1$, we have that (2.7) is bounded by

$$c + c\sum_{n_1} \frac{1}{\langle n_1\rangle^{2s_0+2\epsilon}\langle n\rangle^{2s_0-2s_1+1-3\epsilon}\langle n_1-\gamma_2(n,\tau)\rangle^{1-\epsilon}} \le c,$$

for $4\theta < \epsilon \le 1/3,\ s_0 \ge 0,\ 2s_0 \ge s_1+1-a$ and $s_0 - s_1 \ge 0$. In a similar way to above, in the case of $2|n|/3 < |n_1| \le 3|n|/2$, (2.6) restricted to the above region is bounded by

$$\begin{aligned} &c \sum_{\substack{S=\pm 1 \\ 2|n|/3<|n_1|\le 3|n|/2}} \frac{1}{\langle n_1\rangle^{2s_0-2s_1-2+2a}\langle n-n_1\rangle^{2s_0}\langle n-2n_1-\beta S/\alpha\rangle^{2-2a}} \\ &\quad\times \frac{1}{\langle\langle n\rangle|n_1-\gamma_2(n,\tau)|\rangle^{1-\epsilon}} \\ \le& c + c\sum_{n_1} \frac{1}{\langle n_1\rangle^{2s_0-2s_1+1-\epsilon}\langle n-n_1\rangle^{2s_0}\langle n_1-\gamma_2(n,\tau)\rangle^{1-\epsilon}} \\ \le& c, \end{aligned}$$

for $s_0 \ge 0,\ 2s_0 \ge s_1+1-a$ and $s_0 - s_1 \ge 0$.

(I_4'). We consider I_4'. In the region of $2|n|/5 < |n_1| \le 2|n|/3$, we have that $|n|/3 \le |n-n_1| < 5|n|/3$. Then by using (1.8), we have that I_4' is bounded by

$$c\sum_{S=\pm 1, n_1} \frac{1}{\langle n\rangle^{4s_0-2s_1-1}\langle n-2n_1-\beta S/\alpha\rangle^{2-2a}\langle\langle n\rangle|n_1-\gamma_2(n,\tau)|\rangle^{1-\epsilon}} \le c, \tag{2.8}$$

for $\epsilon \in (4\theta, 1/2)$, $2s_0 \geq s_1 + 1/2$ and $\gamma_2 = \gamma_2(n, \tau)$ is defined in the proof of (I_4).

(I_5). We consider I_5. In a similar way to the proof of (I_4), we have that I_5 is bounded by

$$c \sum_{S=\pm 1, n} \frac{\langle n \rangle^{2s_1+1}}{\langle n_1 \rangle^{2s_0} \langle n - n_1 \rangle^{2s_0} \langle n - 2n_1 - \beta S/\alpha \rangle \langle \tau_1 + \alpha(n-n_1)^2 - \beta S n \rangle^{1-\epsilon}}, \tag{2.9}$$

for $\epsilon \in (2a - 1 + 2\theta, 1/2)$. We divide the sum into the following four regions:

$$\begin{aligned} &|n| < |n_1|/2, && |n_1|/2 \leq |n| < 3|n_1|/2, \\ &3|n_1|/2 \leq |n| < 5|n_1|/2, && 5|n_1|/2 \leq |n|. \end{aligned}$$

In the region of $|n| < |n_1|/2$ or $3|n_1|/2 \leq |n| < 5|n_1|/2$, we have $|n - n_1| \sim |n_1|$, then (2.9) restricted to the above region is bounded by

$$c \sum_{S=\pm 1, n} \frac{1}{\langle n_1 \rangle^{4s_0 - 2s_1 - 1} \langle n - 2n_1 - \beta S/\alpha \rangle \langle \tau_1 + \alpha(n-n_1)^2 - \beta S n \rangle^{1-\epsilon}} \leq c,$$

for $2s_0 \geq s_1 + 1/2 \geq 0$. In the region of $|n_1|/2 \leq |n| < 3|n_1|/2$ we have $|n - 2n_1 - \beta S/\alpha| \geq c|n|$ for sufficiently large $|n|$, then (2.9) restricted to the above region is bounded by

$$\begin{aligned} &c \sum_{\substack{S=\pm 1 \\ |n_1|/2 \leq |n| < 3|n|/2}} \frac{1}{\langle n \rangle^{2s_0 - 2s_1 - 1} \langle n - n_1 \rangle^{2s_0} \langle n - 2n_1 - \beta S/\alpha \rangle} \\ &\qquad \times \frac{1}{\langle \tau_1 + \alpha(n-n_1)^2 - \beta S n \rangle^{1-\epsilon}} \\ &\leq c \sum_{S=\pm 1, n} \frac{1}{\langle n \rangle^{2s_0 - 2s_1} \langle n - n_1 \rangle^{2s_0} \langle \tau_1 + \alpha(n-n_1)^2 - \beta S n \rangle^{1-\epsilon}} \\ &\leq c, \end{aligned}$$

for $s_0 \geq 0$ and $s_0 - s_1 \geq 0$. In the region of $5|n_1|/2 \leq |n|$ we have that $|n - n_1| \sim |n|$ and $|n - 2n_1 - \beta S/\alpha| \sim |n|$, then (2.9) restricted to the above region is bounded by

$$
\begin{aligned}
& c \sum_{\substack{S=\pm 1\\ 5|n_1|/2\le |n|}} \frac{1}{\langle n_1\rangle^{2s_0}\langle n\rangle^{2s_0-2s_1-1}\langle n-2n_1-\beta S/\alpha\rangle\langle \tau_1+\alpha(n-n_1)^2-\beta S n\rangle^{1-\epsilon}}\\
& \le c \sum_{S=\pm 1, n} \frac{1}{\langle n_1\rangle^{2s_0}\langle n\rangle^{2s_0-2s_1}\langle \tau_1+\alpha(n-n_1)^2-\beta S n\rangle^{1-\epsilon}}\\
& \le c,
\end{aligned}
$$

for $s_0 \ge 0,\ s_1 \ge -1/2$ and $s_0 - s_1 \ge 0$. □

Remark 2.1. When $0 \le s_0 - s_1 \le 1$ and $1 \le s_1 + 1 \le 2s_0$, the result in Lemma 2.3 holds even for the case of $\beta/\alpha \in \mathbb{Z}$, which leads to the result without a non resonance condition (see Remark 1.5).

Lemma 2.4. *Assume that β/α is not an integer. Let $0 < \theta < 1/12$. For any s_0 and s_1 satisfying $0 \le s_0 - s_1 \le 1$ and $0 \le s_1 + 1/2 \le 2s_0$, there exists $c > 0$ such that the following estimates hold.*

$$
\begin{aligned}
& \|\langle k\rangle^{s_0} \frac{\widehat{un}(k,\tau)}{\langle \tau+\alpha k^2\rangle}\|_{l_k^2 L_\tau^1} \\
& \le c(\|u\|_{(1,s_0,1/2-\theta)}\|n\|_{(2,s_1,1/2)} + \|u\|_{(1,s_0,1/2)}\|n\|_{(2,s_1,1/2-\theta)}),
\end{aligned} \tag{2.10}
$$

$$
\|\langle k\rangle^{s_1} \frac{\widehat{\partial_x(|u|^2)}(k,\tau)}{\langle |\tau|-\beta|k|\rangle}\|_{l_k^2 L_\tau^1} \le c\|u\|_{(1,s_0,1/2-\theta)}\|u\|_{(1,s_0,1/2)}. \tag{2.11}
$$

Proof of Lemma 2.4. First, we prove (2.10). By using the Schwarz inequality, we have that the left-hand side of (2.10) is bounded by

$$
\begin{aligned}
& \Big(\sum_k \langle k\rangle^{2s_0} \int_{-\infty}^{\infty} \frac{|\widehat{un}(k,\tau)|^2}{\langle \tau+\alpha k^2\rangle^{2(1-a)}} d\tau \int \frac{d\tau}{\langle \tau+\alpha k^2\rangle^{2a}}\Big)^{1/2} \\
\le & \Big(\sum_{k,k_1} \iint_{\mathbb{R}^2} \frac{\langle k\rangle^{2s_0}}{\langle \tau+\alpha k^2\rangle^{2(1-a)}} |\widehat{u}(k_1,\tau_1)|^2 \\
& \qquad \times |\widehat{n}(k-k_1,\tau-\tau_1)|^2 d\tau_1 d\tau \int \frac{d\tau}{\langle \tau+\alpha k^2\rangle^{2a}}\Big)^{1/2}
\end{aligned} \tag{2.12}
$$

where a is defined in Lemma 2.3. We divide the integral region into two regions that $k = k_1$ and that $k \ne k_1$. In the region of $k = k_1$, we have that the contribution of the region of $k = k_1$ to (2.12) is bounded by

$$
c\|u\|_{(1,s_0,0)}\|n\|_{(2,s_1,0)}.
$$

We next consider the region of $k \neq k_1$. We divide the integral region into the following three cases:

$$|\tau+\alpha k^2| \geq \max\{|\tau_1+\alpha k_1^2|, ||\tau-\tau_1|-\beta|k-k_1||\} \text{ and } k \neq k_1, \tag{2.13}$$

$$|\tau_1+\alpha k_1^2| \geq \max\{|\tau+\alpha k^2|, ||\tau-\tau_1|-\beta|k-k_1||\} \text{ and } k \neq k_1, \tag{2.14}$$

$$||\tau-\tau_1|-\beta|k-k_1|| \geq \max\{|\tau+\alpha k^2|, |\tau_1+\alpha k_1^2|\} \text{ and } k \neq k_1. \tag{2.15}$$

In the region of (2.13), we first divide the sum with respect to k_1 into the three cases that $|k_1| < |k|/2$, that $3|k|/2 \leq |k_1|$ and that $|k|/2 \leq |k_1| < 3|k|/2$. In the region of $|k_1| < |k|/2$ or $3|k|/2 \leq |k_1|$, the right-hand side of (1.7) is greater than ck^2 for sufficiently large $|k|$. Then in the region of $|k_1| < |k|/2$ or $3|k|/2 \leq |k_1|$ in (2.13), by using the first estimate in Lemma 2.3, we have that (2.12) restricted to the above integral region is bounded by

$$\begin{aligned}
&c\sup_{k,\tau} \frac{\langle k\rangle^{s_0}}{\langle \tau+\alpha k^2\rangle^{1-a}} \Big(\sum_{k_1}\int_{A_{k,\tau}} \frac{\langle k_1\rangle^{-2s_0}\langle k-k_1\rangle^{-2s_1}}{\langle \tau_1+\alpha k_1^2\rangle^{1-2\theta}\langle |\tau-\tau_1|-\beta|k-k_1|\rangle^{1-2\theta}} d\tau_1\Big)^{\frac12}\\
&\times \Big(\int_{-\infty}^{\infty} \frac{d\tau'}{\langle k^2+|\tau'+\alpha k^2|\rangle^{2a}}\Big)^{1/2} \|u\|_{(1,s_0,1/2-\theta)}\|n\|_{(2,s_1,1/2-\theta)}\\
&\leq c\sup_{k,\tau}(I_1)\|u\|_{(1,s_0,1/2-\theta)}\|n\|_{(2,s_1,1/2-\theta)}\\
&\leq c\|u\|_{(1,s_0,1/2-\theta)}\|n\|_{(2,s_1,1/2-\theta)}.
\end{aligned}$$

On the other hand, in the region of $|k|/2 \leq |k_1| < 3|k|/2$, from the fact $|k-k_1|+|k+k_1+\beta S/\alpha| \geq |2k+\beta S/\alpha|$, the right-hand side of (1.7) is greater than $c|k|$ for sufficiently large $|k|$. Then in the region of $|k|/2 \leq |k_1| < 3|k|/2$ in (2.13), by using the second estimate in Lemma 2.3, we have that (2.12) restricted to the above integral region is bounded by

$$c\sup_{k,\tau}(I_1')\|u\|_{(1,s_0,1/2-\theta)}\|n\|_{(2,s_1,1/2-\theta)} \leq c\|u\|_{(1,s_0,1/2-\theta)}\|n\|_{(2,s_1,1/2-\theta)}.$$

In the region of (2.14), we use the duality argument for (2.12) and so it suffices to show the following inequality restricted to the above integral region

$$\begin{aligned}
&\sup_{k_1,\tau_1} \frac{1}{\langle k\rangle^{s_0}\langle \tau_1+\alpha k_1^2\rangle^{1/2}} \Big(\sum_{k\neq k_1}\int_{-\infty}^{\infty} \frac{\langle k\rangle^{2s_0}}{\langle k-k_1\rangle^{2s_1}\langle \tau+\alpha k^2\rangle^{2(1-a)}}\\
&\times \frac{1}{\langle |\tau-\tau_1|-\beta|k-k_1|\rangle^{1-2\theta}} d\tau\Big)^{1/2} \|u\|_{(1,s_0,1/2)}\|n\|_{(2,s_1,1/2-\theta)}\\
&\leq c\|u\|_{(1,s_0,1/2)}\|n\|_{(2,s_1,1/2-\theta)}.
\end{aligned} \tag{2.16}$$

By using the third estimate in Lemma 2.3, we have that the left-hand side of (2.16) is bounded by

$$c \sup_{k_1,\tau_1} (I_2)\|u\|_{(1,s_0,1/2)}\|n\|_{(2,s_1,1/2-\theta)} \le c\|u\|_{(1,s_0,1/2)}\|n\|_{(2,s_1,1/2-\theta)}.$$

In the case of (2.15), the proof is similar to above by using I_3 in Lemma 2.3. The proof of (2.11) is a similar way to above, by using I_4, I_4' and I_5 in Lemma 2.3. □

Lemma 2.5. *Let s_0, s_1, α, β and θ be the same as in Lemma* 2.4. *Then we have*

$$\|un\|_{(1,s_0,-1/2)} \le c(\|u\|_{(1,s_0,1/2-\theta)}\|n\|_{(2,s_1,1/2)} + \|u\|_{(1,s_0,1/2)}\|n\|_{(2,s_1,1/2-\theta)}), \tag{2.17}$$

$$\|\partial_x(|u|^2)\|_{(2,s_1,-1/2)} \le c\|u\|_{(1,s_0,1/2-\theta)}\|u\|_{(1,s_0,1/2)}. \tag{2.18}$$

Proof of Lemma 2.5. It suffices to show

$$\begin{aligned}&\Big(\sum_k \int_{-\infty}^{\infty} \frac{\langle k\rangle^{2s_0}|\widehat{un}(k,\tau)|^2}{\langle \tau+\alpha k^2\rangle} d\tau\Big)^{1/2} \\ \le & c\Big(\|u\|_{(1,s_0,1/2-\theta)}\|n\|_{(2,s_1,1/2)} + \|u\|_{(1,s_0,1/2)}\|n\|_{(2,s_1,1/2-\theta)}\Big),\end{aligned} \tag{2.19}$$

$$\Big(\sum_k \int_{-\infty}^{\infty} \frac{\langle k\rangle^{2s_1}|\widehat{\partial_x(|u|^2)}(k,\tau)|^2}{\langle |\tau|-\beta|k|\rangle} d\tau\Big)^{1/2} \le c\|u\|_{(1,s_0,1/2-\theta)}\|u\|_{(1,s_0,1/2)}. \tag{2.20}$$

By taking $a = 1/2$ in the proof of Lemmas 2.3, 2.4 and ignoring $\int d\tau/\langle \tau + \alpha k^2\rangle^{2a}$ in (2.12), we obtain the estimates (2.19) and (2.20) in a similar way to the proof of Lemma 2.4. □

Proposition 2.6. (i) *If s_0 and s_1 do not satisfy $s_0 - s_1 \le 1$ and $s_1 \ge -1/2$, then the estimate*

$$\|un\|_{(1,s_0,1-b)} \le c\|u\|_{(1,s_0,b)}\|n\|_{(2,s_1,b)} \tag{2.21}$$

fails for any constants $c > 0$ and $b \in \mathbb{R}$.

(ii) *If s_0 and s_1 do not satisfy $0 \le s_0 - s_1$ and $2s_0 \ge s_1 + 1/2$, then the estimate*

$$\|\partial_x(u\overline{v})\|_{(2,s_1,1-b)} \le c\|u\|_{(1,s_0,b)}\|v\|_{(1,s_0,b)} \tag{2.22}$$

fails for any constants $c > 0$ and $b \in \mathbb{R}$.

Proof of Proposition 2.6. The proof is based on the argument similar to Kenig-Ponce-Vega [14,15]. First, we prove (i). We take for $N \in \mathbb{N}$,

$$\begin{cases} \widehat{u_1}(k,\tau) = \delta(k+N)\psi(\tau+\alpha N^2), \\ \widehat{n_1}(k,\tau) = \delta(k-2N)\psi(|\tau|-2\beta N), \end{cases}$$

where $\delta(\cdot)$ denotes a function such that $\delta(k) = 1$ if $k = 0$ and $\delta(k) = 0$ if $k \neq 0$. Simple calculations yield

$$|\widehat{u_1 n_1}(k,\tau)| \geq c\delta(k-N)\psi((\tau+\alpha N^2 - 2\beta N)/2).$$

Substituting this into (2.21) we have

$$\frac{N^{s_0}}{N^{1-b}} \leq cN^{s_0}N^{s_1}, \tag{2.23}$$

which shows the necessity for $s_1 \geq b-1$. Similarly, taking

$$\begin{cases} \widehat{u_2}(k,\tau) = \delta(k+N)\psi(\tau+\alpha N^2+2\beta N), \\ \widehat{n_2}(k,\tau) = \delta(k-2N)\psi(|\tau|-2\beta N), \end{cases}$$

$$\begin{cases} \widehat{u_3}(k,\tau) = \delta(k)\psi(\tau), \\ \widehat{n_3}(k,\tau) = \delta(k-N)\psi(|\tau|-\beta N), \end{cases}$$

$$\begin{cases} \widehat{u_4}(k,\tau) = \delta(k)\psi(\tau+\alpha N^2+\beta N), \\ \widehat{n_4}(k,\tau) = \delta(k-N)\psi(|\tau|-\beta N), \end{cases}$$

and substituting these into (2.21), we obtain

$$N^{s_0} \leq cN^{s_0}N^{s_1}N^b, \ \frac{N^{s_0}}{N^{2(1-b)}} \leq cN^{s_1} \text{ and } N^{s_0} \leq cN^{s_1}N^{2b}, \tag{2.24}$$

according to $(\widehat{u_j}, \widehat{n_j})$, $j = 2,3,4$, respectively. The relations (2.23) and (2.24) show the necessity for $s_1 \geq \max\{b-1,-b\} \geq -\frac{1}{2}$, $s_1 - s_0 \geq 2\max\{b-1,-b\} \geq -1$. This completes the proof of (i).

In order to prove (ii), we next take

$$\begin{cases} \widehat{u_5}(k,\tau) = \delta(k-N)\psi(\tau+\alpha N^2), \\ \widehat{v_5}(k,\tau) = \delta(k+N)\psi(\tau+\alpha N^2), \end{cases}$$

$$\begin{cases} \widehat{u_6}(k,\tau) = \delta(k-N)\psi(\tau+\alpha N^2 - 2\beta N), \\ \widehat{v_6}(k,\tau) = \delta(k+N)\psi(\tau+\alpha N^2), \end{cases}$$

$$\begin{cases} \widehat{u_7}(k,\tau) = \delta(k)\psi(\tau), \\ \widehat{v_7}(k,\tau) = \delta(k-N)\psi(\tau+\alpha N^2), \end{cases}$$

$$\begin{cases} \widehat{u_8}(k,\tau) = \delta(k)\psi(\tau+\alpha N^2 + \beta N), \\ \widehat{v_8}(k,\tau) = \delta(k-N)\psi(\tau+\alpha N^2), \end{cases}$$

and we substitute these $(\widehat{u_j}, \widehat{v_j})$, $j = 5,6,7,8$, into (2.22). Simple calculations yield

$$\frac{N^{s_1}N}{N^{1-b}} \le cN^{2s_0}, \quad N^{s_1}N \le cN^{2s_0}N^b, \quad \frac{N^{s_1}N}{N^{2(1-b)}} \le cN^{s_0}, \quad N^{s_1}N \le cN^{s_0}N^{2b}, \tag{2.25}$$

according to $(\widehat{u_j}, \widehat{v_j})$, $j = 5,6,7,8$, respectively. The relations (2.25) show the necessity for

$$2s_0 - s_1 - 1/2 \ge \max\{b - 1/2, 1/2 - b\} \ge 0,$$

$$s_0 - s_1 \ge \max\{2b-1, 1-2b\} \ge 0.$$

This completes the proof of (ii). □

Lemma 2.7. *For any* $s \in \mathbb{R}$, $0 < \epsilon < 1/2$, $\delta \in (0,1]$ *and* $0 < \theta' < \theta < 1/2$, *we have*

$$\|\psi_\delta(\cdot)F\|_{(i,s,1/2)} \le c_\epsilon \delta^{-\epsilon} \|F\|_{(i,s,1/2)}, \tag{2.26}$$

$$\|\psi_\delta(\cdot)F\|_{(i,s,1/2-\theta)} \le c\delta^{\theta-\theta'} \|F\|_{(i,s,1/2)}, \tag{2.27}$$

where $i = 1, 2$.

Proof of Lemma 2.7. To prove (2.26), using [13, Theorem A.12], we have

$$\|\psi_\delta(\cdot)F\|_{(i,s,1/2)} \le \|D_t^{1/2}\psi_\delta(\cdot)\|_{L_t^p} \|W(\cdot)F\|_{H_x^s(L_t^{p'})} + \|\psi_\delta(\cdot)\|_{L_t^\infty} \|F\|_{(i,s,1/2)}, \tag{2.28}$$

where $1/p + 1/p' = 1$, $1 < p < \infty$ and $W(t)$ is defined as follows:

$$(W(\cdot)F)(x,t) = \begin{cases} \sum_k e^{i(kx+\alpha k^2 t)} \mathcal{F}_x F(k,t), & \text{if } i = 1, \\ \sum_k e^{i(kx \pm \beta kt)} \mathcal{F}_x F(k,t), & \text{if } i = 2. \end{cases}$$

Using Sobolev inequality, we have that the right-hand side of (2.28) is bounded by

$$\|\psi_\delta(\cdot)\|_{H_t^{1/2+\epsilon}}\|F\|_{(i,s,1/2-\epsilon)} + \|F\|_{(i,s,1/2)} \le c\delta^{-\epsilon}\|F\|_{(i,s,1/2)}.$$

To prove (2.27), using the Sobolev inequality with respect to the time variable, we have

$$\|\psi_\delta(\cdot)F\|_{(i,s,1/2-\theta)} \le c\|\psi_\delta(\cdot)\|_{H_t^{1/2-\theta+\theta'}}\|F\|_{(i,s,1/2-\theta')} \le c\delta^{\theta-\theta'}\|F\|_{(i,s,1/2)},$$

for $i = 1, 2$. □

3. Proof of Theorem 1.1. We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. We put $r_0 = \|u_0\|_{H^{s_0}}$, $r_1 = \|n_0\|_{H^{s_1}}$ and $r_2 = \|n_1\|_{H^{s_1-1}}$ where $0 \le s_0 - s_1 \le 1$ and $0 \le s_1 + 1/2 \le 2s_0$. For $\delta \in (0,1]$, we define

$$\begin{aligned}\mathfrak{B}_{s_0,s_1}(r_0,r_1,r_2) = \{(u,n,\partial_t n) \in X_{s_0}\times Y_{s_1}\times Y_{s_1-1} : \|u\|_{X_{s_0}} \le 2cr_0,\\ \|n\|_{Y_{s_1}} \le 2cr_1,\quad \|\partial_t n\|_{Y_{s_1-1}} \le 2cr_2\},\end{aligned}$$

$$\Phi_\delta(u,n)(t) = \psi(t)S(t)u_0 - i\psi(t)\int_0^t S(t-\lambda)(\psi_\delta u\psi_\delta n)(\lambda)d\lambda,$$

$$\Psi_\delta(u)(t) = \psi(t)\partial_t V(t)n_0 + \psi(t)V(t)n_1 + \beta^2\psi(t)\int_0^t V(t-\lambda)\partial_x^2(|\psi_\delta u|^2)(\lambda)d\lambda.$$

By Lemmas 2.1, 2.4, 2.5 and 2.7, we have

$$\|\Phi_\delta(u,n)\|_{X_{s_0}} \le cr_0 + c\delta^\mu\|u\|_{X_{s_0}}\|n\|_{Y_{s_1}},$$

$$\|\Psi_\delta(u)\|_{Y_{s_1}} \sim \|\partial_t\Psi_\delta(u)\|_{Y_{s_1-1}} \le c(r_1+r_2) + c\delta^\mu\|u\|^2_{X_{s_0}},$$

for some $\mu > 0$. Similarly, for $(u,n,\partial_t n)$, $(\widetilde{u},\widetilde{n},\partial_t\widetilde{n}) \in \mathfrak{B}_{s_0,s_1}(r_0,r_1,r_2)$, we have

$$\|\Phi_\delta(u,n) - \Phi_\delta(\widetilde{u},\widetilde{n})\|_{X_{s_0}} \le c\delta^\mu(r_0+r_1)(\|u-\widetilde{u}\|_{X_{s_0}} + \|n-\widetilde{n}\|_{Y_{s_1}}),$$

$$\|\Psi_\delta(u) - \Psi_\delta(\widetilde{u})\|_{Y_{s_1}} \sim \|\partial_t(\Psi_\delta(u) - \Psi_\delta(\widetilde{u}))\|_{Y_{s_1-1}} \le c\delta^\mu r_0\|u-\widetilde{u}\|_{X_{s_0}}.$$

Thus, we conclude that if we choose $\delta > 0$ sufficiently small, then $\Phi_\delta \times \Psi_\delta \times \partial_t\Psi_\delta : (u, n, \partial_t n) \mapsto (\Phi_\delta(u, n), \Psi_\delta(u), \partial_t\Psi_\delta(u))$ is a contraction map. Then, we obtain the unique local existence results in $X_{s_0} \times Y_{s_1} \times Y_{s_1-1}$ by the contraction argument.

For s_0 and s_1 satisfying $0 \le s_0 - s_1 \le 1$ and $0 \le s_1 + 1/2 \le 2s_0$, to prove the uniqueness results, we use the argument similar to [5]. We put

$$\|u\|_{X_s^T} = \inf\{\|w\|_{X_s} : w \in X_s \text{ s.t., } w(t) = u(t) \text{ for } t \in [0, T] \text{ in } H_x^s\},$$

$$\|n\|_{Y_s^T} = \inf\{\|w\|_{Y_s} : w \in Y_s \text{ s.t., } w(t) = n(t) \text{ for } t \in [0, T] \text{ in } H_x^s\}.$$

Let (u_1, n_1) and (u_2, n_2) be the solutions of the initial value problem (1.1) with the same initial data. We can assume that there exists $M > 0$ such that

$$\|\psi_T(\cdot)u_1\|_{X_{s_0}}, \|\psi_T(\cdot)u_2\|_{X_{s_0}} \le M,$$

$$\|\psi_T(\cdot)n_1\|_{Y_{s_1}}, \|\psi_T(\cdot)n_2\|_{Y_{s_1}} \le M,$$

where $T > 0$ is the existence time of the solution. By the definition, for any $\epsilon > 0$, there exist $(\omega, \phi) \in X_{s_0} \times Y_{s_1}$ such that

$$\omega(t) = \psi_T(t)(u_1 - u_2)(t) \text{ for } t \in [0, T'],$$

$$\phi(t) = \psi_T(t)(n_1 - n_2)(t) \text{ for } t \in [0, T'],$$

$$\|\omega\|_{X_{s_0}} \le \|\psi_T(\cdot)(u_1 - u_2)\|_{X_{s_0}^{T'}} + \epsilon,$$

$$\|\phi\|_{Y_{s_1}} \le \|\psi_T(\cdot)(n_1 - n_2)\|_{Y_{s_1}^{T'}} + \epsilon,$$

where $0 \le T' \le \min\{T, 1\}$ is determined later. We put

$$\widetilde{\omega}(t) = \psi(t)\int_0^t S(t-\lambda)U(\lambda)d\lambda,$$

$$\widetilde{\phi}(t) = \psi(t)\int_0^t V(t-\lambda)N(\lambda)d\lambda,$$

where

$$U(t) = \psi_{T'}(t)\omega(t)\psi_{T'}(t)\psi_T(t)n_1(t) + \psi_{T'}(t)\psi_T(t)u_2(t)\psi_{T'}(t)\phi(t),$$

$$N(t) = \psi_{T'}(t)\psi_T(t)\overline{u_1(t)}\psi_{T'}(t)\omega(t) + \psi_{T'}(t)\psi_T(t)u_2(t)\psi_{T'}(t)\overline{\omega(t)}.$$

Then, we have $\widetilde{\omega}(t) = \omega(t)$, $\widetilde{\phi}(t) = \phi(t)$ and $\partial_t\widetilde{\phi}(t) = \partial_t\phi(t)$ for $t \in [0, T']$. Using the same argument as above, there exists $0 < \mu < 1/12$ such that

$$\|\psi_T(\cdot)(u_1 - u_2)\|_{X_{s_0}^{T'}} \leq cT'^{\mu} M(\|\omega\|_{X_{s_0}} + \|\phi\|_{Y_{s_1}})$$
$$\leq cT'^{\mu}(\|\psi_T(\cdot)(u_1 - u_2)\|_{X_{s_0}^{T'}} + \|\psi_T(\cdot)(n_1 - n_2)\|_{Y_{s_1}^{T'}} + 2\epsilon),$$

$$\|\psi_T(\cdot)(n_1 - n_2)\|_{Y_{s_1}^{T'}} \sim \|\psi_T(\cdot)\partial_t(n_1 - n_2)\|_{Y_{s_1-1}^{T'}} \leq cT'^{\mu} M\|\omega\|_{X_{s_0}}$$
$$\leq cT'^{\mu}(\|\psi_T(\cdot)(u_1 - u_2)\|_{X_{s_0}^{T'}} + \epsilon).$$

By choosing $0 < T'$ sufficiently small, we have

$$\psi_T(t)u_1(t) = \psi_T(t)u_2(t),$$

$$\psi_T(t)n_1(t) = \psi_T(t)n_2(t),$$

$$\psi_T(t)\partial_t n_1(t) = \psi_T(t)\partial_t n_2(t),$$

for $t \in [0, T']$. Repeating this procedure, we have the uniqueness result. □

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