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H. WEYL'S BOUNDARY VALUE PROBLEMS FOR DIFFERENTIAL FORMS

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Abstract. In this paper we give solutions of boundary value problems for differential forms, generalizing Poisson's equation for functions. We focus on problems which for instance give an easy access to decomposition results like those of Hodge, Kodaira and Morrey. The search for general solutions and the proof of appropriate a priori estimates will be the central part of this publication.

1. Introduction

With notice of the results in [4] and [17], we will treat boundary value problems for differential forms in a comprehensive manner given by

$$\begin{aligned}
&\triangle f = g \;\text{ in }\; \Omega,\\
&\nu \wedge f = \xi \;\text{ and }\; \nu \wedge \delta f = \vartheta \;\text{ on }\; \partial\Omega\\
&\triangle f^* = g^* \;\text{ in }\; \Omega,\\
&(\nu, f^*) = \xi^* \;\text{ and } (\nu, df^*) = \vartheta^* \;\text{ on }\; \partial\Omega.
\end{aligned}$$

The data have to fulfill regularity and integrability conditions. There, the set $\Omega \subset \mathbb{R}^n$, $n > 2$, is bounded and has a smooth boundary. The exact definition will be dealt with in the following section. H. Weyl [17] gave, via potential and Fredholm theory, explicit solutions for Helmholtz' wave equation on the mentioned boundary value conditions. But it turned out that the solutions of the homogeneous problems depend on the topology of Ω. That is why H. Weyl's procedure is not adequate to deliver the general solutions of the problems. Since in general each of the homogeneous problems has several solutions, we add side conditions to uniquely determine the solutions. Moreover, suitable regularity properties and a priori inequalities

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for these problems will be presented. We will show that differential forms f with Euclidian components in $C^{2,\lambda}(\bar{\Omega})$ satisfy the a priori estimates

$$\|f\|_{C^{2,\lambda}(\bar{\Omega})} \leq$$
$$c\Big(\|\triangle f\|_{C^{0,\lambda}(\bar{\Omega})} + \|\nu \wedge f\|_{C^{2,\lambda}(\partial\Omega)} + \|\nu \wedge \delta f\|_{C^{1,\lambda}(\partial\Omega)} + \sum_{i=1}^{B_{n-r}} |Y_i[f]|\Big)$$
$$\|f\|_{C^{2,\lambda}(\bar{\Omega})} \leq$$
$$c\Big(\|\triangle f\|_{C^{0,\lambda}(\bar{\Omega})} + \|(\nu, f)\|_{C^{2,\lambda}(\partial\Omega)} + \|(\nu, df)\|_{C^{1,\lambda}(\partial\Omega)} + \sum_{i=1}^{B_r} |Z_i[f]|\Big).$$

Here, B_i are Betti numbers with regard to Ω, and $Y_i[f]$, $Z_i[f]$ are quantities which depend on the topology of Ω. Analogously, we get the L^p estimates

$$\|f\|_{W^{2,p}(\Omega)} \leq$$
$$c\Big(\|\triangle f\|_{L^p(\Omega)} + \|\nu \wedge f\|_{W^{2-\frac{1}{p},p}(\partial\Omega)} + \|\nu \wedge \delta f\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} + \sum_{i=1}^{B_{n-r}} |Y_i[f]|\Big)$$
$$\|f\|_{W^{2,p}(\Omega)} \leq$$
$$c\Big(\|\triangle f\|_{L^p(\Omega)} + \|(\nu, f)\|_{W^{2-\frac{1}{p},p}(\partial\Omega)} + \|(\nu, df)\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} + \sum_{i=1}^{B_r} |Z_i[f]|\Big).$$

It is obvious that these are the appropriate estimates for our boundary value problems.

Consequently the L^2-decomposition of Hodge, Kodaira and Morrey (see [11]) easily follows in L^p and C^λ (cf. the end of section 3). Morrey proved this L^2-decomposition, by using the weak solvability of boundary value problems like ours. The reader who is interested in applications of elliptic theory to boundary value problems for differential forms will gain a lot of dealing with [13]. However, this indirect approach does not deliver the optimal a priori estimates for our boundary value problems. Georgescu [6] gave a priori estimates relating to Hilbert spaces. But those inequalities are not optimal with regard to the data. In [2], special cases of the mentioned a priori estimates are proved. There, the topological restriction that B_{n-r} or B_r respectively vanishes was necessary.

2. Definitions and requirements

If not stated otherwise, we require that $n > 2$, $r \in \mathbb{N}_0, r \leq n$, $k \in \mathbb{N}_0$, $\lambda \in (0,1)$, $p \in (1,\infty)$ and $\sigma \in \mathbb{R}_+ \cup \{0\}$. Furthermore, $\Omega \subset \mathbb{R}^n$ is a bounded set consisting of a finite number of arcwise-connected components Ω_i, where $\overline{\Omega_i} \cap \overline{\Omega_j} = \emptyset$, if $i \neq j$ and $\partial\Omega_i$ belongs to C^∞. $\hat{\Omega}$ denotes the complement of $\bar{\Omega}$; i.e., $\hat{\Omega} := \mathbb{R}^n \backslash \overline{\Omega}$. B_l (and $\hat{B}_l$) means the Betti number concerning the simplicial homology group with regard to l-cycles of the set Ω (and $\hat{\Omega}$ respectively). Throughout this paper, we presuppose that each Betti number B_l is finite.

For the r-form f and 1-form v, we define a product $vf := (v, f)$ by

$$(v, f) := \sum_{i=1}^{n} v_i f_{ij_1,\ldots,j_{r-1}} dx_{j_1} \wedge \cdots \wedge dx_{j_{r-1}}, \quad \text{if} \quad r \in \mathbb{N} : r \leq n,$$

and $(v, f) := 0$, if $r = 0$. Let f and h be r-forms on $\mathbb{R}^n$, $(f, h) = (f, h)_r$ the Euclidian inner product, g an $(n - r)$-form and $\wedge$ Grassmann's wedge product. The Hodge operator $*$ is given by

$$(*f, g)_{n-r} = (f \wedge g, d^n x)_n.$$

The exterior derivative d for differentiable r-forms is defined as usual. We call the operator $\delta := (-1)^{n(r+1)} * d*$ codifferential.

Let f be an arbitrary r-form defined on $\partial\Omega$, and let ν be the 1-form with components which are equal to those of the exterior normal field with unit length. We call the differential form νf the normal component and the differential form $\tau f := \nu \wedge f$ the tangential component of f.

As usual $G(x, x')$ denotes the fundamental solution of $\triangle$. The 1-forms

$$\Gamma(x, x') := d_x\, G(x, x') \quad \text{and} \quad \Gamma'(x, x') := d_{x'}\, G(x, x')$$

are given by means of exterior derivatives with regard to x (i.e., d_x) or x' (i.e., $d_{x'}$) respectively.

We write $f \in X^r(\Omega)$, if f is an r-form on Ω with Euclidian components in the space X. For the sake of simplicity, let $W^{0,p}(\partial\Omega)^r := L^p(\partial\Omega)^r$. In addition to the commonly used spaces, we introduce

$$\begin{aligned}
C^{k,\lambda}_\tau(\partial\Omega)^r &:= \{f \in C^{k,\lambda}(\partial\Omega)^r : \tau f = 0\},\\
W^{\sigma,p}_\tau(\partial\Omega)^r &:= \{f \in W^{\sigma,p}(\partial\Omega)^r : \tau f = 0\},\\
C^{k,\lambda}_\nu(\partial\Omega)^r &:= \{f \in C^{k,\lambda}(\partial\Omega)^r : \nu f = 0\},\\
W^{\sigma,p}_\nu(\partial\Omega)^r &:= \{f \in W^{\sigma,p}(\partial\Omega)^r : \nu f = 0\},
\end{aligned}$$

$$\begin{aligned}
\mathcal{Y}(\Omega)^r &:= \{f \in C^0(\bar{\Omega})^r \cap C^1(\Omega)^r : df = 0 \text{ and } \delta f = 0 \text{ with } \tau f = 0\},\\
\mathcal{Z}(\Omega)^r &:= \{f \in C^0(\bar{\Omega})^r \cap C^1(\Omega)^r : df = 0 \text{ and } \delta f = 0 \text{ with } \nu f = 0\}.
\end{aligned}$$

Elements f of the spaces $\mathcal{Y}(\hat{\Omega})^r$ or $\mathcal{Z}(\hat{\Omega})^r$ fulfill equal conditions to those of $\mathcal{Y}(\Omega)^r$ or $\mathcal{Z}(\Omega)^r$ respectively and in addition have the property that the functions $f_{i_1\dots i_r}(x)$ uniformly converge to 0 for $|x|\to\infty$ with regard to all directions. This implies that $f_{i_1\dots i_r}(x)\in\mathcal{O}(|x|^{-n+1})$, if $|x|\to\infty$. We call the elements of $\mathcal{Y}(\Omega)^r$ and $\mathcal{Y}(\hat{\Omega})^r$ *Dirichlet fields* and the elements of $\mathcal{Z}(\Omega)^r$ and $\mathcal{Z}(\hat{\Omega})^r$ *Neumann fields.*

3. Boundary value problems and a priori estimates for differential forms

In this section, we solve integral equations for differential forms in order to find solutions and to deduce a priori inequalities for our boundary value problems. The following results serve to prove these estimates.

Theorem 1. a) *Let $f\in C_\nu^{k,\lambda}(\partial\Omega)^r$, $x,x'\in\partial\Omega$ and*

$$U_{k,\lambda,r}f(x) := \int_{\partial\Omega}(\nu(x),\nu(x)\wedge(\Gamma'(x,x'),\nu(x')\wedge f(x')))\,d\omega_{x'}\,,$$

$$V_{k,\lambda,r}f(x) := (-1)^r\int_{\partial\Omega}G(x,x')\cdot(\nu(x),\nu(x)\wedge(\nu(x')\wedge f(x')))\,d\omega_{x'}\,,$$

$$U^*_{k,\lambda,r}f(x) := \int_{\partial\Omega}(\nu(x),\Gamma(x,x')\wedge f(x'))\,d\omega_{x'}\,,$$

$$V^*_{k,\lambda,r}f(x) := (-1)^{r-1}\int_{\partial\Omega}G(x,x')\cdot(\nu(x),f(x'))\,d\omega_{x'}\,.$$

Then

$$U_{k,\lambda,r}\in\mathcal{L}(C_\nu^{k,\lambda}(\partial\Omega)^r,C_\nu^{k+1,\lambda}(\partial\Omega)^r),\ \ V_{k,\lambda,r}\in\mathcal{L}(C_\nu^{k,\lambda}(\partial\Omega)^r,C_\nu^{k+1,\lambda}(\partial\Omega)^{r+1}),$$

$$U^*_{k,\lambda,r}\in\mathcal{L}(C_\nu^{k,\lambda}(\partial\Omega)^r,C_\nu^{k+1,\lambda}(\partial\Omega)^r),\ \ V^*_{k,\lambda,r}\in\mathcal{L}(C_\nu^{k,\lambda}(\partial\Omega)^r,C_\nu^{k+1,\lambda}(\partial\Omega)^{r-1}).$$

b) *If $k+\lambda>\sigma$, then uniquely determined extensions of those operators defined on $W_\nu^{\sigma,p}(\partial\Omega)^r$ exist. These operators have the properties*

$$U_{\sigma,p,r}\in\mathcal{L}(W_\nu^{\sigma,p}(\partial\Omega)^r,W_\nu^{\sigma+1,p}(\partial\Omega)^r),\ \ V_{\sigma,p,r}\in\mathcal{L}(W_\nu^{\sigma,p}(\partial\Omega)^r,W_\nu^{\sigma+1,p}(\partial\Omega)^{r+1}),$$

$$U^*_{\sigma,p,r}\in\mathcal{L}(W_\nu^{\sigma,p}(\partial\Omega)^r,W_\nu^{\sigma+1,p}(\partial\Omega)^r),\ \ V^*_{\sigma,p,r}\in\mathcal{L}(W_\nu^{\sigma,p}(\partial\Omega)^r,W_\nu^{\sigma+1,p}(\partial\Omega)^{r-1}).$$

Proof. To prove the assertion with regard to the operators U and U^*, we confine ourselves to the nontrivial cases for $r<n$. By means of

$$(\Gamma'(x,x'),\nu(x')\wedge f(x')) = f(x')(\nu(x'),\Gamma'(x,x'))-\nu(x')\wedge(\Gamma'(x,x'),f(x')) \quad (1)$$

in addition to

$$Uf(x) = (\nu(x),\nu(x)\wedge(\int_{\partial\Omega}f(x')\cdot(\nu(x'),\Gamma'(x,x'))\,d\omega_{x'}$$

$$+\int_{\partial\Omega}((\nu(x)-\nu(x'))\wedge(\Gamma'(x,x'),f(x')))\,d\omega_{x'})) \tag{2}$$

we have shown that the operator U consists of more familiar terms. Analogously the equation

$$\begin{aligned}&(\nu(x),\Gamma(x,x')\wedge f(x'))\\&=f(x')(\nu(x),\Gamma(x,x'))-\Gamma(x,x')\wedge(\nu(x)-\nu(x'),f(x'))\end{aligned} \tag{3}$$

delivers a suitable decomposition for U^*:

$$\begin{aligned}U^*f(x)=&\int_{\partial\Omega}f(x')(\nu(x),\Gamma(x,x'))\,d\omega_{x'}\\&-\int_{\partial\Omega}(\Gamma(x,x')\wedge(\nu(x)-\nu(x'),f(x')))\,d\omega_{x'}.\end{aligned} \tag{4}$$

For a detailed proof of the required estimates the reader will be referred to [2], [7] and [15]. With these results, the inequalities for the operators V and V^* will also be given. □

Definition 1. Let $r\in\mathbb{N}, r\le n$, $k_1,k_2\in\mathbb{N}_0$ and $\sigma_1,\sigma_2\in\mathbb{R}^+\cup\{0\}$. Referring to the operators of Theorem 1, let

$$W_{k_1,k_2,\lambda,r}:=\begin{pmatrix}U_{k_1,\lambda,r} & -V_{k_2,\lambda,r-1}\\ 0 & U_{k_2,\lambda,r-1}\end{pmatrix},\ W_{\sigma_1,\sigma_2,p,r}:=\begin{pmatrix}U_{\sigma_1,p,r} & -V_{\sigma_2,p,r-1}\\ 0 & U_{\sigma_2,p,r-1}\end{pmatrix},$$

$$W^*_{k_1,k_2,\lambda,r}:=\begin{pmatrix}U^*_{k_1,\lambda,r} & 0\\ -V^*_{k_1,\lambda,r} & U^*_{k_2,\lambda,r-1}\end{pmatrix},\ W^*_{\sigma_1,\sigma_2,p,r}:=\begin{pmatrix}U^*_{\sigma_1,p,r} & 0\\ -V^*_{\sigma_1,p,r} & U^*_{\sigma_2,p,r-1}\end{pmatrix},$$

$$M_{k_1,k_2,\lambda,r}:=\begin{pmatrix}U^*_{k_1,\lambda,r-1} & -V^*_{k_2,\lambda,r}\\ 0 & U^*_{k_2,\lambda,r}\end{pmatrix},\ M_{\sigma_1,\sigma_2,p,r}:=\begin{pmatrix}U^*_{\sigma_1,p,r-1} & -V^*_{\sigma_2,p,r}\\ 0 & U^*_{\sigma_2,p,r}\end{pmatrix}.$$

We abbreviate: $W_r:=W_{k_1,k_2,\lambda,r}$, $W^*_r:=W^*_{k_1,k_2,\lambda,r}$ and $M_r:=M_{k_1,k_2,\lambda,r}$.

Remark 1. If $r\in\mathbb{N}, r<n$, $\sigma=0$ and $p=2$, then the operators $U_{\sigma,p,r}$ and $U^*_{\sigma,p,r}$ as well as the operators $V_{\sigma,p,r}$ and $V^*_{\sigma,p,r}$ are adjoint operators with respect to the inner product $\langle\cdot,\cdot\rangle : L^2(\partial\Omega)^r\times L^2(\partial\Omega)^r\longrightarrow\mathbb{R}$ given by

$$\langle f,g\rangle:=\int_{\partial\Omega}(f(x),g(x))\,d\omega_x=\frac{1}{r!}\sum_{j_i\in\{1,\dots,n\}}\int_{\partial\Omega}f_{j_1\cdots j_r}(x)g_{j_1\cdots j_r}(x)\,d\omega_x\,.$$

Lemma 1. *Let $r\in\mathbb{N}$, $r<n$. We presuppose that $\psi\in C^0_\nu(\partial\Omega)^r$, $\varphi\in C^0_\nu(\partial\Omega)^{r-1}$, $\psi^*\in C^0_\nu(\partial\Omega)^{r-1}$, $\varphi^*\in C^0_\nu(\partial\Omega)^r$ and set $\phi:=(\psi,\varphi)$, $\phi^*:=(\psi^*,\varphi^*)$. Moreover, let*

$$f_\phi(x):=-\int_{\partial\Omega}(\varphi(x')\wedge\nu(x')\cdot G(x,x'))\,d\omega_{x'}+\int_{\partial\Omega}(\Gamma'(x,x'),\nu(x')\wedge\psi(x'))\,d\omega_{x'}$$

$$f^*_{\phi^*}(x) := \int_{\partial\Omega} \varphi^*(x') \cdot G(x,x')\, d\omega_{x'} \;-\int_{\partial\Omega} (\psi^*(x') \wedge \Gamma(x,x'))\, d\omega_{x'}\,,$$

where $x \in \mathbb{R}^n \backslash \partial\Omega$. These r-forms are harmonic differential forms, ϕ is a solution of the inhomogeneous integral equation

$$(\pm\frac{1}{2}I + W_r)\phi = \begin{pmatrix} (\nu, \nu \wedge f) \\ (-1)^{r-1}(\nu, \nu \wedge \delta f) \end{pmatrix}$$

and ϕ^* of the inhomogeneous integral equation

$$(\mp\frac{1}{2}I + M_r)\phi^* = \begin{pmatrix} (-1)^r(\nu, f^*) \\ (\nu, df^*) \end{pmatrix}.$$

The upper signs are used if we approximate in Ω along the direction of the exterior normal, and the lower signs are used if we approximate in $\hat{\Omega}$ opposite to this direction.

Proof. Here we give a short outline of the proof of [2], Lemma E4, which in general follows the arguments of [17]. By means of

$$\int_{\partial\Omega} (\Gamma'(x,x'), \nu(x') \wedge \psi(x'))\, d\omega_{x'} = -\delta \int_{\partial\Omega} (\nu(x') \wedge \psi(x') \cdot G(x,x'))\, d\omega_{x'} \quad (5)$$

$$\int_{\partial\Omega} (\Gamma(x,x') \wedge \psi^*(x'))\, d\omega_{x'} = d \int_{\partial\Omega} \psi^*(x') \cdot G(x,x')\, d\omega_{x'} \quad (6)$$

the equations

$$\delta f(x) = (-1)^{r-1} \int_{\partial\Omega} (\Gamma'(x,x'), \nu(x') \wedge \varphi(x'))\, d\omega_{x'} \quad (7)$$

$$df^*(x) = \int_{\partial\Omega} (\Gamma(x,x') \wedge \varphi^*(x'))\, d\omega_{x'} \quad (8)$$

are inferred. Let v be a suitable extension of ν, and let x be a point outside of the set $\partial\Omega$. Then we get, with regard to the form f,

$$(v(x), v(x) \wedge f(x)) = -(V\varphi)(x) + (U\psi)(x) \quad (9)$$

$$(v(x), v(x) \wedge \delta f(x)) = (-1)^{r-1}(U\varphi)(x)\,. \quad (10)$$

Analogously, we obtain, with regard to the form f^*,

$$(v(x), f^*(x)) = (-1)^{r-1}(V^*\varphi^*)(x) + (-1)^r(U^*\psi^*)(x), \quad (11)$$

$$(v(x), df^*(x)) = (U^*\varphi^*)(x)\,. \quad (12)$$

Let $0 < s, s^* < n$, $\gamma \in C^0(\partial\Omega)^{s-1}$, $\kappa := \nu \wedge \gamma$ and $\kappa^* \in C^0_\nu(\partial\Omega)^{s^*}$. By means of the equations

$$v(x) \wedge (\Gamma'(x,x'), \kappa(x')) = -(\Gamma'(x,x'), v(x) \wedge \kappa(x')) + \kappa(x')(v(x), \Gamma'(x,x')) \tag{13}$$

$$(v(x), \Gamma(x,x') \wedge \kappa^*(x')) = -\Gamma(x,x') \wedge (v(x), \kappa^*(x')) + \kappa^*(x')(v(x), \Gamma(x,x')) \tag{14}$$

and a Taylor expansion of the factors $v_i(x)\nu_j(x') - v_j(x)\nu_i(x')$ the asserted integral equations are proven. □

Definition 2. Provided that f is a integrable r-form on $\partial\Omega$, we set

$$Y_i[f] := -\int_{\partial\Omega} (f, \tau\hat{z}^i)\, d\omega\,,\ i = 1, \ldots, B_{n-r},$$
$$Z_i[f] := -\int_{\partial\Omega} (f, \nu\hat{y}^i)\, d\omega\,,\ i = 1, \ldots, B_r,$$

where $\{\hat{z}^i\}_{i=1,\ldots,B_{n-r}}$ denotes a basis of $\mathcal{Z}(\hat{\Omega})^{r-1}$ and $\{\hat{y}^i\}_{i=1,\ldots,B_r}$ denotes a basis of $\mathcal{Y}(\hat{\Omega})^{r+1}$.

Theorem 2. *Let $r \in \mathbb{N}$, $r < n$, $k \in \mathbb{N}$ and $f \in C^{k,\lambda}(\bar{\Omega})^r$. There exist positive constants $c = c(n, r, k, \lambda, \Omega)$ such that*

$$\|f\|_{C^{k,\lambda}(\bar{\Omega})} \le c(\|df\|_{C^{k-1,\lambda}(\bar{\Omega})} + \|\delta f\|_{C^{k-1,\lambda}(\bar{\Omega})} + \|\nu \wedge f\|_{C^{k,\lambda}(\partial\Omega)} + \sum_{i=1}^{B_{n-r}} |Y_i[f]|)$$

$$\|f\|_{C^{k,\lambda}(\bar{\Omega})} \le c(\|df\|_{C^{k-1,\lambda}(\bar{\Omega})} + \|\delta f\|_{C^{k-1,\lambda}(\bar{\Omega})} + \|(\nu, f)\|_{C^{k,\lambda}(\partial\Omega)} + \sum_{i=1}^{B_r} |Z_i[f]|).$$

Moreover, there exist positive constants $c = c(n, r, k, p, \Omega)$ such that

$$\|f\|_{W^{k,p}(\Omega)} \le$$
$$c(\|df\|_{W^{k-1,p}(\Omega)} + \|\delta f\|_{W^{k-1,p}(\Omega)} + \|\nu \wedge f\|_{W^{k-\frac{1}{p},p}(\partial\Omega)} + \sum_{i=1}^{B_{n-r}} |Y_i[f]|)$$
$$\|f\|_{W^{k,p}(\Omega)} \le$$
$$c(\|df\|_{W^{k-1,p}(\Omega)} + \|\delta f\|_{W^{k-1,p}(\Omega)} + \|(\nu, f)\|_{W^{k-\frac{1}{p},p}(\partial\Omega)} + \sum_{i=1}^{B_r} |Z_i[f]|),$$

where $W^{0,p}(\Omega) := L^p(\Omega)$.

The result is a generalization of results in [1] and [16]. With respect to r and k the estimates of [2] and [3] are special cases of the inequalities presented above. The suitable boundary value problems will be found in [10] and for $n = 3$ in [9] and [14].

Proof. According to [10], a form $f \in C^{1,\lambda}(\bar{\Omega})^r$ is decomposable as follows:

$$f(x) = d\,(\int_\Omega (\delta f)(x')\cdot G(x,x')\,dx' - \int_{\partial\Omega} (\nu f)(x')\cdot G(x,x')\,d\omega_{x'}) \\ + \delta(\int_\Omega (df)(x')\cdot G(x,x')\,dx' - \int_{\partial\Omega} (\nu\wedge f)(x')\cdot G(x,x')\,d\omega_{x'}), \quad (15)$$

where $x \in \Omega$. From potential theory the integral operators given by

$$G_\Omega f(x) := \int_\Omega G(x,x')f(x')\,dx' \text{ and } G_{\partial\Omega} f(x) := \int_{\partial\Omega} G(x,x')f(x')\,d\omega_{x'}$$

are well-known. We take from [2] and [12] that the operators G_Ω belong to $\mathcal{L}(C^{k-1,\lambda}(\bar{\Omega}), C^{k+1,\lambda}(\bar{\Omega}))$ or $\mathcal{L}(W^{k-1,p}(\Omega), W^{k+1,p}(\Omega))$ and from [2] and [18] that the operators $G_{\partial\Omega}$ are in $\mathcal{L}(C^{k,\lambda}(\partial\Omega), C^{k+1,\lambda}(\bar{\Omega}))$ or $\mathcal{L}(W^{k-\frac{1}{p},p}(\partial\Omega), W^{k+1,p}(\Omega))$. As shown in [10], to each basis $\{e_i^*\}_{i=1,\ldots,B_{n-r}}$ of $\mathcal{N}(\frac{1}{2}I + U^*_{0,\lambda,r-1})$ we get forms $\hat{z}^j \in \mathcal{Z}(\hat{\Omega})^{r-1}$ such that

$$\int_{\partial\Omega} (e_i^*(x), \hat{z}^j(x))\,d\omega_x = \delta_{ij}\,,\ i,j \in \{1,\ldots,B_{n-r}\}\,. \quad (16)$$

If $h \in \mathcal{R}(\frac{1}{2}I + U^*_{0,\lambda,r-1})$, then there is a solution $g \in C^{1,\lambda}_\nu(\partial\Omega)^{r-1}$ of

$$(\frac{1}{2}I + U^*_{0,\lambda,r-1})g = h\,,\ \text{where} \int_{\partial\Omega} (g,\hat{z}^i)\,d\omega = Y_i\,,\ i = 1,\ldots,B_{n-r}\,. \quad (17)$$

Additional, we define $g_0 := \sum_{i=1}^{B_{n-r}} Y_i e_i^*$ and $g_1 := g - g_0$. Banach's open mapping theorem delivers the estimates

$$\|g_1\|_{C^{k,\lambda}(\partial\Omega)} \le c\|(\frac{1}{2}I + U^*_{k,\lambda,r-1})g_1\|_{C^{k,\lambda}(\partial\Omega)} \quad (18)$$

$$\|g_1\|_{W^{k-\frac{1}{p},p}(\partial\Omega)} \le c\|(\frac{1}{2}I + U^*_{k-\frac{1}{p},p,r-1})g_1\|_{W^{k-\frac{1}{p},p}(\partial\Omega)}\,. \quad (19)$$

We derive then from our conclusions and [10] the inequalities

$$\|(\nu,f)\|_{C^{k,\lambda}(\partial\Omega)} \le c(\|df\|_{C^{k-1,\lambda}(\partial\Omega)} + \|\delta f\|_{C^{k-1,\lambda}(\partial\Omega)} \\ + \|\nu\wedge f\|_{C^{k,\lambda}(\partial\Omega)} + \sum_{i=1}^{B_{n-r}} |Y_i|) \quad (20)$$

$$\|(\nu,f)\|_{W^{k-\frac{1}{p},p}(\partial\Omega)} \le c(\|df\|_{W^{k-1,p}(\partial\Omega)} + \|\delta f\|_{W^{k-1,p}(\partial\Omega)}$$

$$+ \|\nu \wedge f\|_{W^{k-\frac{1}{p},p}(\partial\Omega)} + \sum_{i=1}^{B_{n-r}} |Y_i|) \,. \tag{21}$$

As can easily be shown, the corresponding dual results are included in the latter ones. □

Definition 3. Let $r \in \mathbb{N}$, $r < n$ and either $(\Omega', B'_r) = (\Omega, B_r)$ or $(\Omega', B'_r) = (\hat{\Omega}, \hat{B}_r)$. As usual, $\hat{\Omega}'$ denotes the set $\mathbb{R}^n \backslash \overline{\Omega'}$. Moreover, $\{\hat{z}^i\}_{i=1,\dots,B'_{n-r}}$ is a basis of $\mathcal{Z}(\hat{\Omega}')^{r-1}$ and $\{\hat{y}^i\}_{i=1,\dots,B'_r}$ is a basis of $\mathcal{Y}(\hat{\Omega}')^{r+1}$.

a) We presuppose that $\xi \in C^{2,\lambda}_\tau(\partial\Omega')^{r+1}$, $\vartheta \in C^{1,\lambda}_\tau(\partial\Omega')^r$, $Y_i \in \mathbb{R}$ and

$$g \in \{g \in C^{0,\lambda}(\bar{\Omega}')^r \cap L^2(\Omega')^r : \int_{\Omega'} (g,y)\, dx = \int_{\partial\Omega'} (\vartheta, y)\, d\omega \text{ for all } y \in \mathcal{Y}(\Omega')^r\}.$$

Then the boundary problem

$$\begin{aligned} &\triangle f = g \quad \text{in} \quad \Omega' \\ &\nu \wedge f = \xi \quad \text{and} \quad \nu \wedge \delta f = \vartheta \quad \text{on} \quad \partial\Omega' \\ &-\int_{\partial\Omega'} (f, \tau\hat{z}^i)\, d\omega = Y_i, \quad i = 1, \dots, B'_{n-r}, \end{aligned}$$

will be called a $\triangle$-*Dirichlet problem* or $\triangle_{\mathcal{D},\lambda,r}(\Omega', g, \xi, \vartheta, Y_i)$. If the topological constraint is not taken into account, we speak of a $\triangle$-Dirichlet problem as well. For this the abbreviation $\triangle_{\mathcal{D},\lambda,r}(\Omega', g, \xi, \vartheta)$ will be used.

b) We presuppose that $\xi^* \in C^{2,\lambda}_\nu(\partial\Omega')^{r-1}$, $\vartheta^* \in C^{1,\lambda}_\nu(\partial\Omega')^r$, $Z_i \in \mathbb{R}$ and

$$g^* \in \{g^* \in C^{0,\lambda}(\bar{\Omega}')^r \cap L^2(\Omega')^r : \int_{\Omega'} (g^*, z) dx = \int_{\partial\Omega'} (\vartheta^*, z) d\omega \text{ for all } z \in \mathcal{Z}(\Omega')^r\}.$$

Then the boundary problem

$$\begin{aligned} &\triangle f^* = g^* \quad \text{in} \quad \Omega' \\ &(\nu, f^*) = \xi^* \quad \text{and} \quad (\nu, df^*) = \vartheta^* \quad \text{on} \quad \partial\Omega' \\ &-\int_{\partial\Omega'} (f, \nu\hat{y}^i)\, d\omega = Z_i,\ i = 1, \dots, B'_r, \end{aligned}$$

will be called a $\triangle$-*Neumann problem* or $\triangle_{\mathcal{N},\lambda,r}(\Omega', g^*, \xi^*, \vartheta^*, Z_i)$. Without the topological constraint, we also speak of a $\triangle$-Neumann problem and use the abbreviation $\triangle_{\mathcal{N},\lambda,r}(\Omega', g^*, \xi^*, \vartheta^*)$.

Remark 2. a) The solutions of $\triangle$-Neumann problems are always solutions of $\triangle$-Dirichlet problems and vice versa.

b) The solutions of the homogeneous Dirichlet problems $\triangle_{\mathcal{D},\lambda,r}(\Omega, 0, 0, 0)$ are just Dirichlet fields $\mathcal{Y}(\Omega)^r$ and the solutions of the homogeneous

Neumann problems $\triangle_{\mathcal{N},\lambda,r}(\Omega,0,0,0)$ are Neumann fields $\mathcal{Z}(\Omega)^r$. This can easily be shown by Green's integral formulas. Therefore, we ascertain that the dimension of the linear space of solutions concerning the first problem is given by B_{n-r}, and those relating to the latter is given by B_r (cf. [5], [10]).

Theorem 3. *Let $r \in \mathbb{N}$, $r < n$, $\lambda \in (0,1)$ and $p \in (1,\infty)$. It is assumed that $g \in C^{0,\lambda}(\bar{\Omega})^r$, $\xi \in C^{2,\lambda}_\tau(\partial\Omega)^{r+1}$, $\vartheta \in C^{1,\lambda}_\tau(\partial\Omega)^r$, $g^* \in C^{0,\lambda}(\bar{\Omega})^r$, $\xi^* \in C^{2,\lambda}_\nu(\partial\Omega)^{r-1}$ and $\vartheta^* \in C^{1,\lambda}_\nu(\partial\Omega)^r$.*

a) *The problems $\triangle_{\mathcal{D},\lambda,r}(\Omega,g,\xi,\vartheta,Y_i)$ and $\triangle_{\mathcal{N},\lambda,r}(\Omega,g^*,\xi^*,\vartheta^*,Z_i)$ respectively have uniquely determined solutions $f \in C^{2,\lambda}(\bar{\Omega})^r$.*

b) *Any r-form $f \in C^{2,\lambda}(\bar{\Omega})^r$ satisfies the estimates*

$$\|f\|_{C^{2,\lambda}(\bar{\Omega})} \le$$
$$c(\|\triangle f\|_{C^{0,\lambda}(\bar{\Omega})} + \|\nu\wedge f\|_{C^{2,\lambda}(\partial\Omega)} + \|\nu\wedge\delta f\|_{C^{1,\lambda}(\partial\Omega)} + \sum_{i=1}^{B_{n-r}} |Y_i[f]|),$$
$$\|f\|_{C^{2,\lambda}(\bar{\Omega})} \le$$
$$c(\|\triangle f\|_{C^{0,\lambda}(\bar{\Omega})} + \|(\nu, f)\|_{C^{2,\lambda}(\partial\Omega)} + \|(\nu, df)\|_{C^{1,\lambda}(\partial\Omega)} + \sum_{i=1}^{B_r} |Z_i[f]|),$$
$$\|f\|_{W^{2,p}(\Omega)} \le$$
$$c(\|\triangle f\|_{L^p(\Omega)} + \|\nu\wedge f\|_{W^{2-\frac{1}{p},p}(\partial\Omega)} + \|\nu\wedge\delta f\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} + \sum_{i=1}^{B_{n-r}} |Y_i[f]|),$$
$$\|f\|_{W^{2,p}(\Omega)} \le$$
$$c(\|\triangle f\|_{L^p(\Omega)} + \|(\nu, f)\|_{W^{2-\frac{1}{p},p}(\partial\Omega)} + \|(\nu, df)\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} + \sum_{i=1}^{B_r} |Z_i[f]|),$$

where $c = c(n,r,\lambda,\Omega)$ and $c = c(n,r,p,\Omega)$ respectively are positive constants.

Proof. Let $\tilde{\xi} \in C^{2,\lambda}_\tau(\partial\Omega)^{r+1}$, $\tilde{\xi}^* \in C^{2,\lambda}_\nu(\partial\Omega)^{r-1}$, $\tilde{\vartheta} \in C^{1,\lambda}_\tau(\partial\Omega)^r$ and $\tilde{\vartheta}^* \in C^{1,\lambda}_\nu(\partial\Omega)^r$. The differential form $f_{\phi|\Omega}$, defined in Lemma 1, is a solution of $\triangle_{\mathcal{D},\lambda,r}(\Omega,0,\tilde{\xi},\tilde{\vartheta})$, if ϕ satisfies the integral equation

$$(\frac{1}{2}I + W_{2,1,\lambda,r})\phi = \begin{pmatrix} (\nu,\tilde{\xi}) \\ (-1)^{r-1}(\nu,\tilde{\vartheta}) \end{pmatrix}. \tag{22}$$

Furthermore, the differential form $f^*_{\phi^*}|_\Omega$ from Lemma 1 is a solution of $\triangle_{\mathcal{N},\lambda,r}(\Omega, 0, \tilde{\xi}^*, \tilde{\vartheta}^*)$, if ϕ^* satisfies the integral equation

$$(-\frac{1}{2}I + M_{2,1,\lambda,r})\phi^* = \begin{pmatrix} (-1)^r \tilde{\xi}^* \\ \tilde{\vartheta}^* \end{pmatrix}. \tag{23}$$

Supposing $\phi^* = (\psi^*, \varphi^*) \in \mathcal{N}(-\frac{1}{2}I + M_{2,1,\lambda,r})$ we get

$$(\frac{1}{2}I - U^*_{2,\lambda,r-1})\psi^* + V^*_{1,\lambda,r}\varphi^* = 0 \tag{24}$$

$$(\frac{1}{2}I - U^*_{1,\lambda,r})\varphi^* = 0. \tag{25}$$

Therefore, the form $f^*_{\phi^*}|_\Omega$ is a solution of $\triangle_{\mathcal{N},\lambda,r}(\Omega, 0, 0, 0)$ and thus a Neumann field. Referring to the equation

$$df^*_{\phi^*}(x) = \int_{\partial\Omega} (\Gamma(x, x') \wedge \varphi^*(x'))\, d\omega_{x'} \tag{26}$$

and [10], Satz 7.2, we conclude that the form $df^*_{\phi^*}|_{\hat{\Omega}}$ is a Dirichlet field. Moreover, the just-mentioned theorem yields $(\nu, df^*_{\phi^*}) = \varphi^*$. Because of that, the integrability condition for the Neumann field $f^*_{\phi^*}|_\Omega$,

$$\int_{\partial\Omega} ((\nu, df^*_{\phi^*}), z)\, d\omega = 0 \text{ for all } z \in \mathcal{Z}(\Omega)^r, \tag{27}$$

entails

$$\int_{\partial\Omega} (\varphi^*, z)\, d\omega = 0 \text{ for all } z \in \mathcal{Z}(\Omega)^r. \tag{28}$$

The only solution of the equations (25) and (28) is $\varphi^* = 0$. Let

$$\tilde{f}^* := f^*_{\phi^*}|_\Omega + z, \text{ where } z \in \mathcal{Z}(\Omega)^r \text{ and } Z_i[z] = Z_i[\tilde{f}^*],\ i = 1, \ldots, B_r. \tag{29}$$

The form $\tilde{f}^*$ is another solution of $\triangle_{\mathcal{N},\lambda,r}(\Omega, 0, 0, 0)$ and therefore a further Neumann field. Without loss of generality, we assumed that

$$Z_i[f^*_{\phi^*}|_\Omega] = -\int_{\partial\Omega} (f^*_{\phi^*}, \nu\hat{y}^i)\, d\omega = 0 \text{ for all } \hat{y}^i \in \mathcal{Y}(\hat{\Omega})^{r+1},\ i = 1, \ldots, B_r. \tag{30}$$

Consequently, the Neumann field $f^*_{\phi^*}|_\Omega$ vanishes (cf. [10]). Thus, the form $U^*_{2,\lambda,r-1}\psi^*$ vanishes, and the integral equation (24) gives $\psi^* = 0$. Quite analogously, we can deal with our second problem. For this, let

$$\tilde{f} := f_\phi|_\Omega + y, \text{ where } y \in \mathcal{Y}(\Omega)^r \text{ and } Y_i[y] = Y_i[\tilde{f}],\ i = 1, \ldots, B_{n-r}. \tag{31}$$

By means of Theorem 2, we get the estimates

$$\|z\|_{C^{2,\lambda}(\bar{\Omega})} \leq c \sum_{i=1}^{B_r} |Z_i[\tilde{f}^*]| \tag{32}$$

$$\|y\|_{C^{2,\lambda}(\bar{\Omega})} \leq c \sum_{i=1}^{B_{n-r}} |Y_i[\tilde{f}]| \tag{33}$$

and analogous results for the $W^{2,p}$-norms. From Fredholm theory we gather that

$$\begin{pmatrix} (\nu, \tilde{\xi}) \\ (-1)^{r-1}(\nu, \tilde{\vartheta}) \end{pmatrix} \in \mathcal{R}(\frac{1}{2}I + W_{0,0,2,r})$$

if and only if

$$\int_{\partial\Omega} [((\nu, \tilde{\xi}), \alpha^*) + (-1)^{r-1}((\nu, \tilde{\vartheta}), \beta^*)] \, d\omega = 0 \tag{34}$$

for all $\begin{pmatrix} \alpha^* \\ \beta^* \end{pmatrix} \in \mathcal{N}(\frac{1}{2}I + W_r^*)$. Obviously, the integral equation

$$(\frac{1}{2}I + W_r^*) \begin{pmatrix} \alpha^* \\ \beta^* \end{pmatrix} = 0 \tag{35}$$

is equivalent to the equation

$$(\frac{1}{2}I + M_r) \begin{pmatrix} \beta^* \\ \alpha^* \end{pmatrix} = 0. \tag{36}$$

This one yields the Neumann field $f^*_{(\beta^*,\alpha^*)}|_{\hat{\Omega}}$ and the Dirichlet field $df^*_{(\beta^*,\alpha^*)}|_{\Omega}$. The integrability condition

$$\int_{\partial\Omega} ((\nu, df^*_{(\beta^*,\alpha^*)}), \hat{z}) \, d\omega = 0 \quad \text{for all} \quad \hat{z} \in \mathcal{Z}(\hat{\Omega})^r \tag{37}$$

of the problem $\triangle_{\mathcal{N},\lambda,r}(\hat{\Omega}, 0, 0, 0)$ entails that α^* vanishes. Therefore, in the condition (34) one may replace $(\alpha^*, \beta^*) \in \mathcal{N}(\frac{1}{2}I + W_r^*)$ by $\alpha^* = 0$ and $\beta^* \in \mathcal{N}(\frac{1}{2}I - U^*_{k,\lambda,r-1})$. To each of these forms β^* exist $y \in \mathcal{Y}(\Omega)^r$, where $\beta^* = -(\nu, y)$ and vice versa. By means of $\nu \wedge \tilde{\vartheta} = 0$, we get

$$\langle(\nu, \tilde{\vartheta}), (\nu, y)\rangle = \langle\nu \wedge (\nu, \tilde{\vartheta}), y\rangle = \langle\tilde{\vartheta}, y\rangle, \tag{38}$$

and therefore

$$\langle(\nu, \tilde{\vartheta}), \beta^*\rangle = 0 \quad \text{for all } \beta^* \in \mathcal{N}(\frac{1}{2}I - U^*_{k,\lambda,r-1}) \tag{39}$$

$$\iff \int_{\partial\Omega} (\tilde{\vartheta}, y) \, d\omega = 0 \quad \text{for all } y \in \mathcal{Y}(\Omega)^r.$$

The latter equation corresponds to our integrability condition. As can easily be shown, our second problem is connected with this one by the Hodge mapping. Let

$$\mathcal{C} := \Big\{\phi = (\psi, \varphi) \in C_\nu^{2,\lambda}(\partial\Omega)^{r-1} \times C_\nu^{1,\lambda}(\partial\Omega)^r : \\ \|\phi\| = \|\psi\|_{C^{2,\lambda}(\partial\Omega)} + \|\varphi\|_{C^{1,\lambda}(\partial\Omega)}, Y_i[f_{\phi|\Omega}] = 0, i = 1, \dots, B_{n-r}\Big\}$$

$$\mathcal{C}^* := \Big\{\phi^* = (\psi^*, \varphi^*) \in C_\nu^{2,\lambda}(\partial\Omega)^{r-1} \times C_\nu^{1,\lambda}(\partial\Omega)^r : \\ \|\phi^*\| = \|\psi^*\|_{C^{2,\lambda}(\partial\Omega)} + \|\varphi^*\|_{C^{1,\lambda}(\partial\Omega)}, Z_i[f^*_{\phi^*|\Omega}] = 0, i = 1, \dots, B_r\Big\}.$$

We consider the operator $G_{\partial\Omega,k,\lambda}$, which is defined by

$$G_{\partial\Omega,k,\lambda} f(x) := \int_{\partial\Omega} G(x, x') f(x')\, d\omega_{x'}\,. \tag{40}$$

It belongs to $\mathcal{L}(C^{k,\lambda}(\partial\Omega), C^{k+1,\lambda}(\bar{\Omega}))$. The analogously explained operator $G_{\partial\Omega,k,p}$, which is defined on $W^{k-\frac{1}{p},p}(\partial\Omega)$, is in $\mathcal{L}(W^{k-\frac{1}{p},p}(\partial\Omega), W^{k+1,p}(\Omega))$. Consequently, the operator $W_{2,1,\lambda,r}$, seen as mapping in $\mathcal{L}(\mathcal{C}, \mathcal{C})$, is compact (cf. [8], Satz 5.9 and [2], proof of Satz E 2). We gather from the topological restriction, given by the definition of the Banach space $\mathcal{C}$, that

$$\mathcal{N}(\frac{1}{2}I + W_{2,1,\lambda,r}) = \{0\}. \tag{41}$$

Then, Banach's open mapping theorem gives the estimate

$$\|\psi\|_{C^{2,\lambda}(\partial\Omega)} + \|\varphi\|_{C^{1,\lambda}(\partial\Omega)} \le c(\|\tilde{\xi}\|_{C^{2,\lambda}(\partial\Omega)} + \|\tilde{\vartheta}\|_{C^{1,\lambda}(\partial\Omega)}). \tag{42}$$

The properties of the operator $G_{\partial\Omega,k,\lambda}$ yield

$$\|f_\phi\|_{C^{2,\lambda}(\bar{\Omega})} \le c(\|\psi\|_{C^{2,\lambda}(\partial\Omega)} + \|\varphi\|_{C^{1,\lambda}(\partial\Omega)}). \tag{43}$$

To deduce the equivalent L^p estimates, we define the Banach spaces

$$\mathcal{D} := \Big\{\phi = (\psi, \varphi) \in W_\nu^{2-\frac{1}{p},p}(\partial\Omega)^r \times W_\nu^{1-\frac{1}{p},p}(\partial\Omega)^{r-1} : \\ \|\phi\| = \|\psi\|_{W^{2-\frac{1}{p},p}(\partial\Omega)} + \|\varphi\|_{W^{1-\frac{1}{p},p}(\partial\Omega)}, Y_i[f_{\phi|\Omega}] = 0, i = 1, \dots, B_{n-r}\Big\},$$

$$\mathcal{D}^* := \Big\{\phi^* = (\psi^*, \varphi^*) \in W_\nu^{2-\frac{1}{p},p}(\partial\Omega)^r \times W_\nu^{1-\frac{1}{p},p}(\partial\Omega)^{r-1} : \\ \|\phi^*\| = \|\psi^*\|_{W^{2-\frac{1}{p},p}(\partial\Omega)} + \|\varphi^*\|_{W^{1-\frac{1}{p},p}(\partial\Omega)}, Z_i[f^*_{\phi^*|\Omega}] = 0, i = 1, \dots, B_r\Big\}$$

and refer to the properties of $G_{\partial\Omega,k,p}$. Therefore, it follows that

$$\|\psi\|_{W^{2-\frac{1}{p},p}(\partial\Omega)} + \|\varphi\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} \le c(\|\tilde{\xi}\|_{W^{2-\frac{1}{p},p}(\partial\Omega)} + \|\tilde{\vartheta}\|_{W^{1-\frac{1}{p},p}(\partial\Omega)}), \tag{44}$$

$$\|f_\phi\|_{W^{2,p}(\Omega)} \le c(\|\psi\|_{W^{2-\frac{1}{p},p}(\partial\Omega)} + \|\varphi\|_{W^{1-\frac{1}{p},p}(\partial\Omega)}). \tag{45}$$

As is commonly known, the form

$$u(x) := \int_\Omega G(x,x')g(x')\,dx'\,,\ \text{where } x \in \Omega \text{ and } g \in C^{0,\lambda}(\bar\Omega)^r, \tag{46}$$

is a solution of $\triangle u = g$. Moreover, the operator $G_{\Omega,0,\lambda}$ given by

$$G_{\Omega,0,\lambda} f(x) := \int_\Omega G(x,x')f(x')\,dx' \tag{47}$$

belongs to $\mathcal{L}(C^{0,\lambda}(\bar\Omega), C^{2,\lambda}(\bar\Omega))$. The analogously explained operator $G_{\Omega,0,p}$ defined on $L^p(\Omega)$ belongs to $\mathcal{L}(L^p(\Omega), W^{2,p}(\Omega))$. From $G_{\Omega,0,\lambda} \in \mathcal{L}(C^{0,\lambda}(\bar\Omega),$ $C^{2,\lambda}(\bar\Omega))$ it follows that

$$\|u\|_{C^{2,\lambda}(\bar\Omega)} \le c\,\|g\|_{C^{0,\lambda}(\bar\Omega)}. \tag{48}$$

Besides, we infer that

$$\|u|_{\partial\Omega}\|_{C^{2,\lambda}(\partial\Omega)} \le c\,\|g\|_{C^{0,\lambda}(\bar\Omega)} \text{ and } \|\delta u|_{\partial\Omega}\|_{C^{1,\lambda}(\partial\Omega)} \le c\,\|g\|_{C^{0,\lambda}(\bar\Omega)}. \tag{49}$$

For the equivalent L^p estimates, we refer to the properties of the operator $G_{\Omega,0,p}$ and to those of traces. Choosing $\tilde\xi$ and $\tilde\vartheta$ such that

$$\xi = \nu \wedge u|_{\partial\Omega} - \tilde\xi \ \text{ and } \ \vartheta = \nu \wedge \delta u|_{\partial\Omega} - \tilde\vartheta, \tag{50}$$

the form $f := u - \tilde f$ is a solution of the problem $\triangle_{\mathcal{D},\lambda,r}(\Omega, g, \xi, \vartheta)$. The integrability condition

$$\int_\Omega (g,y)\,dx = \int_{\partial\Omega} (\nu \wedge \delta f, y)\,d\omega \text{ for all } y \in \mathcal{Y}(\Omega)^r \tag{51}$$

delivers the condition for the homogeneous problem

$$0 = \int_{\partial\Omega} (\nu \wedge \delta \tilde f, y)\,d\omega \text{ for all } y \in \mathcal{Y}(\Omega)^r. \tag{52}$$

Let $p^{-1} + q^{-1} = 1$ and $i = 1, \ldots, B_{n-r}$. Clearly, we have

$$|Y_i[u]| \le c\|u\|_{L^p(\partial\Omega)}\|\hat z^i\|_{L^q(\partial\Omega)}, \tag{53}$$

$$|Y_i[u]| \le c\|u\|_{C^0(\partial\Omega)}\|\hat z^i\|_{C^0(\partial\Omega)}. \tag{54}$$

Furthermore, we get

$$\|u\|_{W^{2-\frac{1}{p},p}(\partial\Omega)} \le c\|u\|_{W^{2,p}(\Omega)} \tag{55}$$

and make use of the inequalities

$$\|u\|_{W^{2,p}(\Omega)} \le c\|g\|_{L^p(\Omega)}, \tag{56}$$

$$\|u\|_{C^{2,\lambda}(\Omega)} \le c\|g\|_{C^{0,\lambda}(\bar\Omega)}. \tag{57}$$

Finally, our second problem can be dealt with quite analogously. □

By means of Theorem 3, we are able to prove decompositions of the spaces $L^p(\Omega)^r$, where $p \in (1, \infty)$ and $r \in \mathbb{N}$, $r < n$. Our results are generalizations of the decompositions of Kodaira and Morrey (cf. [11], Theorem 7.7.7) from Hilbert to Banach spaces. Therefore, decompositions will be given, similar to those presented in [2], Satz F 1 and Satz F 2, yet without using the topological restriction of those. The following method of decomposing was already used in [11], Theorem 7.7.8 and [13].

Lemma 2. *Let $p \in (1, \infty)$ and $r \in \mathbb{N}$, $r < n$. Then*

$$\begin{aligned} L^p(\Omega)^r &= \{df : f \in W^{1,p}(\Omega)^{r-1} \textit{ with } \tau f = 0\} \\ &\oplus \{\delta g : g \in W^{1,p}(\Omega)^{r+1} \textit{ with } \nu g = 0\} \\ &\oplus \overline{\{h \in C^\infty(\bar{\Omega})^r : dh = 0 \textit{ and } \delta h = 0\}}^{\|\cdot\|_p}. \end{aligned}$$

Proof. This will be shown using a solution $\Phi_{\mathcal{D}}$ of the problem

$$\triangle_{\mathcal{D},\lambda,r}(\Omega, \psi - \sum_{i=1}^{B_{n-r}} a_i y^i, 0, 0)$$

and a solution $\Phi_{\mathcal{N}}$ of

$$\triangle_{\mathcal{N},\lambda,r}(\Omega, \psi - \sum_{i=1}^{B_r} b_i z^i, 0, 0),$$

where ψ is a form of $C^\infty(\bar{\Omega})^r$, $\{y^i\}_{i=1,\dots,B_{n-r}}$ denotes a basis of $\mathcal{Y}(\Omega)^r$ and $\{z^i\}_{i=1,\dots,B_r}$ denotes a basis of $\mathcal{Z}(\Omega)^r$. The values $a_i \in \mathbb{R}$, $b_i \in \mathbb{R}$ are chosen such that $\psi - \sum_{i=1}^{B_{n-r}} a_i y^i \in (\mathcal{Y}(\Omega)^r)^\perp$ and $\psi - \sum_{i=1}^{B_r} b_i z^i \in (\mathcal{Z}(\Omega)^r)^\perp$. By

$$\kappa := \psi - d\delta\Phi_{\mathcal{D}} - \delta d\Phi_{\mathcal{N}}$$

forms $\kappa \in C^\infty(\bar{\Omega})^r$ with $d\kappa = 0$ and $\delta\kappa = 0$ are defined. Finally, we make use of [2], Lemma F 2 and the a priori estimates given in Theorem 3. □

Remark 3. An analogous decomposition for $C^{0,\lambda}(\Omega)^r$, with $\lambda \in (0,1)$ and $r \in \mathbb{N} : r < n$, can easily be shown.

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