

## SYMMETRY AND NONEXISTENCE RESULTS FOR EMDEN-FOWLER EQUATIONS IN CONES

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**Abstract.** The purpose of this paper is to state some qualitative properties of the solutions to the Emden-Fowler equation  $\Delta u + r^\sigma u^p = 0$  in a cone with Dirichlet boundary conditions. Namely one can show that every solution has the same symmetry as the cone in some sense; furthermore it is possible to extend the nonexistence results for regular solutions to this equation already stated by C. Bandle and M. Essen in [2]. For this one needs to establish some asymptotics for the solutions as  $r \rightarrow 0$  or  $r \rightarrow \infty$ , relying on methods used by Veron in [25] for similar equations, but in different geometries, and then use a special form of the moving-planes method on a sphere in the spirit of [22].

### 1. INTRODUCTION

The so-called Emden-Fowler equation

$$\Delta u + r^\sigma u^p = 0, \quad u(x) > 0 \quad \text{in } \Omega \quad (1.1)$$

where  $\Omega$  is a domain in  $\mathbb{R}^N$ ,  $r = |x|$ ,  $N \geq 3$  and  $\sigma \in \mathbb{R}$ ,  $p > 1$ , has attracted the attention of many authors because of its physical relevance in various contexts (cf. Wong [26]). One is mainly interested in establishing qualitative properties of solutions to (1.1), such as existence, asymptotic behaviour, and symmetry. The study of the radial case, which was initiated by Fowler in his classical paper [14], has been completed by Bandle and Marcus in [4].

When  $\Omega = \mathbb{R}^N$ ,  $\mathbb{R}^N \setminus \{0\}$ ,  $B_R \setminus \{0\}$  or  ${}^c B_R$  the situation is now well understood thanks to the works of Gidas, Spruck and Veron [15], [25]. They established very general results concerning asymptotics of solutions to (1.1) near an isolated singularity or at infinity, and gave nonexistence results.

When  $\Omega = \mathbb{R}^N$  and  $\sigma = 0$ , the radial symmetry of all solutions to (1.1) has been proved under various additional assumptions on  $u$  at infinity in the critical  $p = (N + 2)/(N - 2)$ , [11], or supercritical case  $(N + 2)/(N - 2) <$

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$p < (N + 1)/(N - 3)$ ; cf. [27], [28]. When  $\Omega$  is the half-space  $\{x_N > 0\}$ , monotonicity and symmetry results are also available, symmetry in this context meaning  $u = u(x_N)$  (cf. [12], [6]).

The case where  $\Omega$  is a cone is of particular interest; it has been considered for equation (1.1) and its corresponding evolution counterpart  $\partial_t u = \Delta u + r^\sigma u^p$  by Bandle, Essen, Levine, Meyer and others [1], [2], [3], [21]. Several asymptotics and nonexistence results are given there. On the other hand, no symmetry statement seems to be known for (1.1) in cones with a Dirichlet boundary condition.

Our purpose is thus to state symmetry results for equation (1.1), and to extend the nonexistence theorems in [1], [2]. As is usual in the unbounded domain setting, one has to study carefully the behaviour of  $u$  near infinity, and near zero too, since we shall allow it to be a singular point for  $u$ . This will be done through an appropriate generalization of asymptotics given in [2]. Actually, we will consider a slightly more general class of problems than (1.1), namely

$$\Delta u + \frac{\lambda}{r^2} u + r^\sigma u^p = 0, \quad u > 0 \quad \text{in } \Omega \quad (1.2)$$

where  $\lambda \geq 0$ ,  $\sigma \in \mathbb{R}$  and  $p > 1$ , since essentially the same devices will work for this equation as already noticed in [21], [25].

## 2. MAIN RESULTS

Let  $N \geq 3$ , and  $\omega$  be a domain on  $S^{N-1}$  (the unit sphere in  $\mathbb{R}^N$ ) having a piecewise- $C^1$  boundary. Let  $(r, \theta)$  denote the polar coordinates in  $\mathbb{R}^N \setminus \{0\}$  ( $r = |x|$ ,  $\theta = x|x|^{-1} \in S^{N-1}$ ). We define a cone in  $\mathbb{R}^N$  to be a set of the type  $\Omega = \{x \in \mathbb{R}^N : r = |x| > 0, \theta = x|x|^{-1} \in \omega\}$ . We consider classical solutions to (1.2) satisfying the Dirichlet boundary condition, namely  $u \in C^2(\Omega) \cap C^0(\overline{\Omega} \setminus \{0\})$  such that

$$u = 0 \quad \text{on } \partial\Omega \setminus \{0\}. \quad (2.1)$$

Thus zero may be a singular point for  $u$ . We shall say that a solution is regular if furthermore  $u$  is continuous at zero ( $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ ).

Our aim is to see how symmetry on the domain is reflected in the solution, in the spirit of works by H. Berestycki and L. Nirenberg [8]. We say that  $\xi$  is a direction of symmetry for  $\Omega$  if  $s_\xi(\Omega) = \Omega$  where  $s_\xi$  is the reflection with respect to the hyperplane  $T_\xi = \{x : x \cdot \xi = 0\}$ . We say that  $u$  has the same symmetry as  $\Omega$  if  $u(s_\xi(x)) = u(x)$  for all  $x \in \Omega$  and all  $\xi$  directions of symmetry for  $\Omega$ .

We have then the following:

**Theorem 2.1.** *Let  $\Omega$  be a convex cone in  $\mathbb{R}^N$  such that  $\overline{\Omega} \setminus \{0\} \subset \{x : x \cdot \eta > 0\}$  for some  $\eta \in S^{N-1}$ . Let  $u$  be a solution to (1.2)–(2.1) with  $p \neq (N + 2 + 2\sigma)/(N - 2)$  satisfying*

$$|x|^{\frac{\sigma+2}{p-1}}u(x) \in L^\infty(\Omega); \tag{2.2}$$

*then  $u$  has the same symmetry as  $\Omega$  in the above sense.*

Note that this result holds for any solution to (1.2), i.e., regular or singular at zero. The condition on  $\Omega$  ensures that it is included in some half-space. The case of a half-space requires a special treatment, since it has extra invariances. When  $\Omega$  is not included in a half-space, our symmetry argument no longer works. As we shall see shortly, this is related to the fact that one cannot in general find a geodesic on  $S^{N-1}$  that does not intersect  $\omega = \Omega \cap S^{N-1}$ . The same trouble arises in the study of the symmetry of equations in domains of  $S^{N-1}$  that are not included in a half-sphere. Actually little seems to be known in that case (see [23]).

The growth condition (2.2), often referred to in the literature as the “slow decay” assumption, is natural. Indeed, all known solutions satisfy (2.2) (see below). Besides, this condition is necessary for the symmetry to hold when (1.2) is considered in  $\mathbb{R}^N$  (cf. the counterexample of Zou [27]). Moreover, this is the condition under which the asymptotics of Veron are established in  $\mathbb{R}^N \setminus \{0\}$ ,  $B_R \setminus \{0\}$ , or  ${}^cB_R$ . It is automatically satisfied in these geometries if  $p$  is subcritical ( $p < (N + 2)/(N - 2)$ ); cf. Veron, Theorem 6.2 in [25]. Whether this statement, or an analogue, still holds true for cones is still open.

The restriction  $p \neq (N + 2 + 2\sigma)/(N - 2)$  is linked to our method of proof. In this case, as noticed in [25], it seems more difficult to establish asymptotics, due to the conservation of some energy functional (see the proof of Theorem 2.1 below). The behaviour of  $u$  for such  $p$  seems to involve topological properties of the domain (see Egnell [13]).

We now turn to a nonexistence theorem. When considered for the general class of solutions (i.e., regular or singular), a quite general result is known. In the sequel,  $p^*$  will denote

$$p^* = \max\left(1 - \frac{(\sigma + 2)}{\gamma_-}, 1 - \frac{(\sigma + 2)}{\gamma_+}\right),$$

where  $\gamma_\pm$  are the roots of  $\gamma(\gamma + N - 2) = \lambda_1$ , first eigenvalue of the Laplace-Beltrami operator  $-\Delta_\theta$  in  $H_0^1(\omega)$ :

$$\gamma_\pm = -\left(\frac{N - 2}{2}\right) \pm \sqrt{\lambda_1 + \left(\frac{N - 2}{2}\right)^2}. \tag{2.3}$$

We shall call  $\psi$  the normalized eigenfunction of  $-\Delta_\theta$  in  $\omega$  under Dirichlet boundary conditions:

$$\Delta_\theta \psi + \lambda_1 \psi = 0, \quad \text{in } \omega, \quad \psi = 0 \quad \text{on } \partial\omega, \quad \int_\omega \psi(\theta) d\theta = 1.$$

Then (see Theorem 2.3 in [2]): if  $p \leq p^*$ , there are no solutions to (1.1); and if  $p^* < p < (N + 1)/(N - 3)$  ( $+\infty$  if  $N = 3$ ), there are separate variable solutions. On this question, see also [7]. An analogous statement holds true for equation (1.2); see [21]. Namely, if

$$\frac{\sigma + 2}{p - 1} \left( \frac{\sigma + 2}{p - 1} + 2 - N \right) + \lambda \geq \lambda_1,$$

there are no solutions to (1.2). On the other hand, if  $p \in (1, (N + 1)/(N - 3))$  ( $+\infty$  if  $N = 3$ ), and

$$\frac{\sigma + 2}{p - 1} \left( \frac{\sigma + 2}{p - 1} + 2 - N \right) + \lambda < \lambda_1,$$

there are separate variable solutions. Naturally, this condition reduces to the one above when  $\lambda = 0$ .

Concerning regular solutions, whether they existed at all was not clear. The question was raised by C. Bandle in [1], [2], where a partial negative answer is given in two dimensions. Our purpose is to make use of somewhat recent developments in the moving-planes method in unbounded domains (see [5]) in order to get a general result. We have the following:

**Theorem 2.2.** *Let  $\Omega$  be a convex cone in  $\mathbb{R}^N$  such that  $\overline{\Omega} \setminus \{0\} \subset \{x : x \cdot \eta > 0\}$  for some  $\eta \in S^{N-1}$ . Then there are no regular solutions (i.e., continuous at zero) to (1.1)–(2.1) when  $\sigma \geq 0$ ,  $p > 1$ .*

When  $\sigma \leq -2$  and  $p^* < p < (N + 1)/(N - 3)$  ( $+\infty$  if  $N = 3$ ), there are known regular solutions (see the separate-variable-solutions section below). When  $\sigma \in (-2, 0)$ , the moving-plane arguments fail and this case seems to require a deeper analysis.

### 3. SEPARATE VARIABLE SOLUTIONS

For certain ranges of the parameters  $\lambda$ ,  $\sigma$ ,  $p$ , there are known separate variable solutions to (1.2)–(2.1). Since they are of utmost importance in the asymptotic study, we shall describe briefly how to construct them. If we set  $u(r, \theta) = r^{-(\sigma+2)/(p-1)} \alpha(\theta)$ , a simple computation shows that  $u$  is a solution of (1.2)–(2.1) if and only if  $\alpha(\theta) > 0$  satisfies

$$\begin{cases} \Delta_\theta \alpha + \nu \alpha + \alpha^p = 0 & \text{in } \omega \\ \alpha = 0 & \text{on } \partial\omega, \end{cases} \tag{3.1}$$

with  $\nu = (\sigma + 2)/(p - 1)((\sigma + 2)/(p - 1) + 2 - N) + \lambda$ . A necessary condition for (3.1) to be solvable is obviously  $\nu < \lambda_1$ . This, together with  $p < (N + 1)/(N - 3)$  ( $+\infty$  if  $N = 3$ ), turns out to be a sufficient condition, as shown in [3] with a variational method. Note that in the case where  $\lambda = 0$ , these conditions simply read  $p \in (1, (N + 1)/(N - 3))$  and  $p > p^*$ , as mentioned earlier. Let us finally point out that these solutions are regular if  $\sigma \leq -2$ , and singular if  $\sigma > -2$ .

#### 4. PROOF OF THEOREM 1

In contrast with some works where the knowledge of an asymptotic expansion of the solution is assumed in order to get the moving-plane method to work, the trouble here is to control the behaviour of  $u$  at infinity from the rather mild “slow decay” assumption (2.2). Thus we shall first classify the possible different behaviours of  $u$  at infinity, and next show that the asymptotic profiles of  $u$  are symmetric. We shall then conclude by an adaptation of the classical moving-plane method (as presented by H. Berestycki and L. Nirenberg in [8]) to the symmetry of  $u$ . The following proposition classifies the possible behaviours of  $u$  at infinity:

**Proposition 4.1.** *Let the assumptions of Theorem 1 be fulfilled, and let  $r_n$  be a positive sequence tending to zero (respectively  $+\infty$ ). Then, either*

- i) *there exists a subsequence  $r_{n_j}$  such that  $\lim_{j \rightarrow \infty} r_{n_j}^{(\sigma+2)/(p-1)} u(r_{n_j}, \theta) = \alpha(\theta)$  uniformly in  $C^1(\omega)$ , where  $\alpha$  is a positive solution of (3.1), or*
- ii)  *$\lim_{r \rightarrow 0} r^{-\gamma+} u(r, \theta)$  (respectively  $\lim_{r \rightarrow \infty} r^{-\gamma-} u(r, \theta) = c_0 \psi(\theta)$ ) uniformly in  $C^1(\omega)$ , where  $c_0 > 0$  is a constant.*

This is an extension of the previous results by C. Bandle and M. Essen [2]. Note that we only know that i) happens for some subsequence of  $r_n$ , and this is just what is needed to get the symmetry of  $u$ . Whether this is true for all  $r$ 's (i.e.,  $r^{(\sigma+2)/(p-1)} u(r, \theta) \rightarrow \alpha(\theta)$ ) seems to be a much more involved question.

In particular, one cannot apply the powerful result of Simon [24] in this direction, since it requires uniform analyticity of the nonlinearity  $s \rightarrow s^p$  on some neighbourhood of the range of  $u$ . This condition is not fulfilled here, apart from the very particular case when  $p$  is an integer. Another favorable case is when one knows a priori that (3.1) admits a unique positive solution.

One of the main features of equation (1.2) is that it can be transformed into an autonomous equation, which allows us to use methods based on energy. Here, we shall rely heavily on the methods used by Veron in [25]. If

one sets  $u(r, \theta) = r^{-(\sigma+2)/(p-1)}v(t, \theta)$ ,  $t = \ln r$ , then  $v$  satisfies

$$v_{tt} + \delta v_t + \Delta_\theta v + \nu v + v^p = 0, \tag{4.1}$$

with  $\delta = N - 2(\sigma + p + 1)/(p - 1)$  and  $\nu = (\sigma + 2)/(p - 1)((\sigma + 2)/(p - 1) + 2 - N) + \lambda$ . We then define the associated “energy” functional,

$$E(\eta) = \int_\omega \left( \frac{1}{2} |\nabla \eta|^2 - \frac{\nu}{2} \eta^2 - \frac{1}{p+1} \eta^{p+1} \right) d\theta. \tag{4.2}$$

Multiplying (4.1) by  $v_t$  and integrating results in

$$\delta \int_\omega v_t^2 = \frac{d}{dt} \left[ E(v)(t) - \frac{1}{2} \int_\omega v_t^2 \right]. \tag{4.3}$$

Note that  $\delta \neq 0$  since  $p \neq (N + 2 + 2\sigma)/(N - 2)$  from the assumption.

Let then  $F$  be defined by

$$F(t) = \int_0^t \int_\omega v_t^2 d\theta.$$

Relation (4.3) then reads

$$\delta F(t) = E(t) - \frac{1}{2} F'(t) + \frac{1}{2} F'(0) - E(0). \tag{4.4}$$

Now, the slow-decay assumption (2.2) implies that  $v$  is bounded on  $\mathbb{R} \times \omega$ . By elliptic estimates, we infer that  $\partial_t^\alpha \nabla_\theta^\beta v$  is also bounded on  $\mathbb{R} \times \omega$ . So  $E$  and  $F'$  are bounded in  $\mathbb{R} \times \omega$ , and so is  $F$  thanks to (4.4) since  $\delta \neq 0$ . This means that

$$\int_{-\infty}^{+\infty} \int_\omega v_t^2 d\theta < +\infty.$$

It is then standard to infer

$$\int_{-\infty}^{+\infty} \int_\omega |\partial_t^\alpha \nabla_\theta^\beta v|^2 d\theta < \infty \tag{4.5}$$

for all  $\alpha, \beta$ ,  $\alpha \geq 1$ ,  $\alpha + |\beta| \leq 3$  (cf. [25]), so that

$$v_t(t, \cdot) \longrightarrow 0 \quad \text{and} \quad v_{tt}(t, \cdot) \longrightarrow 0 \quad \text{in} \quad L^2(\omega). \tag{4.6}$$

We next introduce the  $\omega$ -limit sets of (4.1):

$$\Gamma^- = \bigcap_{t \leq 0} \overline{\bigcup_{\tau \leq t} \{v(\tau, \cdot)\}}^{C^1(\omega)}, \quad \text{respectively} \quad \Gamma^+ = \bigcap_{t \geq 0} \overline{\bigcup_{t \leq \tau} \{v(\tau, \cdot)\}}^{C^1(\omega)}.$$

It is well known that from the above results  $\Gamma^-$  and  $\Gamma^+$  are nonvoid, connected, compact subsets of  $C_0^1(\omega) = \{u \in C^1(\omega) : u = 0 \text{ on } \partial\omega\}$  (cf. [10] Theorem 9.1.8 for instance).

We have the following alternative:

1)  $0 \notin \Gamma^-$  (respectively  $0 \notin \Gamma^+$ ). In particular,

$$0 \notin \tilde{\Gamma}^- := \bigcap_{n \in \mathbb{N}} \overline{\bigcup_{k \geq n} \{v(t_k, \cdot)\}}^{C^1(\omega)} \subset \Gamma^-,$$

where  $t_n = \ln r_n$ . Moreover,  $\tilde{\Gamma}^-$  is a nonvoid, compact subset of  $C_0^1(\omega)$  for the same reasons that  $\Gamma^-$  is. Hence there exist  $\alpha \in \tilde{\Gamma}^-$ ,  $\alpha \neq 0$ , and a subsequence  $t_{n_j}$  of  $t_n$  such that  $v(t_{n_j}, \cdot) \rightarrow \alpha(\cdot)$  in  $C_0^1(\omega)$ . By (4.6), we know that  $\alpha$  satisfies equation (3.1). Clearly  $\alpha \geq 0$ ,  $\alpha \neq 0$  requires  $\nu < \lambda_1$  in (3.1) (multiply by the principal eigenfunction  $\psi$  and integrate by parts). Hence the elliptic operator  $-(\Delta_\theta + \nu)$  has positive principal eigenvalue, and so  $\Delta_\theta \alpha + \nu \alpha \leq 0$ ,  $\alpha \geq 0$ ,  $\alpha \neq 0$  imply, by the strong maximum principle, that  $\alpha > 0$  in  $\omega$ . Thus we have reached part i) of Proposition (4.1).

2)  $0 \in \Gamma^-$  (respectively  $0 \in \Gamma^+$ ). In this case, there exists a sequence  $t_n \rightarrow -\infty$  (respectively  $+\infty$ ) such that  $v(t_n, \cdot) \rightarrow 0$  in  $C^1(\omega)$ . So  $E(t_n) \rightarrow 0$ , and using (4.6) we know that  $\int_\omega v_t^2 d\theta \rightarrow 0$ ,  $t \rightarrow \pm\infty$ . Relation (4.4) then implies that  $\lim_{n \rightarrow \infty} F(t_n)$  exists (since  $\delta \neq 0$ ). Since  $F$  is monotonic,  $\lim_{t \rightarrow -\infty} F(t)$  (respectively  $t \rightarrow +\infty$ ) exists and, in turn, so does  $\lim_{t \rightarrow -\infty} E(t)$  (respectively  $t \rightarrow +\infty$ ).

Now  $E(t_n) \rightarrow 0$  yields  $\lim_{t \rightarrow -\infty} E(t) = 0$  ( $t \rightarrow +\infty$ ). Hence  $E(\alpha) = 0$  for all  $\alpha$  in  $\Gamma^-$  (respectively  $\Gamma^+$ ). Besides, using (4.6) as above, we know that every element  $\alpha$  of the  $\omega$ -limit set  $\Gamma^-$  satisfies equation (3.1). But (3.1) together with  $E(\alpha) = 0$  yields

$$\left(\frac{1}{2} - \frac{1}{p+1}\right) \int_\omega \alpha^{p+1} = 0,$$

and so  $\alpha \equiv 0$ . This means  $\Gamma^- = \{0\}$  (respectively  $\Gamma^+ = \{0\}$ ). From the compactness of  $\Gamma^\pm$ , we infer  $v(t, \cdot) \rightarrow 0$  as  $t \rightarrow -\infty$  ( $+\infty$ ) in  $C^1(\omega)$ .

In the special case when  $\lambda = 0$ , Theorem 3.3 and Corollary 3.6 in [2] readily imply at that point part ii) of Proposition 4.1. This relies on a local analysis of  $u$  near zero with the help of a Green's representation formula. However, these arguments do not seem to extend straightforwardly to the case  $\lambda > 0$ . Thus we shall use a different method, based again on change of function and energy estimates.

We have the following lemma:

**Lemma 4.1.** *In the above setting, let us assume that  $0 \in \Gamma^-$  (respectively  $\Gamma^+$ ). Then necessarily  $\lambda = 0$  and we have part ii) in Proposition 4.1.*

The first statement says that  $\lambda > 0$  cannot happen when  $0 \in \Gamma^-$  (respectively  $\Gamma^+$ ), and so we could get ii) from [2]. However our method of proof

for Lemma 4.1 allows us to rederive ii) directly, which we shall do for the sake of completeness.

Here we will use another interesting feature of equation (1.2) in cones. If one performs a Kelvin transform on  $u$ , i.e., defines  $\bar{u}(x) = |x|^{2-N}u(x|x|^{-2})$ , then  $\bar{u}$  satisfies

$$\Delta \bar{u} + \frac{\lambda}{r^2} \bar{u} + r^{\sigma'} \bar{u}^p = 0$$

in the same domain  $\Omega$ , with  $\sigma' = -\sigma + p(N - 2) - (N + 2)$ . Hence we can restrict ourselves to the case when the sequence  $r_n$  tends to zero, being careful about the fact that the conditions we need on  $\sigma$  in the proof must be invariant under  $\sigma \rightarrow \sigma'$ . This will be clearly indicated below.

Once again we make use of an auxiliary function  $w(t, \theta) = r^{-\gamma_+}u(r, \theta)$  with  $t = \ln r$ , and  $\gamma_+$  given by (2.3). A simple computation yields

$$w_{tt} + (\gamma_+ - \gamma_-)w_t + \Delta_\theta w + (\lambda + \lambda_1)w + e^{\beta t}w^p = 0, \tag{4.7}$$

with  $\beta = \sigma + 2 + (p - 1)\gamma_+$ . This equation is not autonomous; however, we shall show we can treat it as a perturbation of the kind of equation we considered previously. Since we are in the case  $r_n \rightarrow 0$ , our aim is to study (4.7) when  $t \rightarrow -\infty$ . By a result of Min'Chi [21] which generalizes Theorem 2.3 in [2] when  $\lambda \neq 0$ , we know that a necessary condition for equation (1.2) to admit a solution (regular or singular) is

$$\gamma_- < -\frac{\sigma + 2}{p - 1} < \gamma_+. \tag{4.8}$$

In particular this gives  $\beta = \sigma + 2 + (p - 1)\gamma_+ > 0$ . This inequality is invariant under Kelvin transform; indeed,

$$\begin{aligned} \beta' &= \sigma' + 2 + (p - 1)\gamma_+ = -(\sigma + 2) + 2 - N + p(N - 2) + (p - 1)\gamma_+ \\ &> (p - 1)\gamma_- + 2 - N + p(N - 2) + (p - 1)\gamma_+ > 0, \end{aligned} \tag{4.9}$$

since  $\gamma_+ + \gamma_- = -(N - 2)$ . Hence, when  $t \rightarrow -\infty$ ,  $e^{\beta t}w^p$  is a perturbation term. By a result of Min'Chi [21], one has

$$C_1 r^{\gamma_+} \leq \sup_{\theta \in \omega} u(r, \theta) \leq C_2 r^{\gamma_+} \quad \text{for } r \leq r_0 \tag{4.10}$$

( $C_1, C_2$  positive constants) whenever one knows  $r^{(\sigma+2)/(p-1)}u(r, \cdot) = v(r, \cdot) \rightarrow 0$  as  $r \rightarrow 0$ , which is the case here. Hence,  $w$  is bounded on  $(-\infty, 0) \times \omega$ . Reasoning then in a similar way as above, we define

$$E_1(\eta) = \int_\omega \frac{1}{2} |\nabla_\theta \eta|^2 - \left(\frac{\lambda + \lambda_1}{2}\right) \eta^2 d\theta,$$



and  $F_1(t) = \int_0^t w_t^2 d\theta$ , which satisfy

$$\delta_1 F_1(t) = E_1(t) - \frac{1}{2} F_1'(t) + C_0 + \frac{1}{p+1} e^{\beta t} \int_{\omega} w^{p+1}(t, \theta) d\theta, \tag{4.11}$$

with  $\delta_1 = \gamma_+ - \gamma_- > 0$ , and  $C_0 = \frac{F'(0)}{2} - E(0) - \frac{1}{p+1} \int_{\omega} w^{p+1}(0, \theta) d\theta$ .

Since by (4.10)  $w$  is bounded on  $\mathbb{R} \times \omega$  and  $e^{\beta t} \rightarrow 0$  as  $t \rightarrow -\infty$ , the last term tends to zero. Retracing the above proof shows that we arrive at the same conclusion for  $w$  as for  $v$ . More precisely, the  $\omega$ -limit set

$$\Gamma_1 = \bigcap_{t \leq 0} \bigcup_{\tau \leq t} \overline{\{w(\tau, \cdot)\}}^{C^1(\omega)}$$

is a nonempty, connected, compact subset of  $C_0^1(\omega)$ , and every element  $\alpha$  of  $\Gamma_1$  satisfies the limiting equation; i.e.,

$$\begin{cases} \Delta_{\theta} \alpha + (\lambda + \lambda_1) \alpha = 0 & \text{in } \omega \\ \alpha = 0 & \text{on } \partial\omega. \end{cases} \tag{4.12}$$

There are now two cases to consider.

a)  $\lambda = 0$ : It is well known (see [9] for instance) that all solutions of (4.12) are given by  $\{c_0\psi\}$ ,  $c_0 \in \mathbb{R}$ , where  $\psi$  is the normalized eigenfunction of  $-\Delta_{\theta}$  introduced previously. Now, (4.10) clearly implies  $c_0 > 0$ . Besides, the constant  $c_0$  is uniquely determined from  $u$  (cf [2]); it is given by

$$c_0 = \|\psi\|_2^{-2} \lim_{r \rightarrow 0} r^{-\gamma_+} \int_{\omega} u(r, \theta) \psi(\theta) d\theta.$$

Hence  $\Gamma_1$  is the singleton  $\{c_0\psi\}$ , and, by compactness,  $w(t, \cdot) \rightarrow c_0\psi(\cdot)$  in  $C_0^1(\omega)$  as  $t \rightarrow -\infty$ , which gives part i) of Proposition (4.1) in the case  $\lambda = 0$ .

b)  $\lambda > 0$ : In this case, equation (4.12) admits no nonnegative solutions apart from 0, as is well known (multiply by  $\psi$  and integrate by parts for instance). Thus  $\Gamma_1 = \{0\}$ , and, by compactness, we infer  $w(t, \cdot) \rightarrow 0$  in  $C_0^1(\omega)$  when  $t \rightarrow -\infty$ . But (4.10) says that  $\sup_{\theta \in \omega} w(t, \theta) \geq C_1 > 0$  for  $t \leq t_0$ , a contradiction. Hence we have reached the conclusion of Lemma 4.1, and of Proposition 4.1.

With these asymptotics in hand, we can now turn to symmetry properties. Consider  $\xi$  a direction of symmetry for  $\Omega$ . We shall use reflections with respect to an appropriate one-parameter family of hyperplanes containing  $T_{\xi}$ , and study the difference between  $u$  and the reflected function with respect to the hyperplanes. The aim is then to show that this difference function is identically zero when the hyperplane reaches  $T_{\xi}$ .

The unit sphere  $S^{N-1}$  can be parametrized with  $\theta = (\cos \varphi, \sin \varphi \Theta)$ ,  $\Theta \in S^{N-2}$ ,  $\varphi \in [-\pi, \pi]$ . From the assumption on  $\Omega$ , we know that  $\omega = \Omega \cap$

$S^{N-1}$  is contained in some half-sphere on  $S^{N-1}$ . Applying some Euclidean transform in  $\mathbb{R}^N$ , we can assume without loss of generality that  $\omega \subset \{\theta = (\cos \varphi, \sin \varphi \Theta) : \varphi \in (-\pi/2, \pi/2), \Theta \in S^{N-2}\}$ , and that  $\xi = (0, 1)$  in this parametrization (i.e.,  $\varphi = \pi/2, \Theta = 1$ ).

We next define the continuous one-parameter family of hyperplanes  $\varphi \rightarrow T_\varphi = \{x \cdot \theta(\varphi) = 0\}$  with  $\theta(\varphi) = (\sin \varphi, \cos \varphi)$ . Then  $T_0 = T_\xi$ , and from the convexity assumption in the  $\xi$  direction, we know that the reflection of

$$\Sigma_\varphi = \{\theta = (\cos \zeta, \sin \zeta \Theta) \in \omega : \zeta \leq \varphi\}$$

with respect to the hyperplane  $T_\varphi$  remains in  $\Omega$  for all  $\varphi$  from  $\varphi_0 = \inf\{\varphi \in (-\pi/2, \pi/2) : \exists \Theta, (\cos \varphi, \sin \varphi \Theta) \in \omega\}$  to 0.

Hence we can define the difference function  $w_\varphi(r\theta) = u(r\theta_\varphi) - u(r\theta)$  in the cap  $\Sigma_\varphi$  for all  $\varphi \in (\varphi_0, 0)$ , where  $\theta_\varphi$  denotes the reflection of  $\theta$  with respect to the hyperplane  $T_\varphi$ . This function satisfies a linear equation

$$\begin{cases} \Delta_\theta w_\varphi + c(r, \theta)w_\varphi = 0 & \text{in } \Sigma_\varphi \\ w_\varphi \geq 0 & \text{on } \partial\Sigma_\varphi, \end{cases} \tag{4.13}$$

where  $c(r, \theta)$  is given by  $\frac{u(r\theta_\varphi)^p - u(r\theta)^p}{u(r\theta_\varphi) - u(r\theta)}$  if  $u(r\theta_\varphi) - u(r\theta) \neq 0$ , and 0 otherwise. Now  $\theta \rightarrow c(r, \theta)$  is clearly bounded in the region  $\theta \in \Sigma_\varphi, 0 < r_0 < r < R_0$ , since  $u$  is bounded there.

The aim here is to show that  $w_\varphi \geq 0$  for all  $\varphi \in (\varphi_0, 0)$ , which gives by a continuity argument  $w_0 \geq 0$ . Doing the same process the other way round (i.e., with  $\varphi \rightarrow -\varphi$ ) then gives  $w_0 \equiv 0$ , and so  $u(s_\xi(x)) = u(x)$ , which is the conclusion of Theorem 2.1.

For this, define  $\bar{\varphi} = \sup\{\varphi \in (\varphi_0, 0) : w_\zeta(r, \theta) \geq 0 \text{ for all } r > 0, \theta \in \Sigma_\zeta \text{ and all } \zeta \in (\varphi_0, \varphi)\}$ . We proceed in two steps:

- a)  $\bar{\varphi}$  is well-defined, and
- b)  $\bar{\varphi} = 0$ .

In these two steps, we make use of the following lemma, whose proof is postponed for the sake of clarity:

**Lemma 4.2.** *Given  $\varepsilon > 0$ , there exist  $r_0 = r_0(\varepsilon) > 0$  and  $R_0 = R_0(\varepsilon) > 0$  such that for all  $\varphi \in (\varphi_0, -\varepsilon)$ ,  $w_\varphi(r, \theta) \geq 0$  in  $\{(r, \theta) : \theta \in \Sigma_\varphi, r < r_0 \text{ or } R_0 < r\}$ .*

This lemma contains all the information we have on  $u$  at infinity and near zero from our previous study.

**Step a)** Let us fix  $\varepsilon > 0$ , and define  $\tilde{\Sigma}_\varphi = \{x : |x|^{-1}x \in \Sigma_\varphi\}$ . Let us denote by  $D_\varphi$  the domains  $D_\varphi = \tilde{\Sigma}_\varphi \cap \{x : r_0 < |x| < R_0\}$ , where  $r_0$  and  $R_0$  are

given by Lemma 4.2. It is clear that

$$\|c\|_{L^\infty(D_\varphi)} \leq p \|u\|_{L^\infty(\Omega \cap \{x:r_0 < |x| < R_0\})}^{p-1} =: M,$$

for all  $\varphi \in (\varphi_0, -\varepsilon)$ . Now, it is well known that the maximum principle for a linear elliptic operator with bounded coefficients holds true without any assumption on the sign of the zero-order coefficient, provided the domain is small enough with respect to Lebesgue measure (see for instance [9]). More precisely, we have  $\delta = \delta(N, M)$  such that the maximum principle holds true for  $\Delta + c(r, \theta)$  in every domain having Lebesgue measure less than  $\delta$ . Now, if one chooses  $\varphi_1$  close to  $\varphi_0$  so that  $|\tilde{\Sigma}_{\varphi_1} \cap \{(r, \theta) : r_0 < r < R_0\}| < \delta$  ( $|\cdot| =$  Lebesgue measure in  $\mathbb{R}^N$ ), we get from (4.13) and Lemma 4.2 that  $w \geq 0$  in  $\Sigma_\varphi$ , for all  $0 < \varphi < \varphi_1$ . Thus  $\bar{\varphi}$  is well defined.

**Step b)** Suppose for contradiction that  $\bar{\varphi} < 0$ , and let  $\varepsilon = |\bar{\varphi}|/2$ . Let us define  $r_0, R_0$ , and  $\delta$  as above. We next take  $K \subset \tilde{\Sigma}_{\bar{\varphi}}$  to be a compact set such that

$$|\{x = (r, \theta) \in \tilde{\Sigma}_{\bar{\varphi}} : r_0 < r < R_0\} \setminus K| < \frac{\delta}{2}.$$

By the strong maximum principle, we know that either  $w_\varphi > 0$  or  $w_\varphi \equiv 0$  in  $\tilde{\Sigma}_{\bar{\varphi}}$ . But  $w_\varphi \equiv 0$  would imply that  $u$  vanishes at interior points in  $\Omega$ , which is excluded by the assumption  $u > 0$  in  $\Omega$ . Hence  $w_\varphi > 0$  in  $\tilde{\Sigma}_{\bar{\varphi}}$  and  $\inf_K w_{\bar{\varphi}} > \eta > 0$ . Using a continuity argument, we know there exists  $\varepsilon' < \varepsilon$  such that one has  $\inf_K w_{\bar{\varphi}} > \eta/2$  for all  $\varphi \in (\bar{\varphi}, \bar{\varphi} + \varepsilon')$ . Since  $\varepsilon' < \varepsilon$ , we already know that  $w_\varphi \geq 0$  in  $\{x \in \tilde{\Sigma}_\varphi : |x| < r_0 \text{ or } R_0 < |x|\}$  for all  $\varphi \in (\bar{\varphi}, \bar{\varphi} + \varepsilon')$ . Now let us call  $\tilde{\Sigma}'_\varphi$  the complement in  $\tilde{\Sigma}_\varphi$  of  $K \cap \{x \in \Sigma_\varphi : |x| < r_0 \text{ or } R_0 < |x|\}$ . It is clear that  $w_\varphi \geq 0$  on  $\partial\tilde{\Sigma}'_\varphi$  from the above considerations, and that  $w_\varphi$  satisfies  $\Delta w_\varphi + c(r, \theta)w_\varphi = 0$  in  $\Sigma'_\varphi$ . Now,  $|\tilde{\Sigma}'_\varphi| < \delta$  implies by the maximum principle in small domains that  $w_\varphi \geq 0$  in  $\Sigma'_\varphi$ . Hence  $w_\varphi \geq 0$  in the whole domain  $\Sigma_\varphi$  for all  $\varphi \in (\bar{\varphi}, \bar{\varphi} + \varepsilon')$ , contradicting the supremum definition of  $\bar{\varphi}$ . Thus, one necessarily has  $\bar{\varphi} = 0$ , which is the desired result.

**Proof of Lemma 4.2.** We first state some symmetry properties of the functions  $\psi$  and  $\alpha$  defined on  $\omega \subset S^{N-1}$ . One can view them as functions of  $\theta = (\varphi, \Theta)$  with the help of the parametrization introduced previously. We define the difference functions  $V_\zeta(\theta) = \alpha(\theta_\zeta) - \alpha(\theta)$  and  $W_\zeta(\theta) = \psi(\theta_\zeta) - \psi(\theta)$ , where  $\theta_\zeta$  is the reflection of  $\theta$  with respect to the hyperplane  $T_\zeta$ . One has the following properties:

**Lemma 4.3.** *The functions  $\psi$  and  $\alpha$  are symmetric with respect to the hyperplane  $T_0$ ; i.e.,  $V_0 \equiv 0$  and  $W_0 \equiv 0$ . Besides, one has the following*

properties:  $V_\zeta > 0, W_\zeta > 0$  in  $\Sigma_\zeta$  and

$$\frac{\partial V_\zeta}{\partial \varphi} < 0, \quad \frac{\partial W_\zeta}{\partial \varphi} < 0$$

on  $T_\zeta \cap \partial \Sigma_\zeta$ , for all  $\zeta \in (\varphi_0, 0)$ .

Let us sketch briefly the argument, which can be found essentially in [23]. The functions  $\psi$  and  $\alpha$  both satisfy semilinear elliptic equations in the form  $\Delta_\theta \chi + g(\chi) = 0, \chi > 0$  in  $\omega$ , and  $\chi = 0$  on  $\partial \omega$ , with some given  $C^1$  function  $g$ . So the difference functions  $U, W$  are well defined in the caps  $\Sigma_\varphi \cap \omega$ , and satisfy linear equations in the form  $\Delta_\theta Z + c(\varphi, \Theta)Z = 0$  in  $\omega, Z \geq 0$  on  $\partial \omega$ , where  $c$  is some bounded function. When  $\varphi$  is close to  $\varphi_0$ , the caps  $\Sigma_\varphi \cap \omega$  have small  $(N - 1)$ -dimensional measure. The maximum principle in small domains (which is available for any elliptic operator) then tells us that there exist  $\delta = \delta(\|c\|_\infty, N)$  such that the maximum principle holds true for  $\Delta_\theta + c(\varphi, \Theta)$  in all domains with  $(N - 1)$ -dimensional measure less than  $\delta$ . This implies that  $Z \geq 0$  in  $\Sigma_\varphi \cap \omega$  for all  $\varphi, \varphi_0 < \varphi < \varphi_0 + \varepsilon$ . Define next  $\bar{\varphi}$  as previously to be  $\sup\{\varphi : Z_\zeta \geq 0 \text{ for all } \varphi_0 < \zeta < \varphi\}$ . Suppose for contradiction that  $\bar{\varphi} < 0$ . Take  $K$  a compact subset of  $S^{N-1}$  such that the measure  $H^{N-1}(\Sigma_\varphi \setminus K)$  is less than  $\delta/2$ . Then for  $\bar{\varphi} < \varphi < \bar{\varphi} + \varepsilon_1, H^{N-1}(\Sigma_\varphi \setminus K) < \delta$ , and the maximum principle holds in  $\Sigma_\varphi \setminus K$ . Now, we know  $Z_{\bar{\varphi}} \geq 0$  by the definition of  $\bar{\varphi}$ . The strong maximum principle then states that either  $Z_{\bar{\varphi}} > 0$  in  $\Sigma_{\bar{\varphi}}$ , or  $Z_{\bar{\varphi}} \equiv 0$ . the last case is ruled out since it would provide existence of interior zeroes for  $\chi$  (remember  $\chi > 0$  in  $\Sigma_\varphi$ ). Hence the positive minimum of  $Z_{\bar{\varphi}}$  is reached in  $K$ , and, by a continuity argument,  $Z_\varphi$  will remain positive in  $K$  for  $\bar{\varphi} < \varphi < \bar{\varphi} + \varepsilon_2$ . Taking  $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$ , we infer by the maximum principle that  $Z_\varphi \geq 0$  for  $\bar{\varphi} < \varphi < \bar{\varphi} + \varepsilon$ , contradicting the supremum definition of  $\bar{\varphi}$ .

That  $\frac{\partial V}{\partial \varphi} < 0$  and  $\frac{\partial W}{\partial \varphi} < 0$  is then a consequence of Hopf's lemma applied at the boundary of  $\Sigma_\varphi$ .

We now turn to the proof of Lemma 4.2. Arguing by contradiction, one can construct sequences  $\varphi_n \in (\varphi_0, -\varepsilon_0)$  ( $\varepsilon_0$  fixed), and  $(r_n, \theta_n) \in \tilde{\Sigma}_{\varphi_n}$  such that  $w_{\varphi_n}(r_n, \theta_n) < 0$  and  $r_n \rightarrow 0$  or  $+\infty$ . Up to the extraction of a subsequence, we can assume  $\varphi_n \rightarrow \bar{\varphi} \in [\varphi_0, -\varepsilon_0]$ , and  $\theta_n \rightarrow \bar{\theta} \in \bar{\Sigma}_{\bar{\varphi}}$ .

There are two cases to consider:

1)  $\bar{\theta} \in \Sigma_{\bar{\varphi}}$ : thanks to the asymptotics in Proposition 4.1,  $w_{\varphi_n}(r_n, \theta_n) < 0$  implies  $\alpha(\bar{\theta}_{\bar{\varphi}}) - \alpha(\bar{\theta}) \leq 0$  (respectively  $\psi(\bar{\theta}_{\bar{\varphi}}) - \psi(\bar{\theta}) \leq 0$ ), which contradicts  $V_{\bar{\varphi}}(\bar{\theta}) > 0$  (respectively  $W_{\bar{\varphi}}(\bar{\theta}) > 0$ ) (see Lemma 4.3).

2)  $\bar{\theta} \in \partial \Sigma_{\bar{\varphi}}$ : if  $\bar{\theta} \in \partial \Sigma_{\bar{\varphi}} \setminus T_{\bar{\varphi}}$ , we can reach the same contradiction as above, since we know that  $V_{\bar{\varphi}} > 0$  (respectively  $W_{\bar{\varphi}} > 0$ ) on this set by

the boundary conditions on  $u$ . On the other hand, if  $\bar{\theta} \in \partial\Sigma_{\bar{\varphi}} \cap T_{\bar{\varphi}}$ , by considering the quotient between  $u(r_n, (\theta_n)_{\varphi_n}) - u(r_n, \theta_n)$  and the geodesic distance from  $(\theta_n)_{\varphi_n}$  to  $\theta_n$ , and noting that the asymptotics in Proposition 4.1 can be differentiated, we get a contradiction with the second part of Lemma 4.3.

5. PROOF OF THEOREM 2.2

This nonexistence result relies on the following idea. By averaging equation (1.1) in a certain way, we know that some average of  $u$  tends to zero when  $r \rightarrow \infty$ . But the moving-planes method applied to the regular function  $u$  tells us that  $u$  is an increasing function of  $r$  along some directions, yielding a contradiction. This idea of averaging  $u$  was already used in [2], but that  $u$  is increasing was only proved under severe restrictions (namely  $N = 2$  and the aperture of the cone is bigger than  $\pi/2$ ).

Let us define  $\bar{u}$ :

$$\bar{u}(r) = \int_{\omega} u(r, \theta) \psi(\theta) d\theta.$$

Then  $\bar{u}$  clearly satisfies  $L\bar{u} \leq 0$ , where

$$L = \frac{\partial^2}{\partial r^2} + \frac{N-1}{r} \frac{\partial}{\partial r} - \frac{\lambda_1}{r^2}.$$

Thanks to inequality (2.11) in [2], we know that this implies  $\bar{u} \leq Cr^{-\frac{\sigma+2}{p-1}}$  for some positive constant  $C$ . Since by assumption  $\sigma \geq 0$ , we have  $\bar{u} \rightarrow 0$  when  $r \rightarrow \infty$ .

We next have the following lemma:

**Lemma 5.1.** *Under the assumptions of Theorem 2.2, there exists a nonempty open set  $\omega_0 \subset\subset \omega$  such that  $u$  is increasing along each direction  $\xi \in \omega_0$ .*

This is an adaptation of Theorem 7 in [5] for the nonautonomous case. For this we need again to use the moving-plane method. But, in contradistinction to the previous study, we shall deal only with bounded domains by reflecting the apex of the cone into its interior.

Since the cone  $\Omega$  is convex and included in some half-space, a simple geometrical argument gives the existence of a nonempty open set of direction  $\omega_0 \subset\subset \omega$  such that  $\Omega$  can be written as a coercive epigraph over the hyperplanes  $T_{\xi}$ ,  $\xi \in \omega_0$ ; i.e.,

$$\Omega = \{y = (y_1, \dots, y_{N-1}, y_N) = (y', y_N) : y_N > \Phi(y')\}$$

for some continuous, nonnegative function  $\Phi$  with  $\lim_{|y'| \rightarrow +\infty} \Phi(y') = +\infty$ . Here the  $y_N$  axis is along  $\xi$ , and  $(y_1, \dots, y_N)$  are orthonormal coordinates.

We now fix such a  $\xi$  and perform the Euclidean change of variable from  $(x_1, \dots, x_N)$  to  $(y_1, \dots, y_N)$  (which leaves the Laplacian invariant). In these coordinates,  $T_\xi$  has the equation  $y_N = 0$ , and we shall consider the one-parameter family of hyperplanes  $T_\lambda = \{y_N = \lambda\}$  for positive  $\lambda$ , and define as usual  $y_\lambda$  to be the reflection of  $y$  with respect to  $T_\lambda$ , and the difference functions  $w_\lambda(y) = u_\lambda(y) - u(y)$  (with  $u_\lambda(y) = u(y_\lambda)$ ) in the cap

$$\Sigma_\lambda = \{y \in \Omega : y_N < \lambda\}.$$

It is clear that all these sets are bounded, and that  $|\Sigma_\lambda| \rightarrow 0$  as  $\lambda \rightarrow 0$ .

Subtracting equation (1.1) written at  $y$  and at  $y_\lambda$ , we obtain that difference functions  $w_\lambda$  satisfy

$$\Delta w_\lambda + r_\lambda^\sigma \left[ \frac{u_\lambda^p - u^p}{u_\lambda - u} \right] w_\lambda = (r^\sigma - r_\lambda^\sigma) u^p \leq 0,$$

since  $|y_\lambda| > |y|$  for  $y \in \Sigma_\lambda$  and  $\sigma \geq 0$ . As previously, the differential quotient is meant to be zero if not defined; it will from now on be denoted by  $c(\lambda, y)$ . Taking into account the Dirichlet boundary condition on  $u$ , one has

$$\begin{cases} \Delta w_\lambda + c(\lambda, y) w_\lambda = 0 & \text{in } \Sigma_\lambda \\ w_\lambda \geq 0 & \text{on } \partial \Sigma_\lambda. \end{cases} \quad (5.1)$$

Since  $u$  is regular,  $c$  is a bounded function with  $\|c\|_\infty \leq M(\Lambda)$  for all  $\lambda \leq \Lambda$ .

The purpose here is again to prove that  $w_\lambda \geq 0$  for all  $\lambda > 0$  by standard arguments. Let us introduce  $\lambda_0 = \sup\{\lambda : w_\mu \geq 0 \text{ for all } 0 < \mu \leq \lambda\}$ . We proceed in two steps.

**a)  $\lambda_0$  is well defined.** By this we mean that the set over which  $\lambda_0$  is defined is nonempty. Given  $M > 0$ , the maximum principle in small domains gives  $\delta = \delta(M, N) > 0$  such that in all domains having Lebesgue measure less than  $\delta$  the maximum principle is true for all operators  $\Delta + c(y)$  provided  $\|c\|_\infty < M$ . Take then  $M = M(1)$  defined above and define the corresponding  $\delta$ . Since  $|\Sigma_\lambda| \rightarrow 0$  as  $\lambda \rightarrow 0$ , there exists  $\lambda \leq 1$  such that  $|\Sigma_\mu| \leq \delta$  for all  $\mu \leq \lambda$ . From (5.1) and the maximum principle in small domains, we infer that  $w_\mu \geq 0$  in  $\Sigma_\mu$  for all such  $\mu$ . Hence  $\lambda_0$  is well defined.

Note that here we used the fact that  $u$  is continuous at zero, this condition being required in the maximum principle.

**a)  $\lambda_0 = +\infty$ .** Since we already used this argument, we simply sketch the proof. Suppose for contradiction that  $\lambda_0$  is finite. We take some compact set  $K$  filling sufficiently the cap  $\Sigma_{\lambda_0}$ , and deduce that  $w_\lambda$  remains nonnegative in some right neighbourhood of  $\lambda_0$ , contradicting the supremum definition of  $\lambda_0$ .

At this point, we know that  $w_\lambda \geq 0$  in  $\Sigma_\lambda$  for all  $\lambda > 0$ . From the strong maximum principle, we deduce that  $w_\lambda > 0$ , since  $w_\lambda \equiv 0$  for some  $\lambda$  is ruled out (otherwise it would provide interior zeroes for  $u$ ). This, the fact that  $w_\lambda$  vanishes on  $T_\lambda \cap \Sigma_\lambda$  and Hopf's lemma give the fact that  $\frac{\partial w_\lambda}{\partial \nu} < 0$  on  $T_\lambda \cap \Sigma_\lambda$ , where  $\nu$  is the exterior normal to  $\Sigma_\lambda$ . Writing this inequality at  $p$ , the intersection of  $T_\lambda$  with the  $y_N$  axis, and noticing that  $\nu = \xi$  at this point gives  $\frac{\partial u}{\partial \xi}(0, \lambda) > 0$  for all  $\lambda$ , and the proof of Lemma 4.3 is over.

Collecting the information we have gathered on  $u$ , we get the inequality

$$\int_{\omega_0} u(r, \theta) \psi(\theta) d\theta \leq \int_{\omega} u(r, \theta) \psi(\theta) d\theta \leq Cr^{-\frac{\sigma+2}{p-1}}.$$

Since the first integral on the left is an increasing function of  $r$ , we get a contradiction if we let  $r \rightarrow +\infty$ . Hence there is no regular positive solution of (1.1) when  $\sigma \geq 0$ . This concludes the proof of Theorem 2.1.

**Examples.** Let  $u$  be a solution of (1.2) in the cone  $\Omega = \{x = (x_1, \dots, x_N) : x_k > 0 \text{ for all } k\}$  having a slow decay behaviour (2.2). Then Theorem 2.1 implies that  $u$  is invariant under any permutation of the variables, i.e.,  $u(x_{\sigma(1)}, \dots, u_{\sigma(N)}) = u(x_1, \dots, x_N)$  for all permutations  $\sigma$ .

Another interesting example is when  $\Omega \cap S^{N-1}$  is a geodesic disk contained in some half-sphere. It can be parametrized by  $(\rho, \Theta)$ , where  $\rho$  is the geodesic distance to the center of the disk, and  $\Theta \in S^{N-2}$ . Theorem 2.1 then states that  $u$  does not depend on  $\Theta$ ; i.e.,  $u(r, \rho, \Theta) = u(r, \rho)$ .

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