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ON THE CLOSED SOLUTION TO SOME NONHOMOGENEOUS EIGENVALUE PROBLEMS WITH p-LAPLACIAN*

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Abstract. We deal with the Dirichlet, Neumann and periodic eigenvalue problems for the equation

$$(|u'|^{p-2}u')' + \lambda |u|^{q-2}u = 0, \quad \text{on } (0,T),$$

where $T > 0$, $\lambda > 0$, and $p, q > 1$. For those problems we obtain a complete description of the spectra and a closed form representation of the corresponding eigenfunctions. As an application of our results we present sharp Poincaré and Wirtinger inequalities for the imbeddings $W_0^{1,p}(0,T)$ into $L^q(0,T)$ and $W_T^{1,p}(0,T)$ into $L^q(0,T)$, respectively.

1. Introduction. In this paper we deal with eigenvalue problems for the equation

$$(E) \qquad (\phi_p(u'))' + \lambda \phi_q(u) = 0, \quad \text{on } (0,T)$$

under Dirichlet, Neumann and periodic boundary value conditions. Here T is a positive real number, λ is a real parameter, p and q are real numbers greater than 1, not necessarily equal, and for $m \in \{p, q\}$,

$$\phi_m(s) = \begin{cases} |s|^{m-2}s, & \text{for } s \in \mathbb{R} \setminus \{0\} \\ 0 & \text{for } s = 0. \end{cases}$$

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In (E), $' = \frac{d}{dt}$. Also through the paper for $m > 1$, $m^* := \frac{m}{m-1}$.

By a solution of (E) we mean a function $u \in C^1[0,T]$ for which $\phi_p(u') \in C^1[0,T]$ and (E) is satisfied.

A simple integration argument shows that under Dirichlet, Neumann or periodic boundary value conditions, a necessary condition for (E) to have a nontrivial solution is that λ be nonnegative. Thus we will assume in the rest of the paper that $\lambda \geq 0$.

The homogeneous Dirichlet boundary value problem for (E) was studied in [10]. There the existence of infinitely many multi-node solutions was proved by using subdifferential operators method and phase plane analysis combined with symmetry properties of the solutions.

In this paper we deal with equation (E) by using a totally different, and rather elementary, approach. We study first the initial value problem for equation (E), and obtain its solution in closed form in terms of incomplete gamma functions. With this at hand we can provide an explicit form of the whole spectrum for equation (E), under the various boundary conditions mentioned above. In particular, for the Dirichlet boundary value problem, some new interesting comparatively different qualitative behavior of the spectrum for the cases $p > q$, $p < q$ and the well known case $p = q$ can be observed in Figure 1 to Figure 5, in Section 3.

Our method allows us for example to show, in a simple and explicit form, that all of our eigenvalues and eigenfunctions are of Ljusternik-Schnirelmann type as well as to obtain Poincaré and Wirtinger inequalities related to our Dirichlet and periodic problem which in turn provide sharp Poincaré and Wirtinger inequalities for the imbeddings $W_0^{1,p}(0,T)$ into $L^q(0,T)$ and $W_T^{1,p}(0,T)$ into $L^q(0,T)$, respectively.

This paper is organized as follows. In Section 2, we study the initial value problem associated to (E) and give a representation for its general solution. Section 3 is dedicated to the complete study of the Dirichlet, Neumann and periodic eigenvalue problems, thus in particular, we complete and simplify the above mentioned results in [10] for the Dirichlet eigenvalue problem. In Section 4 we make the connection of the eigenvalues and eigenfunctions found in Section 3 with those given by the Ljusternik-Schnirelmann theory. Finally in Section 5 we present the Poincaré and Wirtinger inequalities related to our Dirichlet and periodic problem. In both cases we calculate the best possible constants in terms of beta functions and provide a closed form for the extremals in terms of incomplete beta functions.

2. The associated initial value problem. In this section we will study the initial value problem

$$(IV) \qquad \begin{cases} (\phi_p(u'))' + \lambda\phi_q(u) = 0 \\ u(t_0) = a\,, \quad u'(t_0) = b\,, \quad t_0 \in \mathbb{R}\,, \end{cases}$$

for any $a \in \mathbb{R}$ and $b \in \mathbb{R}$. We first prove

Proposition 2.1. *For any $\lambda \in [0, +\infty)$, the initial value problem (IV) has a unique solution defined in $\mathbb{R}$.*

Proof. We note that without loss of generality we can assume that $t_0 = 0$ and $\lambda > 0$ (the equation is autonomous and if $\lambda = 0$ the assertion follows from direct integration). The existence of a local solution in a small interval around zero is a direct application of Schauder fixed point theorem. We show now that this local solution is unique by examining the four different possibilities: (i) $a = 0,\ b = 0$, (ii) $a = 0,\ b \neq 0$, (iii) $a \neq 0,\ b = 0$, and (iv) $a \neq 0,\ b \neq 0$. We do this for $t > 0$ only.

(i) follows immediately from the fact that any solution to (IV) satisfies

$$\frac{|u'(t)|^p}{p^*} + \lambda\frac{|u(t)|^q}{q} = \frac{|b|^p}{p^*} + \lambda\frac{|a|^q}{q}, \tag{2.1}$$

in its domain of definition.

To show (ii) let us assume that u and v are two local solutions to (IV) such that $a = 0$, $b \neq 0$. Then

$$\begin{aligned} \phi_p(u'(t)) - \phi_p(v'(t)) &= \lambda\int_0^t [\phi_q(v(\tau)) - \phi_q(u(\tau))]\,d\tau \\ &= \lambda\int_0^t \tau^{q-1}[\phi_q(\frac{v(\tau)}{\tau}) - \phi_q(\frac{u(\tau)}{\tau})]\,d\tau. \end{aligned} \tag{2.2}$$

Since $\frac{u(\tau)}{\tau} \to b$ and $\frac{v(\tau)}{\tau} \to b$ as $\tau \to 0$ we have that for t small and $\tau \leq t$ $\frac{u(\tau)}{\tau} \to b$, $\frac{v(\tau)}{\tau} \to b$, ($u'(t)$, and $v'(t)$ are in a small subinterval where both functions $\phi_{p^*}(t)$ and $\phi_q(t)$ are C^1). Hence there are two positive constants K_1 and K_2 such that

$$K_1|u'(t) - v'(t)| \leq K_2\lambda\int_0^t \tau^{q-1}|\frac{v(\tau)}{\tau} - \frac{u(\tau)}{\tau}|\,d\tau,$$

which implies

$$K_1||w'||_\varepsilon \leq K_2\lambda\varepsilon^{q-1}||w||_\varepsilon,$$

where $w = u - v$, and $||\cdot||_\varepsilon$ indicates the sup norm in the interval $[0, \varepsilon]$. Then combining this last expression with that fact that

$$w(t) = \int_0^t w'(\tau)\, d\tau,$$

and that $w(0) = 0$, $w'(0) = 0$, we find, for ε small enough, the contradiction

$$(K_1 - K_2 \lambda \varepsilon^q)||w'||_\varepsilon \leq 0.$$

The case (iii) is reduced to case (ii). Indeed, rewriting the equation of (IV) in the equivalent system form

$$\begin{cases} u' = \phi_{p^*}(v) \\ v' = -\lambda \phi_q(u), \end{cases} \tag{2.3}$$

we find from the second equation of this system that $\phi_{q^*}(\frac{v'}{\lambda}) = -u$. Combining then with the first of (2.3), we obtain that v satisfies

$$(\phi_{q^*}(v'))' + \lambda^{q^*-1}\phi_{p^*}(v) = 0, \quad \text{on } (0, T),$$

together with the initial conditions: $v(0) = 0$ and $v'(0) = -\lambda\phi_q(a) \neq 0$. Thus (iii) is reduced to (ii).

Finally for case (iv) we consider again (2.3). Since in this case $u(0) \neq 0$ and $v(0) \neq 0$, we have that for t small the right hand of (2.3) stays in a subinterval where the functions ϕ_{p^*} and ϕ_q are C^1. Hence from the classical theory for o.d.e. the solution is locally unique.

Thus in all the cases the solution to (IV) is locally unique. Since from (2.1) any solution to (IV) is defined in all $\mathbb{R}$, it is clear that our previous argument also implies global uniqueness to the solution to (IV). □

The rest of this section will be dedicated to obtain the general solution to (IV) in a closed form. For this it will be convenient to find first the solution to the initial value problem

$$(IV)_0 \qquad \begin{cases} (\phi_p(u'))' + \lambda\phi_q(u) = 0 \\ u(0) = 0 \quad u'(0) = \alpha, \end{cases}$$

where without loss of generality we may assume $\alpha > 0$.

Let u be a solution to $(IV)_0$ and let $t(\alpha)$ be the first zero point of $u'(t)$. On $(0, t(\alpha))$ u satisfies $u(t) > 0$ and $u'(t) > 0$, and thus from (2.1)

$$\frac{u'(t)^p}{p^*} + \lambda\frac{u(t)^q}{q} = \lambda\frac{R^q}{q} = \frac{\alpha^p}{p^*}, \tag{2.4}$$

where $R = u(t(\alpha))$. Solving for u' and integrating , we find

$$\Big(\frac{q}{\lambda p^*}\Big)^{\frac{1}{p}} \int_0^t \frac{u'(s)\,ds}{(R^q - u(s)^q)^{\frac{1}{p}}} = t, \tag{2.5}$$

which after some changes of variables can be written as

$$t = \Big(\frac{q}{\lambda p^*}\Big)^{\frac{1}{p}} \frac{1}{R^{\frac{q-p}{p}}} \int_0^{\frac{u}{R}} \frac{ds}{(1 - s^q)^{\frac{1}{p}}}. \tag{2.6}$$

For $\sigma \in [0, \frac{q}{2}]$, let us set

$$\operatorname{arcsin}_{pq}(\sigma) := \frac{q}{2} \int_0^{\frac{2\sigma}{q}} \frac{ds}{(1 - s^q)^{\frac{1}{p}}}, \tag{2.7}$$

and note that this integral converges for all $\sigma \in [0, \frac{q}{2}]$.

Substituting $s = z^{\frac{1}{q}}$ in (2.7), we obtain

$$\operatorname{arcsin}_{pq}(\sigma) = \frac{1}{2}\tilde{B}(\frac{1}{q}, \frac{1}{p^*}, (\frac{2\sigma}{q})^q), \tag{2.8}$$

where $\tilde{B}(\frac{1}{q}, \frac{1}{p^*}, y)$ denotes the incomplete beta function

$$\tilde{B}(\frac{1}{q}, \frac{1}{p^*}, y) = \int_0^y z^{\frac{1}{q}-1}(1 - z)^{-\frac{1}{p}}\,dz,$$

see [1].

We note that $\tilde{B}(\frac{1}{q}, \frac{1}{p^*}, y)$ can also be expressed in terms of hypergeometric functions, see ([1], p. 263), we find

$$\operatorname{arcsin}_{pq}(\sigma) = \frac{1}{2} q \sigma^{\frac{1}{q}} F(\frac{1}{q}, \frac{1}{p}, 1 + \frac{1}{q}; \sigma),$$

where $F(a, b, c; \sigma)$ denotes the hypergeometrical function, see [1] and also [12]. Next, setting $\sigma = \frac{q}{2}$ in (2.8), we obtain

$$\pi_{pq} := 2 \operatorname{arcsin}_{pq}(\frac{q}{2}) = B(\frac{1}{q}, \frac{1}{p^*}),$$

where B denotes the beta function.

Remark 2.1. From the properties of the beta function we immediately find that

$$\pi_{pq} = \pi_{q^*p^*}. \tag{2.9}$$

We have that $\arcsin_{pq} : [0, \frac{q}{2}] \mapsto [0, \frac{\pi_{pq}}{2}]$, and is strictly increasing. Let us denote its inverse by $\sin_{pq}$. Then, $\sin_{pq} : [0, \frac{\pi_{pq}}{2}] \mapsto [0, \frac{q}{2}]$ and is strictly increasing.

We extend $\sin_{pq}$ to all $\mathbb{R}$ (and still denote this extension by $\sin_{pq}$) in the following form: for $t \in [\frac{\pi_{pq}}{2}, \pi_{pq}]$, we set $\sin_{pq}(t) = \sin_{pq}(\pi_{pq} - t)$, then for $t \in [-\pi_{pq}, 0]$, we define $\sin_{pq}(t) = -\sin_{pq}(-t)$, and finally we extend $\sin_{pq}$ to all $\mathbb{R}$ as a $2\pi_{pq}$ periodic function.

Then, it is simple matter to verify that $\sin_{pq}$ is the unique (global) solution to the initial value problem

$$\begin{aligned} &(\phi_p(u'))' + \frac{2^q}{p^* q^{q-1}} \phi_q(u) = 0 \\ &u(0) = 0, \quad u'(0) = 1. \end{aligned} \tag{2.10}$$

For later use let us define $\cos_{pq}(t) := \frac{d}{dt} \sin_{pq}(t)$. Then from (2.1), (2.10), we have that

$$|\cos_{pq}(t)|^p + \left(\frac{2}{q}\right)^q |\sin_{pq}(t)|^q = 1 \quad \text{for all } t \in \mathbb{R}. \tag{2.11}$$

From (2.6) and (2.7), we find that

$$t = \frac{2}{(\lambda p^*)^{\frac{1}{p}} q^{\frac{1}{p^*}}} R^{\frac{p-q}{p}} \arcsin_{pq}\left(\frac{qu}{2R}\right), \tag{2.12}$$

and hence

$$u(t) = \frac{2R}{q} \sin_{pq}\Big(\frac{(\lambda p^*)^{\frac{1}{p}} q^{\frac{1}{p^*}}}{2} R^{\frac{q-p}{p}} t\Big), \tag{2.13}$$

for all $t \in \mathbb{R}$.

From (2.4) we can solve for R in terms of α to obtain $R = \left(\frac{q}{\lambda p^*}\right)^{\frac{1}{q}} \alpha^{\frac{p}{q}}$, and thus

$$R^{\frac{q-p}{p}} = \left(\frac{q}{\lambda p^*}\right)^{\frac{q-p}{pq}} \alpha^{\frac{q-p}{q}}.$$

Substituting this expression in (2.13), and setting

$$A_{pq}(\alpha, \lambda) = \frac{(\lambda p^*)^{\frac{1}{q}} q^{\frac{1}{q^*}}}{2} \alpha^{\frac{q-p}{q}},$$

we obtain that the general solution to $(IV)_0$ can be written as

$$u(t) = \frac{\alpha}{A_{pq}(|\alpha|, \lambda)} \sin_{pq}(A_{pq}(|\alpha|, \lambda)t), \tag{2.14}$$

for all $t \in \mathbb{R}$. For fixed λ this solution is $\tau(\alpha)$-periodic, with

$$\tau(\alpha) = \frac{2\pi_{pq}}{A_{pq}(\alpha, \lambda)} = 4t(\alpha).$$

We now proceed to evaluate the general solution to (IV). Keeping in mind that the differential equation involved is autonomous, we can modify (2.14) by a shifting to

$$u(t) = \frac{\alpha}{A_{pq}(|\alpha|, \lambda)} \sin_{pq}(A_{pq}(|\alpha|, \lambda)(t - t_0) + \delta), \quad t \in \mathbb{R}, \tag{2.15}$$

and still have a solution to the equation of (IV). Thus to satisfy the initial conditions of (IV), we have to solve (uniquely) for α and δ in terms of the initial conditions a and b from the equations

$$\begin{aligned} u(t_0) = a &= \frac{\alpha}{A_{pq}(|\alpha|, \lambda)} \sin_{pq}(\delta), \\ u'(t_0) = b &= \alpha \cos_{pq}(\delta). \end{aligned} \tag{2.16}$$

Taking into account (2.11), from (2.16) we get

$$\Big(\frac{2}{q}\Big)^q \Big(\frac{|a| A_{pq}(|\alpha|, \lambda)}{|\alpha|}\Big)^q + \Big(\frac{|b|}{|\alpha|}\Big)^p = 1,$$

and so

$$|\alpha|^p = \frac{|a|^q \lambda p^*}{q} + |b|^p. \tag{2.17}$$

Thus, from (2.16) and (2.17), we can solve uniquely for $\alpha \in \mathbb{R}$ and $\delta \in [0, \pi_{pq})$. Therefore (2.15) effectively provides a representation for the general solution to (IV) in terms of the parameters α and δ supplemented by (2.16).

We note that in (2.15) α does not necessarily measure the slope of the solution at t_0 for problem (IV).

3. The eigenvalue problems. In this section we give a full description of the set of eigenvalues and corresponding eigenfunctions for the equation (E) under Dirichlet, Neumann an periodic boundary conditions. We only will treat in full detail the case of Dirichlet problem. Thus we first consider the problem

$$(E_d) \qquad \begin{cases} (\phi_p(u'))' + \lambda \phi_q(u) = 0, & \text{on } (0, T) \\ u(0) = 0, \quad u(T) = 0. \end{cases}$$

We have

Theorem 3.1. *For any given* $\alpha \neq 0$, *the set of eigenvalues of problem* (E_d) *is given by*

$$\lambda_n(\alpha) = \Big(\frac{2n\pi_{pq}}{T}\Big)^q \frac{|\alpha|^{p-q}}{p^* q^{q-1}} \quad \textit{for each} \quad n \in \mathbb{N}, \tag{3.1}$$

with corresponding eigenfunctions

$$u_{n,\alpha}(t) = \frac{\alpha T}{n\pi_{pq}} \sin_{pq}(\frac{n\pi_{pq}}{T}\ t). \tag{3.2}$$

Proof. For $\alpha \in \mathbb{R}$ given, by imposing that u in (2.14) satisfies the boundary conditions in (E_d), we obtain that λ is an eigenvalue of (E_d) if and only if

$$\frac{1}{2} p*^{\frac{1}{q}} q^{\frac{1}{q^*}} \lambda^{\frac{1}{q}} |\alpha|^{\frac{q-p}{q}} T = n\pi_{pq}, \quad n \in \mathbb{N} \tag{3.3}$$

and hence (3.1) follows. The expression for the eigenfunctions follows then directly from (2.14). □

Next let us define the spectrum of (E_d), denoted by $\mathcal{S}(E_d)$, to be the set of all pairs (α, λ) for which problem (E_d) has a nontrivial solution. If for each $n \in \mathbb{N}$ we set $\mathcal{S}_n = \{(\alpha,\lambda) | (\alpha,\lambda) \text{ satisfies } (3.3)\}$, then

$$\mathcal{S}(E_d) = \cup_1^\infty \mathcal{S}_n.$$

We call $\mathcal{S}_n$ a generalized eigencurve for problem (E_d) and we note that to each $\mathcal{S}_n$ we can associate the function $t \mapsto \frac{T}{n\pi_{pq}} \sin_{pq}(\frac{n\pi_{pq}}{T}\ t)$.

We will distinguish next among the cases $q > p$, $q < p$, and $q = p$, this last case being very well known, see for example [4, 5, 6, 7, 8, 10].

In the case $q > p$ we have the following geometrical interpretation of $\mathcal{S}(E_d)$:

For the case $q < p$ we can distinguish three different qualitative behaviors as we show in the next three diagrams:

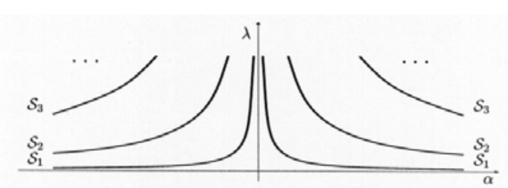

Figure 1. $\lambda_n = n^3 |\alpha|^{-1}, \ n = 1, 2, 3, \ldots$

(i) $q < p - 1$

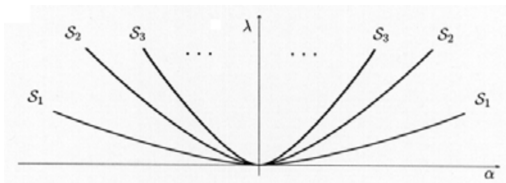

Figure 2. $\lambda_n = n^{1.5}|\alpha|^{1.5}, \ n = 1, 2, 3, \ldots$

(ii) $q = p - 1$

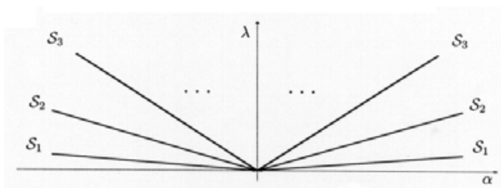

Figure 3. $\lambda_n = n^2|\alpha|, \ n = 1, 2, 3, \ldots$

(iii) $p - 1 < q < p$

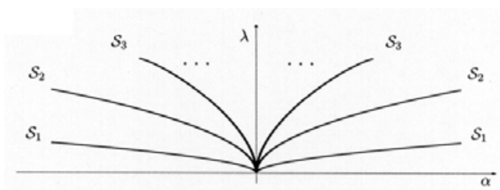

Figure 4. $\lambda_n = n^{1.5}|\alpha|^{0.5}, \ n = 1, 2, 3, \ldots$

The case $p = q$ is a singular one in the sense that the homogeneity of (E_d) implies that λ_n is independent of α.

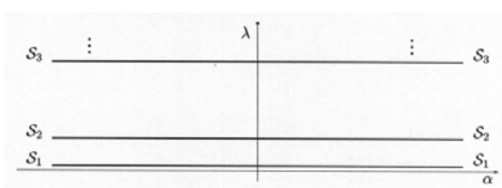

Figure 5. $\lambda_n = n^3, \ n = 1, 2, 3, \ldots$

Remark 3.1. Let $n_0 \in \mathbb{N}$ and $\alpha_0 \neq 0$, be fixed, then

- $q > p$ implies that

$$\lim_{|\alpha| \to \infty} \lambda_{n_0}(\alpha) = 0, \quad \lim_{|\alpha| \to 0} \lambda_{n_0}(\alpha) = \infty, \quad \lim_{n \to \infty} \lambda_n(\alpha_0) = \infty.$$

- $q < p$ implies that

$$\lim_{|\alpha|\to\infty} \lambda_{n_0}(\alpha) = \infty, \quad \lim_{|\alpha|\to 0} \lambda_{n_0}(\alpha) = 0, \quad \lim_{n\to\infty} \lambda_n(\alpha_0) = \infty.$$

- $q = p$ implies that

$$\lambda_{n_0}(\alpha) = \lambda_{n_0}, \text{ for all } \alpha \neq 0, \quad \lim_{n\to\infty} \lambda_n = \infty.$$

Moreover, for $q < p$, we have that

- $q < p - 1$ implies that $\frac{d}{d\alpha}\lambda_{n_0}|_{\alpha=0} = 0$,
- $q = p - 1$ implies that $\frac{d}{d\alpha}\lambda_{n_0} = \left(\frac{2n_0\pi_{pq}}{T}\right)^q \frac{1}{p^* q^{q-1}}$,
- $p - 1 < q < p$ implies that $\frac{d}{d\alpha}\lambda_{n_0}|_{\alpha=\pm 0} = \pm\infty$.

Let us observe that for any $\alpha \neq 0$ and any $n \in \mathbb{N}$ we have that $\frac{\lambda_n(\alpha)}{\lambda_1(\alpha)} = n^q$, and that the zero points of the eigenfunctions associated with the generalized eigencurve $\mathcal{S}_n$ do not depend on α and they form a sequence of equidistant points in $(0, T)$, see [10].

Remark 3.2. Let us denote by $\mu_n = \mu_n(\beta)$ the eigenvalues of the dual problem to (E_d)

$$(E_d)^* \qquad \begin{cases} (\phi_{q^*}(u'))' + \mu\phi_{p^*}(u) = 0 & \text{on } (0, T) \\ u(0) = 0, \quad u(T) = 0. \end{cases}$$

Then, from (3.1), it follows that

$$\mu_n(\beta) = \left(\frac{2n\pi_{q^*p^*}}{T}\right)^{p^*} \frac{|\beta|^{q^*-p^*}}{q\, p^{*p^*-1}} \quad \text{for each} \quad n \in \mathbb{N}.$$

Hence, from (2.9), we have that the following symmetry holds:

$$\lambda_n^{\frac{1}{q}}\left((p^*)^{\frac{1}{p}}\right) = \mu_n^{\frac{1}{p^*}}\left((q)^{\frac{1}{q^*}}\right). \tag{3.4}$$

In particular when $p = q$, (3.4) reads as

$$\lambda_n^{\frac{1}{p}} = \mu_n^{\frac{1}{p^*}}, \tag{3.5}$$

(cf. [8]).

For later use we consider the solutions $w_n(t) = \frac{1}{n}\sin_{pq}(nt)$, for $n \in \mathbb{N}$, to the boundary value problem

$$(E_d^n) \qquad \begin{cases} (\phi_p(w_n'))' + \frac{(2n)^q}{p^* q^{q-1}}\phi_q(w_n) = 0 \quad \text{on } (0, \pi_{pq}) \\ w_n(0) = 0, \quad w_n(\pi_{pq}) = 0. \end{cases}$$

Note that these solutions w_n are normalized so that $w_n'(0) = 1$.

We are interested here in computing the norms $||w_n||_{L^q}^q$, and $||w_n'||_{L^p}^p$. Multiplying the equation in (E_d^n) by w_n and integrating over $[0, \pi_{pq}]$, yields

$$\int_0^{\pi_{pq}} |w_n'(t)|^p \, dt = \frac{(2n)^q}{p^* q^{q-1}} \int_0^{\pi_{pq}} |w_n(t)|^q \, dt. \tag{3.6}$$

Since on the other hand w_n also satisfies

$$\frac{|w_n'(t)|^p}{p^*} + \frac{(2n)^q}{p^* q^{q-1}} \frac{|w_n(t)|^q}{q} = \frac{1}{p^*},$$

for all $t \in \mathbb{R}$, by integrating this expression on $[0, \pi_{pq}]$, we find

$$\frac{1}{p^*}\int_0^{\pi_{pq}} |w_n'(t)|^p \, dt + \frac{(2n)^q}{p^* q^q}\int_0^{\pi_{pq}} |w_n(t)|^q \, dt = \frac{\pi_{pq}}{p^*}. \tag{3.7}$$

Thus, from (3.6) and (3.7), we obtain

$$\int_0^{\pi_{pq}} |w_n(t)|^q \, dt = \frac{\pi_{pq} p^* q^q}{(2n)^q (p^* + q)}, \tag{3.8}$$

and

$$\int_0^{\pi_{pq}} |w_n'(t)|^p \, dt = \frac{\pi_{pq} q}{(p^* + q)}. \tag{3.9}$$

In particular, from (3.8) and (3.9), it follows respectively that

$$\int_0^T |\sin_{pq}(\frac{n\pi_{pq}}{T}t)|^q \, dt = \frac{T p^* q^q}{(2)^q (p^* + q)}, \tag{3.10}$$

and

$$\int_0^T |\frac{d}{dt}\sin_{pq}(\frac{n\pi_{pq}}{T}t)|^p \, dt = \frac{n^p {\pi_{pq}}^p q}{T^{p-1}(p^* + q)}. \tag{3.11}$$

Next let us consider the Neumann problem

$$(E_n) \qquad \begin{cases} (\phi_p(v'))' + \nu\phi_q(v) = 0 \quad \text{on } (0,T) \\ v'(0) = 0, \quad v'(T) = 0. \end{cases}$$

Observe that $v_0 = 1$ is a normalized eigenfunction corresponding to the eigenvalue $\nu_0 = 0$. Also observe that any (non constant) eigenfunction of (E_n) must change sign.

Since the equation in (E_n) is autonomous, Proposition 2.1 implies that positive eigenvalues of (E_n) coincide with those of the Dirichlet problem and the corresponding eigenfunctions are simply translations of those given by (3.1). Namely, we have

Theorem 3.2. *For any given* $\alpha \neq 0$, *the set of eigenvalues of the problem* (E_n) *is given by* $\nu_0 = 0$ *and* $\nu_n(\alpha) = \lambda_n(\alpha)$ *(as given by* (3.1)*). The corresponding eigenfunctions are* $v_0(t) = c$, $c \in \mathbb{R} \setminus \{0\}$, *and*

$$v_{n,\alpha}(t) = \frac{\alpha T}{n\pi_{pq}} \sin_{pq}(\frac{n\pi_{pq}}{T}\ (t - \frac{T}{2n})) \quad n \in \mathbb{N}.$$

Finally, let us consider the periodic problem

$$(E_p) \qquad \begin{cases} (\phi_p(w'))' + \delta\phi_q(w) = 0 \quad \text{on } (0,T) \\ w(0) = w(T), \quad w'(0) = w'(T). \end{cases}$$

Similar observations as above yield

Theorem 3.3. *For any given* $\alpha \neq 0$, *the set of eigenvalues of the problem* (E_p) *is given by* $\delta_0 = 0$ *and* $\delta_n(\alpha) = \lambda_{2n}(\alpha)$ *(as given by* (3.1)*). The corresponding eigenfunctions are* $w_0(t) = c$, $c \in \mathbb{R} \setminus \{0\}$, *and any translation of* $w_{n,\alpha}(t) = u_{2n,\alpha}(t)$ *(as given by* (3.2)*), i.e.,*

$$w_{n,\alpha}(t) = \frac{\alpha T}{2n\pi_{pq}} \sin_{pq}(\frac{2n\pi_{pq}}{T}\ (t - t_n)) \quad n \in \mathbb{N},$$

where $t_n \in \mathbb{R}$ *is arbitrary.*

Remark 3.3. Note that to the contrary of problems (E_d) and (E_n) the set of all generalized eingencurves of (E_p) is *smaller* but , on the other hand, the eigenspace associated with every eigencurve of (E_p) is *larger* than those corresponding to (E_d) and (E_n).

4. Connection to Ljusternik-Schnirelmann theory. On light of the results of Section 3 and since our eigenvalue problems are variational it is natural to ask about the relationship of the eigenvalues and eigenfunctions obtained in that section with those provided by the Ljusternik-Schnirelmann theory. It is the purpose of this section to show that these two sets of eigenvalues (and eigenfunctions) coincide. Again we will only treat the case of the Dirichlet problem in full detail.

Let $K \subset W_0^{1,p}(0,T)$ be a compact symmetric (with respect to the origin) subset such that $0 \notin K$. Define the genus of K as

$$\gamma(K) := \inf\{n \in \mathbb{N} : \text{there exists } h : K \mapsto \mathbb{R}^n \backslash \{0\}, h \text{ is continuous and odd}\}.$$

Also define the general level sets (for $r > 0$)

$$S_r = \{u \in W_0^{1,p}(0,T) \mid \frac{1}{p}\int_0^T |u'(t)|^p \, dt = r^p\}.$$

For $n \in \mathbb{N}$ and $r > 0$, let us set

$$\Lambda_{n,r} = \{K \subset S_r \mid \gamma(K) \geq n\}.$$

Then

$$c_n(r) = \frac{1}{r^p} \sup_{K \in \Lambda_{n,r}} \inf_{u \in K} \frac{1}{q}\int_0^T |u(t)|^q \, dt,$$

are Ljusternik-Schnirelmann critical levels, (see [2], Th. 6.6.11). We have the following

Theorem 4.1. *Let $\alpha = \alpha(r)$ be given by*

$$\alpha = \Big(\frac{(p^* + q)p}{Tq}\Big)^{\frac{1}{p}} r, \quad r > 0. \tag{4.1}$$

Then for any $n \in \mathbb{N}$,

$$c_n(r) = \frac{p}{q\lambda_n(\alpha)},$$

where the λ_n, $n \in \mathbb{N}$, are given by (3.1). *The critical level $c_n(r)$ is achieved just at $\{u_n, -u_n\}$, where*

$$u_n(t) = \frac{T^{\frac{1}{p^*}}}{n\pi_{pq}}\Big(\frac{(p^* + q)p}{q}\Big)^{\frac{1}{p}} r \sin_{pq}(\frac{n\pi_{pq}}{T}t), \quad n \in \mathbb{N}. \tag{4.2}$$

Proof. Since for any $n \in \mathbb{N}$ and $r > 0$ the value $\frac{p}{qc_n(r)}$ is an eigenvalue of (E_d) and by the uniqueness of the solutions to the i.v.p (Proposition 2.1), we have that

$$\{c_n(r)\}_{n=1}^{\infty} \subset \{\frac{p}{q\lambda_n(\alpha)}\}_{n=1}^{\infty}, \tag{4.3}$$

where α and r satisfy (4.1).

Note that the relation (4.1) between α and r follows from the fact the eigenfunctions obtained via Ljusternik-Schnirelmann belong to S_r.

Let now $r > 0$ be given and $w_{1,\alpha}, \cdots, w_{n,\alpha}$ be the "bumps" of the eigenfunction $u_{n,\alpha}$ associated with the eigenvalue $\lambda_n(\alpha)$, with α given by (4.1); i.e., for $k = 1, \dots, n$, $w_{k,\alpha}(t) = u_{n,\alpha}(t)$, $t \in [z_k, z_{k+1}]$, $w_{k,\alpha}(t) = 0$, otherwise, where z_k is the k-th zero point of $u_{n,\alpha}$ in $[0, T)$. Consider the set $K_n = \text{span}\{w_{1,\alpha}, \cdots, w_{n,\alpha}\} \cap S_r$. Then $K_n \subset \Lambda_{n,r}$ and so $c_n(r) = \frac{p}{q\lambda_n(\alpha)}$. Hence equality holds in (4.3) and (4.2) follows from (3.2) and (4.1). $\square$

In a similar form for the Neumann and periodic problem one can show a one to one correspondence between the eigenvalues (ν_n and δ_n) with the corresponding critical Ljusternik-Schnirelmann levels.

5. Some sharp Poincaré and Wirtinger inequalities. It is well known fact that for any $p > 1$, $q > 1$ the imbedding $W_0^{1,p}(0,T)$ into $L^q(0,T)$ is compact. We present next a proof based on our Theorem 3.1 for a sharp Poincaré inequality for this case. We emphasize that this result was already known, see e.g. [9], p.90, [11], p.357, and [3]. Thus the merit of our proof is that it is a different and direct one based on our results.

Theorem 5.1. *For any $u \in W_0^{1,p}(0,T)$ the following inequality holds.*

$$\left(\int_0^T |u(t)|^q \, dt\right)^{\frac{1}{q}} \le \frac{T^{\frac{1}{q}+\frac{1}{p^*}}}{2\pi_{pq}} (p^*)^{\frac{1}{q}} (q)^{\frac{1}{p^*}} (p^* + q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_0^T |u'(t)|^p dt\right)^{\frac{1}{p}}. \tag{5.1}$$

The equality in (5.1) holds if and only if $u(t) = C \sin_{pq}(\frac{\pi_{pq}}{T} t)$, where $C \in \mathbb{R}$ is arbitrary.

Proof. A standard compactness argument implies that

$$\frac{1}{c_{pq}} = \inf_{u \in W_0^{1,p} \setminus \{0\}} \frac{\left(\int_0^T |u'(t)|^p \, dt\right)^{\frac{1}{p}}}{\left(\int_0^T |u(t)|^q \, dt\right)^{\frac{1}{q}}}, \tag{5.2}$$

is achieved for some $u_{pq} \in W_0^{1,p}(0,T)$. Lagrange's multiplier techniques then yields that u_{pq} must be an eigenfunction of (E_d). By the structure of the quotient in (5.2) this eigenfunction can be taken to be nonnegative (and normalized in a suitable form), thus from Theorem 3.1, it follows that

$$u_{pq}(t) = \sin_{pq}(\frac{\pi_{pq}}{T}t).$$

Then from (3.10)–(3.11), we obtain

$$c_{pq} = \frac{\left(\int_0^T [\sin_{pq}(\frac{\pi_{pq}}{T}t)]^q dt\right)^{\frac{1}{q}}}{\left(\int_0^T [\frac{d}{dt}\sin_{pq}(\frac{\pi_{pq}}{T}t)]^p dt\right)^{\frac{1}{p}}} = \frac{T^{\frac{1}{q}+\frac{1}{p^*}}}{2\pi_{pq}}(p^*)^{\frac{1}{q}}(q)^{\frac{1}{p^*}}(p^*+q)^{\frac{1}{p}-\frac{1}{q}}. \quad (5.3)$$

From (5.2) and (5.3), (5.1) follows.

Remark 4.1. Taking into account the symmetry relation $c_{pq} = c_{q^*p^*}$ we find that c_{pq} as in (5.3) also gives the best Poincaré constant for the imbedding $W_0^{1,q^*}(0,T)$ into $L^{p^*}(0,T)$. For $p = q = 2$ the best constant reduces to the well known value $\frac{T}{\pi}$.

Let $W_T^{1,p}(0,T)$ be the space of functions in $L^p(0,T)$ with weak derivate $u' \in L^p(0,T)$, and such that $u(0) = u(T)$. In a similar form as above we now have the following sharp Wirtinger inequality, cf. [9], p.74.

Theorem 5.2. *For any $u \in W_T^{1,p}(0,T)$ satisfying $\int_0^T u(t)\,dt = 0$ the following inequality holds.*

$$\left(\int_0^T |u(t)|^q\,dt\right)^{\frac{1}{q}} \leq \frac{T^{\frac{1}{q}+\frac{1}{p^*}}}{4\pi_{pq}}(p^*)^{\frac{1}{q}}(q)^{\frac{1}{p^*}}(p^*+q)^{\frac{1}{p}-\frac{1}{q}}\left(\int_0^T |u'(t)|^p\,dt\right)^{\frac{1}{p}}. \quad (5.4)$$

The equality in (5.4) *holds if and only if $u(t) = C\sin_{pq}(\frac{2\pi_{pq}}{T}t)$, where $C \in \mathbb{R}$ is arbitrary.*

Remark 4.2. Noting that the constant in (5.4) is symmetric when changing the roles played by p and q^*, and q and p^* respectively. For $p = q = 2$ the best constant reduces to the well known value $\frac{T}{2\pi}$.

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