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ASYMPTOTICS OF RADIAL OSCILLATORY SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS

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Abstract. We study radially symmetric oscillatory solutions of semilinear elliptic equations of the form

$$\Delta u + \phi(|x|, u) = 0 \quad \text{in} \quad R^n \ (n \geq 2)$$

where $\phi(r, u)$ is a nonnegative function having the form $\sum_i c_i r^{\nu_i} |u|^{p_i - 1} u$ with $c_i > 0$. Under certain resrictions on the exponents ν_i and p_i (roughly speaking, $2\nu_i + n + 2 \geq (2 - n)p_i$ for all i where a strict inequality holds for at least one i), we show that all radial solutions must oscillate, i.e., change their signs infinitely many times. Moreover, we provide accurate estimates on the frequencies and amplitudes of these oscillatory solutions. These results are sharp in the sense that positive solutions exist when restrictions on these exponents are removed.

1. Introduction

Semilinear elliptic equations of the type

$$\triangle u + \phi(|x|, u) = 0 \quad \text{in} \quad R^n \tag{1.1}$$

arise in various branches of applied mathematics and have many origins such as the prescribed curvature problems in Riemannian geometry and the Lane–Emden–Fowler equation and the Matukuma equation in astrophysics. In addition to extensive studies on positive solutions ([8, 9, 10, and the references therein]), research on asymptotic behaviors of oscillatory solutions has also attracted much attention recently; see for example, [2, 3, 4, 5, 11, 12, 13] for second-order semilinear differential equations and [6] for quasilinear elliptic equations.

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It is well-known that (1.1) does not always have positive solutions. Under suitable conditions (cf. [1, 3, 4, 11, 13]), radial solutions to (1.1) must oscillate, i.e., change sign infinitely many times. In such a case, one is eager to know as precisely as possible the periods and amplitudes of oscillations. In [3], asymptotic behaviors of radial solutions of (1.1) was studied for $\phi = |x|^\mu |u|^{q-1}u + |x|^\nu |u|^{p-1}u$, i.e., for the initial value problem

$$u'' + (n-1)u'/r + (\, r^\mu \,|u|^{q-1} + b\, r^\nu \,|u|^{p-1})u = 0, \qquad u'(0) = 0, \qquad (1.2)$$

for $n \geq 3$ and

$$b > 0,\ \mu > -2,\ \nu > -2,\ 1 < q < \frac{n+2+2\mu}{n-2},\ 1 < p = \frac{n+2+2\nu}{n-2}. \quad (1.3)$$

With certain Pohazaev identities [15] and some rather involved calculations, the authors in [3] established the following result:

Proposition A [3]. *Assume* (1.3). *Then any nontrivial solution* u *to* (1.2) *must oscillate infinitely many times and there exist positive constants* c_1, c_2, *and* c_3 *such that*

$$\begin{aligned}
&\lim_{r\to\infty} r^{\frac{2(n-1)(q+1)-2\mu}{q+3}} \Big\{ \tfrac{1}{2}u'^2(r) + \tfrac{1}{q+1} r^\mu |u(r)|^{q+1} + \tfrac{b}{p+1}\, r^\nu |u(r)|^{p+1} \Big\} = c_1, \\
&\lim_{k\to\infty} \rho_k^{\frac{2n-2+\mu}{q+3}} |u(\rho_k)| = \{(q+1)c_1\}^{\frac{1}{q+1}}, \qquad (1.4)\\
&\lim_{k\to\infty} r_k^{\frac{(n-1)(q+1)-\mu}{q+3}} |u'(r_k)| = \sqrt{2c_1}, \\
&c_2 \leq \liminf_{k\to\infty} r_{k+1}^{\frac{2\mu-(n-1)(q-1)}{q+3}} \big(r_{k+1} - r_k\big) \leq \limsup_{k\to\infty} r_{k+1}^{\frac{2\mu-(n-1)(q-1)}{q+3}} \big(r_{k+1} - r_k\big) \leq c_3
\end{aligned}$$

where $\{r_k\}$ *and* $\{\rho_k\}$ *are zeroes of* u *and* u' *respectively.*

This paper is to extend and improve Proposition A in several substantial ways.

First of all, we shall include the case when the spatial dimension n is equal to 2 and allow more general nonlinearity of ϕ, the exponents p and q, and the initial condition than those in [3]. More precisely, we consider the initial value problem

$$\left\{ \begin{array}{l} u_{rr} + \frac{n-1}{r} u_r + \sum_{i=1}^I a_i\, r^{\mu_i}\, |u|^{q_i-1}u + \sum_{j=1}^J b_j\, r^{\nu_j}\, |u|^{p_j-1}u = 0 \quad \forall\, r > 0, \\ \lim_{r\searrow 0}\, (n-2) r^{n-2} u^2(r) = 0 \end{array} \right. \qquad (1.5)$$

where $n \geq 2$ and $a_i, b_j, q_i, p_j, \mu_i, \nu_j$ are all constants having the following properties:

(A1) All $a_1, \dots, a_I, b_1, \dots, b_J$ are positive,

$$q_1 \geq 1, \quad \text{and} \quad q_i > -1, \quad p_j > -1 \quad \forall \ i = 2, \cdots, I, \ j = 1, \cdots, J;$$

(A2) There exists a positive constant β such that

$$\beta = \frac{2\mu_i + n + 2 - (n-2)q_i}{q_i + 3} > \frac{2\nu_j + n + 2 - (n-2)p_j}{p_j + 3} \geq 0 \quad \forall i, \ j.$$

One notices that the requirements $n \geq 3$, $\nu > -2$, and $\mu > -2$ in Proposition A are removed. Furthermore, the condition $1 < p = (n+2+2\nu)/(n-2)$ is relaxed to $0 \leq 2\nu_j + n + 2 - (n-2)p_j$ and the initial condition $u'(0) = 0$ is relaxed to account for possible singular solutions.

Secondly, the method in this paper is simpler and more elementary than those used in [3]. We first make a Sturm–Liouville type change of variables which is similar in essence to the classical treatment in finding zeroes and amplitudes of oscillations for the Bassel functions; see, for example, Olver [14]. With this transformation, the subtleties in finding the Pohazaev identities become apparent, and many of the tedious calculations in [3] are greatly simplified. After establishing a limit analogous to (1.4), we use a "blow-up" argument to study the periods and amplitudes of oscillations; that is, we study the asymptotic behavior of u in any interval of two consecutive zeroes, to obtain a recursive relation between ρ_{k+1} and ρ_k. This recursive relation can be asymptotically expanded to arbitrarily high order, enabling one to obtain as much detailed asymptotic behavior as one wishes; here we shall carry out only the first- and second-order asymptotic expansions.

Finally and most importantly, our results here are sharper than Proposition A. In particular, we shall prove that the constants c_2 and c_3 in Proposition A are the same constant; indeed, we provide for them a closed formula depending on c_1 and the constants q_i and μ_i. Moreover, we shall provide more comprehensive information for the asymptotic behavior of the solutions than that in Proposition A.

To state our result, we shall use the following positive constants:

$$\alpha := \tfrac{1}{2}(\beta + n - 2) > 0,$$
$$\delta_j := \frac{p_j + 3}{2} - \frac{2\nu_j + n + 2 - (n-2)p_j}{2\beta}, \quad \delta := \min\{2, \delta_1, \dots, \delta_J\} > 0.$$

For each positive constant A, we also denote by W^A the solution to the autonomous ODE

$$\begin{cases} \frac{d^2}{dt^2} W^A + \sum_{i=1}^{I} a_i |W^A|^{q_i - 1} W^A = 0, \quad t \in (-\infty, \infty), \\ W^A(0) = A, \qquad \frac{d}{dt} W^A(0) = 0. \end{cases} \tag{1.6}$$

Observe that on the w–w_t phase plane, the trajectory is a closed curve given by

$$\beta^2 w_t^2 + 2F(w) = 2F(A); \qquad F(w) := \sum_{i=1}^{I} \frac{a_i}{1+q_i} |w|^{1+q_1}.$$

Consequently, W^A is periodic with period T^A given by

$$T^A = 2\sqrt{2}\beta \int_0^A \Big(F(A) - F(w)\Big)^{-1/2} \, dw \, . \tag{1.7}$$

Our main result is the following:

Theorem 1.1. *Assume* **(A1)** *and* **(A2)**. *Let u be an arbitrary nontrivial solution of* (1.5). *Then there exists a positive constant A depending on u such that*

$$u(r) = r^{-\alpha}\Big\{W^A(r^\beta + \omega(r)) + o(1)\Big\} \qquad \text{as } r \to \infty$$

where $\omega = r^\beta \cdot o(1)$ and $o(1)$, $r\frac{d}{dr}o(1) \to 0$ as $r \to \infty$. Consequently, denoting by $\{r_k\}$ and $\{\rho_k\}$ the zeroes of u and u' respectively, we have

$$\lim_{k\to\infty} \rho_k^\alpha |u(\rho_k)| = A, \qquad \lim_{k\to\infty} \big\{\beta r_k^{\alpha+1-\beta} u'(r_k)\big\}^2 = 2F(A),$$
$$\lim_{k\to\infty} \beta\, r_k^{\beta-1}(r_{k+1} - r_k) = \lim_{k\to\infty} \beta\, \rho_k^{\beta-1}(\rho_{k+1} - \rho_k) = T^A/2.$$

In addition there hold the following estimates for ω.

(1) *If $\delta > 1$, then $\omega(r) = B + o(1)$ for some constant $B \in [0, T^A)$;*
(2) *If $\frac{1}{2} < \delta \le 1$, then $\omega(r) = r^\beta \sum_{j=1}^J z_j r^{-\beta\delta_j} + B + o(1)$ for some constants $B, z_1, \ldots, z_J$. (Here $r^\beta r^{-\beta\delta_j}$ should be replaced by $\ln r$ if $\delta_j = 1$.)*
(3) *If $0 < \delta \le \frac{1}{2}$, then $\omega(r) = r^\beta\{\sum_{j=1}^J z_j r^{-\beta\delta_j} + O(r^{-2\beta\delta})\}$. (Here $r^\beta \cdot O(r^{-2\delta\beta})$ should be replaced by $O(\ln r)$ if $\delta = \frac{1}{2}$.)*

Remark 1.1. The definition of δ_j implies that $\delta = 2$ under the assumption (1.3).

In the next section, we shall make a change of variables from (r, u) to (t, w) and derive certain energy identities. Then in Section 3, we prove that as $t \to \infty$, the trajectory (w, w_t) approaches the trajectory of (1.6); i.e., $\beta^2 w_t^2 + 2F(w) = 2F(A) + o(1)$ for some $A > 0$. Finally, we prove Theorem 1.1 in §4.

In the sequel, $\sum_{i=1}^I$ and $\sum_{j=1}^J$ will be shortened to $\sum_i$ and $\sum_j$ respectively.

2. A Sturm–Liouville transformation

It is convenient to use a Sturm–Liouville-type transformation $(r, u) \to (t, w)$ via

$$t = r^{\beta}, \qquad u(r) = r^{-\alpha} w(t). \tag{2.1}$$

Under this transformation, equation (1.5) can be written as, $\forall\, t > 0$,

$$\beta^2 w_{tt} + \sum_i a_i |w|^{q_i - 1} w + \sum_j b_j t^{-\delta_j} |w|^{p_j - 1} w + \tfrac{1}{4}[\beta^2 - (n-2)^2] t^{-2} w = 0. \tag{2.2}$$

In obtaining this, we have used the following identities:

(1) $\beta - 2\alpha + n - 2 = 0$;
(2) $\alpha(\alpha - n - 2) = \frac{1}{4}[\beta^2 - (n-2)^2]$;
(3) $\mu_i - \alpha(q_i - 1) - 2\beta + 2 = \frac{q_i+3}{2}[\frac{2\mu_i + n + 2 - (n-2)q_i}{q_i+3} - \beta] = 0$ for all i;
(4) $\nu_j - \alpha(p_j - 1) - 2\beta + 2 = \frac{p_i+3}{2}[\frac{2\nu_j + n + 2 - (n-2)p_j}{p_j+3} - \beta] = -\beta\delta_j$ for all j.

We remark that the initial condition (1.5) is equivalent to

$$\lim_{t \searrow 0} (n-2) t^{-1} w^2(t) = 0 \tag{2.3}$$

since $t^{-1} w^2(t) = r^{2\alpha - \beta} u^2(r) = r^{n-2} u^2(r)$. For simplicity, we introduce the following notation:

$$F(w) := \sum_i \tfrac{a_i}{q_i+1} |w|^{q_i+1}, \qquad f(w) := \tfrac{\partial}{\partial w} F = \sum_i a_i |w|^{q_i - 1} w,$$

$$F_1(t, w) := \sum_j \tfrac{b_j}{p_j+1} t^{-\delta_j} |w|^{p_j+1} \qquad f_1(t, w) := \tfrac{\partial}{\partial w} F_1 = \sum_j b_j t^{-\delta_j} |w|^{p_j - 1} w,$$

$$F_2(t, w) := \tfrac{1}{8}[\beta^2 - (n-2)^2] t^{-2} w^2, \quad f_2(t, w) := \tfrac{\partial}{\partial w} F_2 = \tfrac{1}{4}[\beta^2 - (n-2)^2] t^{-2} w,$$

$$\begin{aligned}
Q_1 &:= \tfrac{1}{2}\beta^2 w_t^2 + F(w) + F_1(t, w), \\
Q_2 &:= Q_1 + F_2 = \tfrac{1}{2}\beta^2 w_t^2 + F(w) + F_1(t, w) + F_2(t, w), \\
Q_3 &:= t Q_2 - \tfrac{1}{2}\beta^2 w w_t = t\Big(\tfrac{1}{2}\beta^2 w_t^2 + F + F_1 + F_2\Big) - \tfrac{1}{2}\beta^2 w w_t \\
&= \tfrac{1}{2} t \beta^2 \big[w_t - \tfrac{w}{2t}\big]^2 + tF + tF_1 - \tfrac{1}{8}(n-2)^2 t^{-1} w^2.
\end{aligned}$$

Note that equation (2.2) can be written as

$$\beta^2 w_{tt} + f(w) + f_1(t, w) + f_2(t, w) = 0, \quad t > 0. \tag{2.4}$$

Also, for any solution w of (2.4), the following identities hold:

$$\frac{d}{dt}Q_1 = \tfrac{\partial}{\partial t}F_1(t,w) - f_2 w_t, \tag{2.5}$$

$$\frac{d}{dt}Q_2 = \tfrac{\partial}{\partial t}F_1(t,w) + \tfrac{\partial}{\partial t}F_2(t,w), \tag{2.6}$$

$$\frac{d}{dt}Q_3 = [F + \tfrac{1}{2}wf] + \sum_{k=1}^{2}\Big[F_k + t\tfrac{\partial}{\partial t}F_k + \tfrac{1}{2}wf_k\Big]. \tag{2.7}$$

3. Limit of the energy as $t \to \infty$

In this section, we shall prove the following theorem.

Theorem 3.1. *Assume* **(A1), (A2)** *and that w is any nontrivial solution of* (2.2) *satisfying* (2.3). *Then there exists a positive constant A such that as $t \to \infty$,*

$$\tfrac{1}{2}\beta^2 w_t^2 + F(w) - F(A) = O(t^{-2}) - F_1(t,w) - \int_t^\infty \tfrac{\partial}{\partial t}F_1(s,w(s))ds = O(t^{-\delta}). \tag{3.1}$$

Proof. We shall prove the theorem in several steps.

Step 1. First we prove that $\limsup_{t\to\infty}\ \{|w(t)| + |w_t(t)|\} < \infty$. It suffices to show that Q_1 is uniformly bounded. To this end, we estimate the right-hand side of (2.5) by $\frac{\partial}{\partial t}F_1 = -\sum_{j=1}^{J}\frac{b_j\delta_j}{p_j+1}t^{-1-\delta_j}|w|^{p_j+1} \le 0$, $|w_t| \le (2Q_1/\beta^2)^{1/2}$, and $|f_2| = ct^{-2}|w| \le ct^{-2}\{(q_1+1)Q_1/a_1\}^{1/(1+q_1)}$ ($c = \frac{1}{4}|\beta^2 - (n-2)^2|$). Thus,

$$\frac{d}{dt}Q_1 \le CQ_1^{1/2+1/(1+q_1)}t^{-2},$$

$C := \frac{1}{8}|\beta^2 - (n-2)^2|\sqrt{2}\beta^{-1}(q_1+1)^{1/(1+q_1)}a_1^{-1/(1+q_1)}$. Since $q_1 \ge 1$, we can divide both sides by $Q_1^{1/2+1/(1+q_1)}$ and integrate the resulting inequality over $[1,t)$ for any $t > 1$ to obtain

$$Q_1^{1/2-1/(1+q_1)}(t) \le Q_1^{1/2-1/(1+q_1)}(1) + C\big[\tfrac{1}{2} - \tfrac{1}{1+q_1}\big][1 - t^{-1}], \quad \forall t > 1.$$

(In case $q_1 = 1$, $\ln Q$ should be used.) Hence, $\limsup_{t\to\infty} Q_1 < \infty$.

Step 2. Next we show that $\liminf_{t\to\infty}\ tQ_1(t) > 0$.

We shall utilize the identity (2.7). For this purpose, we can use the definition of F, F_1, and F_2 to compute

$$\begin{aligned} F_2 + t\tfrac{\partial}{\partial t}F_2 + \tfrac{1}{2}wf_2 &\equiv 0, \\ F + \tfrac{1}{2}wf &= \sum_i \left[\tfrac{1}{1+q_i} + \tfrac{1}{2}\right] a_i |w|^{1+q_i} \geq 0, \\ F_1 + t\tfrac{\partial}{\partial t}F_1 + \tfrac{1}{2}wf_1 &= \sum_j \left[\tfrac{1}{1+p_j} - \tfrac{\delta_j}{1+p_j} + \tfrac{1}{2}\right] b_j\, t^{-\delta_j} |w|^{1+p_j} \geq 0 \end{aligned}$$

since

$$\frac{1}{1+p_j} - \frac{\delta_j}{1+p_j} + \frac{1}{2} = \frac{2\nu_j + n + 2 - (n-2)p_j}{2\beta(1+p_j)} \geq 0 \quad \forall\, j.$$

Hence, $\frac{d}{dt}Q_3(t) \geq 0$; that is, Q_3 is a strictly monotonic function (since w is nontrivial). Using the definition of Q_3, we have

$$\liminf_{t\searrow 0} Q_3(t) \geq -\tfrac{1}{8} \limsup_{t\searrow 0} (n-2)^2 t^{-1} w^2(t) = 0. \tag{3.2}$$

It then follows that $Q_3(t) > Q_3(1) > 0\ \forall\, t > 1$. The assertion that $\liminf_{t\to\infty} tQ_1 > 0$ thus follows from the relation

$$Q_3 = tQ_1 + tF_2 - \tfrac{1}{2}\beta^2 w w_t \leq tQ_1 - O(t^{-1}) + \beta^2[\tfrac{1}{4}tw_t^2 + \tfrac{1}{4}t^{-1}w^2] \leq \tfrac{3}{2}tQ_1 + O(t^{-1})$$

since from Step 1, w is uniformly bounded in $[1, \infty)$.

Step 3. We now show that (3.1) holds for some $A \geq 0$. Since w is bounded, $\frac{\partial}{\partial t}F_2 = O(t^{-3})$, and $\frac{\partial}{\partial t}F_1 = O(t^{-1-\delta})$, we can integrate the identity (2.6) over $[1, \infty)$ to conclude that the limit $\lim_{t\to\infty} Q_2(t) = \lim_{t\to\infty}[\frac{1}{2}\beta^2 w_t^2 + F]$ exists. We denote this limit by $F(A)$ for some $A \geq 0$. In addition, integrating (2.6) over (t, ∞) yields

$$\begin{aligned} Q_1(t) &= F(A) - F_2(t, w(t)) - \int_t^\infty \left[\tfrac{\partial}{\partial t}F_1(s, w(s)) + \tfrac{\partial}{\partial t}F_2(s, w(s))\right] ds \\ &= F(A) - O(t^{-2}) - \int_t^\infty \tfrac{\partial}{\partial t}F_1(s, w(s))\, ds = F(A) + O(t^{-\delta}). \end{aligned} \tag{3.3}$$

The assertion (3.1) thus follows from the definition of Q_1.

Step 4. Finally, we show that $A > 0$. Suppose, on the contrary, that $A = 0$. Then, (3.3) yields $Q_1(t) = O(t^{-\delta})$, which implies, since $Q_1 \geq \frac{a_1}{1+q_1}|w|^{1+q_1}$, that $w(t) = o(t^{-\lambda})$ for $\lambda = \frac{\delta}{2(1+q_1)} > 0$.

For any $\tau > 0$ with $w(\tau) \neq 0$, set $c = |w(\tau)|\tau^{\lambda}$. Then $|w(\tau)| = c\tau^{-\lambda}$. As $w = o(t^{-\lambda})$, there exists $\hat{\tau} \geq \tau$ such that

$$|w(\hat{\tau})| = c\hat{\tau}^{-\lambda}, \qquad |w(t) < ct^{-\lambda} \ \text{ for all } \ t \geq \hat{\tau}.$$

It then follows that for any $j = 1, \dots, J$,

$$\int_{\hat\tau}^{\infty} \delta_j s^{-1-\delta_j} |w(s)|^{p_j+1}\, ds \le \delta_j c^{1+p_j} \int_{\hat\tau}^{\infty} s^{-1-\delta_j-\lambda(1+p_j)}$$
$$= \frac{\delta_j}{\delta_j + \lambda(1+p_j)} \hat\tau^{-\delta_j} \left(c\hat\tau^{-\lambda}\right)^{1+p_j} = \frac{\delta_j}{\delta_j + \lambda(1+p_j)} \hat\tau^{-\delta_j} |w(\hat\tau)|^{1+p_j}.$$

Hence, defining $\theta = \min\left\{\frac{\delta_1}{\delta_1+\lambda(1+p_1)},\ \dots\ , \frac{\delta_J}{\delta_J+\lambda(1+p_J)}\right\} \in (0,1)$ we have

$$\int_{\hat\tau}^{\infty} \left|\frac{\partial}{\partial t} F_1(s, w(s))\right| ds \le \theta F_1(\hat\tau, w(\hat\tau)).$$

Inserting this estimate into (3.3) then gives

$$\tfrac{1}{2}\beta^2 w_t^2(\hat\tau) + F(w(\hat\tau)) + (1-\theta) F_1(\hat\tau, w(\hat\tau)) \le O(\hat\tau^{-2}).$$

Sending $\tau \to \infty$ (so $\hat\tau \to \infty$), we conclude that

$$\liminf_{t\to\infty}\ t^2 Q_1(t) < \infty.$$

But this contradicts the conclusion in Step 2. Hence, we must have $A > 0$. This completes the proof of Theorem 3.1. $\square$

Remark 3.1. The equality in (3.2) is the only place where the initial condition in (1.5) (i.e., (2.3)) is used. It is for the purpose of concluding that $Q_3(1) \ge 0$, or $\lim_{t\to\infty} Q_3 > 0$. On the other hand, suppose $Q_3 < 0$ for all t. Then, for every j, $\frac{a_j}{q_j+1}|w|^{q_j+1} t < \frac{(n-2)^2}{8t} w^2$, so that w does not change sign and $|w|^{q_j-1} < \frac{(q_1+1)(n-2)^2}{8a_j t^2}$ for all t; since w is bounded for large t, the latter cannot happen if there is some j such that $q_j \le 1$. In other words, the initial condition (2.3) is not needed if one assumes that there is a j such that $q_j \le 1$.

4. Asymptotic behavior of the solution as $t \to \infty$

We now study the asymptotic behavior, as $t \to \infty$, of solutions of (2.2).

Lemma 4.1. *Under the assumption of Theorem* 3.1 *the following holds:*

(i) *Both w and w_t have infinitely many zeroes, and the large zeroes of w and w_t interlace.*

(ii) *Denote by $\{t_k\}_{k=0}^{\infty}$ all the local points of maximum of w, arranged in increasing order. Let A be as in* (3.1) *and let W^A be defined as in* (1.6) *and T^A be the period of W^A. Then, as $k \to \infty$,*

$$w(t) = W^A(t - t_k) + O(t^{-\delta}) \qquad \forall\, t \in [t_k, t_k + 2T^A], \tag{4.1}$$
$$t_{k+1} - t_k = T^A + O(t_k^{-\delta}). \tag{4.2}$$

Proof. Equation (3.1) means that on the w–w_t phase plane, the trajectory $(w(t), w_t(t))$ is within an $O(t^{-\delta})$ neighborhood of the curve $\frac{1}{2}\beta^2X^2+F(Y) = F(A)$. Also the differential equation for w can be written as $\beta^2 w_{tt} + f(w) = O(t^{-\delta})$. The assertion of the lemma thus follows from a standard perturbation argument. □

Remark 4.1. Observe that Proposition A follows immediately from Lemma 4.1; in particular, from (4.2) and the fact $\lim_{k\to\infty}(t_k/t_{k+1}) = 1$, we see that the constants c_2 and c_3 in Proposition A are the same constant given by $T^A/(2\beta)$ since $r^\beta_{k+1} - r^\beta_k = \beta\tilde{r}^{\beta-1}(r_{k+1} - r_k)$ for some $\tilde{r} \in (r_k, r_{k+1})$.

The assertion of Lemma 4.1 is local in time. In order to prove Theorem 1.1, one needs to estimate the total phase shift $\sum_{k=0}^\infty(t_{k+1}-t_k-T^A)$. To do this, higher-order terms $O(t^{-\delta})$ have to be taken into account. When $\delta > 1$, the total phase shift $\sum_{k=1}^\infty(r_{k+1} - r_k - T^A)$ is finite, and Lemma 4.1 can be upgraded to a global-in-time version as follows.

Lemma 4.2. *Assume the conditions in Theorem* 3.1 *and also assume that* $\delta > 1$*. Then there exists* $B \in [0, T^A)$ *such that*

$$w(t) = W^A\big(t + B + O(t^{1-\delta})\big) + O(t^{-\delta}) \quad as \quad t \to \infty. \tag{4.3}$$

Proof. We need only to replace t_k in (4.2) by a quantity independent of k. As (4.2) implies $t_{k+1} - t_k = T^A[1 + o(1)]$, $t_k = kT^A[1 + o(1)]$ as $k \to \infty$. Consequently, $|t_{k+1} - t_k - T^A| = O(t_k^{-\delta}) = O(k^{-\delta})$. Since $\delta > 1$, by a comparison test, the series $\sum_{k=0}^\infty(t_{k+1} - t_k - T^A) =: \hat{B}$ is uniformly convergent. Hence, for any $k \ge 1$,

$$\begin{aligned} t_k &= t_0 + kT^A + \sum_{l=0}^{k-1}(t_{l+1} - t_l - T^A) \\ &= t_0 + kT^A + \sum_{l=0}^{\infty}(t_{l+1} - t_l - T^A) - \sum_{l=k}^{\infty}(t_{l+1} - t_l - T^A) \\ &= t_0 + kT^A + \hat{B} - \sum_{l=k}^{\infty} O(l^{-\delta}) = t_0 + kT^A + \hat{B} + O(k^{1-\delta}). \end{aligned}$$

Substituting this relation into (4.1) then gives, for all $t \in [t_k, t_{k+1}]$,

$$\begin{aligned} w(t) &= W^A\Big(t - t_0 - kT^A - \hat{B} + O(t_k^{1-\delta})\Big) + O(t_k^{-\delta}) \\ &= W^A\Big(t + B + O(t^{1-\delta})\Big) + O(t^{-\delta}) \end{aligned}$$

where $B \in [0, T^A)$ is the unique constant such that $B + t_0 + \hat{B}$ is an integer multiple of T^A. As B is independent of k, the assertion of the lemma thus follows. □

Next we consider the case $\delta \le 1$. In this case $\sum_{k=0}^{\infty} |t_{k+1} - t_k - T^A| = \sum_{k=0}^{\infty} |O(k^{-\delta})|$ may diverge, so we need an accurate expression for the $O(t^{-\delta})$ term. To do this, we want to use the first equation in (3.1), where we need to calculate the integral $\int_t^\infty \frac{\partial}{\partial t} F_1$. By Lemma 4.1, this integral can be calculated as follows. For every large k,

$$\int_{t_k}^{t_{k+1}} t^{-1-\delta_j} |w(t)|^{1+p_j}\, dt = \big[t_k^{-1-\delta_j} + O(t_k^{-2-\delta_j})\big]\Big\{\int_0^{T^A} |W^A(t)|^{1+p_j} + O(t_k^{-\delta})\Big\}$$
$$= \bar{A}_j^{1+p_j} \int_{t_k}^{t_{k+1}} \big[t^{-1-\delta_j} + O(t^{-1-2\delta})\big]\, dt$$

where

$$\bar{A}_j := \Big(\frac{1}{T^A}\int_0^{T^A} |W^A|^{1+p_j}\Big)^{1/(1+p_j)}. \tag{4.4}$$

Hence,

$$-\int_{t_k}^{\infty} \tfrac{\partial}{\partial t} F_1(t, w(t))\, dt = \sum_{l=k}^{\infty} \int_{t_l}^{t_{l+1}} \sum_j \tfrac{b_j \delta_j}{1+p_j} t^{-1-\delta_j} |w|^{1+p_j}$$
$$= \sum_j \tfrac{b_j \bar{A}_j^{1+p_j}}{1+p_j} \int_{t_k}^{\infty} \delta_j \big[t^{-1-\delta_j} + O(t^{-1-2\delta})\big] = \sum_j \tfrac{b_j \bar{A}_j^{1+p_j}}{1+p_j} t_k^{-1-\delta_j} + O(t_k^{-2\delta}).$$

Now we can apply the first equation in (3.1) for $t = t_k$ to deduce, as $w_t(t_k) = 0$, that

$$F(w(t_k)) - F(A)O(t_k^{-2}) - F_1(t_k, w(t_k)) - \int_{t_k}^{\infty} \tfrac{\partial}{\partial t} F_1$$
$$= O(t_k^{-2\delta}) - \sum_j \tfrac{b_j}{1+p_j}\Big[A^{1+p_j} - \bar{A}_j^{1+p_j}\Big] t_k^{-\delta_j}.$$

The mean-value theorem then gives, as $\frac{\partial}{\partial w} F(w) = f(w)$,

$$w(t_k) - A = \varepsilon_k + O(t_k^{-2\delta}), \qquad \varepsilon_k := \sum_j \frac{b_j}{1+p_j} \frac{\bar{A}_j^{1+p_j} - A^{1+p_j}}{f(A)} t_k^{-\delta_j} < 0.$$

Therefore,

$$\begin{cases} \beta^2 w_{tt} + f(w) + f_1(t_k, w) = O(t_k^{-2\delta}), \quad t \in [t_k, t_k + 2T^A], \\ w(t_k) = A + \varepsilon_k + O(t_k^{-2\delta}), \quad w_t(t_k) = 0. \end{cases} \tag{4.5}$$

Approximately (in the order of $O(t_k^{-2\delta})$) solving this equation for $t \in [t_k, t_k + 2T^A]$ then gives

$$
\begin{aligned}
&t_{k+1} - t_k \\
&= O(t_k^{-2\delta}) + \int_0^{A+\varepsilon_k} \frac{\sqrt{8}\beta \, dw}{\sqrt{F(A+\varepsilon_k) + F_1(t_k, A+\varepsilon_k) - F(w) - F_1(t_k, w)}} \\
&= T^A + \sum_j \xi_j \, t_k^{-\delta_j} + O(t_k^{-2\delta})
\end{aligned}
$$

where $\xi_1, \dots, \xi_J$ are constants and can be calculated as follows. Notice that

$$
\frac{d}{d\varepsilon} \int_0^{A+\varepsilon} \frac{dw}{\sqrt{F(A+\varepsilon) - F(w)}} \Big|_{\varepsilon=0} = \frac{d}{d\varepsilon} \int_0^1 \frac{(A+\varepsilon)\, d\zeta}{\sqrt{F(A+\varepsilon) - F((A+\varepsilon)\zeta)}} \Big|_{\varepsilon=0}
$$

$$
= \int_0^A \frac{[2F(A) - f(A)] - [2F(w) - wf(w)/A]}{2A[F(A) - F(w)]^{3/2}} dw,
$$

$$
\frac{d}{d\eta} \int_0^A \frac{dw}{\sqrt{F(A) + \eta g(A) - F(w) - \eta g(w)}} \ \Big|_{\eta=0} = \int_0^A \frac{g(w) - g(A)}{2[F(A) - F(w)]^{3/2}} dw.
$$

It then follows, after taking $\eta g = F_1(t_k, w) = \sum_j \frac{b_j}{p_j+1} t_k^{-\delta_j} |w|^{p_j+1}$, that

$$
\begin{aligned}
&\int_0^{A+\varepsilon_k} \frac{\sqrt{8}\beta \, dw}{\sqrt{F(A+\varepsilon_k) + F_1(t_k, A+\varepsilon_k) - F(w) - F_1(t_k, w)}} \\
&= T^A + \varepsilon_k \int_0^A \frac{[2F(A) - f(A)] - [2F(w) - wf(w)/A]}{2A[F(A) - F(w)]^{3/2}} \, dw \\
&\qquad + \sum_j \frac{b_j}{p_j + 1} t_k^{-\delta_j} \int_0^A \frac{w^{p_j+1} - A^{p_j+1}}{2[F(A) - F(w)]^{3/2}} dw + O(t_k^{-2\delta}).
\end{aligned}
$$

Hence,

$$
\xi_j = \frac{\sqrt{8}\beta b_j}{1 + p_j} \int_0^A \frac{A\{w^{1+p_j} - A^{1+p_j}\} + (\bar{A}_j^{1+p_j} - A^{1+p_j})\{[2F(A) - f(A)] - [2F(w) - wf(w)/A]\}}{2A[F(A) - F(w)]^{3/2}} \, dw.
$$

Now we can compute

$$
\begin{aligned}
t_k &= t_0 + kT^A + \sum_{l=0}^{k-1} (t_{l+1} - t_l - T^A) \\
&= t_0 + kT^A + \sum_{l=0}^{k-1} \Big\{ O(t_l^{-2\delta}) + \sum_j \xi_j t_l^{-\delta_j} \Big\} \qquad (4.6)
\end{aligned}
$$

$$= t_0 + kT^A + \sum_{l=0}^{k-1} \Big\{ O(l^{-2\delta}) + \sum_j \xi_j (lT^A)^{-\delta_j} \Big\}.$$

In case $\delta > 1/2$, we have

$$\begin{aligned} t_k &= t_0 + kT^A + \sum_j \ \xi_j \sum_{l=1}^{k-1} (lT^A)^{-\delta_j} + \Big\{ \sum_{l=1}^{\infty} - \sum_{l=k}^{\infty} \Big\} O(l^{-2\delta_j}) \\ &= \hat{B} + kT^A + \sum_j \ \xi_j (1-\delta_j)^{-1} k (kT^A)^{-\delta_j} + O(k^{1-2\delta}) \\ &\quad \big(\text{if } \delta_j = 1, \ (1-\delta_j)^{-1} k (kT^A)^{-\delta_j} \text{ should be replaced by } \ln(kT^A)\big). \end{aligned}$$

Here we have used the fact that $\sum_{l=1}^{k-1} l^{-\delta_j} = B_j + k^{1-\delta_j}/(1-\delta_j) + O(k^{1-2\delta})$ if $\delta_j \neq 1$ and $= B_j + \ln k + O(k^{-1})$ if $\delta_j = 1$.

Similarly, when $\delta \in (0, 1/2]$,

$$t_k = kT^A + \sum_j \ \xi_j (1-\delta_j)^{-1} k (kT^A)^{-\delta_j} + O(k^{1-2\delta})$$

where in case $\delta = 1/2$, one needs to replace $O(k^{1-2\delta})$ by $O(\ln k)$.

Finally, we replace t_k in (4.1) by the above expressions. Note that for $t \in [t_k, t_k + 2T^A]$, $(kT^A)^{1-\delta_j} - t^{1-\delta_j} = O((\min\{kT^A, t\})^{-\delta_j}) |t - kT^A| = O(t^{1-2\delta_j})$. Therefore, we have the following theorem.

Theorem 4.3. *Assume* **(A1), (A2)**, *and that w is a nontrivial solution to* (2.2) *satisfying* (2.3). *Then there exists $A > 0$ such that as $t \to \infty$, the following holds:*

(i) *If $\delta > 1/2$, then for some B and $z_1, \dots z_J$,*

$$w(t) = W^A\Big(t + \sum_j z_j t^{1-\delta_j} + B + O(t^{1-2\delta})\Big) + O(t^{-\delta}), \qquad (4.7)$$

where in case $\delta_j = 1$, $t^{1-\delta_j}$ should be replaced by $\ln t$.

(ii) *If $\delta \in (0, 1/2]$, then*

$$w(t) = W^A\Big(t + \sum_j z_j t^{1-\delta_j} + O(t^{1-2\delta})\Big) + O(t^{-\delta}). \qquad (4.8)$$

Here if $\delta = 1/2$, then $O(t^{1-2\delta})$ should be replaced by $O(\ln t)$.

Clearly, our main Theorem 1.1 follows from Theorem 4.3 and the transformation (2.1).

Remark 4.2. By solving (4.5) one can replace the last term $O(t^{-\delta})$ in (4.7) and (4.8) by

$$\sum_j t^{-\delta_j} W_j\Big(t + \sum_j z_j t^{1-\delta_j} + O(t^{1-2\delta})\Big) + O(t^{-2\delta}),$$

where W_j are calculable functions; therefore, one obtains a complete second-order expansion of the solution.

The next order expansion of the form

$$r_{k+1} = r_k + T^A + \sum_j \xi_j\, t_k^{-\delta_j} + \sum_{j_1,j_2=1}^{J} \xi_{j_1 j_2}\, t_k^{-\delta_{j_1}-\delta_{j_2}} + O(r^{-3\delta}) \tag{4.9}$$

and a similar enhanced version for $w(t)$ can be obtained as follows. Instead of the leading order expansion for w and $r_{k+1} - r_k$ given in Lemma 4.1, we substitute the enhanced $O(t^{-2\delta})$-order expression of w and $t_{k+1} - t_k$ into the first equation in (3.1) (replacing the $O(t^{-2})$-term by its exact expression $-F_2(t,w) - \int_t^\infty F_2$ if necessary) to obtain an $O(r_k^{-3\delta})$-order expression $w(t_k) = A + \varepsilon_k + \varepsilon_k^{(2)} + O(r^{-3\delta})$, where $\varepsilon_k^{(2)}$ depends only on r_k and known quantities. Then one can solve a refined version of (4.5) to obtain an $O(r_k^{-3\delta})$-order solution (depending on r_k) and hence obtain a $O(r_k^{-3\delta})$-order recursive relation (4.9) between r_{k+1} and r_k. Following the same calculation as in (4.6) one can solve this recursive relation to obtain an explicit $O(r^{1-3\delta})$-order expression for r_k, and therefore an $O(t^{1-3\delta})$-order explicit expression for the phase shift of $w(t)$ from that of W^A.

In a similar manner, one can obtain desired higher-order expansions, provided sufficient differentiabilities for F, F_1, F_2 are available.

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