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EXISTENCE OF NON-TOPOLOGICAL SOLUTIONS FOR A NONLINEAR ELLIPTIC EQUATION ARISING FROM CHERN-SIMONS-HIGGS THEORY IN A GENERAL BACKGROUND METRIC

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Abstract. In this paper we show the existence of non-topological 0-vortex and 1-vortex solutions for a nonlinear elliptic equation arising from Chern-Simons-Higgs theory in a general background metric $(g_{\mu\nu}) = diag(1, -k(x), -k(x))$ with decay $k(x) = O(|x|^{-l})$ for some $l > 2$ at infinity.

1. Introduction and main result

Following the paper [10] (see also [6], [7]), we briefly explain the self-dual Chern-Simon-Higgs theory in a general background metric. The action in the (2+1)-dimensional relativistic Abelian Chern-Simons Higgs theory under the background metric $(g_{\mu\nu}) = diag(1, -k(x), -k(x))$ is defined as follows:

$$S = \int_{t_1}^{t_2} dt \int_{\mathbf{R}^2} dx \big\{ \mathcal{L}_{CS} + \sqrt{g}(D^\mu \phi)(D_\mu \phi)^* - \sqrt{g} V(|\phi|) \big\}$$

for $t_1 < t_2$, where $\mathcal{L}_{CS} = (\kappa/4)\epsilon^{\mu\nu\rho} F_{\mu\nu} A_\rho$ is the Chern-Simons term, ϕ is a complex scalar field, ϕ^* is a complex conjugate of ϕ, $(x_\mu) = (t, x_1, x_2)(\mu = 0, 1, 2)$, A_μ is a vector field, $D_\mu \phi = (\partial_\mu - iA_\mu)\phi$, $D^\mu \phi = g^{\mu\nu} D_\nu \phi$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $(g^{\mu\nu}) = (g_{\mu\nu})^{-1}$, $g = det(g_{\mu\nu})$, $\kappa > 0$ is a coupling constant, $k(x)$ is a positive function, and $\epsilon^{\mu\nu\rho}$ is a totally skew-symmetric tensor such that $\epsilon^{012} = 1$. Here we used the Einstein summation convention. Note that, by the gauge transformation $\phi \to \phi e^{-i\omega}$, $A_\mu \to A_\mu - \partial_\mu \omega$, $\mathcal{L}_{CS}$ changes

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into $\mathcal{L}_{CS} - \partial_\mu(\omega\epsilon^{\mu\nu\rho}\partial_\nu A_\rho)$ but the action S is invariant under certain decay assumption on $\omega\epsilon^{\mu\nu\rho}\partial_\nu A_\rho$. The Euler-Lagrange equations for this system are as follows:

$$\frac{1}{\sqrt{g}}D_\mu(\sqrt{g}g^{\mu\nu}D_\nu\phi) = -\frac{\partial V}{\partial\phi^*}, \quad F_{\mu\nu} = \frac{1}{\kappa}\epsilon_{\mu\nu\rho}J^\rho\sqrt{g},$$

where $\epsilon_{\mu\nu\rho} = g_{\mu\mu'}g_{\nu\nu'}g_{\rho\rho'}\epsilon^{\mu'\nu'\rho'}$ and $(J^\rho) = (\rho, \mathbf{j})$ is the matter current density with $J^\rho = i(\phi(D^\rho\phi)^* - \phi^*(D^\rho\phi))$. In this paper we are only concerned with static solutions. Then $F_{12} = (1/\kappa)\epsilon_{012}J^0\sqrt{g}$ yields the following Chern-Simons Gauss law:

$$\rho(x)\sqrt{g} = -2k(x)|\phi|^2A_0 = \kappa F_{12}. \tag{1}$$

Here $\rho(x)$ is the electronic charge density. Let Φ be the total magnetic flux, namely

$$\Phi = \int_{\mathbf{R}^2} F_{12}\,dx.$$

Then (1) implies

$$Q \equiv \int_{\mathbf{R}^2} \rho(x)\sqrt{g}\,dx = \kappa\Phi. \tag{2}$$

Taking into account this Chern-Simons Gauss law, the energy for static solutions is represented by the following(see e.g. [5]):

$$\begin{aligned} E &= \int_{\mathbf{R}^2}\left\{(D^\mu\phi)(D_\mu\phi)^* + V(|\phi|)\right\}\sqrt{g}\,dx. \\ &= \int_{\mathbf{R}^2}\left\{|D_1\phi|^2 + |D_2\phi|^2 + \frac{\kappa^2F_{12}^2}{4k(x)|\phi|^2} + k(x)V(|\phi|)\right\}dx. \end{aligned}$$

Now, we take the special Higgs potential $V(|\phi|) = (1/\kappa^2)|\phi|^2(|\phi|^2-1)^2$. Then we have the following Bogomolnyi decomposition under certain decay assumptions to some quantities:

$$\begin{aligned} E = &\int_{\mathbf{R}^2}\left\{|(D_1 \pm iD_2)\phi|^2 + \left|\frac{\kappa F_{12}}{2\sqrt{k(x)}\phi} \pm \frac{\sqrt{k(x)}}{\kappa}\phi^*(|\phi|^2-1)\right|^2\right\}dx \\ &\pm \int_{\mathbf{R}^2} F_{12}\,dx. \end{aligned} \tag{3}$$

If we fix the total magnetic flux Φ, it follows that (ϕ, A_μ) is the global minimizer of E if and only if (ϕ, A_μ) satisfies

$$(D_1 \pm iD_2)\phi = 0, \tag{4}$$

$$F_{12} \pm \frac{2k(x)}{\kappa^2}|\phi|^2(|\phi|^2 - 1) = 0. \tag{5}$$

As in self-dual models in many gauge theory, the study of this system can be reduced to the one of certain second order scalar nonlinear elliptic equation. For example, the reduction for 0-vortex solutions is done as follows. Writing $\phi = he^{i\omega} = e^{(1/2)u+i\omega}$ with smooth real-valued functions u and ω, (4) yields

$$A_i = -\partial_i\omega \pm \epsilon_{ij}\partial_j(\log h),$$

where ϵ_{ij} is the totally skew-symmetric tensor such that $\epsilon_{12} = 1$. Thus, $F_{12} = -\{\partial_1(\partial_1 h/h) + \partial_2(\partial_2 h/h)\} = -\Delta(\log h)$. Therefore, by (5), we obtain

$$\Delta u = \frac{4k(x)}{\kappa^2}e^u(e^u - 1). \tag{6}$$

We use the notation $\lambda = 4/\kappa^2$ throughout this paper.

Remark 1. The formula (5) in the case of plus sign yields

$$\Phi = \int_{\mathbf{R}^2} F_{12}\,dx = \frac{1}{2}\int_{\mathbf{R}^2}\left\{\frac{4k(x)}{\kappa^2}e^u(1 - e^u)\right\}dx. \tag{7}$$

Later, we will find solutions to (6) with the boundary condition $u(x) \to -\infty$ as $|x| \to \infty$ (i.e. non-topological solution) in the form $u = u_0 + w, u_0 = \log(1 + |x|^2)^{\alpha/2}$, where w satisfies $\int_{\mathbf{R}^2}\Delta w\,dx = 0$. Since $\int_{\mathbf{R}^2}\Delta u_0\,dx = 2\pi\alpha$, we have the relation $\Phi = -\alpha\pi$. So, to prescribe the magnetic flux Φ is equivalent to prescribe the number α. In a similar way, we have $\Phi = -\alpha\pi + 2N\pi$ for N-vortex solution.

From the energy finiteness of solutions, the following two type of boundary conditions can be considered: the one is $|\phi| \to 1$ as $|x| \to \infty$ (which is called a topological solution), the other is $|\phi| \to 0$ as $|x| \to \infty$ (which is called a non-topological solution), although another type of boundary condition might be considered if $\int_{\mathbf{R}^2}\sqrt{g}\,dx < +\infty$. Hence, for solution u to (6), we call u is topological if and only if $u(x) \to 0$ as $|x| \to \infty$; u is non-topological if and only if $u(x) \to -\infty$ as $|x| \to \infty$. We also say ϕ is a N-vortex solution $(N \geq 0)$ for prescribed vortices $\{p_i\}_{i=1}^l$, if

$$|\phi| \sim c_i|x - p_i|^{n_i}(|x - p_i| \to 0), c_i > 0, N = \sum_{i=1}^{l} n_i.$$

For a $N-$vortex solution, the system can be reduced into the study of

$$\Delta u = \frac{4k(x)}{\kappa^2} e^u(e^u - 1) + 4\pi \sum_{i=1}^{l} n_i \delta(x - p_i).$$

Furthermore, by using $w_0(x) = -\sum_{i=1}^{l} n_i \log(1 + |x - p_i|^{-2})$ which tends to zero at infinity, the problem can be reduced to find a solution v of

$$\Delta v = \frac{4k(x)}{\kappa^2} e^{w_0+v}(e^{w_0+v} - 1) + 4\sum_{i=1}^{l} \frac{n_i}{(1 + |x - p_i|^2)^2} \tag{8}$$

with the asymptotic behaviour $v(x) \to 0$ for a topological solution, $v(x) \to -\infty$ for a non-topological solution, respectively. We recall several known results on existence of non-topological and topological solutions. First, for the case $(g_{\mu\nu}) = diag(1, -1, -1)$(i.e., $k(x) \equiv 1$), existence of arbitrary N-vortex topological solution was shown by Wang [13], Spruck-Yang [12], and Φ, Q, E are all quantized: $\Phi = 2\pi N$, $Q = 2\pi N\kappa$, $E = 2\pi N$. They also proved the asymptotic behaviour $(|\phi|^2 - 1), |F_{12}| \sim O(e^{-(c/|\kappa|)|x|})$ at infinity, The existence of radially symmetric N-vortex non-topological solutions was proved by Spruck,-Yang [11], Chen-Hastings-McLeod-Yang, especially in [4] they showed that for every $\beta > 2N + 4, N \geq 0$, there exist a solution such that $|\phi|^2, |F_{12}| \sim O(|x|^{-\beta}), |D_j\phi|^2 = O(|x|^{-(2+\beta)})$ at infinity. In this case, we have $\Phi = 2\pi N + \pi\beta$, $Q = \kappa\Phi$, $E = \Phi$. Very recently, Chae and Imanuvilov [3] proved the existence of multivortex non-topological solutions. Next, Schiff [10] studied self-dual Chern-Simon-Higgs theory in a general background metric and proved that for the case $4k(|x|)/\kappa^2 = \beta^2/|x|^2$ with $\beta > 0$, $u(|x|) = -\log(\lambda|x|^\beta + 1)$ satisfies (6) for any $\lambda > 0$ (β-vortex non-topological solution). In [4] they also studied for certain radially symmetric $k(x) = k(|x|)$ the uniqueness of N- vortex topological radially symmetric solution for the prescribed N; existence of N-vortex non-topological radially symmetric solutions for certain range for β.

The purpose of this paper is to show existence of 0-vortex and 1-vortex non-topological solutions via variational method for certain general $k(x)$, not necessary radially symmetric.

In the following, we assume $k(x) \not\equiv 0$ and $k(x)$ is a non-negative Hölder continuous function. Our main result is as follows.

Theorem 1. *Suppose $k(x)$ satisfies $k(x) = O(1/|x|^l)$ as $|x| \to \infty$ for some $l > 2$.* (i) *Let $-4 < \alpha < \min(0, l - 4)$. Then there exists a constant $\lambda_0 > 0$*

such that for every $\lambda > \lambda_0$ (6) *has a solution* u *satisfying*

$$\lim_{|x|\to\infty}(u(x) - \alpha \log|x|) = C_0$$

for some constant C_0.

(ii) *Let* $p \in \mathbf{R}^2$, $w_0(x) = -\log(1 + |x-p|^{-2})$ *and* $-2 < \alpha < \min(0, l-4)$. *Then there exists a constant* $\lambda_0 > 0$ *such that for every* $\lambda > \lambda_0$ *we have a solution* v *to*

$$\Delta v = \lambda k(x) e^{w_0+v}(e^{w_0+v} - 1) + \frac{4}{(1 + |x-p|^2)^2} \tag{9}$$

satisfying

$$\lim_{|x|\to\infty}(v(x) - \alpha \log|x|) = C_1$$

for some constant C_1.

Corollary 1. *Suppose* $k(x)$ *satisfies* $k(x) = O(1/|x|^l)$ *as* $|x| \to \infty$ *for some* $l > 2$. *Let* $N = 0$ *or 1 and* $-2(2-N) < \alpha < \min(0, l-4)$. *Then there exist non-topological* N*-vortex solution with* $\Phi = 2\pi N - \pi\alpha$, $Q = \kappa(2\pi N - \pi\alpha)$, $E = 2\pi N - \pi\alpha$ *for sufficiently small coupling constant* κ.

Remark 2. Actually, we can prove the following: there exists a critical parameter $\lambda_c \geq (-8\pi(\alpha - 2N))/\int_{\mathbf{R}^2} k(x)\,dx$ such that there exists a solution for every $\lambda > \lambda_c$ and no solution for $0 < \lambda < \lambda_c$. Since we need to combine the result here with subsolution-supersolution method, we will treat it in elsewhere.

We show Theorem 1 via variational method based on several results on the weighted Sobolev spaces $W^2_{s,\delta}$ (see [9]). The equation (6) has some similarity to the Gauss curvature equation, but a difficult problem to determine the sign of the Lagrange multiplier occurs due to the nonlinearity in (6). We overcome this difficulty by using the idea of Caffarelli and Yang [1], in which they employed their idea to periodic problem.

Remark 3. In [8] Matsuda proved the following result for $k(x) = O(|x|^{-l}$ with $l > 2$ by using the Leray-Schauder's fixed point theorem. "Fix $\lambda > 0$ in (6). There exists sufficiently small constant $\alpha_0 > 0$ such that for every $0 > \alpha > -\alpha_0$ (6) has a solution satisfying the asymptotic behavior as in Theorem 1". Moreover, he also discussed the existence problem under certain mild condition for the decay on $k(x)$ (e.g. $\int_{\mathbf{R}^2} k(x)\,dx < +\infty$) via subsolution-supersolution method.

Existence problem for non-topological solutions to (8) with non-radially symmetric $k(x) = O(|x|^{-l})$ with $0 \le l \le 2$ is open, as far as I know. Theorem 1(i) and (ii) give the existence of non-topological 0-vortex and 1-vortex solutions, respectively. However, our method does not work for N-vortex solutions for $N \ge 2$, because of its relation to the sharp constant of Moser-Trudinger's inequality(Lemma 1).

Thoroughout this paper we use the notation $\int g\,dx = \int_{\mathbf{R}^2} g(x)\,dx$. $f \in C_0(\mathbf{R}^2)$ means that f is continuous on $\mathbf{R}^2$ and satisfies $f(x) \to 0$ as $|x| \to \infty$.

2. Preliminaries

In this section, we recall several known results on weighted Sobolev spaces and Moser-Trudinger's inequality and Poincaré's inequality adapted in this setting. The weighted Sobolev spaces $W^2_{s,\delta}$ are defined as the closure of C_0^∞ with respect to the norm:

$$\|u\|^2_{W^2_{s,\delta}} = \sum_{|\beta| \le s} \|(1+|x|)^{|\beta|+\delta} |D_x^\beta u|\|^2_{L^2}$$

We use the notation $L^2_\delta = W^2_{0,\delta}$. The following results are well-known (see [2], [9]).

(i) If $s' > s, \delta' > \delta$, then we have a compact embedding: $W^2_{s',\delta'} \subset W^2_{s,\delta}$.

(ii) If $s > 1, \delta > -1$, then $W^2_{s,\delta} \subset C_0(\mathbf{R}^2)$.

(iii) $u \in L^2_\delta, \Delta u \in L^2_{\delta+2}$ implies $u \in W^2_{2,\delta}$.

(iv) Let $-1 < \delta < 0$. Then $\Delta : W^2_{2,\delta} \to L^2_{\delta+2}$ is the bijection to the range $\{f \in L^2_{\delta+2}; \int f\,dx = 0\}$.

We also need the following two technical lemmas.

Lemma 1. *Let $d\mu = h(x)\,dx$ with $h(x) \sim (1+|x|)^{-(2+\epsilon)}$, $\epsilon > 0$, and $0 < \beta < \min(4\pi, 2\pi\epsilon)$. Then there exists a constant C such that*

$$\int e^{a|\nu|}\,d\mu \le C \exp\left(\frac{a^2}{4\beta}\|\nabla \nu\|^2_{L^2}\right)$$

for every $a > 0$ and $\nu \in \widetilde{\mathcal{H}}$. Here $\mathcal{H}$ is the closure of C_0^∞ with respect to $\|\nu\|^2_{\mathcal{H}} = \int |\nabla \nu|^2\,dx + \int \nu^2\,d\mu$ and $\widetilde{\mathcal{H}} = \{\nu \in \mathcal{H}; \int \nu\,d\mu = 0\}$.

Lemma 2. *Let $\eta > 0$. Then there exists a constant $C = C(\eta)$ such that*

$$\|\nu\|^2_{L^2_{-1-\eta}} \le C(\eta)\|\nabla \nu\|_{L^2}$$

holds for every $\nu \in \widetilde{\mathcal{H}}$.

We refer to [9] for the proof of these lemmas.

3. Proof of Theorem 1

In a similar way as in [1], we consider certain minimizing problem as follows. First we consider the case (i) and show the existence of non-topological 0-vortex solutions. At the end of this section, we just mention that how to modify the proof for the case (ii). Let $\alpha < 0$ and take $u_0(x) = \log(1+|x|^2)^{\alpha/2}$. Then

$$f(x) \equiv -\Delta u_0(x) = \frac{-2\alpha}{(1+r^2)^2}(\geq 0).$$

Consider the measure $d\mu = f(x)\,dx$. Then we have

$$\int d\mu = \int f(x)\,dx = -2\pi\alpha.$$

Let $\mathcal{H}$ be the closure of $C_0^\infty(\mathbf{R}^2)$ with respect to the norm

$$\|w\|_{\mathcal{H}}^2 = \{\int |\nabla w|^2\,dx + \int w^2\,d\mu < +\infty\} \text{ and } \widetilde{\mathcal{H}} = \{w \in \mathcal{H}; \int w\,d\mu = 0\}.$$

Now $u = w + u_0$ is a solution to (6) if and only if w satisfies

$$\Delta w + \lambda k(x) e^{u_0+w}(1 - e^{u_0+w}) = f.$$

We will find a solution w in the class $\mathcal{H}$. Decompose $w \in \mathcal{H}$ into $w = \nu + c, \nu \in \widetilde{\mathcal{H}}$ with a constant c. If we assume

$$\int \Delta w\,dx = 0,$$

then ν and c should satisfy

$$e^{2c}\int k(x)e^{2(u_0+\nu)}\,dx - e^c\int k(x)e^{u_0+\nu}\,dx + (-2\pi\alpha)/\lambda = 0.$$

Thus the following condition is necessary:

$$(\int k(x)e^{u_0+\nu}\,dx)^2 + \frac{8\pi\alpha}{\lambda}\int k(x)e^{2(u_0+\nu)}\,dx \geq 0. \tag{10}$$

Let $\mathcal{H}_* = \{\nu \in \widetilde{\mathcal{H}}; \nu$ satisfies the condition above$\}$. Define the constant $c = c(\nu)$ as follows:

$$e^{c(\nu)} = \frac{\int k(x)e^{u_0+\nu}\,dx + \sqrt{(\int k(x)e^{u_0+\nu}\,dx)^2 + \frac{8\pi\alpha}{\lambda}\int k(x)e^{2(u_0+\nu)}\,dx}}{2\int k(x)e^{2(u_0+\nu)}\,dx}. \tag{11}$$

Then consider the following minimizing problem:

$$\sigma = \inf_{\nu\in\mathcal{H}_*} I(\nu), \tag{12}$$

where

$$I(\nu) = \int \frac{1}{2}|\nabla \nu|^2 + \frac{\lambda}{2}k(x)e^{2(u_0+\nu+c(\nu))} - \lambda k(x)e^{u_0+\nu+c(\nu)}\,dx - 2\pi\alpha c(\nu). \quad (13)$$

We show Theorem 1 by proving the existence of the minimizer of this problem in an interior of $\mathcal{H}_*$.

Proof of Theorem 1 (i) (Step 1). We first show that for large λ the minimizer must be in an interior of $\mathcal{H}_*$.

Lemma 3. *There exists a constant* $M = M(\alpha)$ *such that*

$$I(\nu) \geq -(-2\pi\alpha)\log\lambda - M(\alpha), \nu \in \partial\mathcal{H}_*.$$

Proof. Let $\nu \in \partial\mathcal{H}_*$, namely $\nu \in \widetilde{\mathcal{H}}$ satisfies

$$(\int k(x)e^{u_0+\nu}\,dx)^2 = -\frac{8\pi\alpha}{\lambda}\int k(x)e^{2(u_0+\nu)}\,dx \geq 0.$$

Then

$$e^{c(\nu)} = \frac{-4\pi\alpha}{\lambda(\int k(x)e^{u_0+\nu}\,dx)}.$$

This implies $I(\nu) = 3\pi\alpha - 2\pi\alpha c(\nu) + \frac{1}{2}\|\nabla\nu\|^2$, where $\|\nabla\nu\|^2 = \int|\nabla\nu|^2\,dx$. On the otherhand, by the assumption $\alpha < l-4$, we have $k(x)e^{u_0(x)} \leq Cf(x)$ for some constant C. It follows from Lemma 1 that

$$\int k(x)e^{u_0+\nu}\,dx \leq C\int e^{\nu}\,d\mu \leq Ce^{\frac{\|\nabla\nu\|^2}{4\beta}}$$

for every $\beta \in (0, 4\pi)$. Since $c(\nu) = \log(-4\pi\alpha) - \log\lambda - \log(\int ke^{u_0+\nu}\,dx)$, there exists a constant M such that

$$c(\nu) \geq \log(-4\pi\alpha) - \log\lambda - \frac{\|\nabla\nu\|^2}{4\beta} - M.$$

Thus we obtain

$$I(\nu) \geq (\frac{1}{2} + \frac{\pi\alpha}{2\beta})\|\nabla\nu\|^2 - (-2\pi\alpha)\log\lambda - M(\alpha)$$

for some constant $M(\alpha)$ depending only on α. By the assumption $\alpha > -4$, we can take $\beta \in (0, 4\pi)$ so that $\beta > -\pi\alpha$. Therefore, we complete the proof. □

Next take λ_0 sufficiently large such that

$$(\int k(x)e^{u_0}\,dx)^2 + \frac{8\pi\alpha}{\lambda}\int k(x)e^{2u_0}\,dx > 0.$$

Then 0 belongs to the interior of $\mathcal{H}_*$ for every $\lambda \geq \lambda_0$.

Lemma 4. *There exist positive constants $C_j = C_j(\alpha), j = 1, 2$ such that $I(0) \le -C_1\lambda + C_2$ for $\lambda \ge \lambda_0$.*

Proof. By (11) we have

$$e^{c(0)} = \frac{\int k(x)e^{u_0}\,dx + \sqrt{(\int k(x)e^{u_0}\,dx)^2 + \frac{8\pi\alpha}{\lambda}\int k(x)e^{2u_0}\,dx}}{2\int k(x)e^{2u_0}\,dx}. \tag{14}$$

Hence

$$I(0) = -\frac{\lambda}{2}\int ke^{u_0}e^{c(0)}\,dx + \pi\alpha - 2\pi\alpha c(0).$$

Note that by (14) we have

$$\frac{\int ke^{u_0}\,dx}{\int ke^{2u_0}\,dx} > e^{c(0)} > \frac{\int ke^{u_0}\,dx}{2\int ke^{2u_0}\,dx}.$$

It follows that

$$I(0) < -\frac{\lambda}{4}\frac{(\int ke^{u_0}\,dx)^2}{\int ke^{2u_0}\,dx} + (-2\pi\alpha)\log\left(\frac{\int ke^{u_0}\,dx}{\int ke^{2u_0}\,dx}\right) + \pi\alpha$$

and complete the proof. □

Now, we have by Lemma 3 and 4, taking λ_0 sufficiently large if necessary again, $I(0) < -1 + I(\nu)$, $\nu \in \partial\mathcal{H}_*$ for $\lambda \ge \lambda_0$. Hence, we can conclude that if there exists a minimizer ν_0 to the minimizing problem, then ν_0 belongs to the interior of $\mathcal{H}_*$.

(**Step 2**). We show the existence of the minimizer for the minimizing problem (12).

Lemma 5. *There exist positive constants δ, C such that $I(\nu) \ge \delta\|\nabla\nu\|_{L^2}^2 - C$ for every $\nu \in \mathcal{H}_*$.*

Proof. Let $\nu \in \mathcal{H}_*$. By (10) we have

$$e^{c(\nu)} \ge \frac{\int ke^{u_0+\nu}\,dx}{2\int ke^{2(u_0+\nu)}\,dx} \ge \frac{-4\pi\alpha}{\lambda\int ke^{u_0+\nu}\,dx}.$$

This implies

$$\begin{aligned} I(\nu) \quad &\ge \quad \frac{1}{2}\|\nabla\nu\|^2 + \lambda\left(\frac{1}{2}\int ke^{2(u_0+\nu+c(\nu))}\,dx - \int ke^{u_0+\nu+c(\nu)}\,dx\right) \\ &+ \quad (-2\pi\alpha)\log(\frac{-4\pi\alpha}{\lambda}) - (-2\pi\alpha)\log(\int ke^{u_0+\nu}\,dx). \end{aligned}$$

By the assumption $l > 2$, we have $\int k\,dx < +\infty$ and

$$\int ke^{u_0+\nu+c(\nu)}\,dx \leq \frac{1}{2}\int ke^{2(u_0+\nu+c(\nu))}\,dx + \frac{1}{2}\int k\,dx.$$

Therefore, we obtain by using Lemma 1

$$I(\nu) \geq \frac{1}{2}(1+\frac{\pi\alpha}{\beta})\|\nabla\nu\|^2 - M(\alpha,\lambda) - \frac{1}{2}\lambda\int k\,dx$$

for some constant $M(\alpha,\lambda)$ depending only on α and λ. This completes the proof. $\square$

Once we obtain this estimate, Lemma 1 and the standard argument yield the existence of the minimizer ν_0. Here we note $\mathcal{H} \hookrightarrow W^2_{1,-1-\epsilon}$ for every $\epsilon < 1$ and $L^2(d\mu) = W^2_{0,-2}$. Hence we have the compactness embedding $\widetilde{\mathcal{H}} \hookrightarrow L^2(d\mu)$ by Lemma 2 and the results in section 2. By Step 1, ν_0 belongs to the interior of $\mathcal{H}_*$ for sufficiently large $\lambda > 0$. Since ν_0 belongs to the interior of $\mathcal{H}_*$, we have

$$\begin{aligned}
&\langle I'(\nu_0),\psi\rangle \\
&= \int \nabla\nu_0\cdot\nabla\psi\,dx + \int\{\lambda k(x)(e^{2(u_0+\nu_0+c(\nu_0))} - e^{u_0+\nu_0+c(\nu_0)}) + \gamma f\}\psi\,dx \\
&+ \left[\int\{\lambda k(x)(e^{2(u_0+\nu_0+c(\nu_0))} - e^{u_0+\nu_0+c(\nu_0)}) + f\}\,dx\right]\frac{dc(\nu_0+\epsilon\psi)}{d\epsilon}|_{\epsilon=0} = 0
\end{aligned}$$

for every $\psi \in \mathcal{H}$. Here $\gamma \in \mathbf{R}$ is the Lagrange multiplier. Using the definition of $c = c(\nu_0)$, we have $[\cdots] = 0$. This implies

$$\int \nabla\nu_0\cdot\nabla\psi\,dx + \int\{\lambda k(x)(e^{2(u_0+\nu_0+c(\nu_0))} - e^{u_0+\nu_0+c(\nu_0)}) + \gamma f\}\psi\,dx = 0$$

for every $\psi \in \mathcal{H}$. By taking $\psi = 1 \in \mathcal{H}$, we obtain $\gamma = 1$. Hence, we obtain

$$-\Delta\nu_0 + \lambda k(x)(e^{2(u_0+\nu_0+c(\nu_0))} - e^{u_0+\nu_0+c(\nu_0)}) + f = 0.$$

Let $g = \lambda k(x)(e^{2(u_0+\nu_0+c(\nu_0))} - e^{u_0+\nu_0+c(\nu_0)}) + f$. Since it is easy to show that $\int g\,dx = 0$ and $g \in L^2_{2+\delta}$ for $0 > \delta > -1$ by using Lemma 1, we have $\nu_0 \in W^2_{2,\delta} \subset C_0(\mathbf{R}^2)$. Therefore $\nu_0(x) \to 0$ as $|x| \to \infty$. Thus $U(x) = u_0 + \nu_0 + c(\nu_0)$ satisfies $-\Delta U + \lambda k(x)e^U(e^U - 1) = 0$ with $U(x) \sim \alpha\log|x|$ as $|x| \to \infty$. This conclude the desired result. $\square$

Proof of Theorem 1 (ii). We use the same functions u_0 and $f = -\Delta u_0$ as in the case (i). Writing $v = u_0 + w$, we reduce the problem (9) to find a solution w to

$$\Delta w + \lambda k(x) e^{u_0+w_0+w}(1 - e^{u_0+w_0+w}) = f + \frac{4}{(1+|x-p|^2)^2} \equiv \widetilde{f}.$$

Since $\int \widetilde{f}\,dx = -2\pi\alpha + 4\pi = -2\pi(\alpha - 2)$, we can prove Theorem 1(ii) in the same way as in the proof of Theorem 1(i) by using $d\mu = \widetilde{f}\,dx$ and replacing u_0 to $u_0 + w_0$ and α to $\alpha - 2$. The assumption $-2 < \alpha$ allows us to take $\beta \in (0, 4\pi)$ so that $\beta > -\pi(\alpha - 2)$. □

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