Differential and Integral Equations Volume 14, Number 8, August 2001, Pages 989–1004

GEVREY CLASS REGULARITY FOR PARABOLIC EQUATIONS

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Abstract. We consider the regularity of parabolic equations. We obtain that the solution belongs to Gevrey class 2 up to the boundary if functions in the equation belong to Gevrey class 2 in all dependent variables.

1. Introduction

Consider a second-order parabolic equation for $v(x,t)$ in a domain Q in R^{n+1}, $x \in R^n$, $t \in R^1$:

$$v_t - F(D^2 v, Dv, v, x, t) = 0.$$

Here $Dv = (v_{x_1}, \dots, v_{x_n}) = (v_1, \dots, v_n)$, $D^2 v = (v_{11}, v_{12}, \dots, v_{nn})$, and $F_{v_{ij}}$ is assumed to be positive definite. It is well known that if F is analytic in all its dependent variables, then a solution v is necessarily analytic in the space variables x (Friedman [1]). It is also known that if F is analytic in its space variables and is of Gevrey class 2 in time, then a solution v is analytic in its space variables and is of Gevrey class 2 in time up to the boundary (Kinderlehrer and Nirenberg [2]). In our study of the exact boundary controllability problem of linear heat equations, we need a Gevrey-class-2 regularity of the solution at the boundary under the assumption that all coefficients of heat equations belong to Gevrey class 2 in both space variables x and time variable t. So in this work, we assume that F is of Gevrey class 2 in the space variables x and the time t, then we obtain that a solution v must belong to Gevrey class 2 in x and t up to the boundary.

This result is purely local, valid in a neighborhood of a noncharacteristic boundary point. Assuming the boundary to be analytic there we may make a simple analytic transformation to straighten the boundary to a hyperplane

Accepted for publication July 2000.
AMS Subject Classifications: 35G20, 35K55, 35K60, 93B05, 93C10.

$x_n = 0$. For simplicity we treat only the Dirichlet problem: v is given on $x_n = 0$; further, by subtraction of a suitable function which agrees with v there we may suppose $v = 0$ on $x_n = 0$. Theorem 2.1 asserts that the solution is then of Gevrey class 2 in the space variables and in time.

The method we use here is base on the work of Kinderlehrer and Nirenberg [2] with some modifications. The method for showing Gevrey class regularity proceeds roughly as follows. Firstly we get L^2 norm estimations of solutions of some linear parabolic equations and apply them to estimate all derivatives of solutions v in L^2 norms. With the aid of a Sobolev inequality, we can obtain estimations in L^∞ norms of all derivatives of the solution. The most difficult step is to obtain the L^2 norm estimations of all derivatives of the solution. We remark that even though the conclusion of Gevrey class regularity is weaker than the analyticity regularity in [2], all lemmas and theorems in [2] cannot imply directly the lemmas and theorems in this paper because the lemmas and theorems here hold under weaker assumptions. For convenience, we use the same notation as [2].

The paper is organized as follows. Section 2 contains the statement of the main theorem and L^2 norm estimation of the solution of a linear parabolic equation. The proof of the main theorem is then given in Sections 3–5 . The most difficult L^2 norm estimations are given in Section 4 and Section 5.

2. A local smoothness theorem for nonlinear parabolic equations

Let $v(x, t) \in C^\infty(\bar{Q})$ be a solution of the problem

$$-F(D^2v,\, Dv,\, v,\, x,\, t) + v_t = 0 \qquad \text{in} \qquad Q, \tag{2.1}$$

$$v = 0 \qquad \text{on} \qquad \Sigma, \tag{2.2}$$

where $G = \{x \in R^n : x_n > 0,\ |x| < 1\}$, $Q = G \times (0,1)$, $\Sigma = \{(x,t) \in \partial Q : x_n = 0\}$, and F is a C^∞ function; here $Dv = (v_{x_1}, \ldots, v_{x_n}) = (v_1, \ldots, v_n)$, $D^2v = (v_{11}, \ldots, v_{nn})$. The values assumed by v on $\partial Q - \Sigma$ are not relevant to us. We shall denote $\partial/\partial t$ by ∂_t, and space derivatives by D.

Firstly, we define the Gevrey class δ function.

Definition 2.1. Let Ω be a subset of R^n and $\delta > 0$. A C^∞ function f in Ω is said to be of Gevrey class δ in Ω (in short, $f \in \gamma^\delta(\Omega)$) if there exist positive constants C and H such that

$$|D_x^\alpha f(x)| \le C H^{\delta|\alpha|} (\delta|\alpha|)!$$

for all multi-indices α and for all $x \in \Omega$, where $\alpha! = \Gamma(\alpha + 1)$ and Γ is the usual gamma function and $|\alpha| = \alpha_1 + \cdots + \alpha_n$ for $\alpha = (\alpha_1, \ldots, \alpha_n)$.

Now, we state the first main result of this paper.

Theorem 2.1. *Let* $v \in C^\infty(\bar{Q})$ *be a solution of* (2.1)–(2.2). *Assume that*

(1) $(F_{v_{ij}})$ *is a positive definite form for* $(x,t) \in \bar{Q}$,

(2) F *is a Gevrey-class-*2 *function of* v_{ij}, v_i, v, x, t *in the range of these arguments for* $(x,t) \in \bar{Q}$. *More precisely, we assume that in the range of these arguments, for some constants* C,κ,

$$|D^\lambda \partial_t^j F| \leq C\kappa^{2|\lambda|+2j}(2|\lambda|+2j)!, \tag{2.3}$$

for all multi-indices λ *and* $j = 0, 1, 2, \ldots$. *Here* D^λ *represents differentiation of order* $|\lambda|$ *of* F *with respect to all but the* t *variables. Then for each* σ, $0 < \sigma < \frac{1}{2}$, $v(x,t)$ *is of second Gevrey class of* x *and* t *in* $\{(x,t) : x_n \geq 0, |x| < 1-\sigma, \sigma < t < 1\}$. *In fact the derivatives of* v *satisfy*

$$|D^\lambda \partial_t^j v| \leq C' H_0^{2|\lambda|+2j}(2|\lambda|+2j)! \tag{2.4}$$

for some constants C', H_0 *and for all* λ *and* $j = 0, 1, 2, \ldots$.

Before we proceed to the proof, we state some well-known results which were stated in [2] and will be used in the proof.

Consider the linearized problem. Suppose that v is a fixed solution of (2.1)–(2.2) and F and v satisfy the hypotheses of the theorem. Let $a_{ij} = F_{v_{ij}}(D^2v, Dv, v, x, t)$, $b_i = -F_{v_i}(D^2v, Dv, v, x, t)$, $b_0 = -F_v(D^2v,\, Dv, v, x, t)$, and define $Lw = -a_{ik}w_{ik} + b_i w_i + b_0 w$, where the summation convention is understood.

There are some well-known estimates for linear parabolic equations; see [3]: If $w \in C^2(\bar{Q})$ is a solution of

$$\begin{aligned} Lw + w_t &= f \qquad \text{in} \qquad Q, \\ w &= 0 \qquad \text{on} \qquad \partial_p Q, \end{aligned}$$

where $\partial_p Q$ denotes the parabolic boundary of Q, then

$$\|D^2 w\|_{L^2(Q)} + \|Dw\|_{L^2(Q)} + \|w_t\|_{L^2(Q)} \leq c_0 \|f\|_{L^2(Q)}, \tag{2.5}$$

where c_0 depends on L, Q, etc., but not w. Using this well-known result, in [2] they derive a corresponding (also known) local estimate, (2.9) below.

Let $\zeta_0(\xi) \in C^\infty(R^1)$ be a function satisfying $0 \leq \zeta_0 \leq 1$, and

$$\begin{aligned} \zeta_0 &= 1 \qquad \text{for} \qquad \xi \leq 0, \\ \zeta_0 &= 0 \qquad \text{for} \qquad \xi \geq 1. \end{aligned}$$

Given $\sigma > 0$ and $0 < h < \sigma$, we define

$$\zeta(x,t) = \zeta_{\sigma,h}(x,t) = \zeta_0\big(\frac{\sigma - t}{h}\big)\zeta_0\big(\frac{|x| - (1-\sigma)}{h}\big). \tag{2.6}$$

Such ζ has the properties $\zeta = 1$ for $|x| \le 1-\sigma$ and $\sigma \le t \le 1$, supp$\zeta \subset \{(x,t) : |x| \le 1-(\sigma-h),\ \ \sigma-h \le t\}$, and

$$\sum_{|\alpha|=s} |D^\alpha \zeta| + |\partial_t^s \zeta| \le C_N h^{-s} \quad \text{for all } s \le N,$$

where C_N depends on N. Furthermore,

$$\zeta_{\sigma,h'} \le \zeta_{\sigma,h} \quad \text{for} \quad h' < h. \tag{2.7}$$

Suppose now that w is a solution of

$$\begin{aligned} Lw + w_t &= f \qquad \text{in} \qquad Q, \\ w &= 0 \qquad \text{on} \qquad \Sigma. \end{aligned} \tag{2.8}$$

Then $\zeta^2 w$ is a solution of

$$\begin{aligned} L(\zeta^2 w) + (\zeta^2 w)_t &= g \qquad \text{in} \qquad Q, \\ w &= 0 \qquad \text{on} \qquad \partial_p Q, \end{aligned}$$

where $g = \zeta^2 f - 2\zeta a_{ik}(\zeta_i w_k + \zeta_k w_i) + (L\zeta^2 - b_0\zeta^2 + (\zeta^2)_t)w$. By (2.5) it is easy to obtain (here $\|\cdot\|$ denotes $L^2(Q)$ norm)

$$\begin{aligned} &\|D^2(\zeta^2 w)\| + \|D(\zeta^2 w)\| + \|\partial_t(\zeta^2 w)\| \\ &\qquad \le \text{const.}[\|\zeta^2 f\| + h^{-1}\|\zeta D w\| + h^{-2}\|w\|_{L^2(\text{supp}\zeta)}]. \end{aligned}$$

It is helpful to eliminate the first-order terms in w on the right. Multiplying the equation for w in (2.8) by $\zeta^2 w$ and integrating by parts we find after a standard procedure that

$$\|\zeta D w\|^2_{L^2(Q)} \le \text{const.}\{\|w\|_{L^2(\text{supp}\zeta)}\|\zeta^2 f\|_{L^2(Q)} + h^{-2}\|w\|^2_{L^2(\text{supp}\zeta)}\}.$$

Inserting this into the preceding inequality we obtain

$$\begin{aligned} &\|\zeta^2 D^2 w\|_{L^2(Q)} + \|\zeta^2 D w\|_{L^2(Q)} + \|\zeta^2 w_t\|_{L^2(Q)} \\ &\qquad \le c_0\{\|\zeta^2 f\|_{L^2(Q)} + h^{-2}\|w\|_{L^2(\text{supp}\zeta)}\}, \end{aligned}$$

for a constant c_0 independent of h.

The proof of Theorem 2.1 will be given in Sections 3–5.

3. Beginning of the proof of Theorem 2.1

We first study the equations satisfied by $v^{(j)} = \partial_t^j v$ and $D^\lambda v^{(j)}$. For $\lambda = (\lambda_1, \ldots, \lambda_{n-1}, 0)$, i.e., D^λ a spatial derivative tangential to Σ, we introduce the notation $T^\lambda = \partial_1^{\lambda_1} \cdots \partial_{n-1}^{\lambda_{n-1}}$, $\lambda = (\lambda_1, \ldots, \lambda_{n-1})$. Differentiating equation (2.1) with respect to t yields

$$Lv^{(1)} + v_t^{(1)} = F_t \quad \text{in} \quad Q.$$

Applying ∂_t^{j-1} to this equation gives

$$Lv^{(j)} + v_t^{(j)} = f_j,$$
$$f_j = \sum_{s=1}^{j-1} \binom{j-1}{s} \Big\{ \partial_t^s a_{ik} v_{ik}^{(j-s)} - \partial_t^s b_i v_i^{(j-s)} - \partial_t^s b_0 v^{(j-s)} \Big\} + \partial_t^{j-1} F_t.$$

Applying, next, $D^\eta = \partial_n^k T^\lambda$ we obtain

$$LD^\eta v^{(j)} + D^\eta v_t^{(j)} = f_{j,\eta}, \tag{3.1}$$
$$f_{j,\eta} = \sum_{\substack{\eta'+\eta''=\eta \\ |\eta'|>0}} \binom{\eta}{\eta'} \Big\{ D^{\eta'} a_{ik} D^{\eta''} v_{ik}^{(j)} - D^{\eta'} b_i D^{\eta''} v_i^{(j)} - D^{\eta'} b_0 D^{\eta''} v^{(j)} \Big\} + D^\eta f_j.$$

In cases $k = 0$ the function $D^\eta v = T^\lambda v$ is also subject to the boundary condition

$$T^\lambda v^{(j)} = 0 \quad \text{on} \quad \Sigma.$$

When $j = 0$, the equation for $D^\lambda v$ has a slightly different form.

We shall prove the theorem by successive estimation of the L^2 norms of the derivatives of v. Let $Q_\sigma = \{(x,t) : x_n > 0,\ |x| < 1 - \sigma,\ \sigma < t < 1\}$, $\sigma < \frac{1}{2}$, $h_k = h_k(\sigma) = \begin{cases} \frac{\sigma}{k} & \text{for } k \geq 1, \\ 0 & \text{for } k = 0, \end{cases}$ $\|\phi\|_\sigma = \|\phi\|_{L^2(Q_\sigma)}$, $\|\phi\| = \|\phi\|_{L^2(Q_0)}$. Then we have $Q_{\sigma-h_k} \subset Q_{\sigma-h_l}$ when $k \geq l$.

Our aim is to prove the following inequalities: for each σ, $0 < \sigma < \frac{1}{2}$, and ζ given by (2.6), $\zeta = \zeta_{\sigma, h_{2j+2|\lambda|+2k}(\sigma)}$, there exist constants $C_0 > 0$, $M > 1$, and $N > 1$ such that

$$\sum_{|\alpha| \leq 2} \|\zeta^2 D^\alpha T^\lambda \partial_n^k v^{(j)}\| + \|\zeta^2 T^\lambda \partial_n^k v_t^{(j)}\|$$
$$\leq C_0 N^{2k} (2j + 2|\lambda| + 2k - \mu)! \Big(\frac{M}{\sigma}\Big)^{(2j+2|\lambda|+2k-\mu)^+}, \tag{3.2}$$

for all j, k, and $\lambda = (\lambda_1, \ldots, \lambda_{n-1}, 0)$, where $\mu = 2\nu + 5$, $\nu = [\frac{1}{2}n] + 1$, and $l^+ = \max(l, 0)$. It is understood that $k! = 1$ if $k \leq 0$.

We introduce

Condition $(\mathbf{I_q})$ *Inequality* (3.2) *holds whenever* $2j + 2k + 2|\lambda| \leq q - 1$.

Remark. The letters C, C_1, C_ν will denote various constants independent of M, N, σ, C_0. We will use the notation: for $l, m = 0, \dots, n$,

$$v_{ml} = v_{lm} = \begin{cases} \frac{\partial}{\partial x_l}\frac{\partial}{\partial x_m} v & \text{if } l,\ m > 0, \\ \frac{\partial}{\partial x_m}\ v & \text{if } l = 0,\ m > 0, \\ v & \text{if } l = m = 0. \end{cases}$$

Theorem 2.1 follows easily from (3.2) once we have expressed these estimates in L^∞ norms rather than L^2 norms. This is done with the aid of a well-known Sobolev inequality, which we use in the following form; here we set $\|w\|_{\infty,\sigma} = \|w\|_{L^\infty(Q_\sigma)}$.

Lemma 3.1. *Suppose that $w \in C^\infty(\bar{Q})$ and there exist constants $M, N \geq 1$, $H_0 > 0$ and some positive integers $p \leq \tilde{p}$ such that for $\zeta = \zeta_{\sigma, h_{p+2a+2b+2|\beta|}(\sigma)}$,*

$$\|\zeta^2 \partial_t^a \partial_n^b T^\beta w\| \leq H_0 N^{2b} (\tilde{p} + 2a + 2b + 2|\beta| - \mu)! \big(\frac{M}{\sigma}\big)^{(p+2a+2b+2|\beta|-\mu)^+},$$

whenever $a + b + |\beta| \leq \nu$ and for every positive $\sigma < \frac{1}{2}$. Then

$$\|w\|_{\infty, \sigma - h_p(\sigma)} \leq C H_0 N^{2\nu} (\tilde{p} + 2\nu - \mu)! \big(\frac{M}{\sigma} v\big)^{(p+2\nu-\mu)^+}, \quad 0 < \sigma < \frac{1}{2}.$$

Proof. According to Sobolev's inequality, for $\tilde{\zeta} = \zeta_{\sigma - h_p(\sigma), h_{p+2\nu}(\sigma - h_p(\sigma))}$,

$$\begin{aligned} \|w\|_{\infty, \sigma - h_p} &\leq \|\tilde{\zeta}^{2+\nu} w\|_\infty \\ &\leq \text{const.}\Big\{\|\partial_t^\nu (\tilde{\zeta}^{2+\nu} w)\| + \|\partial_n^\nu (\tilde{\zeta}^{2+\nu} w)\| + \sum_{|\beta| = \nu} \|T^\beta (\tilde{\zeta}^{2+\nu} w)\|\Big\}, \end{aligned} \tag{3.3}$$

where $\|\cdot\|$ denotes the L^2 norm. Firstly, we consider the time derivatives. We have

$$\partial_t^\nu (\tilde{\zeta}^{2+\nu} w) = \sum_0^\nu \binom{\nu}{\alpha} \partial_t^\alpha w \partial_t^{\nu-\alpha} (\tilde{\zeta}^{2+\nu}).$$

Now

$$\begin{aligned} |\partial_t^{\nu-\alpha} (\tilde{\zeta}^{2+\nu})| &\leq C \tilde{\zeta}^2 |h_{p+2\nu}(\sigma - h_p(\sigma))|^{\alpha - \nu} = C \tilde{\zeta}^2 \Big| \frac{p + 2\nu}{\sigma(1 - 1/p)} \Big|^{\nu - \alpha} \\ &\leq C_\nu \zeta_\alpha^2 \big(\frac{p}{\sigma}\big)^{\nu - \alpha}, \end{aligned}$$

since $\tilde{\zeta} \leq \zeta_\alpha \equiv \zeta_{\sigma - h_p(\sigma), h_{p+2\alpha}(\sigma - h_p(\sigma))}$ by (2.7). Thus

$$\|\partial_t^\nu (\tilde{\zeta}^{2+\nu} w)\| \leq C_\nu \sum_0^\nu \binom{\nu}{\alpha} \big(\frac{p}{\sigma}\big)^{\nu - \alpha} \|\zeta_\alpha^2 \partial_t^\alpha w\|$$

$$\leq C_\nu H_0 \sum_0^\nu \binom{\nu}{\alpha} \big(\frac{p}{\sigma}\big)^{\nu-\alpha} (\tilde{p}+2\alpha-\mu)! \big(\frac{M}{\sigma-h_p}\big)^{(p+2\alpha-\mu)^+}$$

$$\leq C_\nu H_0 \big(\frac{M}{\sigma}\big)^{(p+2\nu-\mu)^+} \big(\frac{p}{p-1}\big)^{(p+2\nu-\mu)^+} \sum_0^\nu \binom{\nu}{\alpha} (\tilde{p}+2\alpha-\mu)! p^{\nu-\alpha}.$$

Clearly $|\frac{p}{p-1}|^{(p+2\nu-\alpha)^+} \leq C$ for all p, and for $\alpha \leq \nu$, $p \leq \tilde{p}$, we have $(\tilde{p}+2\alpha-\mu)! p^{\nu-\alpha} \leq C(\tilde{p}+2\nu-\mu)!$, since $\tilde{p} \leq C(\tilde{p}-\mu)$ for $\tilde{p} > \mu$. Hence we find

$$\|\partial_t^\nu (\tilde{\zeta}^{2+\nu} w)\| \leq C H_0 \big(\frac{M}{\sigma}\big)^{(p+2\nu-\mu)^+} (\tilde{p}+2\nu-\mu)!.$$

The other terms in (3.3) can be estimated in a similar way; factors of $N^{2\nu}$ will enter, and we obtain the desired conclusion. □

Now, we prove the main theorem.

Proof of Theorem 2.1. Assume that (3.2) holds, i.e., condition $(\mathrm{I_q})$ holds for all $q = 0, 1, \ldots$. Then we apply the Lemma 3.1 to the functions $\partial_t^s \partial_n^r T^\gamma v_{lm}$, $l, m = 0, \ldots, n$, with $H_0 = C_0 N^{2r}$, $p = \tilde{p} = 2s + 2r + 2|\gamma|$; then

$$\|\partial_t^s \partial_n^r T^\gamma v_{lm}\|_{\infty, \sigma - h_{2s+2r+2|\gamma|}} \tag{3.4}$$
$$\leq C C_0 N^{2r+2\nu} (2s+2r+2|\gamma|+2\nu-\mu)! \big(\frac{M}{\sigma}\big)^{(2s+2r+2|\gamma|+2\nu-\mu)^+}$$

provided $p + 2\nu \leq q - 1$. We see from (3.4) that, for all derivatives,

$$\|\partial_t^s \partial_n^r T^\gamma v\|_{\infty,\sigma} \leq C C_0 N^{2r+2\nu} (2s+2r+2|\gamma|+2\nu-\mu)! \big(\frac{M}{\sigma}\big)^{(2s+2r+2|\gamma|+2\nu-\mu)^+},$$

from which (2.4) follows with $C' = C C_0 N^{2\nu}$, and $H_0 = \frac{NM}{\sigma}$, so Theorem 2.1 is proved. □

To prove (3.2), we need some inequalities, which can be proved in way similar way to the proof in the appendix of [2]. Let μ, N be fixed nonnegative integers with $\mu \geq N+2$. There exists a constant $C_{\mu,N}$ such that if j is a nonnegative integer, λ is a multi-index, and $s = 2j + 2|\lambda| \geq 2N+2$, then for every integer $r \geq s$,

$$\sum_{\substack{2i+2|\lambda'| \leq [s/2] \\ i \leq j \\ \lambda'+\lambda''=\lambda}} \binom{j}{i} \binom{\lambda}{\lambda'} [(r-s+2i+2|\lambda'|+N-\mu)!(2j-2i+2|\lambda''|-\mu)!$$
$$+ (2i+2|\lambda'|+N-\mu)!(r-2i-2|\lambda'|-\mu)!] \leq C_{\mu,N}(r-\mu)!, \tag{3.5}$$

and

$$\sum_{\substack{2i+2|\lambda'|\leq[s/2]\\ i\leq j-1\\ \lambda'+\lambda''=\lambda}} \binom{j-1}{i}\binom{\lambda}{\lambda'}[(r-s+2i+2|\lambda'|+N-\mu)!(2j-2i+2|\lambda''|-\mu)!$$

$$+(2i+2|\lambda'|+N-\mu)!(r-2i-2|\lambda'|-\mu)!] \leq C_{\mu,N}(r-\mu)!. \tag{3.6}$$

To establish (3.2) we shall need bounds for the derivatives of the functions $F_t,\ F_{x_i}$ and $a_{ij},\ b_i$, which are defined in Section 2. According to the hypothesis (2.3), for any one of these functions, which for convenience we denote by $g(v_{lm},\ x,\ t)$, there are constants $C',\ K$ such that

$$|g^{(\tau,j)}| \leq C'K^{2j+2|\tau|}(2j+2|\tau|-\mu)!. \tag{3.7}$$

Here $g^{(\tau,j)} = \prod_{l,m,i} \partial_{v_{lm}}^{\tau_{lm}} \partial_{x_i}^{\tau_i} \partial_t^j g,\ |\tau| = \sum \tau_{lm} + \sum \tau_i$.

We shall prove $(\mathrm{I_q})$ by induction. First we fix $C_0 \geq 1$ so that $|T^\lambda \partial_n^k v^{(j)}| \leq C_0$ in Q for $2j+2|\lambda|+2k \leq 2\mu+2$, and so that (3.2) holds with $M=N=1$ for $2j+2|\lambda|+2k \leq 2\mu+2$; i.e., $(\mathrm{I_q})$ holds for $q=2\mu+3$ for any $M,\ N \geq 1$. In our induction proof of (3.2), it is convenient to prove in addition the following:

There exists a constant $\bar{C} \geq 1$ such that for g satisfying (3.7) and for $\zeta = \zeta_{\sigma,h_{2a+2k+2|\gamma|}(\sigma)}$, we have

Condition$(\mathbf{J_q})$: *Inequality*

$$\|\zeta^2 \partial_t^a \partial_n^k T^\gamma g^{(\tau,j)}\| \leq \bar{C}K^{2j+2|\tau|}N^{2k}(2a+2k+2|\gamma|+2|\tau|+2j-\mu)! \times \Big(\frac{M}{\sigma}\Big)^{(2a+2k+2|\gamma|-\mu)^+} \tag{3.8}$$

holds provided $2a+2k+2|\gamma|+2|\tau|+2j \leq q-2$, *where* $\bar{C}$ *depends on* K *but not on* M *or* N.

Using induction on q we shall prove, for suitable $C_0, \bar{C}, M, N$:

Proposition 3.1. *Assume* $(\mathrm{I_q})$ *and* $(\mathrm{J_q})$ *hold for some* $q \geq 2\mu+3$; *then* $(\mathrm{J_{q+1}})$ *and* $(\mathrm{I_{q+1}})$ *hold.*

4. A crucial step in the proof

To prove Proposition 3.1, we need the following lemma:

Lemma 4.1. *Assume* $(\mathrm{I_q})$ *and* $(\mathrm{J_q})$ *hold for some* $q \geq 2\mu+3$; *then* $(\mathrm{J_{q+1}})$ *holds.*

Proof. We prove it by induction on q. We may choose $\bar{C}$ so large that $(\mathrm{J_q})$ holds with $M=N=1$ (and so also for $M,N \geq 1$) in case $q \leq 2\mu+4$. Furthermore, for $\bar{C} \geq C'$, where $\bar{C}$ depends on K, but not on q, it is easy to see that (3.8) holds with $M = N = 1$ for all q provided $2a+2k+2|\gamma| \leq 2\mu+2$. This follows directly from (3.7) and the fact that the derivatives of v of order at most $2\mu + 2$ are bounded. We shall always suppose $\bar{C} \geq C'$.

Thus we have to prove (3.8) for $2a + 2k + 2|\gamma| + 2|\tau| + 2j = q - 1$, $s = 2a + 2k + 2|\gamma| - 2 > 2\mu$; i.e., $2j + 2|\tau| < q - 2\mu - 3$, $q > 2\mu + 3$, and we shall use (backward) induction on $2j + 2|\tau|$; i.e., suppose (3.8) holds for $2j + 2|\tau| = r \leq q - 2\mu - 3$; we wish to prove it for $2j + 2|\tau| = r - 2$.

There are various cases to consider.

Case (i) $k \geq 0$. We write (using summation on $l, m = 0, \dots, n$)

$$\partial_t^a \partial_n^k T^\gamma g^{(\tau,j)} = \partial_t^a \partial_n^{k-1} T^\gamma [g_{v_{lm}}^{(\tau,j)} \partial_n v_{lm} + g_{x_n}^{(\tau,j)}]. \tag{4.1}$$

The first term $\partial_t^a \partial_n^{k-1} T^\gamma (g_{v_{lm}}^{(\tau,j)} \partial_n v_{lm})$ is equal to

$$\sum_{\gamma''+\gamma'=\gamma} \sum_{b=0}^{a} \sum_{i=0}^{k-1} \binom{\gamma}{\gamma'} \binom{a}{b} \binom{k-1}{i} (T^{\gamma'} \partial_t^b \partial_n^i g_{v_{lm}}^{(\tau,j)})(T^{\gamma''} \partial_t^{a-b} \partial_n^{k-i} v_{lm}).$$

Thus for $\zeta = \zeta_{\sigma, h_{2a+2k+2|\gamma|}}$, we have

$$\|\zeta^2 \partial_t^a \partial_n^k T^\gamma g^{(\tau,j)}\| \leq I_1 + I_2 + I_3, \tag{4.2}$$

where $I_1 = \|\zeta^2 g_{v_{lm}}^{(\tau,j)} \partial_t^a \partial_n^k T^\gamma v_{lm}\|$,

$$I_2 = \sum_{\substack{\gamma'+\gamma''=\gamma \\ 0<|\gamma'|+b+i \\ \leq|\gamma|+a+k-1}} \sum_{0}^{a} \sum_{0}^{k-1} \binom{\gamma}{\gamma'} \binom{a}{b} \binom{k-1}{i} \|(\zeta^2 T^{\gamma'} \partial_t^b \partial_n^i g_{v_{lm}}^{(\tau,j)})(T^{\gamma''} \partial_t^{a-b} \partial_n^{k-i} v_{lm})\|,$$

and $I_3 = \|\zeta^2 \partial_t^a \partial_n^{k-1} T^\gamma g_{x_n}^{(\tau,j)}\|$. By (backward) induction and the fact that $\zeta \leq \zeta_{\sigma, h_{2a+2k-2+2|\gamma|}}$, we see that

$$\begin{aligned} I_3 \quad &\leq \quad \bar{C} K^{2j+2|\tau|+2} N^{2k-2} (2a + 2k - 2 + 2|\gamma| + 2|\tau| + 2j + 2 - \mu)! \\ &\qquad \times \Big(\frac{M}{\sigma}\Big)^{2a+2k-2+2|\gamma|-\mu} \\ &\leq \quad \frac{K^2}{M} \bar{C} K^{2j+2|\tau|} N^{2k} (q - 1 - \mu)! \Big(\frac{M}{\sigma}\Big)^{2a+2k+2|\gamma|-\mu}. \end{aligned}$$

Next, by (3.7) and property ($\mathrm{I_q}$) we obtain

$$\begin{aligned} I_1 \quad &\leq \quad C' K^{2j+2|\tau|+2} (2j + 2|\tau| + 2 - \mu)! C_0 N^{2k} (2a + 2k + 2|\gamma| - \mu)! \\ &\qquad \times \Big(\frac{M}{\sigma}\Big)^{2a+2k+2|\gamma|-\mu} \end{aligned}$$

$$\leq \quad \frac{C'C_0K^2}{\bar{C}}\bar{C}K^{2j+2|\tau|}N^{2k}(q-1-\mu)!\big(\frac{M}{\sigma}\big)^{2a+2k+2|\gamma|-\mu}.$$

Since $a!b! \leq (a+b-1)!$ for $a, b > 0$. Consider finally I_2; we have $I_2 \leq I_2' + I_2''$, where

$$\begin{aligned} I_2' &= \sum_{0<2|\gamma'|+2b+2i\leq[s/2]} \binom{\gamma}{\gamma'}\binom{a}{b}\binom{k-1}{i} \\ &\qquad \times\|T^{\gamma'}\partial_t^b\partial_n^i g_{v_{lm}}^{(\tau,j)}\|_{\infty,\sigma-h_{2a+2k+2|\gamma|}} \times \|\zeta^2 T^{\gamma''}\partial_t^{a-b}\partial_n^{k-i}v_{lm}\|, \\ I_2'' &= \sum_{[s/2]<2|\gamma'|+2b+2i\leq s} \binom{\gamma}{\gamma'}\binom{a}{b}\binom{k-1}{i}\|\zeta^2 T^{\gamma'}\partial_t^b\partial_n^i g_{v_{lm}}^{(\tau,j)}\| \\ &\qquad \times\|T^{\gamma''}\partial_t^{a-b}\partial_n^{k-i}v_{lm}\|_{\infty,\sigma-h_{2a+2k+2|\gamma|}}, \end{aligned}$$

for $a' + b' + |\beta'| \leq \nu$ and $2|\gamma'| + 2b + 2i \leq [s/2]$; by $(\mathrm{J_q})$ we have

$$\begin{aligned} &\|\zeta^2\partial_t^{a'}\partial_n^{b'}T^{\beta'}(T^{\gamma'}\partial_t^b\partial_n^i g_{v_{lm}}^{(\tau,j)})\| = \|\zeta^2\partial_t^{a'+b}\partial_n^{b'+i}T^{\beta'+\gamma'}g_{v_{lm}}^{(\tau,j)}\| \\ &\leq \bar{C}K^{2j+2+2|\tau|}N^{2b'+2i}(2a'+2b+2b'+2i+2|\beta'|+2|\gamma'|+2|\tau| \\ &\qquad +2j+2-\mu)! \times \big(\frac{M}{\sigma}\big)^{(2a'+2b+2b'+2i+2|\beta'|+2|\gamma'|-\mu)^+}. \end{aligned}$$

It follows from Lemma 3.1 with $H_0 = \bar{C}K^{2j+2+2|\tau|}N^{2i}$, $\tilde{p} = 2b+2i+2|\gamma'| + 2|\tau| + 2j + 2$, $p = 2b + 2i + 2|\gamma'|$, for $2|\gamma'| + 2b + 2i \leq [s/2]$,

$$\begin{aligned} &\|T^{\gamma'}\partial_t^b\partial_n^i g_{v_{lm}}^{(\tau,j)}\|_{\infty,\sigma-h_{2a+2k+2|\gamma|}} \leq \|T^{\gamma'}\partial_t^b\partial_n^i g_{v_{lm}}^{(\tau,j)}\|_{\infty,\sigma-h_p} \\ &\leq C\bar{C}K^{2j+2+2|\tau|}N^{2i+2\nu}(2b+2i+2|\gamma'|+2|\tau|+2j+2+2\nu-\mu)! \\ &\qquad \times\big(\frac{M}{\sigma}\big)^{(2b+2i+2|\gamma'|+2\nu-\mu)^+}. \end{aligned}$$

Inserting this estimate into I_2' and using $(\mathrm{I_q})$

$$\begin{aligned} I_2' \leq &\sum_{0<2|\gamma'|+2b+2i\leq[s/2]} \binom{\gamma}{\gamma'}\binom{a}{b}\binom{k-1}{i}C\bar{C}K^{2j+2+2|\tau|}N^{2i+2\nu} \\ &\times(2b+2i+2|\gamma'|+2|\tau|+2j+2+2\nu-\mu)!\big(\frac{M}{\sigma}\big)^{(2b+2i+2|\gamma'|+2\nu-\mu)^+} \\ &\times C_0N^{2k-2i}(2a-2b+2k-2i+2|\gamma|-2|\gamma'|-\mu)! \\ &\times\big(\frac{M}{\sigma}\big)^{2a+2k+2|\gamma|-2b-2i-2|\gamma'|-\mu}. \end{aligned}$$

Since $(2b+2i+2|\gamma'|+2\nu-\mu)^+ + 2a + 2k + 2|\gamma| - 2b - 2i - 2|\gamma| - \mu \leq 2a + 2k + 2|\gamma| - \mu - 1$, by setting $k - 1 = l$ so that $s = 2a + 2l + 2|\gamma|$ and using (3.5) with $r = q - 3$ and with $\mu - 2$ in place of μ, we obtain

$$I_2' \le \frac{CC_0K^2N^{2\nu}}{M}\bar{C}K^{2j+2|\tau|}N^{2k}\big(\frac{M}{\sigma}\big)^{2a+2k+2|\gamma|-\mu}(q-1-\mu)!.$$

Consider next I_2''. From (3.4) and the induction hypothesis we find

$$\begin{aligned} I_2'' \le & \sum_{[s/2]<2|\gamma'|+2b+2i\le s} \binom{\gamma}{\gamma'}\binom{a}{b}\binom{k-1}{i} CC_0N^{2k-2i+2\nu} \\ & \times (2|\gamma|-2|\gamma'|+2a-2b+2k-2i+2\nu-\mu)! \\ & \times \big(\frac{M}{\sigma}\big)^{(2|\gamma|-2|\gamma'|+2a-2b+2k-2i+2\nu-\mu)^+}\bar{C}K^{2j+2+2|\tau|}N^{2i} \\ & \times (2|\gamma'|+2b+2i+2|\tau|+2j+2-\mu)!\big(\frac{M}{\sigma}\big)^{2|\gamma'|+2b+2i-\mu}; \end{aligned}$$

letting $k-1=l$,

$$\begin{aligned} I_2'' \le & \frac{CC_0K^2N^{2\nu}}{M}\bar{C}K^{2j+2|\tau|}N^{2k}\big(\frac{M}{\sigma}\big)^{2a+2k+2|\tau|-\mu} \sum_{[s/2]<2|\gamma'|+2b+2i<s} \binom{\gamma}{\gamma'}\binom{a}{b}\binom{l}{i} \\ & \times (2|\gamma|-2|\gamma'|+2a-2b+2l-2i+2\nu+2-\mu)! \\ & \times (q-3+2|\gamma'|+2b+2i-2a-2l-2|\gamma|+2-\mu)! \\ \le & \frac{CC_0K^2N^{2\nu}}{M}\bar{C}K^{2j+2|\tau|}N^{2k}\big(\frac{M}{\sigma}\big)^{2a+2k+2|\gamma|-\mu}(q-1-\mu)!. \end{aligned}$$

Replace γ' by $\gamma-\gamma'$, b by $a-b$ and i by $l-i$ in the last inequality and use (3.5).

Now, we combine the estimates for $I_1,\ I_2 \le I_2' + I_2''$, and I_3 in (4.2) and obtain the estimate

$$\begin{aligned} \|\zeta^2\partial_t^a T^\gamma \partial_n^k g^{(\tau,j)}\| \le & \Big(\frac{C'C_0K^2}{\bar{C}} + \frac{CC_0K^2N^{2\nu}+K^2}{M}\Big) \\ & \times \bar{C}K^{2j+2|\tau|}N^{2k}(q-1-\mu)!\big(\frac{M}{\sigma}\big)^{2a+2k+2|\gamma|-\mu}, \end{aligned}$$

and thus (3.8) holds in this case if we choose $\bar{C}, M$ so large that

$$\frac{C'C_0K^2}{\bar{C}} \le \frac{1}{2} \qquad \text{and} \qquad \frac{CC_0K^2N^{2\nu}+K^2}{M} \le \frac{1}{2}\,. \tag{4.3}$$

Case (i) is finished.

Case (ii) $k=0$, $|\gamma|>0$. This case is treated in just the same manner as case (i) and we obtain the same result.

Case (iii) $k+|\gamma|=0$, then $2a+2|\tau|+2j=q-1$, $s=2a-2$. Though it differs in some details, this case is much the same as case (i). We write

$\partial_t^a g^{(\tau,j)} = \partial_t^{a-1}[g_{v_{lm}}^{(\tau,j)}\partial_t v_{lm} + g^{(\tau,j+1)}]$ and we see as in case (i) that, for $\zeta = \zeta_{\sigma,h_{2a}}$,

$$\|\zeta^2\partial_t^a g^{(\tau,j)}\| \le \tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3, \tag{4.4}$$

where

$$\tilde{I}_1 = \|\zeta^2 g_{v_{lm}}^{(\tau,j)}\partial_t^a v_{lm}\|, \quad \tilde{I}_2 = \sum_{0<b\le a-1}\binom{a-1}{b}\|\zeta^2(\partial_T^b g_{v_{lm}}^{(\tau,j)})(\partial_t^{a-b}v_{lm})\|,$$
$$\tilde{I}_3 = \|\zeta^2\partial_t^{a-1}g^{(\tau,j+1)}\|.$$

As before, we find easily by induction

$$\tilde{I}_3 \le \frac{K^2}{M}\bar{C}K^{2j+2|\tau|}(q-1-\mu)!\Big(\frac{M}{\sigma}\Big)^{2a-\mu},$$
$$\tilde{I}_1 \le \frac{C'C_0K^2}{\bar{C}}\bar{C}K^{2j+2|\tau|}(q-1-\mu)!\Big(\frac{M}{\sigma}\Big)^{2a-\mu}.$$

Next $\tilde{I}_2 \le \tilde{I}_2' + \tilde{I}_2''$, where

$$\tilde{I}_2' = \sum_{0<b\le[\frac{a-1}{2}]}\binom{a-1}{b}\|\partial_t^b g_{v_{lm}}^{(\tau,j)}\|_{\infty,\sigma-h_{2a}}\cdot\|\zeta^2\partial_t^{a-b}v_{lm}\|,$$
$$\tilde{I}_2'' = \sum_{0\le b\le[\frac{a-1}{2}]}\binom{a-1}{b}\|\zeta^2\partial_t^{a-1-b}g_{v_{lm}}^{(\tau,j)}\|\cdot\|\partial_t^{b+1}v_{lm}\|_{\infty,\sigma-h_{2a}}.$$

As in case (i), we obtain

$$\begin{aligned}\tilde{I}_2' &\le \frac{CC_0K^2N^{2\nu}}{M}\bar{C}K^{2j+2|\tau|}\Big(\frac{M}{\sigma}\Big)^{2a-\mu}\sum_{0<b\le[\frac{a-1}{2}]}\binom{a-1}{b}\\ &\qquad\times(q-1-2a+2+2b+2\nu-\mu)!(2a-2b-\mu)!\\ &\le \frac{CC_0K^2N^{2\nu}}{M}\bar{C}K^{2j+2|\tau|}\Big(\frac{M}{\sigma}\Big)^{2a-\mu}(q-1-\mu)!,\end{aligned}$$

by (3.6). Finally, we have as before that

$$\tilde{I}_2'' \le \frac{CC_0K^2N^{2\nu}}{M}\bar{C}K^{2j+2|\tau|}\Big(\frac{M}{\sigma}\Big)^{2a-\mu}(q-1-\mu)!.$$

Inserting these estimates in (4.4) we find as before

$$\|\zeta^2\partial_t^a g^{(\tau,j)}\| \le \Big(\frac{C'C_0K^2}{\bar{C}} + \frac{CC_0K^2N^{2\nu}+K^2}{M}\Big)\bar{C}K^{2j+2|\tau|}(q-1-\mu)!\Big(\frac{M}{\sigma}\Big)^{2a-\mu},$$

and thus (3.8) holds provided

$$\frac{C'C_0K^2}{\bar{C}} \le \frac{1}{2} \qquad \text{and} \qquad \frac{CC_0K^2N^{2\nu}+K^2}{M} \le \frac{1}{2}\,. \tag{4.5}$$

If we require $\bar{C} \ge C'$ and M to satisfy (4.3), (4.5), we see that Lemma 4.1 is proved. □

With the aid of Lemma 3.1, (J_{q+1}) yields the estimate with $H_0 = \bar{C}N^{2i}$, $\tilde{p} = 2a+2i+2|\gamma| = p$,

$$\begin{aligned}\|\partial_t^a\partial_n^i T^\gamma g\|_{\infty,\sigma-h_{2a+2i+2|\gamma|}} \le\ & C\bar{C}N^{2i+2\nu}(2a+2i+2|\gamma|+2\nu-\mu)! \\ & \times \big(\frac{M}{\sigma}\big)^{(2a+2i+2|\gamma|+2\nu-\mu)^+},\end{aligned} \tag{4.6}$$

provided $2a+2i+2|\gamma|+2\nu \le q-1$.

5. Completion of the proof

Now, we prove a special case of Proposition 3.1.

Proposition 5.1. *Assume* (I_q) *and* (J_q) *hold for some* $q \ge 2\mu+3$. *Then* (3.2) *is true for* $k=0$ *and* $2j+2|\lambda| \le q$; *i.e.,*

$$\sum_{|\alpha|\le 2} \|\zeta^2 D^\alpha T^\lambda v^{(j)}\| + \|\zeta^2 T^\lambda v_t^{(j)}\| \le C_0(2j+2|\lambda|-\mu)!\big(\frac{M}{\sigma}\big)^{(2j+2|\lambda|-\mu)^+}.$$

Firstly, we prove a lemma which will be used to prove Proposition 3.1.

Lemma 5.1. *Assume* (I_q) *and* (J_{q+1}) *hold for some* $q \ge 2\mu+3$. *Then for* $f_{j,\eta}$ *in* (3.2), $2j+2k+2|\lambda| = q$, *and* $\zeta = \zeta_{\sigma,h_q}$, *we have*

$$\|\zeta^2 f_{j,\eta}\| \le M^{-1}CC_0\bar{C}N^{2\nu}(q-\mu)!\big(\frac{M}{\sigma}\big)^{q-\mu}N^{2k}.$$

Proof. We consider only a typical term of $f_{j,\eta}$ with $j>0$, say,

$$I = \sum_{s=1}^{j-1}\binom{j-1}{s}D^\eta[(\partial_t^s a_{lm})v_{lm}^{(j-s)}].$$

Then $\|\zeta^2 I\| \le A+B$, where $\eta'+\eta''=\eta$,

$$A = \sum_{\substack{1\le s\le j-1\\ 2|\eta'|+2s\le[\frac{q}{2}]}} \binom{j-1}{s}\binom{\eta}{\eta'}\|D^{\eta'}\partial_t^s a_{lm}\|_{\infty,\sigma-h_q} \times \|\zeta^2 D^{\eta''}v_{lm}^{(j-s)}\|,$$

$$B = \sum_{\substack{1\le s\le j-1\\ [\frac{q}{2}]<2|\eta'|+2s\le q-2}} \binom{j-1}{s}\binom{\eta}{\eta'} \|\zeta^2 D^{\eta'}\partial_t^s a_{lm}\| \times \|D^{\eta''} v_{lm}^{(j-s)}\|_{\infty,\sigma-h_q}.$$

Using (4.6), $(\mathrm{I_q})$ and (3.6), we find

$$\begin{aligned} A &\le M^{-1}C_0 C\bar{C}N^{2k+2\nu}\big(\frac{M}{\sigma}\big)^{q-\mu} \sum_{\substack{1\le s\le j-1\\ 2|\eta'|+2s\le [\frac{q}{2}]}} \binom{j-1}{s}\binom{\eta}{\eta'} \\ &\quad\times (2|\eta'|+2s+2\nu-\mu)!(2|\eta''|+2j-2s-\mu)! \\ &\le M^{-1}C_0 C\bar{C}N^{2k+2\nu}\big(\frac{M}{\sigma}\big)^{q-\mu}(q-\mu)!. \end{aligned}$$

Similarly, using $(\mathrm{J_{q+1}})$, (3.4) and (3.5) we find

$$\begin{aligned} B &\le M^{-1}CC_0\bar{C}N^{2k+2\nu}\big(\frac{M}{\sigma}\big)^{q-\mu} \sum_{\substack{0\le s\le j-2\\ 2|\eta'|+2s<[\frac{q}{2}]-2}} \binom{j-1}{s}\binom{\eta}{\eta'} \\ &\quad\times (2|\eta|-2|\eta'|+2j-2-2s-\mu)!(2|\eta'|+2s+2+2\nu-\mu)! \\ &\le M^{-1}CC_0\bar{C}N^{2k+2\nu}\big(\frac{M}{\sigma}\big)^{q-\mu}(q-\mu)!. \end{aligned}$$

Thus

$$\|\zeta^2 I\| \le M^{-1}CC_0\bar{C}N^{2\nu}\big(\frac{M}{\sigma}\big)^{q-\mu}(q-\mu)!N^{2k},$$

and Lemma 5.1 is proved for $j>0$. The case $j=0$ is treated similarly. □

Proof of Proposition 5.1. According to Lemma 4.1 we know that $(\mathrm{J_{q+1}})$ holds. Now we apply (2.9) to the equation (3.1), which is satisfied by $T^\lambda v^{(j)}, 2j+2|\lambda|=q$; then for $\zeta=\zeta_{\sigma,h_q}$, we have

$$J \equiv \sum_{|\alpha|\le 2} \|\zeta^2 D^\alpha T^\lambda v^{(j)}\| + \|\zeta^2 T^\lambda v_t^{(j)}\| \le C_1\big\{\|\zeta^2 f_{j,\lambda}\| + \frac{1}{h_q^2}\|T^\lambda v^{(j)}\|_{\sigma-h_q}\big\},$$

for a constant $C_1>0$. Now, we consider the case $j>0$, and the case $j=0$ can be treated similarly. For $\tilde{\zeta}=\zeta_{\sigma-h_q,h_{q-2}(\sigma-h_q)}$,

$$\begin{aligned} \|T^\lambda v^{(j)}\|_{\sigma-h_q} &\le \|\tilde{\zeta}^2 T^\lambda v^{(j)}\| = \|\tilde{\zeta}^2 T^\lambda v_t^{(j-1)}\| \\ &\le C_0(q-\mu-2)!\big(\frac{M}{\sigma-h_q}\big)^{q-\mu-2} \le eC_0(q-\mu-2)!\big(\frac{M}{\sigma}\big)^{q-\mu-2}. \end{aligned}$$

Thus,

$$\frac{1}{h_q^2}\|T^\lambda v^{(j)}\|_{\sigma-h_q} \le eC_0(q-\mu-2)!\frac{q^2}{\sigma^2}\big(\frac{M}{\sigma}\big)^{q-\mu-2} \le \frac{C_0C}{M^2}(q-\mu)!\big(\frac{M}{\sigma}\big)^{q-\mu}.$$

Therefore,

$$J \le C_1\big\{\|\zeta^2 f_{j,\lambda}\| + \frac{CC_0}{M^2}(q-\mu)!\big(\frac{M}{\sigma}\big)^{q-\mu}\big\}. \tag{5.1}$$

Then from Lemma 5.1, we have

$$J \le (M^{-1}CC_1\bar{C}N^{2\nu} + C_1CM^{-2})C_0(q-\mu)!\big(\frac{M}{\sigma}\big)^{q-\mu}.$$

We thus obtain the desired conclusion of Proposition 5.1 provided M is chosen so large that

$$M^{-1}CC_1\bar{C}N^{2\nu} + C_1CM^{-2} \le 1; \tag{5.2}$$

we add this to our requirements on M. $\square$

We finally complete the proof of Proposition 3.1 by verifying (3.2) for $k > 0$, $2j + 2k + 2|\lambda| = q$. We prove this result by induction on k.

Proposition 5.2. *Assume* ($\mathrm{I_q}$) *and* ($\mathrm{J_{q+1}}$) *hold for some* $q \ge 2\mu + 3$. *For* $\zeta = \zeta_{\sigma,h_{2j+2k+2|\lambda|}}$, *assume*

$$\|\zeta^2\partial_n^{k+2}T^\lambda v^{(j)}\| \le C_0N^{2k}(2j+2k+2|\lambda|-\mu)!\big(\frac{M}{\sigma}\big)^{2j+2k+2|\lambda|-\mu} \tag{5.3}$$

is true for $2k+2j+2|\lambda| = q$, $k \le k_0 - 1$; *then* (5.3) *also holds for* $2k+2j+2|\lambda| = q$, $\quad k = k_0$.

Proof. Let $2k + 2j + 2|\lambda| = q$ and $k = k_0$. We can express $\partial_n^{k_0+2}T^\lambda v^{(j)}$ in terms of quantities we already estimated by solving $\partial_n^{k_0+2}T^\lambda v^{(j)}$ in (3.1) with $D^\eta = \partial_n^{k_o}T^\lambda$; i.e.,

$$-\partial_n^{k_0+2}T^\lambda v^{(j)} = \tag{5.4}$$

$$\frac{1}{a_{nn}}\Big[\sum_{l+m<2n} a_{lm}\partial_n^{k_0}T^\lambda v_{lm}^{(j)} - \sum_{l=0}^{n} b_l\partial_n^{k_0}T^\lambda v_l^{(j)} - \partial_n^{k_0}T^\lambda v^{(j+1)}\Big] + \frac{1}{a_{nn}}f_{j,\eta}.$$

Consider a typical term: $\frac{a_{ln}}{a_{nn}}\partial_n^{k_0}T^\lambda v_{ln}^{(j)}$ with $l < n$. For $\zeta = \zeta_{\sigma,h_q}$, $\tilde{\zeta} = \zeta_{\sigma-h_q,h_q(\sigma-h_q)}$,

$$\|\zeta^2\frac{a_{ln}}{a_{nn}}\partial_n^{k_0}T^\lambda v_{ln}^{(j)}\| \le C\|\tilde{\zeta}^2\partial_n^{k_0}T^\lambda v_{ln}^{(j)}\| = C\|\tilde{\zeta}^2\partial_n^{k_0-1}T^\tau v_{nn}^{(j)}\| \tag{5.5}$$

$$= C\|\tilde{\zeta}^2\partial_n^{k_0+1}T^\tau v^{(j)}\| \le \frac{CC_0}{N}(q-\mu)!\big(\frac{M}{\sigma}\big)^{q-\mu}N^{2k_0},$$

since we write $\partial_n^{k_0} T^\lambda v_{ln}^{(j)}$ as $\partial_n^{k_0-1} T^\tau v_{nn}^{(j)}$ with $|\tau| = |\lambda|+1$, and use the induction hypothesis. The other terms in the bracket in (5.4) can be estimated in the same way.

Next, by Lemma 5.1, with a different constant C,

$$\|\zeta^2 a_{nn}^{-1} f_{j,\eta}\| \le M^{-1} C C_0 \bar{C} N^{2\nu} (q-\mu)! \big(\frac{M}{\sigma}\big)^{q-\mu} N^{2k_0}. \tag{5.6}$$

Combining the estimates (5.5) and (5.6), we find

$$\|\zeta^2 \partial_n^{k_0+2} T^\lambda v^{(j)}\| \le \big(\frac{C}{N} + \frac{C\bar{C}N^{2\nu}}{M}\big) C_0 (q-\mu)! \big(\frac{M}{\sigma}\big)^{q-\mu} N^{2k_0},$$

which yields (5.3) for $k = k_0$ if

$$\frac{C}{N} \le \frac{1}{2} \quad \text{and} \quad \frac{C\bar{C}N^{2\nu}}{M} \le \frac{1}{2}\,. \tag{5.7}$$

We add these to all our previous conditions on N and M. □

By Proposition 5.1 we know that (5.3) holds for $k = 0$; hence, it will follow from Proposition 5.2 that (5.3) holds for all j, k, λ with $2j + 2k + 2|\lambda| = q$. Together with Proposition 5.1 this yields ($\mathrm{I_q}$), and thus *completes the proof of* Proposition 3.1.

Thus for suitable choices of C_0, $\bar{C}$, N and M we have proved Proposition 5.2 and hence all the other propositions. These constants are chosen in the following way. Having fixed C_0 we next fix $\bar{C}$ sufficiently large so as to satisfy also (4.3) and (4.5). Then we fix N so that the first inequality in (5.7) holds and, finally, we fix M to satisfy (5.7) as well as (4.3),(4.5) and (5.2).

The proof of Theorem 2.1 is complete.

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