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COMPARISON RESULTS FOR SOLUTIONS OF ELLIPTIC PROBLEMS VIA STEINER SYMMETRIZATION

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Abstract. We consider the Dirichlet problem for a class of linear elliptic equations, whose model is

$$-\Delta u-\sum_{i=1}^{n}(b_i(y)u)_{x_i}-\sum_{j=1}^{m}\Big(\widetilde{b}_j(y)u\Big)_{y_j}+\sum_{i=1}^{n}d_i(y)u_{x_i}$$
$$+\sum_{j=1}^{m}\widetilde{d}_j(y)u_{y_j}+c(y)u=f(x,y)\quad \text{in } G,$$

where $G=G'\times G''$ is an open, bounded and connected subset of $\mathbb{R}^N=\mathbb{R}^n\times\mathbb{R}^m$, the coefficients $b_i(y)$, $\widetilde{b}_j(y)$, $d_i(y)$, $\widetilde{d}_j(y)$ and $c(y)$ are in $L^\infty(G)$ and the datum $f(x,y)$ belongs to $L^p(G)$ with $p>\frac{N}{2}$. We prove some comparison results by using Steiner symmetrization.

1. Introduction

Let G be an open and bounded subset of $\mathbb{R}^N$, and let us consider the following problem:

$$\begin{cases}-\sum_{i,j=1}^{N}\left(a_{ij}(x)u_{x_i}\right)_{x_j}=f & \text{in } G\\ u=0 & \text{on } \partial G\end{cases}\tag{1.1}$$

with ellipticity condition

$$\sum_{i,j=1}^{N}a_{ij}(x)\xi_i\xi_j\geq|\xi|^2,\quad \forall\xi\in\mathbb{R}^N.\tag{1.2}$$

In [18] (see also [20]) it was proved that the L^p norm of the solution u of (1.1) can be estimated in terms of the same norm of the solution v of the

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following spherically symmetric problem:

$$\begin{cases} -\triangle v = f^\star & \text{in } G^\star \\ v = 0 & \text{on } \partial G^\star, \end{cases}$$

where $G^\star$ is the ball of $\mathbb{R}^N$centered at the origin having the same measure of G and $f^\star$ is the Schwarz symmetrization of f, i.e., the spherically symmetric function from $\mathbb{R}^N$ into $[0,+\infty)$, whose level sets $\{x \in G^\star : f^\star(x) > t\}$ are concentric balls with the same measure as the level sets $\{x \in G : |f(x)| > t\}$ of $|f|$. More precisely, in order to obtain such a priori estimates, Talenti proves a pointwise comparison result of the type[1]

$$u^*(s) \leq v^*(s), \quad \forall s \in [0, |G|_N]. \tag{1.3}$$

We recall that if u is a measurable function defined in G the decreasing rearrangement u^* is the function

$$u^*(s) = \sup\{t \geq 0 : \mu(t) \geq s\}, \quad s \in [0, |G|_N], \tag{1.4}$$

where $\mu(t) = |\{x \in G : |u(x)| > t\}|_N$, $t \geq 0$, is the distribution function of u. The function u^* is the unique function, decreasing in $[0,+\infty)$, with the same distribution function as u. Moreover, the Schwarz-symmetrized $u^\star$ of u is

$$u^\star(x) = u^*(\omega_N |x|^N), \quad x \in G^\star, \tag{1.5}$$

where ω_N is the measure of the unit ball of $\mathbb{R}^N$.

The comparison result of [18] has been generalized in many directions: by introducing lower-order terms (see e.g. [5] and [6]), by studying the same problem for nonlinear operators (see e.g. [14] and [19]), for parabolic operators (see e.g. [4], [6] and [16]) and for operators which satisfy a weaker ellipticity condition (see e.g. [4]).

In all those papers the estimates of the L^p norms or Orlicz norms of u are a consequence of an inequality such as (1.3) or a weaker one,

$$\int_0^s u^*(\sigma)d\sigma \leq \int_0^s v^*(\sigma)d\sigma \;\forall s \in [0, |G|_N]. \tag{1.6}$$

In any case the comparison results are obtained by using Schwarz symmetrization, so that the differential problems can lose properties that arise from symmetries of the data with respect to axes, planes for instance. So one can ask if comparison results which preserve the symmetries of the data

[1] We will denote by $|E|_N$ the Lebesgue measure of a measurable subset E of $\mathbb{R}^N$.

hold. In order to obtain this kind of results it is natural to use a partial-symmetrization process such as Steiner symmetrization (see e.g. [7], [13] and [17]).

To introduce Steiner symmetrization we need some definitions and notation.

Let $G = G' \times G''$ be an open, bounded and connected subset of $\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m$ $(n + m = N)$ and let u be a measurable function defined on G; if (x, y) is a point in G, for every fixed $y \in G''$, we consider the decreasing rearrangement of the function $u(\cdot, y) : x \to u(x, y) \in \mathbb{R}$, that is,

$$u^*(s, y) = \sup\{t \geq 0 : \mu(t, y) \geq s\} \quad s \in \left[0, \left|G'\right|_n\right], \tag{1.7}$$

where $\mu(t, y) = \left|\{x \in G' : u(x, y) > t\}\right|_n$, with $t \geq 0$, is the distribution function of $u(\cdot, y)$.

The Steiner rearrangement of u with respect to x is the function $u^{\#}(x, y)$ defined as follows:

$$u^{\#}(x, y) = u^*(\omega_n |x|^n, y), \tag{1.8}$$

with $(x, y) \in G^{\#} = B \times G''$, where B is the ball of $\mathbb{R}^n$ centered on zero whose measure is $|G'|_n$.

In this paper by using Steiner symmetrization we prove a comparison result for linear elliptic problems whose model is

$$\begin{cases} -\Delta u - \sum\limits_{i=1}^{n} (b_i(y)u)_{x_i} - \sum\limits_{j=1}^{m} \left(\widetilde{b}_j(y)u\right)_{y_j} + \\ + \sum\limits_{i=1}^{n} d_i(y)u_{x_i} + \sum\limits_{j=1}^{m} \widetilde{d}_j(y)u_{y_j} + c(y)u = f(x, y) \quad \text{in } G \\ u = 0 \quad \text{on } \partial G, \end{cases} \tag{1.9}$$

where the coefficients $b_i(y)$, $\widetilde{b}_j(y)$, $d_i(y)$, $\widetilde{d}_j(y)$ and $c(y)$ belong to $L^\infty(G)$, and the datum $f(x, y)$ is in $L^p(G)$ with $p > \frac{N}{2}$.

More precisely, in Section 3 we prove the following inequality:

$$\int_0^s u^*(\sigma, y)d\sigma \leq \int_0^s v^*(\sigma, y)d\sigma \quad \forall s \in \left[0, \left|G'\right|_n\right], \tag{1.10}$$

where u is the weak solution of problem (1.9) and v is the weak solution of a problem whose data are symmetrized in the sense of Steiner, that is,

$$\begin{cases} -\Delta v - \sum\limits_{j=1}^{m} \left(\widetilde{b}_j(y)v\right)_{y_j} + \sum\limits_{j=1}^{m} \widetilde{d}_j(y)v_{y_j} + c(y)v = f^{\#} \quad \text{in } G^{\#} \\ v = 0 \quad \text{on } \partial G^{\#}. \end{cases} \tag{1.11}$$

The estimate (1.10) allows us to get an a priori estimate of the Orlicz norm of u. Moreover, we prove the result also for a more general elliptic linear operator.

The same result has been proved in [1] and [2] when the second-order elliptic operator does not contain lower-order terms.

Finally, in Section 4 we will also obtain an inequality involving the gradients of u and v, which gives the "energy estimate"

$$\int_G |\nabla u|^2 \, dx \, dy \leq \int_{G^\#} |\nabla v|^2 \, dx \, dy \tag{1.12}$$

when $\widetilde{b}_j(y) = \widetilde{d}_j(y) = c(y) = 0$, for $j = 1, \ldots, m$.

We would like to thank F. Brock for giving precious advice during some conversations.

2. Preliminary results and notation

Let $G = G' \times G''$ be an open, bounded subset of $\mathbb{R}^n \times \mathbb{R}^m$, and let u be an analytic function on $\overline{G}$ such that $u > 0$ on G and $u = 0$ on ∂G.

Let us define the following function:

$$F(s, y) = \int_0^s u^*(\sigma, y) d\sigma \tag{2.1}$$

where $s \in \left[0, \left|G'\right|_n\right]$ and $y \in G''$.

Since u has no flat zone (in fact u is analytic), it is known that

$$F(s, y) = \int_{\{u>t\}} u(x, y) dx \tag{2.2}$$

where $\{u > t\}$ denotes the set $\left\{x \in G' : u(x, y) > t\right\}$ and t is chosen such that $\left|\left\{x \in G' : u(x, y) > t\right\}\right| = s$; that is, $t = u^*(s, y)$.

Now we will recall some results, contained in [2] and [6] (see also [10] and [16]), which will be used in the next section. These results evaluate the derivatives of the function $F(s, y)$ with respect to y.

Lemma 2.1. *Let $F(s, y)$ be the function defined in (2.2). Then it results that*

$$\frac{\partial F}{\partial y_k} = \int_{u>t} \frac{\partial u}{\partial y_k} \, dx, \tag{2.3}$$

$$\frac{\partial^2 F}{\partial y_k \partial y_h} \geq \int_{u>t} \frac{\partial^2 u}{\partial y_k \partial y_h} dx, \tag{2.4}$$

where the last inequality is in the sense of matrices.

Theorem 2.1. *If $g \in C^1(G)$ we have*[2]

$$\frac{\partial}{\partial y_k}\int_{u>t} g(x,y)dx = \int_{u>t}\frac{\partial g}{\partial y_k}dx - \int_{u=t} g\alpha_k \, dH^{n-1}(x), \tag{2.5}$$

where

$$\alpha_j(x,y) = \frac{1}{|\nabla_x u|}\left\{\frac{\int_{u=t}\frac{u_{y_j}}{|\nabla_x u|}\, dH^{n-1}(x)}{\int_{u=t}\frac{1}{|\nabla_x u|}dH^{n-1}(x)} - u_{y_j}\right\} \ \forall j \in \{1,\dots,m\} \tag{2.6}$$

and $H^{n-1}(x)$ is the $(n-1)$-dimensional Hausdorff measure.

By definition, the function α_j satisfies the following relation:

$$\int_{u=t}\alpha_j \, dH^{n-1}(x) = 0, \ \forall j \in \{1,\dots,m\}. \tag{2.7}$$

Thanks to formula (2.5) we can specify inequality (2.4); indeed, if one chooses $g = \frac{\partial u}{\partial y_j}$ in (2.5), one easily obtains (see [2] for explicit calculations)

$$\frac{\partial^2 F}{\partial y_j \partial y_h} = \int_{u>t}\frac{\partial^2 u}{\partial y_j \partial y_h}\, dx + \int_{u=t}\alpha_h\alpha_j \, |\nabla_x u| \, dH^{n-1}(x), \tag{2.8}$$

which implies (2.4) since $(\alpha_h\alpha_j)_{hj}$ is a semidefinite positive matrix.

3. Main results

For the sake of simplicity we begin by proving the comparison result in the particular case when the principal part of the operator is the Laplacian.

Theorem 3.1. *Let us consider the following problem:*

$$\begin{cases} Lu = -\Delta u - \sum\limits_{i=1}^{n}(b_i(y)u)_{x_i} - \sum\limits_{j=1}^{m}\left(\widetilde{b}_j(y)u\right)_{y_j} + \\ + \sum\limits_{i=1}^{n} d_i(y)u_{x_i} + \sum\limits_{j=1}^{m}\widetilde{d}_j(y)u_{y_j} + c(y)u = f(x,y) \quad \text{in } G \\ u \in H_0^1(G), \end{cases} \tag{3.1}$$

where $G = G' \times G'$ is an open, bounded and connected subset of $\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m$. We assume that the following conditions hold:

(i) $b_i(y)$, $\widetilde{b}_j(y)$, $d_i(y)$, $\widetilde{d}_j(y)$, $c(y) \in L^\infty(G)$, $\forall i \in \{1,\dots,n\}$ *and* $\forall j \in \{1,\dots,m\}$, *and $f(x,y) \in L^p(G)$ with $p > \frac{N}{2}$;*

[2] $\{u = t\}$ will represent the set $\{x \in G' : u(x,y) = t\}$ for each fixed y.

(ii) $c(y) - \sum_{j=1}^{m} \frac{\partial}{\partial y_j}\widetilde{b}_j(y) \geq 0$ *in* $\mathcal{D}'(G)$ *and* $f(x,y) \geq 0$ *almost everywhere in* $(x,y) \in G$.

Moreover, let v *be the solution of the following problem:*

$$\begin{cases} -\Delta v - \sum_{j=1}^{m} \left(\widetilde{b}_j(y)v\right)_{y_j} + \sum_{j=1}^{m} \widetilde{d}_j(y)v_{y_j} + c(y)v = f^{\#} \quad \text{in } G^{\#} \\ v \in H_0^1(G^{\#}), \end{cases} \tag{3.2}$$

where $G^{\#} = B \times G''$, *where* B *is the ball of* $\mathbb{R}^n$ *centered on zero whose measure is* $|G'|_n$.

Then, if u *is the solution of* (3.1), *we have for almost any* $y \in G''$

$$\int_0^s u^*(\sigma,y)d\sigma \leq \int_0^s v^*(\sigma,y)d\sigma,\ \forall s \in \left[0, \left|G'\right|_n\right] \tag{3.3}$$

where $u^*(\sigma,y)$ *and* $v^*(\sigma,y)$ *are the decreasing rearrangements of the functions* $u(\cdot,y)$ *and* $v(\cdot,y)$ *respectively.*

Proof. Step 1 (the smooth case). In this step we will prove Theorem 3.1 assuming that coefficients b_i, $\widetilde{b}_j$, d_i, $\widetilde{d}_j$ and c and the datum f are analytic so that the solution u of (3.1) is analytic too. Note that in this case equation (3.1) holds pointwise.

We fix y and we consider a noncritical value $u^*(s,y) = t$. By integrating equation (3.1) on the set $\{x : u(x,y) > t\}$ we get

$$\begin{aligned} -\int_{u>t} \Delta_x u\, dx - \int_{u>t} \Delta_y u\, dx - \int_{u>t} \sum_{i=1}^{n} (b_i(y)u)_{x_i}\, dx \\ - \int_{u>t} \sum_{j=1}^{m} \left(\widetilde{b}_j(y)u\right)_{y_j} dx + \int_{u>t} \sum_{i=1}^{n} d_i(y)u_{x_i} dx \\ + \int_{u>t} \sum_{j=1}^{m} \widetilde{d}_j(y)u_{y_j} dx + \int_{u>t} c(y)u\, dx = \int_{u>t} f(x,y)dx, \end{aligned} \tag{3.4}$$

where $\Delta_x u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial^2 x_i}$ and $\Delta_y u = \sum_{j=1}^{m} \frac{\partial^2 u}{\partial^2 y_j}$.

Now let us estimate the various terms in (3.4). The first integral can be estimated from below via the classical isoperimetric inequality (see [8]), the Schwarz inequality and the co-area formula (see [9]). So in a standard way (see e.g. [18]) we have

$$-\int_{u>t} \Delta_x u\, dx = \int_{u=t} |\nabla_x u|\, dH^{n-1}(x) \geq n^2 \omega_n^{\frac{2}{n}} \Big[-\frac{\partial u^*}{\partial s}(s,y) \Big] s^{2-\frac{2}{n}}. \tag{3.5}$$

Then from Lemma 2.1, taking into account that u is a smooth and nonnegative function, we deduce that

$$-\int_{u>t}\sum_{j=1}^{m}\frac{\partial^2 u}{\partial y_j^2}\,dx \geq -\sum_{j=1}^{m}\frac{\partial^2}{\partial y_j^2}\int_0^s u^*(\sigma,y)d\sigma. \tag{3.6}$$

Moreover, it holds that

$$\begin{aligned}-\int_{u>t}\sum_{i=1}^{n}(b_i(y)u)_{x_i}\,dx &= \sum_{i=1}^{n}\int_{u=t} b_i(y)u\frac{u_{x_i}}{|\nabla_x u|}\,dH^{n-1}(x) \qquad (3.7)\\ &= t\sum_{i=1}^{n} b_i(y)\int_{u=t}\frac{u_{x_i}}{|\nabla_x u|}\,dH^{n-1}(x)=0.\end{aligned}$$

Then using Theorem 2.1 and (2.7), we get

$$\begin{aligned}-\int_{u>t}\Big(\widetilde{b}_j(y)u\Big)_{y_j}\,dx &= -\frac{\partial}{\partial y_j}\int_{u>t}\widetilde{b}_j(y)u\,dx-\widetilde{b}_j(y)t\int_{u=t}\alpha_j\,dH^{n-1}(x)\\ &= -\frac{\partial}{\partial y_j}\Big(\widetilde{b}_j(y)\int_{u>t}u\,dx\Big), \qquad (3.8)\end{aligned}$$

for $j\in\{1,\dots,m\}$. From formula (2.3) for $j\in\{1,\dots,m\}$, we have

$$\int_{u>t}\widetilde{d}_j(y)u_{y_j}\,dx=\widetilde{d}_j(y)\frac{\partial}{\partial y_j}\int_{u>t}u\,dx, \tag{3.9}$$

and for divergence theorem it holds that

$$\int_{u>t}\sum_{i=1}^{n}d_i(y)u_{x_i}\,dx=\sum_{i=1}^{n}d_i(y)\int_{u>t}u_{x_i}\,dx=0. \tag{3.10}$$

Finally, the last integral in (3.4) can be estimated by using the classical Hardy inequality for rearrangements (see e.g. [12]),

$$\int_{u>t}f(x,y)dx\leq\int_0^s f^*(\sigma,y)d\sigma, \tag{3.11}$$

where $s=|\{x\in G' : u(x,y)>t\}|_n$.

Collecting estimates (3.5)–(3.11) and taking into account (2.2), we get

$$\begin{aligned}&n^2\omega_n^{\frac{2}{n}}\Big[-\frac{\partial u^*}{\partial s}(s,y)\Big]s^{2-\frac{2}{n}}-\sum_{j=1}^{m}\frac{\partial^2}{\partial y_j^2}\int_0^s u^*(\sigma,y)d\sigma \qquad (3.12)\\ &-\sum_{j=1}^{m}\frac{\partial}{\partial y_j}\Big(\widetilde{b}_j(y)\int_0^s u^*(\sigma,y)d\sigma\Big)+\sum_{j=1}^{m}\widetilde{d}_j(y)\frac{\partial}{\partial y_j}\int_0^s u^*(\sigma,y)d\sigma\end{aligned}$$

$$+ c(y) \int_0^s u^*(\sigma, y) d\sigma \le \int_0^s f^*(\sigma, y) d\sigma.$$

If we set

$$U(s,y) = \int_0^s u^*(\sigma, y) d\sigma, \tag{3.13}$$

then from inequality (3.12) we obtain

$$-n^2 \omega_n^{\frac{2}{n}} s^{2-\frac{2}{n}} \frac{\partial^2 U}{\partial s^2} - \sum_{j=1}^m \frac{\partial^2 U}{\partial y_j^2} - \sum_{j=1}^m \frac{\partial}{\partial y_j} \left(\widetilde{b}_j(y) U(s,y) \right) \tag{3.14}$$
$$+ \sum_{j=1}^m \widetilde{d}_j(y) \frac{\partial U}{\partial y_j} + c(y) U(s,y) \le \int_0^s f^*(\sigma, y) d\sigma$$

where $(s,y) \in [0, |G'|_n] \times G''$. Besides inequality (3.14), function U satisfies the following boundary conditions:

$$\begin{cases} U(0,y) = 0 \quad U_s(|G'|_n, y) = 0, \ \forall y \in G'' \\ U(s,y) = 0 \ \ \forall y \in \partial G'', \ \forall s \in (0, |G'|_n). \end{cases} \tag{3.15}$$

Now we define

$$V(s,y) = \int_0^s v^*(\sigma, y) d\sigma \tag{3.16}$$

where v is the solution of problem (3.2). Handling problem (3.2) in the same way as (3.1) we get in place of inequality (3.14) the following equality:

$$-n^2 \omega_n^{\frac{2}{n}} s^{2-\frac{2}{n}} \frac{\partial^2 V}{\partial s^2} - \sum_{j=1}^m \frac{\partial^2 V}{\partial y_j^2} - \sum_{j=1}^m \frac{\partial}{\partial y_j} \left(\widetilde{b}_j(y) V(s,y) \right) + \tag{3.17}$$
$$+ \sum_{j=1}^m \widetilde{d}_j(y) \frac{\partial V}{\partial y_j} + c(y) V(s,y) = \int_0^s f^*(\sigma, y) d\sigma.$$

Moreover, the function V satisfies the same boundary conditions (3.15) as U; that is,

$$\begin{cases} V(0,y) = 0 \quad V_s(|G'|_n, y) = 0, \ \forall y \in G'' \\ V(s,y) = 0 \ \ \forall y \in \partial G'', \ \forall s \in (0, |G'|_n). \end{cases} \tag{3.18}$$

Finally, if we set

$$Z(s,y) = U(s,y) - V(s,y) = \int_0^s u^*(\sigma, y) d\sigma - \int_0^s v^*(\sigma, y) d\sigma, \tag{3.19}$$

from (3.14) and (3.17) we get

$$-n^2\omega_n^{\frac{2}{n}} s^{2-\frac{2}{n}} \frac{\partial^2 Z}{\partial s^2} - \sum_{j=1}^{m} \frac{\partial^2 Z}{\partial y_j^2} - \sum_{j=1}^{m} \frac{\partial}{\partial y_j}\left(\widetilde{b}_j(y) Z(s,y)\right) \tag{3.20}$$
$$+ \sum_{j=1}^{m} \widetilde{d}_j(y) \frac{\partial Z}{\partial y_j} + c(y) Z(s,y) \leq 0.$$

Moreover, Z satisfies the same boundary conditions as U and V.

From the last inequality, by using the maximum principle, and taking into account (ii), we have

$$Z(s,y) \leq 0 \quad \text{on } G^{\#}, \tag{3.21}$$

which gives formula (3.3).

Step 2 (an approximation argument). By standard density arguments, it is easy to find, $\forall i \in \{1,\ldots,n\}$ and $\forall j \in \{1,\ldots,m\}$, some sequences of analytic functions $\{b_i^k\}$, $\{\widetilde{b}_j^k\}$, $\{d_i^k\}$, $\{\widetilde{d}_j^k\}$ and $\{c^k\}$ uniformly bounded in $L^\infty(G)$, such that

$$c^k(y) - \sum_{j=1}^{m} \frac{\partial}{\partial y_j} \widetilde{b}_j^k(y) \geq 0, \qquad \forall y \in G'', \ \forall k \in \mathbb{N} \tag{3.22}$$

and

$$\begin{cases} b_i^k \to b_i & \text{in } L^q(G)\ \forall 1 \leq q < \infty \\ \widetilde{b}_j^k \to \widetilde{b}_j & \text{in } L^q(G)\ \forall 1 \leq q < \infty \\ d_i^k \to d_i & \text{in } L^q(G)\ \forall 1 \leq q < \infty \\ \widetilde{d}_j^k \to \widetilde{d}_j & \text{in } L^q(G)\ \forall 1 \leq q < \infty \\ c^k \to c & \text{in } L^q(G)\ \forall 1 \leq q < \infty. \end{cases} \tag{3.23}$$

In the same way it is possible to find a sequence of analytic functions $\{f^k\}$, uniformly bounded in $L^p(G)$, such that

$$f^k(x,y) \geq 0, \ \forall (x,y) \in G, \ \forall k \in \mathbb{N} \tag{3.24}$$

and

$$f^k \to f \quad \text{in } L^p(G). \tag{3.25}$$

For example one can obtain these sequences as follows:

$$b_i^k(y) = \frac{k^m}{(4\pi)^{\frac{m}{2}}} \int_{G''} e^{-\frac{k^2}{4}|y-\widehat{y}|^2} b_i(\widehat{y}) d\widehat{y} \ \ \forall k \in \mathbb{N}. \tag{3.26}$$

Analogously, it is possible to approach the other coefficients $\widetilde{b}_j(y)$, $d_i(y)$, $\widetilde{d}_j(y)$ and $c(y)$. Finally the approximate sequence of data $f^k(x,y)$ can be obtained in the following way:

$$f^k(z) = \frac{k^N}{(4\pi)^{\frac{N}{2}}} \int_{\mathbb{G}} e^{-\frac{k^2}{4}|z-\widehat{z}|^2} f(\widehat{z}) d\widehat{z} \quad \forall k \in \mathbb{N}, \tag{3.27}$$

where $z \equiv (x,y) \in \mathbb{R}^N$.

It is easy to check that the approximate coefficients and data are analytic in $\overline{G}$ and they verify conditions (3.22), (3.23), (3.24) and (3.25).

Let $\{u^k\}$ be the solutions of the following sequence of problems:

$$\begin{cases} -\Delta u^k - \sum\limits_{i=1}^{n} \left(b_i^k(y) u^k\right)_{x_i} - \sum\limits_{j=1}^{m} \left(\widetilde{b}_j^k(y) u^k\right)_{y_j} \\ + \sum\limits_{i=1}^{n} d_i^k(y) u^k_{x_i} + \sum\limits_{j=1}^{m} \widetilde{d}_j^k(y) u^k_{y_j} + c^k(y) u^k = f^k(x,y) \quad \text{in } G \\ u^k \in H_0^1(G) \ , \end{cases} \tag{3.28}$$

and $\{v^k\}$ the solutions of the sequence of "symmetrized" problems

$$\begin{cases} -\Delta v^k - \sum\limits_{j=1}^{m} \left(\widetilde{b}_j^k(y) v^k\right)_{y_j} + \sum\limits_{j=1}^{m} \widetilde{d}_j^k(y) v^k_{y_j} + c^k(y) v^k = \left(f^k\right)^{\#} \text{ in } G^{\#} \\ v^k \in H_0^1(G^{\#}). \end{cases} \tag{3.29}$$

By step 1, for the solutions $u^k$and v^k of problems (3.28) and (3.29) the following estimate holds:

$$\int_0^s (u^k)^*(\sigma,y) d\sigma \le \int_0^s (v^k)^*(\sigma,y) d\sigma, \ \forall s \in \left[0, \left|G'\right|_n\right] \text{ and } \forall k \in \mathbb{N}. \tag{3.30}$$

Now, since the sequences of approximate coefficients $\{b_i^k\}$, $\{\widetilde{b}_j^k\}$, $\{d_i^k\}$, $\{\widetilde{d}_j^k\}$ and $\{c^k\}$ are uniformly bounded in $L^\infty(G)$, and the sequence of data $\{f^k\}$ is uniformly bounded in $L^p(G)$, the sequence $\{u^k\}$ is uniformly bounded in $H_0^1(G)$ (see e.g. [11]). So, up to a subsequence, we can say that there exists a function $\widehat{u} \in H_0^1(G)$ such that

$$\begin{cases} u^k \rightharpoonup \widehat{u} \text{ weakly in } H_0^1(G), \\ u^k \to \widehat{u} \text{ in } L^p(G) \ \forall p < \frac{2N}{N-2}, \\ u^k \to \widehat{u} \text{ a.e. in } G. \end{cases} \tag{3.31}$$

Now one can easily verify, passing to the limit in the weak formulation of problem (3.28), that the function $\widehat{u}$ coincides with the solution u of problem (3.1).

Now, by classical results on linear elliptic differential equations (see e.g. [11]), it is easy to show that the sequence u^k is equicontinuous and uniformly bounded in $L^\infty(G)$ so that, up to a subsequence, $u^k \to u$ actually in $L^\infty(G)$. In particular for any fixed $y \in G''$ we have

$$u^k(x,y) \to u(x,y) \text{ in } L^\infty(G'). \tag{3.32}$$

Since rearrangement is a contraction from $L^p(G')$ into $L^p([0,|G'|_n])$ $\forall\ 1 \le p \le \infty$ (see e.g. [15]), for any fixed $y \in G''$, we have

$$\left\|(u^k)^* - (u)^*\right\|_{L^\infty([0,|G'|_n])} \le \left\|u^k - u\right\|_{L^\infty(G')}, \tag{3.33}$$

which gives $\forall y \in G''$

$$\int_0^s (u^k)^*(\sigma,y)d\sigma \to \int_0^s (u)^*(\sigma,y)d\sigma \ \forall s \in [0,|G'|_n]. \tag{3.34}$$

Analogously one can prove that

$$\int_0^s (v^k)^*(\sigma,y)d\sigma \to \int_0^s (v)^*(\sigma,y)d\sigma \ \forall s \in [0,|G'|_n] \text{ and } \forall y \in G'', \tag{3.35}$$

where v^k and v are the solutions of problems (3.29) and (3.2) respectively.

Finally from (3.30), (3.34) and (3.35) we get (3.3). □

Now let us state the comparison result in a more general case.

Theorem 3.2. *Let us consider the solution u the following problem:*

$$\begin{cases} Lu = \widetilde{L}u - \sum_{i=1}^n (b_i(y)u)_{x_i} - \sum_{j=1}^m \left(\widetilde{b}_j(y)u\right)_{y_j} \\ + \sum_{i=1}^n d_i(y)u_{x_i} + \sum_{j=1}^m \widetilde{d}_j(y)u_{y_j} + c(y)u = f(x,y) \quad \text{in } G \\ u \in H_0^1(G), \end{cases} \tag{3.36}$$

where $G = G' \times G'$ is an open, bounded, connected and smooth subset of $\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m$ and

$$\begin{aligned} \widetilde{L}u \;=\; & -\sum_{i,h=1}^n (a_{ih}(x,y)u_{x_i})_{x_h} - \sum_{j,l=1}^m \left(g_{jl}(y)u_{y_j}\right)_{y_l} \\ & -\sum_{i=1}^n \sum_{l=1}^m (q_{il}(y)u_{x_i})_{y_l} - \sum_{i=1}^n \sum_{j=1}^m \left(r_{ji}(y)u_{y_j}\right)_{x_i}. \end{aligned} \tag{3.37}$$

We assume conditions (i) *and* (ii)*; moreover, we suppose that the following hypotheses hold:*

(iii) *a_{ih}, g_{jl}, q_{il}, $r_{ji} \in L^{\infty}(G)$, and they verify the following ellipticity condition:*

$$\sum_{i,h=1}^{n} a_{ih}(x,y)\xi_i\xi_j + \sum_{j,l=1}^{m} g_{jl}(y)\eta_h\eta_l \tag{3.38}$$
$$+\sum_{i=1}^{n}\sum_{l=1}^{m} q_{il}(y)\xi_i\eta_l + \sum_{i=1}^{n}\sum_{j=1}^{m} r_{ji}(y)\eta_j\xi_i \geq |\xi|^2 + \gamma\,|\eta|^2$$

for some $\gamma > 0$, $\forall(\xi,\eta) \in R^n \times R^m$ and almost every $(x,y) \in G$.

Moreover, let v be the solution of the following problem:

$$\begin{cases} -\Delta_x v - \sum_{j,l=1}^{m}\left(g_{jl}(y)v_{y_j}\right)_{y_l} - \sum_{j=1}^{m}\left(\widetilde{b}_j(y)v\right)_{y_j} \\ + \sum_{j=1}^{m}\widetilde{d}_j(y)v_{y_j} + c(y)v = f^{\#} \quad \text{in } G^{\#} \\ v \in H_0^1(G^{\#}). \end{cases} \tag{3.39}$$

Then inequality (3.3) *holds.*

Sketch of the proof. Following the lines of the proof of the previous theorem it is clear that we may consider a problem with analytic coefficients. More precisely, in this case one has also to find some sequences of analytic coefficients $\{a_{ij}^k\}$, $\{g_{jl}^k\}$, $\{q_{il}^k\}$ and $\{r_{ji}^k\}$ which converge, as k goes to infinity, to a_{ij}, g_{jl}, q_{il} and r_{ji} and satisfy $\forall k \in \mathbb{N}$ the ellipticity condition (3.38). One can obtain these sequences as done before (see (3.26) and (3.27)).

Then we integrate equation (3.36) on the level set of its solution u; that is,

$$\int_{u>t} \widetilde{L}u\,dx - \int_{u>t}\sum_{i=1}^{n}(b_i(y)u)_{x_i}\,dx - \int_{u>t}\sum_{j=1}^{m}\left(\widetilde{b}_j(y)u\right)_{y_j}dx \tag{3.40}$$
$$+\int_{u>t}\sum_{i=1}^{n} d_i(y)u_{x_i}dx \int_{u>t}\sum_{j=1}^{m}\widetilde{d}_j(y)u_{y_j}dx + \int_{u>t} c(y)u dx = \int_{u>t} f(x,y)\,dx.$$

We evaluate $\int_{u>t}\widetilde{L}u$ exactly as in [2] (see proof of Theorem 1.1). Then we handle the other integrals in (3.40) according to Theorem 3.1, and setting, as before,

$$Z(s,y) = U(s,y) - V(s,y) = \int_0^s u^*(\sigma,y)d\sigma - \int_0^s v^*(\sigma,y)d\sigma, \tag{3.41}$$

we get

$$-n^2\omega_n^{\frac{2}{n}}s^{2-\frac{2}{n}}\frac{\partial^2 Z}{\partial s^2}-\frac{\partial}{\partial y_j}\big(g_{jl}(y)\frac{\partial Z}{\partial y_l}\big) \tag{3.42}$$
$$-\sum_{j=1}^{m}\frac{\partial}{\partial y_j}\Big(\widetilde{b}_j(y)Z(s,y)\Big)+\sum_{j=1}^{m}\widetilde{d}_j(y)\frac{\partial Z}{\partial y_j}+c(y)Z(s,y)\le 0.$$

Now, taking into account that the function Z satisfies relation (3.42) and the same boundary conditions as U and V (see proof of Theorem 1.1), by the maximum principle, we have $Z\le 0$ on G^*, that is, (3.3). □

Remark 3.1. Theorem 3.2 still holds if u is the solution of the problem (3.1) with coefficients $\widetilde{b}_j(y)=0$, for $j=1,\dots,m$, (or, more in general, $\sum_{j=i}^{m}\frac{\partial\widetilde{b}_j(y)}{\partial y_j}\le 0$) and coefficient $c=c(x,y)$ depending on both variables and nonnegative, and v is the solution of

$$\begin{cases} -\Delta v+\sum\limits_{j=1}^{m}\widetilde{d}_j(y)v_{y_j}=f^{\#} & \text{in } G^{\#}\\ v=0 \quad \text{on } \partial G^{\#}. \end{cases} \tag{3.43}$$

4. Energy estimate

The next result is an estimate which involves the gradient of the functions u and v.

Theorem 4.1. *Let u and v be the weak solutions of problems* (3.1) *and* (3.2) *respectively. Then, under the same hypotheses as Theorem* 3.1, *we have*

$$\int_G|\nabla u|^2dx\,dy+\int_G\Big[c(y)-\frac{1}{2}\sum_{j=1}^{m}\Big(\frac{\partial\widetilde{b}_j(y)}{\partial y_j}+\frac{\partial\widetilde{d}_j(y)}{\partial y_j}\Big)\Big]u^2dx\,dy \tag{4.1}$$
$$\le\int_{G^{\#}}|\nabla v|^2dx\,dy+\int_{G^{\#}}\Big[c(y)-\frac{1}{2}\sum_{j=1}^{m}\Big(\frac{\partial\widetilde{b}_j(y)}{\partial y_j}+\frac{\partial\widetilde{d}_j(y)}{\partial y_j}\Big)\Big]v^2dx\,dy.$$

Proof. If we use u as a test function in problem (3.1), by standard computations, we obtain

$$\int_G|\nabla u|^2\,dx\,dy-\int_G\sum_{i=1}^{n}(b_i(y)u)_{x_i}\,u\,dx\,dy-\int_G\sum_{j=1}^{m}\Big(\widetilde{b}_i(y)u\Big)_{y_j}\,u\,dx\,dy \tag{4.2}$$
$$+\int_G\sum_{i=1}^{n}d_i(y)u_{x_i}u\,dx\,dy+\int_G\sum_{i=1}^{n}\widetilde{d}_j(y)u_{y_j}u\,dx\,dy$$

$$+\int_G c(y)u^2 dx\,dy \le \int_G f(x,y)u\,dx\,dy.$$

Now let us estimate each term in (4.2). We have

$$-\int_G \sum_{i=1}^n (b_i(y)u)_{x_i}\,u\,dx\,dy = \int_G \sum_{i=1}^n b_i(y)uu_{x_i} dx\,dy = 0 \tag{4.3}$$

and

$$-\int_G \sum_{j=1}^m (\widetilde{b}_i(y)u)_{y_j} u\,dx\,dy = \int_G \sum_{j=1}^m \widetilde{b}_j(y)uu_{y_j} dx\,dy = -\frac{1}{2}\int_G \sum_{j=1}^m \frac{\partial \widetilde{b}}{\partial y_j} u^2 dx\,dy. \tag{4.4}$$

In the same way we obtain

$$\int_G \sum_{i=1}^n d_i(y)uu_{x_i} dx\,dy = 0 \tag{4.5}$$

and

$$\int_G \sum_{j=1}^m \widetilde{d}_j(y)uu_{y_j} dx\,dy = -\frac{1}{2}\int_G \sum_{j=1}^m \frac{\partial \widetilde{d}}{\partial y_j} u^2 dx\,dy. \tag{4.6}$$

Finally, the Hardy–Littlewood theorem gives

$$\int_G fu\,dx\,dy \le \int_{G^\#} f^\# u^\# dx\,dy \le \int_{G^\#} f^\# v\,dx\,dy \tag{4.7}$$

where the last inequality is a consequence of (3.3) (see e.g. [3]).

Collecting estimates (4.3)–(4.7), we have

$$\int_G |\nabla u|^2 dx\,dy + \int_G \Big[c(y) - \frac{1}{2}\sum_{j=1}^m \Big(\frac{\partial \widetilde{b}_j(y)}{\partial y_j} + \frac{\partial \widetilde{d}_j(y)}{\partial y_j}\Big)\Big] u^2 dx\,dy \le \int_{G^\#} f^\# v\,dx\,dy. \tag{4.8}$$

Arguing as for (4.8) we easily verify that v satisfies the following equality:

$$\int_{G^\#} |\nabla v|^2 dx\,dy + \int_{G^\#} \Big[c(y) - \frac{1}{2}\sum_{j=1}^m \Big(\frac{\partial \widetilde{b}_j(y)}{\partial y_j} + \frac{\partial \widetilde{d}_j(y)}{\partial y_j}\Big)\Big] v^2 dx\,dy = \int_{G^\#} f^\# v\,dx\,dy. \tag{4.9}$$

Finally, by (4.8) and (4.9), we obtain estimate (4.1). □

Corollary 4.1. *Under the assumptions of Theorem* 4.1, *if coefficients* $\widetilde{b}_j(y) = \widetilde{d}_j(y) = c(y) = 0$, *for* $j = 1, \ldots, m$, *then the energy estimate* (4.1) *becomes*

$$\int_G |\nabla u|^2 \, dx \, dy \le \int_{G^\#} |\nabla v|^2 \, dx \, dy. \tag{4.10}$$

Remark 4.1. Under the hypotheses of Theorem 3.2 one can prove, repeating the arguments used for (4.1), the following estimate:

$$\begin{aligned}&\int_G \Big[\sum_{i=1}^n \Big(\frac{\partial u}{\partial x_i}\Big)^2 + \gamma \sum_{j=1}^m \Big(\frac{\partial u}{\partial y_j}\Big)^2\Big] dx\, dy \\ &+ \int_G \Big[c(y) - \frac{1}{2}\sum_{j=1}^m \Big(\frac{\partial \widetilde{b}_j(y)}{\partial y_j} + \frac{\partial \widetilde{d}_j(y)}{\partial y_j}\Big)\Big] u^2 dx\, dy \\ &\le \int_{G^\#} |\nabla v|^2 dx\, dy + \int_{G^\#} \Big[c(y) - \frac{1}{2}\sum_{j=1}^m \Big(\frac{\partial \widetilde{b}_j(y)}{\partial y_j} + \frac{\partial \widetilde{d}_j(y)}{\partial y_j}\Big)\Big] v^2 dx\, dy.\end{aligned} \tag{4.11}$$

References

[1] A. Alvino, J.I. Diaz, P.L. Lions, and G. Trombetti, *Equations elliptiques et symétrization de Steiner,* C. R. Acad. Sci. Paris Sér I Math., 314 (1992), 1015–1020.

[2] A. Alvino, J.I. Diaz, P.L. Lions, and G. Trombetti, *Elliptic equations and Steiner symmetrization,* Comm. on Pure and Appl. Math., XLIX (1996), 217–236.

[3] A. Alvino, P.L. Lions, and G. Trombetti, *On optimization problems with prescribed rearrangements,* Nonlinear Analysis, Theory, Methods and Applications, 13 (1989), 185–220.

[4] A. Alvino, P.L. Lions, and G. Trombetti, *Comparison results for elliptic and parabolic equations via Schwarz symmetrization*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 7 (1990), 37–65.

[5] A. Alvino and G. Trombetti *Equazioni ellittiche con termini di ordine inferiore e riordinamenti,* Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei, 66 (1979), 194–200.

[6] C. Bandle, "Isoperimetric Inequalities and Applications," Pitman, London, 1980.

[7] A. Baernstein, *A unified approach to symmetrization,* Symposia Matematica, 35 (1995), 47–91.

[8] E. De Giorgi, *Su una teoria generale della misura* $(r-1)$ *dimensionale in uno spazio ad* r *dimensioni,* Ann. Math. Pura Appl., 36 (1954), 191–213.

[9] H. Federer, "Geometric Measure Theory," Springer Verlag, New York, 1969.

[10] V. Ferone and A. Mercaldo, *A second order derivation formula for functions defined integrals,* C. R. Acad. Sci. Paris, t. 326, Serie I (1998), 549–554.

[11] D. Gilbarg and N.G. Trudinger, "Elliptic Partial Differential Equations of Second Order," Springer Verlag, New York–Tokyo, 1983.

[12] G.H. Hardy, J.E. Littlewood, and G. Polya, "Inequalities," Cambridge University Press, 1964.

[13] B. Kawohl, *Rearrangements and convexity of level sets in PDE,* Lecture Notes in Math., 1150 (1985), Springer Verlag, Berlin–New York.

[14] C. Maderna, D. Pagani, and S. Salsa, *Quasilinear elliptic equations with quadratic growth in the gradient*, J. Differential Equations, 97 (1992), 54–70.

[15] J. Mossino, *Inégalités isopérimétriques et applications en physique*, Hermann, Paris, 1984.

[16] J. Mossino, and J.M. Rakotoson, *Isoperimetric inequalities in parabolic equations,* Ann. Scuola Norm. Sup. Pisa Cl. Sci., 13 (1986), 51–73.

[17] G. Polya and G. Szegö, *Isoperimetric inequalities in mathematical physics*, Ann. of Math. Studies, n. 27 (1951), Princeton.

[18] G. Talenti, *Elliptic equations and rearrangements*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., 3 (1976), 697–718.

[19] G. Talenti, *Nonlinear elliptic equations, rearrangements of functions and Orlicz spaces,* Ann. Math. Pura Appl., 120 (1979), 159–184.

[20] H. Weinberger, "Symmetrization in Uniformly Elliptic Problems," Studies in Math Anal., Stanford University Press, 1962.