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ON TWO-DIMENSIONAL HAMILTONIAN TRANSPORT EQUATIONS WITH CONTINUOUS COEFFICIENTS*

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Abstract. We consider two-dimensional autonomous flows with divergence free continuous coefficients. Under a generic assumption of regularity on the set of critical points, we give a proof of uniqueness for the characteristics and for the transport equation in the framework of distributions.

1. Introduction

According to the theory of characteristics, it is well-known that the linear transport equation,

$$\partial_t u(t,x) + a(t,x)\cdot\nabla_x u(t,x) = 0, \tag{1.1}$$

$$u(0,x) = u^0(x), \tag{1.2}$$

where $t\in\mathbb{R}, x\in\mathbb{R}^N$, $u^0:\mathbb{R}^N\to\mathbb{R}$, $a:\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$ and $u:\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$, is related to the system of ordinary differential equations

$$\frac{dX}{ds}(s) = a(s,X(s)), \tag{1.3}$$

$$X(t) = x^0, \tag{1.4}$$

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where $x^0 \in \mathbb{R}^N$ and $t \in \mathbb{R}$, via the relation $u(t,x) = u^0(X(s=0,t,x))$.

Since this system admits a unique (local) solution of class C^1 for all $x^0 \in \mathbb{R}^N$ as soon as $a \in C^1(\mathbb{R}\times\mathbb{R}^N;\mathbb{R}^N)$, it is also easy to prove that under this assumption, the transport equation (1.1), (1.2) admits a unique (local) solution of class C^1 as long as $u^0 \in C^1(\mathbb{R}^N)$.

The assumption that $a \in C^1(\mathbb{R}\times\mathbb{R}^N)$ can easily be relaxed to a Lipschitz-continuous in the x variable. Under the additional assumption that $\mathrm{div}_x\, a = 0$ (or more generally $\mathrm{div}_x\, a \in L^\infty(\mathbb{R}\times\mathbb{R}^N)$), it can even be relaxed to $\nabla_x a \in L^1_{loc}(\mathbb{R}\times\mathbb{R}^N)$; see [6], and [3], [4], for extensions. Note however that in this case, uniqueness does not hold for all $x^0 \in \mathbb{R}^N$ in system (1.3), (1.4), as is the case in [2], [8], [7], but only for almost every $x^0 \in \mathbb{R}^N$.

We prove here that the regularity of a can even be relaxed (generically) to a only continuous when $N = 2$ and a does not depend on t (and still $\mathrm{div}_x\, a = 0$), that is, for autonomous two-dimensional Hamiltonian fields

$$a(x_1,x_2) = \Big(-\frac{\partial H}{\partial x_2}(x_1,x_2), \frac{\partial H}{\partial x_1}(x_1,x_2)\Big), \qquad H \in C^1(\mathbb{R}^2,\mathbb{R}). \tag{1.5}$$

This leads to the Hamiltonian system

$$\frac{dX_1}{dt}(t) = -\frac{\partial H}{\partial x_2}(X_1(t),X_2(t)), \qquad \frac{dX_2}{dt}(t) = \frac{\partial H}{\partial x_1}(X_1(t),X_2(t)), \tag{1.6}$$

$$(X_1(0),X_2(0)) = (x_1^0,x_2^0). \tag{1.7}$$

In Section 2, we prove that under the assumption that $H(x^0)$ is not a critical value of H, there is uniqueness of the solutions to system (1.6), (1.7). Then, we present in Section 3 a generic condition on a such that there is uniqueness of the solutions to system (1.6), (1.7) for almost every $x^0 \in \mathbb{R}^2$. Finally, in Section 4, we prove an analogous result of uniqueness for the transport equation (1.1), (1.2).

2. Uniqueness of a single characteristic curve

The main result of this section is based on the following remark. A one-dimensional autonomous differential equation with continuous coefficient b,

$$Y'(t) = b(Y), \qquad Y(0) = y^0, \tag{2.1}$$

has a unique solution as long as b does not vanish. The reason is that with this assumption, we can divide (2.1) by $b(Y)$ and integrate. Thus Y must be given by the usual formula involving the inverse of the primitive of $1/b$. In two dimensions we have the following result with (1.5).

Theorem 2.1. *Let $H \in C^1(\mathbb{R}^2, \mathbb{R})$ and $x^0 \in \mathbb{R}^2$. If $H(x^0)$ is not a critical value of H, then any two classical solutions (of class C^1) on an interval I containing 0 to system (1.6), (1.7) coincide.*

Proof. Let us first recall the definition of the set of critical points of H

$$M = \{x \in \mathbb{R}^2 : \nabla H(x) = 0\}, \tag{2.2}$$

which is closed. The set of critical values is then defined to be $H(M)$.

We notice that the Hamiltonian H is conserved along any solution $t \in I \mapsto X(t)$:

$$\forall t \in I \qquad H(X(t)) = H(x^0). \tag{2.3}$$

Thus, $H(X(t))$ is never a critical value of H. Therefore, as usual, it is enough to prove that any two solutions coincide in a neighborhood of 0. Thus we just consider one solution $X(t) = (X_1(t), X_2(t))$, and we are going to prove that it must be given by an explicit formula in a neighborhood of 0. But since $H(x^0) \notin H(M)$, we know that $x^0 \notin M$. Thus, either $\frac{\partial H}{\partial x_2}(x^0) \neq 0$ or $\frac{\partial H}{\partial x_1}(x^0) \neq 0$. Let us suppose for example that $\frac{\partial H}{\partial x_2}(x^0) \neq 0$ (the other case can be treated similarly). Thanks to the implicit function theorem, there exists $\eta > 0$ and $\phi \in C^1((x_1^0-\eta, x_1^0+\eta); \mathbb{R})$ such that when $|x_1-x_1^0|+|x_2-x_2^0| < \eta$,

$$H(x_1, x_2) = H(x_1^0, x_2^0) \qquad \Longleftrightarrow \qquad x_2 = \phi(x_1). \tag{2.4}$$

We can also assume that

$$\forall x \in (x_1^0 - \eta, x_1^0 + \eta), \qquad \Phi(x) \equiv -\frac{\partial H}{\partial x_2}(x, \phi(x)) \neq 0. \tag{2.5}$$

Then, there exists $\varepsilon > 0$ such that for $|t| < \varepsilon$, $|X_1(t) - x_1^0| + |X_2(t) - x_2^0| < \eta$. Therefore, the function $t \to X_1(t)$ satisfies for $|t| < \varepsilon$ the equation

$$\frac{dX_1}{dt}(t) = -\frac{\partial H}{\partial x_2}(X_1(t), \phi(X_1(t))) = \Phi(X_1(t)). \tag{2.6}$$

Introducing the primitive Q of $1/\Phi$ such that $Q(x_1^0) = 0$ and its reciprocal Q^{-1} defined in a neighborhood of 0, one gets for $|t|$ small enough

$$X_1(t) = Q^{-1}(t), \qquad X_2(t) = \phi(Q^{-1}(t)). \tag{2.7}$$

Since these expressions depend only on the data, and not on the choice of the solution X, our theorem is proved. □

Remark. In the above proof, the deep property that is used is that in the direction of $a(x)$, H is smoother than expected (it is constant). For a general time-dependent field $a(t, x)$, a natural condition replacing the nonvanishing condition in the autonomous Hamiltonian case would be that $(1, a(t, x))$ is transverse to any hypersurface of $\mathbb{R}\times\mathbb{R}^N$ where a is not smooth. This condition has been introduced in [2] in one dimension. However, we are not able to prove any result with this kind of condition. Only the multidimensional autonomous "multi-Hamiltonian" case,

$$a(x) = \nabla H_1(x)\wedge\cdots\wedge\nabla H_{N-1}(x), \tag{2.8}$$

with $H_i \in C^1$, seems to work the same. Note that (2.8) automatically gives $\operatorname{div} a = 0$ and $a\cdot\nabla H_i = 0$. On the other hand, any sufficiently smooth autonomous field $a(x)$ such that $\operatorname{div} a = 0$ and which does not vanish can locally be written in the form (2.8), but this does not seem to work as far as unsmooth functions a are concerned. Notice also that the time-space analogue of (2.8) for a coefficient $a(t,x)$ satisfying $\operatorname{div}_x a = 0$ is

$$(1, a(t,x)) = \nabla_{tx}H_1(t,x)\wedge\cdots\wedge\nabla_{tx}H_N(t,x), \tag{2.9}$$

which has a canonical solution $(H_1,\dots,H_N) = X$, the flow of a defined by (1.3), (1.4), evaluated at a fixed value s (up to a factor $(-1)^N$ in (2.9)).

We also have a local version of Theorem 2.1. We define

$$M_R = M\cap\overline{B}(0,R) \tag{2.10}$$

for any $R>0$, where M is defined in (2.2).

Proposition 2.2. *Assume that a is bounded on $\mathbb{R}^2$. If $x^0\in\mathbb{R}^2$ is such that for some $R>0$ and $T>0$, $|x^0|\le R$ and $H(x^0)\notin H(M_{R+\|a\|_\infty T})$, then there exists a unique solution $X\in C^1([-T,T])$ to* (1.6), (1.7).

Proof. We notice that any solution satisfies

$$|X(t)-x^0|\le\|a\|_\infty|t|; \tag{2.11}$$

thus $X(t)\in\overline{B}_{R+\|a\|_\infty T}$ for $|t|\le T$. Then the proof of uniqueness is the same as that of Theorem 2.1, and global existence on $[-T,T]$ follows from this a priori bound. □

Finally, we note that uniqueness of the solution also ensures continuous dependence with respect to the initial datum x^0.

Proposition 2.3. *Assume that a is bounded on $\mathbb{R}^2$ and that for any $x^0\in K$ a compact subset of $\mathbb{R}^2$,* (1.6), (1.7) *has a unique solution on a fixed compact interval $I\ni 0$. Then the solution $X(t,x^0)$, $t\in I$, $x^0\in K$, satisfies*

$$X\in C(I\times K). \tag{2.12}$$

Moreover, for any sequence $H_n\in C^\infty(\mathbb{R}^2)$ such that ∇H_n is bounded and $\nabla H_n\to\nabla H$ locally uniformly in $\mathbb{R}^2$, we have

$$X_n\longrightarrow X\quad\textit{uniformly in } I\times K. \tag{2.13}$$

Proof. Let x^n be a sequence in K, $x^n \to x \in K$. Then, $X(\cdot, x^n)$ is compact in $C(I)$ because of (1.6) and of the boundedness of a. Therefore, up to a subsequence, $X(t, x^n) \to \varphi(t)$ uniformly for $t \in I$, for some continuous function φ. Then $X'(t, x^n) = a(X(t, x^n)) \to a(\varphi(t))$ uniformly in I, and thus $\varphi \in C^1(I)$ and satisfies $\varphi'(t) = a(\varphi(t))$, $\varphi(0) = x$. Since, by assumption, this problem has a unique solution, we have $\varphi(t) = X(t, x)$, and it is not necessary to extract any subsequence. We conclude that $X(t, x^n) \to X(t, x)$ uniformly for $t \in I$; thus $X \in C(I \times K)$. The property (2.13) can be proved with similar arguments. □

3. Uniqueness for almost every initial datum

Theorem 2.1 shows that for certain initial data x^0, there is uniqueness for system (1.6), (1.7). We prove here that under a reasonable assumption on H, the set of initial data x^0 such that uniqueness does not hold is small.

We denote by $|\,|$ the Lebesgue measure in $\mathbb{R}$ or $\mathbb{R}^2$.

Theorem 3.1. *Let $H \in C^1(\mathbb{R}^2)$. We denote by $M = \{x \in \mathbb{R}^2 : \nabla H(x) = 0\}$ the set of its critical points. We suppose that $|\partial M| = 0$ and $|H(\partial M)| = 0$. Then there is uniqueness of solutions to system* (1.6), (1.7) *for almost every initial datum $x^0 \in \mathbb{R}^2$.*

Proof. First observe that H is constant on each connected component of $\overset{\circ}{M}$, the interior of M. Since there are at most a countable number of such components, we see that $H(\overset{\circ}{M})$ is an enumerable set. Therefore, the assumption that $|H(\partial M)| = 0$ also implies that $|H(M)| = 0$.

We now observe that since $|H(M)| = 0$, for almost every $x \in \mathbb{R}^2$ such that $H(x) \in H(M)$, we have $\nabla H(x) = 0$; see [1] for example. In other words, $H^{-1}(H(M)) \subset M \cup P$, where $|P| = 0$.

Suppose now that $x^0 \notin H^{-1}(H(M))$. This exactly means that $H(x^0)$ is not a critical value of H. Thanks to Theorem 2.1, we know that uniqueness holds in such a situation.

Suppose on the other hand that $x^0 \in \overset{\circ}{M}$. Then the unique solution to (1.6), (1.7) is the constant function $X(t) = x^0$.

Finally, only those x^0 which lie in $\partial M \cup P$ may give rise to nonunique solutions. But $|\partial M \cup P| = 0$, and the theorem is proved. □

Remark. Note that the fact that $|H(M)| = 0$ is not a consequence of Sard's theorem, because this result needs more regularity on H than is supposed here, namely $H \in C^2$. A counterexample in which $H \in C^1(\mathbb{R}^2)$ and $[0, 1] \subset H(M)$ can be found in [5].

We now show that any "reasonable" C^1 function from $\mathbb{R}^2$ to $\mathbb{R}$ satisfies the assumption of Theorem 3.1.

Theorem 3.2. *Let $H \in C^1(\mathbb{R}^2)$ be a function whose set of critical points M is a countable union of points, C^1 curves and open sets. Then, there is uniqueness of a solution to system* (1.6), (1.7) *for almost every initial datum $x^0 \in \mathbb{R}^2$.*

Proof. Since M is closed, $\partial M = M \setminus \mathring{M}$ is included in the countable union of points and C^1 curves. Therefore, $|\partial M| = 0$. But on a given C^1 curve, one has $\nabla H = 0$; thus H takes only one value there. Therefore, $H(\partial M)$ is countable, and $|H(\partial M)| = 0$. □

Examples. Very simple counterexamples show that even when H has a single critical point, nonuniqueness can occur for an uncountable (but negligible) set of initial data $x^0 \in \mathbb{R}^2$. Let us for example consider the Hamiltonian

$$H(x_1, x_2) = (x_1^2 + x_2^2)^{\alpha/2} x_2, \qquad 0 < \alpha < 1. \tag{3.1}$$

Then $H \in C^1(\mathbb{R}^2)$, $\nabla H(0,0) = 0$, and for $(x_1, x_2) \neq (0,0)$,

$$\begin{aligned} \frac{\partial H}{\partial x_1} &= \alpha (x_1^2 + x_2^2)^{\alpha/2-1} x_1 x_2, \\ \frac{\partial H}{\partial x_2} &= \alpha (x_1^2 + x_2^2)^{\alpha/2-1} x_2^2 + (x_1^2 + x_2^2)^{\alpha/2} > 0. \end{aligned} \tag{3.2}$$

Therefore, $M = \{(0,0)\}$, $H^{-1}(H(M)) = \mathbb{R} \times \{0\}$, $\frac{\partial H}{\partial x_2}(x_1, 0) = |x_1|^\alpha$, and we easily see that nonuniqueness occurs for all initial data in $\mathbb{R} \times \{0\}$.

Another interesting example is given by

$$H(x_1, x_2) = \frac{x_1 x_2}{(x_1^2 + x_2^2)^{\alpha/2}}, \qquad 0 < \alpha < 1, \qquad (x_1, x_2) \neq (0,0), \tag{3.3}$$

and $H(0,0) = 0$. In this case, no flow satisfying the group property, and continuous with respect to $x^0 \in \mathbb{R}^2$, exists.

On the other hand, because of Theorem 3.2, any counterexample, if it exists, to the almost everywhere uniqueness must involve functions whose definition is complex.

4. Uniqueness for the transport equation

The aim of this section is to prove the following theorem. The coefficient a is still defined by (1.5), M is the set of critical points (2.2), and M_R is defined in (2.10).

Theorem 4.1. *Assume that* $|\partial M| = 0$, $|H(\partial M)| = 0$ *and that* a *is bounded. Then for any* $u^0 \in L^1_{loc}(\mathbb{R}^2)$ *there exists a unique* $u \in C(\mathbb{R}, L^1_{loc}(\mathbb{R}^2))$ *solving*

$$\partial_t u + \mathrm{div}_x(au) = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^2, \tag{4.1}$$

$$u(0, \cdot) = u^0. \tag{4.2}$$

Proof of existence. Thanks to Theorem 3.1 and to the boundedness of a, we can define unambiguously $X(., x)$ for almost every $x \in \mathbb{R}^2$. Then $X \in C(\mathbb{R}, L^1_{loc}(\mathbb{R}^2))$. Indeed, $\mathbb{R} \times [H^{-1}(H(M))]^c$ is a subset of the open set $\Omega \equiv \cup_{R,T}(-T, T) \times [B_R \backslash H^{-1}(H(M_{R+\|a\|T}))]$, and from Propositions 2.2 and 2.3, X is continuous in Ω. The flow X is also continuous on $\mathbb{R} \times \overset{\circ}{M}$ because there, $X(t, x) = x$. Thus X is continuous on the subset $\mathbb{R} \times (\overset{\circ}{M} \cup H^{-1}(H(M))^c)$ of $\Omega \cup (\mathbb{R} \times \overset{\circ}{M})$. But $\overset{\circ}{M} \cup H^{-1}(H(M))^c$ has full measure in $\mathbb{R}^2$, and X is locally bounded. Therefore, thanks to Lebesgue's theorem of dominated convergence, we get the continuity of X with value in L^1_{loc}.

From the approximation property (2.13), we also obtain that $X(t, \cdot)$ is measure preserving, that X satisfies the group property and that $u(t, x) \equiv u^0(X(-t, x))$ solves (4.1), (4.2). □

In order to obtain uniqueness, let us state a renormalization lemma.

Lemma 4.2. *Assume that* $|\partial M| = 0$, $|H(\partial M)| = 0$ *and that* a *is bounded. Let* $u \in C(\mathbb{R}, L^1_{loc}(\mathbb{R}^2))$ *solve* (4.1). *Then for any Lipschitz-continuous function* $S : \mathbb{R} \to \mathbb{R}$, $S(u) \in C(\mathbb{R}, L^1_{loc}(\mathbb{R}^2))$ *also solves* (4.1).

Let us postpone the proof of this lemma, and deduce Theorem 4.1.

Proof of uniqueness for Theorem 4.1. As in [6], it is a consequence of Lemma 4.2, because if $u \in C(\mathbb{R}, L^1_{loc}(\mathbb{R}^2))$ solves (4.1) and $u(0) = 0$, we have that $|u| \in C(\mathbb{R}, L^1_{loc}(\mathbb{R}^2))$ solves (4.1), and by integration over well-chosen cones (note that a is assumed to be bounded), this yields $|u| \equiv 0$. □

In order to prove Lemma 4.2 we need an intermediate result.

Lemma 4.3. *Assume that* $|\partial M| = 0$ *and* $|H(\partial M)| = 0$. *Let* $u \in L^1_{loc}((-T, T) \times B_R)$ *with* $T > 0$ *and* $R > 0$, *and assume that for some* $R' > 0$,

$$\partial_t u + \mathrm{div}_x(au) = 0 \tag{4.3}$$

in $(-T, T) \times \Omega_R$, *with* Ω_R *the open set*

$$\Omega_R = B_R \cap \left\{ \overset{\circ}{M} \cup \left[H^{-1}(H(M_{R'}))\right]^c \right\}. \tag{4.4}$$

Then (4.3) *holds true in* $(-T, T) \times B_R$.

Proof. Let us define $K = H(M_{R'})$, which is a compact subset of $\mathbb{R}$ such that $|K| = 0$. We can write $K = \cap V_n$, with V_n open and bounded, $\overline{V}_{n+1} \subset V_n$. Then there exist functions $\chi_n \in C_c^\infty(V_n)$, $0 \leq \chi_n \leq 1$, $\chi_n = 1$ on $\overline{V}_{n+1}$, and then $\chi_n \downarrow \mathbb{1}_K$. Define $\psi_n(x) = \chi_n(H(x)) \in C^1(\mathbb{R}^2)$. For any $\varphi \in C_c^\infty(B_R)$, $(1-\psi_n)\varphi \in C_c^1(B_R)$ and $\mathrm{supp}(1-\psi_n)\varphi \subset H^{-1}(V_{n+1}^c)$; thus $\mathrm{supp}(1-\psi_n)\varphi \subset \Omega_R$. Therefore, for any $\rho \in C_c^\infty((-T,T))$, $\langle \rho(t)(1-\psi_n)\varphi, \partial_t u + \mathrm{div}_x(au)\rangle = 0$. This means that

$$(1-\psi_n)\left(\partial_t u + \mathrm{div}_x(au)\right) = 0 \quad \text{in } (-T,T)\times B_R. \tag{4.5}$$

But we have

$$\psi_n(\partial_t u + \mathrm{div}_x(au)) = \partial_t(\psi_n u) + \mathrm{div}_x(a\psi_n u) - u\, a\cdot\nabla\psi_n, \tag{4.6}$$

and we notice that since $\nabla\psi_n = \chi_n'(H)\nabla H$, we have $a\cdot\nabla\psi_n = 0$. Therefore, by using Lebesgue's theorem, we get from (4.5) that

$$\partial_t u + \mathrm{div}_x(au) = \partial_t[\mathbb{1}_K(H)u] + \mathrm{div}_x[a\mathbb{1}_K(H)u] \quad \text{in } (-T,T)\times B_R. \tag{4.7}$$

Then we notice that $a = (\nabla H)^\perp = 0$ almost everywhere in $H^{-1}(K)$ since $|K| = 0$ (as in the proof of Theorem 3.1); thus $a\mathbb{1}_K(H) = 0$ almost everywhere. Also, since again $\nabla H = 0$ almost everywhere in $H^{-1}(K)$, and $|\partial M| = 0$, we have almost everywhere $\mathbb{1}_K(H) = \mathbb{1}_K(H)\mathbb{1}_M = \mathbb{1}_K(H)\mathbb{1}_{\overset{\circ}{M}}$, and we obtain

$$\partial_t u + \mathrm{div}_x(au) = \partial_t[\mathbb{1}_K(H)\mathbb{1}_{\overset{\circ}{M}} u] \quad \text{in } (-T,T)\times B_R. \tag{4.8}$$

But since (4.3) holds in $(-T,T)\times\Omega_R$ and $B_R\cap\overset{\circ}{M} \subset \Omega_R$, we know that $\partial_t u = 0$ in $(-T,T)\times(B_R\cap\overset{\circ}{M})$, and we conclude that the right-hand side of (4.8) vanishes. ☐

Proof of Lemma 4.2. Let $T > 0$ and $R > 0$. We are going to prove that

$$\partial_t[S(u)] + \mathrm{div}_x[aS(u)] = 0 \tag{4.9}$$

in $(-T,T)\times B_R$ by using Lemma 4.3 applied to $S(u)$, with $R' = R + \|a\|_\infty 2T$. Obviously, (4.9) holds for $x \in \overset{\circ}{M}$ since there $a = 0$. Therefore, it only remains to consider $x \in B_R\backslash H^{-1}(H(M_{R'}))$. Indeed, it is enough to prove that for given $(t^0, x^0) \in (-T,T)\times B_R\backslash H^{-1}(H(M_{R'}))$, there exists a small neighborhood of (t^0,x^0) where (4.9) holds. Since $x_0 \notin M$ (because $R' > R$), there exists a small neighborhood V of x_0 and $K \in C^1(V)$ such that $x \mapsto (H(x), K(x))$ is a C^1 diffeomorphism from V to an open set W. We know that

$$\partial_t u + \frac{\partial}{\partial x_1}\big(-\frac{\partial H}{\partial x_2}u\big) + \frac{\partial}{\partial x_2}\big(\frac{\partial H}{\partial x_1}u\big) = 0 \tag{4.10}$$

in $(-T,T)\times V$; thus for any $\varphi\in C^1_c((-T,T)\times V)$,

$$\iint_{(-T,T)\times V} u\Big(\partial_t\varphi - \frac{\partial H}{\partial x_2}\frac{\partial\varphi}{\partial x_1} + \frac{\partial H}{\partial x_1}\frac{\partial\varphi}{\partial x_2}\Big)\,dt\,dx = 0. \tag{4.11}$$

Let us now perform the change of variables $(h,k)=(H(x),K(x))$, and let us denote $v(t,h,k)=u(t,x)$, $\psi(t,h,k)=\varphi(t,x)$. We have

$$\frac{\partial\varphi}{\partial x_1} = \frac{\partial\psi}{\partial h}\frac{\partial H}{\partial x_1} + \frac{\partial\psi}{\partial k}\frac{\partial K}{\partial x_1}, \qquad \frac{\partial\varphi}{\partial x_2} = \frac{\partial\psi}{\partial h}\frac{\partial H}{\partial x_2} + \frac{\partial\psi}{\partial k}\frac{\partial K}{\partial x_2}; \tag{4.12}$$

thus

$$-\frac{\partial H}{\partial x_2}\frac{\partial\varphi}{\partial x_1} + \frac{\partial H}{\partial x_1}\frac{\partial\varphi}{\partial x_2} = J\frac{\partial\psi}{\partial k}, \qquad J = \frac{\partial H}{\partial x_1}\frac{\partial K}{\partial x_2} - \frac{\partial H}{\partial x_2}\frac{\partial K}{\partial x_1}. \tag{4.13}$$

We can assume that J has a constant sign $\sigma=\pm1$ in V; thus (4.11) yields

$$\iint_{(-T,T)\times W} v\Big(\frac{1}{|J|}\partial_t\psi + \sigma\frac{\partial\psi}{\partial k}\Big)\,dt\,dh\,dk = 0. \tag{4.14}$$

This holds for any $\psi\in C^1_c((-T,T)\times W)$; thus

$$\partial_t\left(\frac{v}{J}\right) + \frac{\partial v}{\partial k} = 0 \qquad \text{in } (-T,T)\times W. \tag{4.15}$$

Now define $w=v/J$, which solves

$$\partial_t w + \frac{\partial}{\partial k}(Jw) = 0 \qquad \text{in } (-T,T)\times W. \tag{4.16}$$

Since there is no differentiation with respect to h in this equation, it is equivalent to saying that for almost every $h\in W_h$, $\partial_t w + \partial_k(Jw) = 0$ in $\mathcal{D}'((-T,T)\times W_k)$, if we take W to be a product $W = W_h\times W_k$, which is always possible, up to a reduction of the size of W. For almost every $h\in W_h$, we now get a one-dimensional equation with autonomous continuous coefficient J that does not vanish. Such a coefficient induces a C^1 flow. Therefore, uniqueness holds for our one-dimensional equation (we refer to [2] for a more general uniqueness result with continuous coefficient in one dimension), except that since we have a local problem, we need to take a smaller W and a short time interval $(t_0-\varepsilon, t_0+\varepsilon)$. Then, uniqueness also holds for equations (4.16), (4.15) and (4.10). But we know thanks to the existence proof of Theorem 4.1 that $u(t_0, X(t_0-t,x))$ is a solution to the transport equation. Therefore, u must be given by

$$u(t,x) = u(t_0, X(t_0-t,x)) \qquad \text{in } \ (t_0-\varepsilon, t_0+\varepsilon)\times V. \tag{4.17}$$

Finally, we have $S(u(t,x)) = g(X(t_0 - t, x))$ with $g = S(u(t_0, .)) \in L^1_{loc}(\mathbb{R}^2)$; thus, again by the existence proof of Theorem 4.1, $S(u)$ also solves (4.10) in $(t_0 - \varepsilon, t_0 + \varepsilon) \times V$. $\square$

Note that this proof is quite intricate because in it, uniqueness is a consequence of a renormalization property, which is in turn a consequence of a uniqueness theorem, but in a *local* setting.

Remark. In the above proof, we use that the flow X is C^1 on curves where $H = cst$ where $a \neq 0$. However, X is not C^1 in all variables where $a \neq 0$, as the following example shows: take $H(x_1, x_2) = A(x_2)$, $A \in C^1$. Then $X = (x_1^0 - tA'(x_2^0), x_2^0)$.

References

[1] L. Boccardo and F. Murat, *Remarques sur l'homogénéisation de certains problèmes quasi-linéaires,* Portugal. Math., 41 (1984), 535–562.

[2] F. Bouchut and F. James, *One-dimensional transport equations with discontinuous coefficients,* Nonlinear Anal., 32 (1998), 891–933.

[3] I. Capuzzo Dolcetta and B. Perthame, *On some analogy between different approaches to first order PDE's with nonsmooth coefficients,* Adv. Math. Sci. Appl., 6 (1996), 689–703.

[4] B. Desjardins, *A few remarks on ordinary differential equations,* Comm. Partial Differential Equations, 21 (1996), 1667–1703.

[5] J. Dieudonné, "Eléments d'analyse," Tome III: Chapitres XVI et XVII (French), Cahiers Scientifiques, Fasc. XXXIII, Gauthier-Villars Ed., Paris, 1970.

[6] R.J. DiPerna and P.-L. Lions, *Ordinary differential equations, transport theory and Sobolev spaces,* Invent. Math., 98 (1989), 511–547.

[7] G. Petrova and B. Popov, *Linear transport equations with p-monotone coefficients,* preprint (1998).

[8] F. Poupaud and M. Rascle, *Measure solutions to the linear multi-dimensional transport equation with non-smooth coefficients,* Comm. Partial Differential Equations, 22 (1997), 337–358.