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GROUPS OF SCALINGS AND INVARIANT SETS FOR HIGHER-ORDER NONLINEAR EVOLUTION EQUATIONS

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Abstract. Consider a nonlinear m^{th} order evolution PDE

$$u_t = \mathbf{A}(u) \equiv A(x, u, u_1, \dots, u_m), \quad u = u(x,t), \quad (x,t) \in Q = (0,1) \times [0,1], \qquad (*)$$

where A is a C^∞ function and $u_t = \partial u/\partial t$, $u_i = \partial^i u/\partial x^i$. If (*) is invariant under a group of scalings with the infinitesimal generator $X = x\frac{\partial}{\partial x} + \mu t \frac{\partial}{\partial t}$ (μ is a constant "scaling order" of $\mathbf{A}$), then the PDE admits exact self-similar solutions depending on the single invariant variable $u(x,t) = \theta(\xi)$, $\xi = x/t^{1/\mu}$, where θ solves a nonlinear m^{th} order ODE associated with the PDE. We prove that when the operator $\mathbf{A}$ is composed of a finite sum of operators with different scaling orders, $\mathbf{A} = \sum \mathbf{A}_i$, and no group of scalings exists, the exact solutions can be constructed via the invariance of the set $S_0 = \{u : u_1 = F(u)/x\}$ of a contact first-order differential structure, where F is a smooth function to be determined. The time-evolution on S_0 is shown to be governed by a first-order dynamical system. We thus observe that besides scaling group properties, the invariance of S_0 specifies new sets of solutions described by first-order ODEs. The approach applies to a class of nonlinear parabolic equations of the second and of the fourth order.

1. Introduction and main result

Consider a nonlinear, one-dimensional, m^{th} order evolution partial differential equation (PDE) of the form

$$u_t = \mathbf{A}(u) \equiv A(x, u, u_1, \dots, u_m), \quad u = u(x,t), \quad (x,t) \in Q = (0,1) \times [0,1], \tag{1.1}$$

where $u_t = \partial u/\partial t$ is the time derivative and u_i denote the i^{th} partial derivatives $\partial^i u/\partial x^i$ with respect to the spatial variable x. The function A is assumed to be smooth, $A \in C^\infty((0,1) \times \mathbf{R}^{m+1})$. The solutions of (1.1) are

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supposed to be smooth, $u \in C^\infty(Q)$, and we assume that equation (1.1) with appropriate boundary conditions defines a smooth flow on C^∞. In what follows we will deal with invariant sets and exact, low-dimensional solutions of (1.1).

1.1. Group of scalings and scaling order for nonlinear operators. Equation (1.1) is invariant under the Lie group of scaling transformations

$$x^* = e^\varepsilon x, \quad t^* = e^{\mu\varepsilon} t, \tag{1.2}$$

where $\mu \in \mathbf{R}$ is a constant, with the infinitesimal generator

$$X = x\frac{\partial}{\partial x} + \mu t \frac{\partial}{\partial t}, \tag{1.3}$$

if the operator $\mathbf{A}$ satisfies the homogeneity condition:

$$A(sx, u, \frac{1}{s}u_1, \dots, \frac{1}{s^m}u_m) \equiv \left(\frac{1}{s}\right)^\mu A(x, u, u_1, \dots, u_m) \quad \text{for all } s > 0. \tag{1.4}$$

See [1] and [11]. From the invariance equation $Xu = 0$ we then obtain that (1.1) admits invariant (self-similar) solutions depending on a single variable (the invariant of the group):

$$u(x,t) = \theta(\xi), \quad \xi = x/t^{1/\mu}. \tag{1.5}$$

On the set of invariant solutions (1.5) the nonlinear PDE (1.1) reduces to an ordinary differential equation (ODE) for the function θ

$$A(\xi, \theta, \theta', \dots, \theta^{(m)}) + \frac{1}{\mu}\theta'\xi = 0. \tag{1.6}$$

We then say that $\mu = \mu(\mathbf{A})$ is the *scaling order* of the operator $\mathbf{A}$. For instance, for the operator with power-like nonlinearities in the derivatives,

$$\mathbf{A}(u) = \phi(u) x^\beta \Pi_{j=1}^m (u_j)^{\alpha_j}, \tag{1.7}$$

where ϕ is a smooth function, the scaling order is given by

$$\mu(\mathbf{A}) = \sum_{j=1}^m j\alpha_j - \beta. \tag{1.8}$$

1.2. Scaling index. We now consider a more general evolution equation of the form

$$u_t = \mathbf{B}(u) \equiv \sum_{i=1}^M \mathbf{A}_i(u), \tag{EE}$$

where the operators $\mathbf{A}_i$ are of the form (1.1) with the scaling orders μ_i in the sense of (1.4).

Definition. $I(\mathbf{B})$, the *scaling index* of operator $\mathbf{B}$, is the cardinal number of the set of the scaling orders of the summands $\mathbf{A}_i$:

$$I(\mathbf{B}) = \#\{\mu_i\} \le M. \tag{1.9}$$

The following properties of the composition of scaling operators are straightforward.

Proposition 1.1. *The scaling orders $\mu_i(\mathbf{A})$ and the scaling index $I(\mathbf{B})$ do not depend on any invertible point transformation of the dependent variable $u = R(v)$, $R \in C^\infty$.*

Proposition 1.2. *The group of scalings* (1.2) *is admitted by the equation (EE) if and only if $I(\mathbf{B}) = 1$ (and $\mu(\mathbf{B}) = \mu$).*

1.3. Algebraic differentiation. If the index of the operator satisfies $I(\mathbf{B}) > 1$, then (EE) does not admit any Lie group (1.2). In this case we begin our construction of invariant sets and exact solutions by introducing the set of functions

$$S_0 = \big\{u \in C^\infty : u_1 = \frac{1}{x}\, F(u)\big\}, \tag{1.10}$$

where F is a C^∞ function to be determined from the invariance condition

$$u(\cdot, 0) \in S_0 \Rightarrow u(\cdot, t) \in S_0 \ \text{ for } t \in (0, 1]. \tag{1.11}$$

The structure of the manifold (1.10) was used in [3, Section 3] in the construction of first-order sign-invariants for nonlinear parabolic equations, i.e., the operators preserving sign on the evolution orbits, and new exact solutions. The contact (first-order) structure of the semilinear equation

$$u_1 = \frac{1}{x}\, F(u)$$

includes the scaling invariant (1.5). This is easily seen by integration (see below).

Such a differential structure of S_0 makes it possible to introduce the following algebraic differentiation in the set, which finally reduces the PDE on S_0 to a system of ODEs.

Lemma 1.3. *Let $u \in S_0$. Then*

$$u_k = \frac{1}{x^k}\mathbf{G}_k(F), \quad k = 1, 2, \ldots, \tag{1.12}$$

where the ordinary differential operators $\mathbf{G}_k$ satisfy the following recursion relation: for any $F \in C^\infty$

$$\mathbf{G}_{k+1}(F) = F\frac{d}{du}\mathbf{G}_k(F) - k\mathbf{G}_k(F), \quad k = 1, \ldots; \quad \mathbf{G}_1(F) = F. \tag{1.13}$$

Proof. Differentiating (1.12), we obtain

$$u_{k+1} = -\frac{k}{x^{k+1}}\mathbf{G}_k(F) + \frac{1}{x^k}\frac{d}{du}\mathbf{G}_k(F)u_1 \equiv \frac{1}{x^{k+1}}\mathbf{G}_{k+1}(F), \tag{1.14}$$

since $u_1 = F(u)/x$ by definition (1.10). □

Let us present first quasilinear polynomial differential operators which are sufficient to deal with equations up to the fifth order:

$$\begin{aligned} &\mathbf{G}_2(F) = F(F' - 1), \qquad (1.15)\\ &\mathbf{G}_3(F) = F\mathbf{G}_2' - 2\mathbf{G}_2 = F[(FF')' - 3F' + 2],\\ &\mathbf{G}_4(F) = F\mathbf{G}_3' - 3\mathbf{G}_3 = F[(F(FF')')' - 6(FF')' + 11F' - 6],\\ &\mathbf{G}_5(F) = F[(F(F(FF')')')' - 10(F(FF')')' + 35(FF')' - 50F' + 24], \text{ etc.} \end{aligned}$$

1.4. Main result: invariant sets for $I(\mathbf{B}) > 1$. Using the rule (1.12) and the scaling properties (1.4) with $\mu = \mu_i$ of operators $\mathbf{A}_i$, we conclude that on S_0

$$\mathbf{B}(u) = \sum_{i=1}^{M} A_i(x, u, \frac{1}{x}\mathbf{G}_1, \dots, \frac{1}{x^m}\mathbf{G}_m) \equiv \sum_{i=1}^{M} x^{-\mu_i} A_i(1, u, \mathbf{G}_1, \dots, \mathbf{G}_m). \tag{1.16}$$

Let $\lambda_1 < \lambda_2 < \cdots < \lambda_I$ be all the distinct scaling orders of the operators $\{\mathbf{A}_i\}$ so that (1.16) takes the form

$$\mathbf{B}(u) = \sum_{j=1}^{I} x^{-\lambda_j} F\mathbf{C}_j(F), \tag{1.17}$$

where the operators $\mathbf{C}_j$ are given by

$$\mathbf{C}_j(F) = \frac{1}{F}\sum_{\{i:\mu_i=\lambda_j\}} \mathbf{A}_i(1, u, \mathbf{G}_1(F), \dots, \mathbf{G}_m(F)), \quad j = 1, \dots, I. \tag{1.18}$$

We now state the main result.

Theorem 1.4. *The set S_0 is invariant if and only if the function $F(u)$ satisfies the system of $I(\mathbf{B})$ ODEs*

$$F\frac{d}{du}\mathbf{C}_j(F) = \lambda_j\mathbf{C}_j(F), \quad j = 1, \dots, I. \tag{1.19}$$

Then the solutions $u(x,t)$ on S_0 take the form

$$v \equiv \int_1^u \frac{dz}{F(z)} = \ln x + H(t), \tag{1.20}$$

where $H(t)$ solves the ODE

$$H' = \sum_{j=1}^{I} d_j e^{\lambda_j H}, \quad t \in (0,1]; \quad d_j = \mathbf{C}_j(F)|_{u=1}, \tag{1.21}$$

which is the evolution equation (EE) on S_0.

The first-order dynamical system (1.21) is always integrated in quadratures.

Proof. In view of (1.20) we have

$$u_t = F(u)H'(t). \tag{1.22}$$

Substituting (1.22) and (1.17) into equation (EE) restricted to S_0, we obtain the equation

$$H'(t) = \sum_{j=1}^{I} x^{-\lambda_j} \mathbf{C}_j(F(u)) \equiv D(x,t). \tag{1.23}$$

Therefore, $D(x,t)$ does not depend on x whence

$$D_x = \sum_{j=1}^{I} \big[F \frac{d}{du} \mathbf{C}_j(F) - \lambda_j \mathbf{C}_j(F)\big] x^{-(\lambda_j+1)} \equiv 0. \tag{1.24}$$

Since the functions $\{x^{-(\lambda_j+1)}\}$ are linearly independent in $\mathbf{R}^I$ we then arrive at the system (1.19), which means that there holds

$$\frac{d}{dv} \mathbf{C}_j(F) \equiv F \frac{d}{du} \mathbf{C}_j(F) = \lambda_j \mathbf{C}_j(F). \tag{1.25}$$

Hence

$$\mathbf{C}_j(F) = d_j e^{\lambda_j v}. \tag{1.26}$$

Substituting (1.26), with $v = \ln x + H$ given by (1.20), into (1.23), we obtain the ODE (1.21). □

1.5. Group of scalings if $I = 1$. In the case $I(\mathbf{B}) = 1$, when the group of scalings exists, one can solve (1.21) explicitly. We set $\lambda_1 = \mu$, $d_1 = d$. Then from the equation with a single exponential function

$$H' = de^{\mu H},$$

one obtains

$$H(t) = -\frac{1}{\mu} \ln t + \text{const}. \tag{1.27}$$

Substituting (1.27) into (1.20) we have that

$$\int_1^u \frac{dz}{F(z)} = \ln \frac{x}{t^{1/\mu}} + \text{const}; \tag{1.28}$$

i.e., u depends on the single scaling group invariant (1.5).

We thus see that the solutions on the invariant set S_0 (in fact, they belong to a one-dimensional subspace invariant under a nonlinear operator; see Section 4) "bifurcate" at $I = 1$ from the classical self-similar solutions invariant under the Lie group of scaling transformations.

In the subsequent sections we apply the above construction to a general parabolic equation of the second (Section 2) and of the fourth order (Section 3). Results on the third-order equations of the Korteveg-de Vries type

$$u_t = \mathbf{B}_3(u) \equiv \phi(u)u_{xxx} + \psi(u)u_x \tag{1.29}$$

(it is the KdV equation if $\phi = 1$ and $\psi(u) = u$) are presented in [5].

In the last section, Section 4, we give further generalizations of the method and establish connections with the method of linear invariant subspaces for nonlinear operators.

2. Nonlinear parabolic equations

2.1. Quasilinear heat equation. Consider a general reaction-diffusion equation of parabolic type

$$u_t = \mathbf{B}_2(u) \equiv \phi(u)u_{xx} + \psi(u)(u_x)^2 + f(u) \equiv \mathbf{A}_1(u) + \mathbf{A}_2(u) + \mathbf{A}_3(u), \tag{2.1}$$

which contains three smooth functions $\phi(u) \geq 0$, $\psi(u)$ and $f(u)$. By a point transformation $u = R(v)$, $R \in C^\infty$, it reduces to a quasilinear equation of the divergent form from the combustion or filtration theory $v_t = (\varphi(v))_{xx} + q(v)$ with two arbitrary, smooth functions φ and q, $\varphi' \geq 0$.

The scaling orders of the three operators in (2.1) are $\mu_1 = \mu_2 = 2$ and $\mu_3 = 0$, so that the scaling index is $I(\mathbf{B}_2) = 2$. Substituting the differential rule (1.12), one can calculate that on S_0

$$\mathbf{B}_2(u) = F\Big\{\frac{1}{x^2}[\phi(F'-1) + \psi F] + \frac{f}{F}\Big\} \equiv F\big[\mathbf{C}_1(F) + \frac{1}{x^2}\mathbf{C}_2(F)\big], \tag{2.2}$$

so that $\lambda_1 = 0$ and $\lambda_2 = 2$. From Theorem 1.4 we have that the invariant conditions (1.19) take the form

$$F\Big(\frac{f}{F}\Big)' = 0, \quad F[\phi(F'-1) + \psi F]' = 2[\phi(F'-1) + \psi F], \tag{2.3}$$

which are equivalent to the system

$$f = \nu F, \quad FF'' - 2F' + 2 + \frac{\phi'}{\phi}F(F'-1) + \frac{\psi}{\phi}F(F'-2) + \frac{\psi'}{\phi}F^2 = 0, \tag{2.4}$$

where $\nu \in \mathbf{R}$ is an arbitrary constant. Equation (2.1) on S_0 reduces to the following ODE for the function $H(t)$ in the exact solution representation (1.20):

$$H' = d_1 + d_2 e^{2H}, \quad t \in (0,1], \tag{2.5}$$

where

$$d_1 = \Big(\frac{f}{F}\Big)(1) = \nu, \quad d_2 = [\phi(F'-1) + \psi F](1). \tag{2.6}$$

It is worth mentioning that (2.3) is a system of two equations with four arbitrary unknown functions ϕ, ψ, f and F. Therefore the invariant sets and exact solutions are constructed for an infinite-dimensional family of the parabolic equations (2.1).

2.2. Radial N-dimensional equation. The quasilinear heat equation in $\mathbf{R}^N$

$$u_t = \mathbf{B}_2(u) \equiv \phi(u)\Delta u + \psi(u)|\nabla u|^2 + f(u) \equiv \mathbf{A}_1(u) + \mathbf{A}_2(u) + \mathbf{A}_3(u)$$

(which reduces to the divergent form $v_t = \Delta\varphi(v) + q(v)$) in the radial case when

$$\Delta u = u_{xx} + \frac{N-1}{x}u_x, \quad |\nabla u| = u_x \geq 0 \ \ (x > 0),$$

exhibits the same scaling orders as (2.1). Then in (2.2) we have that

$$\mathbf{C}_1(F) = \phi(F' + N - 2) + \psi F.$$

Therefore we replace the second equation in (2.4) by the following one:

$$FF'' - 2F' - 2(N-2) + \frac{\phi'}{\phi}F(F' + N - 2) + \frac{\psi}{\phi}F(F'-2) + \frac{\psi'}{\phi}F^2 = 0.$$

See also examples of exact solutions for different types of nonlinearities in [3, Section 3].

2.3. Equation with gradient-dependent diffusivity. The quasilinear equation

$$u_t = \mathbf{B}_2(u) \equiv \phi(u)(u_x)^\alpha u_{xx} + f(u)(u_x)^\beta \equiv \mathbf{A}_1(u) + \mathbf{A}_2(u) \tag{2.7}$$

is known as a model of curve-shortening flows and in the mechanics of non-Newtonian liquids. We have $\mu_1 = \alpha + 2$ and $\mu_2 = \beta$. If $\alpha + 2 = \beta$ then (2.7) is invariant under the scaling group (1.2).

Let $\alpha+2 \neq \beta$. Then on S_0

$$\mathbf{B}_2(u) = F\big[\frac{1}{x^{\alpha+2}}\phi F^{\alpha}(F'-1) + \frac{1}{x^{\beta}}F^{\beta-1}f\big]. \tag{2.8}$$

Therefore the invariant conditions are

$$F[\phi F^{\alpha}(F'-1)]' = (\alpha+2)\phi F^{\alpha}(F'-1), \quad F[F^{\beta-1}f]' = \beta F^{\beta-1}f, \tag{2.9}$$

or, which is the same,

$$\begin{aligned} &FF'' + \alpha(F')^2 - 2(\alpha+1)F' + (\alpha+2) + \frac{\phi'}{\phi}F(F'-1) = 0,\\ &Ff' + [(\beta-1)F' - \beta]f = 0. \end{aligned} \tag{2.10}$$

The ODE on the function H reads

$$H' = d_1 e^{(\alpha+2)H} + d_2 e^{\beta H}, \tag{2.11}$$

with $d_1 = [\phi F^{\alpha}(F'-1)](1)$, $d_2 = (F^{\beta-1}f)(1)$.

2.4. Fully nonlinear equation. We now consider an example of a fully nonlinear parabolic equation,

$$u_t = \mathbf{B}_2(u) \equiv \phi(u)(u_x)^{\alpha}(u_{xx})^{\gamma} + f(u)(u_x)^{\beta}. \tag{2.12}$$

Then $\mu_1 = \alpha + 2\gamma$ and $\mu_2 = \beta$, and we assume that $\mu_1 \neq \mu_2$, so that $I = 2$. On S_0

$$\mathbf{B}_2(u) = F\big[\frac{1}{x^{\alpha+2}}\gamma\phi F^{\alpha+\gamma-1}(F'-1)^{\gamma} + \frac{1}{x^{\beta}}F^{\beta-1}f\big].$$

Therefore in the invariant conditions (2.9) we replace the first equation by

$$F[\phi F^{\alpha+\gamma-1}(F'-1)^{\gamma}]' = (\alpha+2\gamma)\phi F^{\alpha+\gamma-1}(F'-1)^{\gamma},$$

and the ODE for $H(t)$ becomes

$$H' = d_1 e^{(\alpha+2\gamma)H} + d_2 e^{\beta H}, \quad d_1 = [\phi F^{\alpha+\gamma-1}(F'-1)^{\gamma}](1).$$

3. Fourth-order equations

Consider a fourth-order evolution equation of the form

$$u_t = \mathbf{B}_4(u) \equiv \mathbf{B}_2(u) + \rho(u)u_{xxxx}, \tag{3.1}$$

where $\mathbf{B}_2$ is the second-order operator given in (2.1). If $\phi = \rho \equiv -1$, $\psi \equiv 1$ and $f = 0$, (3.1) becomes the Kuramoto-Sivashinskii equation from the theory of flame propagation, which is a fourth-order semilinear parabolic equation.

The analysis of (3.1) is similar to that for the second-order parabolic equation (2.1). The fourth-order term in (3.1) exhibits the scaling order

$\mu_4 = 4$, and $I(\mathbf{B}_4) = 3$, so that on S_0 we add to the right-hand side of (2.2) an extra term of the form

$$\mathbf{B}_4(u) = \mathbf{B}_2(u) + F\big[\frac{1}{x^4}\rho \mathbf{C}_3(F)\big], \quad \mathbf{C}_3(F) = \frac{1}{F}\mathbf{G}_4(F), \tag{3.2}$$

where $\mathbf{G}_4$ is as given in (1.15). Therefore we add to the system (2.4) the third equation

$$F(\rho \mathbf{C}_3)' = 4\rho \mathbf{C}_3, \tag{3.3}$$

and the ODE takes the form

$$H' = d_1 + d_2 e^{2H} + d_3 e^{4H}, \quad d_3 = [\rho \mathbf{C}_3(F)](1). \tag{3.4}$$

4. Generalizations and invariant subspaces

4.1. Invariant sets for operators of even orders. It was shown in [3, Example 4.3] that under certain assumptions on the coefficients, the quasilinear parabolic equation (2.1) admits solutions on the invariant subspace $W = \text{Span}\{\cos x\}$ (or $W = \text{Span}\{\cosh x\}$). Let us prove that this corresponds to the invariant set of a different structure:

$$S_1 = \{u : u_1 = \tan x \, F(u)\}. \tag{4.1}$$

The algebraic differentiation in S_1 is now different from that in S_0. Namely, setting for convenience $w(x) = \frac{1}{\cos^2 x}$, we have

$$u_2 = wF(F' + 1) - FF' \equiv F(w\mathbf{P}_{21} + \mathbf{P}_{20}). \tag{4.2}$$

Denoting $\mathbf{Q}(F) = F[(FF')' + 3F' + 2]$, we obtain

$$u_4 = F(w^2 \mathbf{P}_{42} + w\mathbf{P}_{41} + \mathbf{P}_{40}), \tag{4.3}$$

where

$$\begin{aligned} \mathbf{P}_{42} &= 3\mathbf{Q} + (F\mathbf{Q})' \quad ((\cdot)' = d/du), \\ \mathbf{P}_{41} &= -2\mathbf{Q} - (F\mathbf{Q})' - (FF')' - (F(FF')')', \\ \mathbf{P}_{40} &= (F(FF')')'. \end{aligned} \tag{4.4}$$

We claim that the even derivatives u_{2k} belong to a $(k+1)$-dimensional linear subspace:

$$u_{2k} \in \text{Span}\{1, w, \ldots, w^k\}. \tag{4.5}$$

Therefore the analysis on S_1 is more convenient for the equations of even orders (or for those which contain operators of even scaling orders).

Example 4.1. Consider the fourth-order parabolic equation

$$u_t = \mathbf{B}_4(u) \equiv \phi(u)u_2 + \psi(u)(u_1)^2 + \rho(u)u_4. \tag{4.6}$$

Then since for $u \in S_1$, $(u_1)^2 = wF^2 - F^2$, we have that in S_1

$$\begin{aligned}\mathbf{B}_4(u) &= F[w^2(\rho \mathbf{P}_{42}) + w(\rho \mathbf{P}_{41} + \phi \mathbf{P}_{21} + \psi F) + \rho \mathbf{P}_{40} + \phi \mathbf{P}_{20} - \psi F] \\ &\equiv F(w^2\Phi_2 + w\Phi_1 + \Phi_0) \in \mathrm{Span}\{1, w, w^2\}. \end{aligned} \tag{4.7}$$

The index of $\mathbf{B}_4$ (as usual, the number of linearly independent terms in (4.7)) is equal to 3. Substituting the general solution of $u_1 = \tan x\, F(u)$,

$$v \equiv \int_1^u \frac{dz}{F(z)} = \ln \cos\, x + H(t) \quad (\cos x > 0), \tag{4.8}$$

into (4.6) and using (4.7) we obtain

$$H'(t) = w^2(x)\Phi_2 + w(x)\Phi_1 + \Phi_0. \tag{4.9}$$

In view of the linear independence, we then conclude that the functions $\Phi_2(v)$, $\Phi_1(v)$ and $\Phi_0(v)$ must be exponential ones, $d_2 e^{4v}$, $d_1 e^{2v}$ and d_0 respectively, whence the system of the three equations on F:

$$F\Phi_2' = 4\Phi_2, \quad F\Phi_1' = 2\Phi_1, \quad F\Phi_0' = 0. \tag{4.10}$$

Then (4.9) becomes

$$H' = d_2 e^{4H} + d_1 e^{2H} + d_0, \tag{4.11}$$

with the coefficients

$$d_2 = \Phi_2|_{u=1}, \quad d_1 = \Phi_1|_{u=1}, \quad d_0 = \Phi_0|_{u=1}. \tag{4.12}$$

In fact, (4.10) is a system of three differential equations with four arbitrary functions. Therefore, there exists in infinite-dimensional family of equations (4.6) which admit exact solutions. Observe that if we add to the right-hand side the operator $\kappa(u)u_1u_3$ with the odd derivatives, then it does not change the index of the operator since on S_1, $u_1u_3 \in \mathrm{Span}\{1, w, w^2\}$. A similar analysis is performed for the set driven by the constraint $u_1 = \tanh x\, F(u)$.

Exact solutions on S_1 can be constructed for the third-order equations of the KdV type (1.29); the corresponding solutions on S_0 can be found in [5].

Remark on completeness. Consider the semilinear heat equation

$$u_t = u_{xx} + f(u), \tag{4.13}$$

and the set with the separate-variable constraint $u_x = g(x)F(u)$. It is proved in [6, Theorem 3.2] that the only functions $g(x)$ which can generate invariant sets for nontrivial semilinear equations are functions $1/x$, $\tan x$ and $\tanh x$, which implies completeness of the analysis.

4.2. Operators representation. It can be shown that the set of solutions (1.20) (or (4.8)) can be treated as a one-dimensional subspace invariant under a certain nonlinear operator. The function $C(t) = \log H(t)$ then plays the role of the expansion coefficient in the subspace $W = \text{Span}\{x\}$ (or $W = \text{Span}\{\cos x\}$). The concept of linear subspaces admitted by a nonlinear operator turned out to be useful in the study of the one-dimensional and multidimensional nonlinear evolution equations; see [3]-[4], [7], [10] and references therein. They apply to nonlinear difference operators [6], [10]. It was shown that given a nonlinear operator, the problem of determination of the linear invariant subspaces reduces to a difficult nonlinear eigenvalue problem [9]. On the other hand, for ODEs operators of finite orders, given a linear subspace, the class of operators admitted it can be described in terms of Lie-Bäcklund symmetries of the linear ODE specifying the subspace, [13] and [14]. In the present paper we thus give a new representation of nonlinear operators admitting one-dimensional invariant subspaces of a special structure in the case $I(\mathbf{B}) > 1$. In the limit case $I(\mathbf{B}) = 1$ of the minimally possible scaling index, the *nonlinear invariance* property (linear subspaces invariant under nonlinear operators) gives the same exact invariant solutions as the classical approach based on the concept of invariance under *groups of scalings*, which is essentially a linear one in terms of the construction of infinitesimal generators. We thus observe that there exists a certain extension (for a wider class of evolutions equations) of applications of the classical Lie group analysis, but in a nonlinear manner.

The procedure of deriving low-dimensional (invariant) solutions of nonlinear PDEs via invariance under a Lie group of transformations is well established [1], [11]. It is therefore important to check if other nonscaling group structures of invariant solutions could admit a "nonlinear" extension via invariant linear subspaces or sets, and that the nonlinear invariant solutions could bifurcate at $I = 1$ from the corresponding group-invariant solutions.

4.3. Higher-order evolution. The method of construction of the invariant set S_0 (or S_1) admits a similar extension to nonlinear equations with a higher-order evolution, for example to hyperbolic-like equations $u_{tt} = \mathbf{A}(u)$, $(x, t) \in Q$.

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