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GLOBAL SOLUTIONS TO BOUNDARY VALUE PROBLEMS FOR A NONLINEAR WAVE EQUATION IN HIGH SPACE DIMENSIONS

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Abstract. In this article we consider an initial–boundary value problem for a wave equation in high dimensions with a nonlinear damping term that is not Lipschitz in u_t. We establish the existence and uniqueness of a global solution by using a compactness method and by exploiting the monotonicity property of the nonlinearity.

1. Introduction

Let Ω be an open, bounded, connected domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega = \Gamma$. Let Γ be the union of two disjoint $n-1$-dimensional submanifolds Γ_0 and Γ_1, each of which has a positive Lebesgue measure. In this paper, interest is focussed on the initial–boundary value problem

$$u_{tt} - \Delta u + |u_t|^m \, sgn(u_t) = 0, \quad \text{in } \Omega \times (0,T), \tag{1.1}$$

$$u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x), \quad \text{in } \Omega, \tag{1.2}$$

$$u(x,t) = 0, \quad \text{on } \Gamma_0 \times (0,T), \quad \frac{\partial u}{\partial n}(x,t) = g(x,t), \quad \text{on } \Gamma_1 \times (0,T), \tag{1.3}$$

where $0 < m < 1$, $g \in C^1([0,\infty), L^2(\Gamma_1))$, and $\frac{\partial}{\partial n}$ denotes the outward normal derivative on Γ_1.

Equation (1.1) represents a classical vibrating membrane with resistance that is proportional to the velocity u_t. The analogue of (1.1)–(1.3) in one space dimension has been studied by Ang and Dinh [2]. In [2], the authors established the existence and uniqueness of a global solution by using a standard Galerkin approximation scheme. However, their proof relies heavily on the fact that the equation is in one space dimension. In particular, having

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in hand explicit formulas for the eigenfunctions of the Laplacian $-(\frac{d}{dx})^2$ on the unit interval $(0,1)$, namely $\{\cos\frac{\pi}{2}(2n+1)x,\ n\in\mathbb{N}\}$, was a key ingredient in obtaining the needed estimates for their proof.

One of the pioneering papers in this area was by Lions and Strauss [11], where hyperbolic partial differential equations of this kind were treated. We also note here the work of Lasiecka and Triggiani [8, 9] and Webb [16]. For other results relevant to ours see [3, 4, 7, 12] and the references therein.

In this paper we deal with the initial–boundary value problem (1.1)–(1.3) in all space dimensions. However, due to the lack of smoothness of the nonlinearity in the equation, a standard fixed-point-theorem argument is not applicable. Our argument utilizes an appropriate compactness theorem and the monotonicity of the Nemytskii operator, $f(u')(x) := f(u'(x)) = |u'(x)|^m\, sgn(u'(x))$, to prove the convergence of the Galerkin scheme in the appropriate spaces. Our estimates for the approximate solutions are derived directly from the integral equations associated with the problem.

The plan for this paper is as follows. In Section 2, we present the technical assumptions, definitions, and notation that are needed for the remaining sections of the paper. Sections 3 and 4 are devoted to the solvability of the problem.

2. Preliminaries

In this section we introduce some notation, definitions, and the technical assumptions that are necessary for the remaining sections of the paper. Let $L^2(\Omega)$, $L^2(\Gamma_1)$, etc. denote the standard Lebesgue spaces and $H^s(\Omega)$, $H^s(\Gamma_0)$, $H^s(\Gamma_1)$, ... denote the standard Sobolev spaces. By $H^s(\Gamma)$ we mean the space $H^s(\Gamma_0)\times H^s(\Gamma_1)$, and by $H^s_{0,\Gamma_0}(\Gamma)$, $s\geq 0$, we mean the subspace of $H^s(\Gamma)$ that is given by $H^s_{0,\Gamma_0}(\Gamma)=\{0\}\times H^s(\Gamma_1)$. Also, for $s>1/2$, we set $H^s_{0,\Gamma_0}(\Omega)=\{u\in H^s(\Omega): u\,|_{\Gamma_0}=0\}$, where the evaluation on Γ_0 is taken in the sense of traces.

For $u\in H^s(\Omega)$ we denote by γu the trace operator (whenever defined) on Γ, i.e., $\gamma u = u\,|_{\Gamma}$. Also, we set $\gamma_1 u = u\,|_{\Gamma_1}$.

Let X and Y be Banach spaces. We write $X\hookrightarrow Y$ if X is continuously imbedded in Y. More precisely, the natural injection $i: X\to Y$ is continuous and we identify X with the subspace $i(X)$ of Y. We denote by $\mathcal{L}(X,Y)$ the space of all continuous linear operators from X to Y.

The following assumptions will be valid throughout the paper:

(H1): There exists a function $\alpha\in C^\infty(\overline{\Omega})$ with $\alpha(x)=\begin{cases}0, & x\in\Gamma_0\\ 1, & x\in\Gamma_1.\end{cases}$

(H2): $u^0 \in H^1_{0,\Gamma_0}(\Omega)$, $u^1 \in L^2(\Omega)$.
(H3): $g \in C^1([0,\infty), L^2(\Gamma_1))$.

By introducing the boundary operator $\mathcal{B}u = \left(\alpha\frac{\partial u}{\partial n} + (1-\alpha)u\right)\big|_{\Gamma}$, the boundary conditions in (1.3) can be expressed as follows:

$$\mathcal{B}u = \alpha g \quad \text{on } \Gamma \times (0,T).$$

At this point we introduce two operators: $A : L^2(\Omega) \to L^2(\Omega)$, where $A = -\Delta$ with its domain

$$\mathcal{D}(A) = \{u \in H^2(\Omega) : \mathcal{B}u = 0\},$$

and the Dirichlét–Neumann map $R : L^2(\Gamma) \to L^2(\Omega)$, which is given by the following: $Rg = w$ if and only if

$$\begin{aligned} \Delta w &= 0 \quad \text{in } \Omega,\\ \mathcal{B}w &= \alpha g \quad \text{on } \Gamma. \end{aligned}$$

Since A is positive, self-adjoint, and has a compact inverse, A has an infinite sequence of positive eigenvalues $\{\lambda_n : n = 1,2,\dots\}$ and a corresponding sequence of eigenfunctions $\{e_n : n = 1,2,\dots\}$ that forms an orthonormal basis for $L^2(\Omega)$. Namely, if $u \in L^2(\Omega)$, then $u = \sum_{n=1}^{\infty} u_n e_n$, where the convergence is in $L^2(\Omega)$, with $\|u\|^2_{L^2(\Omega)} = \sum_{n=1}^{\infty} |u_n|^2$ and $u_n = \langle u, e_n\rangle_{L^2(\Omega)}$.

The powers of A are defined as follows: $A^s : \mathcal{D}(A^s) \subseteq L^2(\Omega) \to L^2(\Omega)$, $A^s u = \sum_{n=1}^{\infty} \lambda_n^s u_n e_n$, with the domain of A^s given by

$$\mathcal{D}(A^s) = \{u \in L^2(\Omega) : u = \sum_{n=1}^{\infty} u_n e_n,\ \sum_{n=1}^{\infty} \lambda_n^{2s} |u_n|^2 < \infty\}.$$

We remark here that the results of Grisvard [5] and Seeley [13] give the following characterization for the fractional powers of A:

$$\mathcal{D}(A^s) = \begin{cases} H^{2s}(\Omega);\ 0 \le s < \frac{1}{4} \\ H^{2s}_{0,\Gamma_0}(\Omega);\ \frac{1}{4} < s < \frac{3}{4} \\ \left\{u \in H^{2s}(\Omega) : u|_{\Gamma_0} = 0, \frac{\partial u}{\partial n}|_{\Gamma_1} = 0\right\};\ \frac{3}{4} < s \le 1. \end{cases} \tag{2.1}$$

Moreover, $\mathcal{D}(A^{\frac{1}{4}}) \hookrightarrow H^{\frac{1}{2}}(\Omega)$, $\mathcal{D}(A^{\frac{3}{4}}) \hookrightarrow H^{\frac{3}{2}}_{0,\Gamma_0}(\Omega)$, and the norm $\|u\|_{H^s(\Omega)}$ is equivalent to $(\sum_{n=1}^{\infty} \lambda_n^s |u_n|^2)^{\frac{1}{2}}$. Therefore, we set

$$\|u\|^2_{H^s(\Omega)} = \sum_{n=1}^{\infty} \lambda_n^s |u_n|^2.$$

We now introduce $S(t)$ and $C(t)$, the sine and cosine operators associated with A. $S(t), C(t) : L^2(\Omega) \to L^2(\Omega)$ are given by $S(t) = A^{-\frac{1}{2}} \sin(A^{\frac{1}{2}} t)$ and $C(t) = \cos(A^{\frac{1}{2}} t)$. More specifically, if $u \in L^2(\Omega)$ with $u = \sum_{n=1}^{\infty} u_n e_n$, then

$$S(t)u = \sum_{n=1}^{\infty} \lambda_n^{-1/2} \sin(\lambda_n^{1/2} t) u_n e_n, \text{ and } C(t)u = \sum_{n=1}^{\infty} \cos(\lambda_n^{1/2} t) u_n e_n.$$

Finally, we cast the initial–boundary value problem (1.1)–(1.3) as follows:

$$u'' + Au + f(u') = 0, \quad \text{on } (0,T), \tag{2.2}$$

$$u(0) = u^0, \quad u'(0) = u^1, \tag{2.3}$$

$$\mathcal{B}u(t) = \alpha g(t), \quad \text{on } (0,T), \tag{2.4}$$

where here and later $f(u') := |u'|^m sgn(u')$ and $u'' = (\frac{d}{dt})^2 u(t)$.

Due to the lack of smoothness of the nonlinearity in the equation, one could anticipate that the solution of the initial–boundary value problem (1.1)–(1.3) would not be highly regular. So, it would not be possible to make sense of the Neumann boundary condition in the sense of traces. Therefore, we shall use the weak formulation of the problem to define a solution.

Definition 1. Let $u^0 \in H^1_{0,\Gamma_0}(\Omega)$, $u^1 \in L^2(\Omega)$. We say that u is a regular solution to the initial–boundary value problem (2.2)–(2.4) on $[0,T]$ if $u \in L^2(0,T,H^1_{0,\Gamma_0}(\Omega))$, $u' \in L^2(0,T,L^2(\Omega))$ and u satisfies

$$\begin{aligned} &\left\langle u'(t), \phi \right\rangle_{L^2(\Omega)} - \left\langle u^1, \phi \right\rangle_{L^2(\Omega)} + \int_0^t \left\langle A^{1/2} u(s), A^{1/2} \phi \right\rangle_{L^2(\Omega)} ds \\ &\quad + \int_0^t \Big[- \langle g(s), \gamma_1 \phi \rangle_{L^2(\Gamma_1)} + \left\langle f(u'(s)), \phi \right\rangle_{L^2(\Omega)} \Big] ds = 0, \end{aligned} \tag{2.5}$$

for all $\phi \in H^1_{0,\Gamma_0}(\Omega)$ and almost every $t \in [0,T]$.

In order for us to easily obtain certain estimates, we now derive the integral equations that must be satisfied by a regular solution to the initial–boundary value problem (2.2)–(2.4). Let $v(t) = u(t) - w(t)$, where $w(t) = Rg(t)$. Then, v formally satisfies

$$v'' + Av = -w'' - f(v' + w'), \ \textit{on } (0,T), \tag{2.6}$$

$$v(0) = u^0 - w(0), \quad v'(0) = u^1 - w'(0), \tag{2.7}$$

$$\mathcal{B}v = 0, \quad \text{on } (0,T). \tag{2.8}$$

Thus, by the variation-of-constants formula, we have

$$v(t)=C(t)(u^0-w(0))+S(t)(u^1-w'(0))-\int_0^t S(t-\tau)[w''(\tau)+f(u'(\tau))]\,d\tau. \tag{2.9}$$

Formal integration by parts yields

$$\begin{aligned}u(t)&=C(t)(u^0-Rg(0))+S(t)u^1+Rg(t)\\&\quad-\int_0^t C(t-\tau)Rg'(\tau)\,d\tau-\int_0^t S(t-\tau)f(u'(\tau))\,d\tau.\end{aligned} \tag{2.10}$$

By differentiating (2.10), one has

$$\begin{aligned}u'(t)&=C(t)u^1-AS(t)(u^0-Rg(0))\\&\quad+\int_0^t AS(t-\tau)Rg'(\tau)\,d\tau-\int_0^t C(t-\tau)f(u'(\tau))\,d\tau.\end{aligned} \tag{2.11}$$

Now, let

$$U_0(t)=C(t)(u^0-Rg(0))+S(t)u^1+Rg(t)-Kg'(t),$$
$$V_0(t)=C(t)u^1-AS(t)(u^0-Rg(0))+Lg'(t),$$

where the operators K and L are given by

$$Kg(t)=\int_0^t C(t-\tau)Rg(\tau)\,d\tau \text{ and } Lg(t)=\int_0^t AS(t-\tau)Rg(\tau)\,d\tau.$$

Remark 2.1. It is not too difficult to show that if $u\in L^2(0,T,H^1_{0,\Gamma_0}(\Omega))$, $u'\in L^2(0,T,L^2(\Omega))$, and u satisfies the integral equations

$$u(t)=U_0(t)-\int_0^t S(t-\tau)f(u'(\tau))\,d\tau, \tag{2.12}$$

$$u'(t)=V_0(t)-\int_0^t C(t-\tau)f(u'(\tau))\,d\tau, \tag{2.13}$$

then u is a regular solution to (2.2)–(2.4) in the sense of Definition 1. Moreover, the converse is also valid. The proof of this remark is contained in the Appendix.

We now state our main result.

Theorem 1. *Let $u^0\in H^1_{0,\Gamma_0}(\Omega)$, $u^1\in L^2(\Omega)$, and $g\in C^1([0,\infty),L^2(\Gamma_1))$. Then there exists a unique global solution u to the initial–boundary value problem* (1.1)–(1.3) *such that*

$$u\in C^0([0,\infty);H^1_{0,\Gamma_0}(\Omega)),\quad u'\in C^0([0,\infty);L^2(\Omega)).$$

The proof of Theorem 1 is contained in Sections 3 and 4.

The following regularity results are well-known (for example, see [8, 9]), and thus their proofs are omitted. However, we remark here that statements (iv) and (v) in Lemma 2.1 below are not the sharpest regularity results that one can show for the operators K and L.

Lemma 2.1. *For $s \geq 0$ and $0 < \epsilon < 1/4$, we have*

(i) $C(\cdot) \in \mathcal{L}\left(\mathcal{D}(A^s),\, C([0,T],\, \mathcal{D}(A^s))\right)$,

(ii) $S(\cdot) \in \mathcal{L}\left(\mathcal{D}(A^s),\, C([0,T],\, \mathcal{D}(A^{s+1/2}))\right)$,

(iii) $R \in \mathcal{L}\left(H^s(\Gamma),\, H^{s+3/2}_{0,\Gamma_0}(\Omega)\right)$,

(iv) $K \in \mathcal{L}\left(L^2(0,T,L^2(\Gamma));\, C([0,T],\, \mathcal{D}(A^{3/4-\epsilon}))\right)$,

(v) $L \in \mathcal{L}\left(L^2(0,T,L^2(\Gamma));\, C([0,T],\, \mathcal{D}(A^{1/4-\epsilon}))\right)$.

3. Approximate solutions

In this section, we construct the approximate solutions to the initial–boundary value problem (2.2)–(2.4) and obtain the necessary estimates for the passage to the limit. Let $\{e_k\}_{k=1}^{\infty}$ be the orthonormal basis for $L^2(\Omega)$, as described in Section 2. Let $\mathcal{P}_N$ be the orthogonal projection of $L^2(\Omega)$ onto the linear span of $\{e_1, \ldots, e_N\}$. Let $u_N(t) = \sum_{k=1}^{N} u_{N,k}(t) e_k$ be a regular solution to the Galerkin system associated with the initial–boundary value problem (2.2)–(2.4); i.e., $u_N(t)$ satisfies the initial value problem

$$\frac{d}{dt}\left\langle u_N'(t), e_k\right\rangle_{L^2(\Omega)} + \left\langle A^{1/2} u_N(t), A^{1/2} e_k\right\rangle_{L^2(\Omega)} - \langle g(t), \gamma_1 e_k\rangle_{L^2(\Gamma_1)} + \left\langle \mathcal{P}_N f(u_N'(t)), e_k\right\rangle_{L^2(\Omega)} = 0, \tag{3.1}$$

$$u_{N,k}(0) = u_k^0, \quad u_{N,k}'(0) = u_k^1, \tag{3.2}$$

for $k = 1, 2, \ldots, N$, where $u_k^0 = \left\langle u^0, e_k\right\rangle_{L^2(\Omega)}$ and $u_k^1 = \left\langle u^1, e_k\right\rangle_{L^2(\Omega)}$.

Now, (3.1) and (3.2) are equivalent to

$$u_{N,k}''(t) + \lambda_k u_{N,k}(t) = \langle g(t), \gamma_1 e_k\rangle_{L^2(\Gamma_1)} - \left\langle \mathcal{P}_N f(u_N'(t)), e_k\right\rangle_{L^2(\Omega)} \tag{3.3}$$

$$u_{N,k}(0) = u_k^0, \quad u_{N,k}'(0) = u_k^1, \quad \text{for } k = 1, 2, \ldots, N. \tag{3.4}$$

Since (3.3), (3.4) is an initial value problem for a second-order $N \times N$ system of ordinary differential equations with continuous nonlinearities in the derivatives of the unknown functions $u_{N,k}$, it follows from the theory of ordinary differential equations that (3.3), (3.4) has a maximal solution on the interval $[0, T_N]$, for some $0 < T_N \leq T$. Moreover, $u_{N,k} \in C^2[0, T_N]$, and they satisfy the following integral equation on $[0, T_N]$:

$$u_{N,k}(t) = u_k^0 \cos \lambda_k^{\frac{1}{2}} t + u_k^1 \lambda_k^{-\frac{1}{2}} \sin \lambda_k^{\frac{1}{2}} t + \int_0^t \lambda_k^{-\frac{1}{2}} \sin \lambda_k^{\frac{1}{2}} (t-\tau) G_{N,k}(\tau)\, d\tau, \quad (3.5)$$

for $k = 1, 2, \ldots, N$, and where

$$G_{N,k}(\tau) = \langle g(\tau), \gamma_1 e_k \rangle_{L^2(\Gamma_1)} - \left\langle \mathcal{P}_N f(u_N'(\tau)), e_k \right\rangle_{L^2(\Omega)}.$$

It follows easily from (3.5) and the definitions of the sine and cosine operators that u_N satisfies the following integral equations on $[0, T_N]$:

$$u_N(t) = U_{N,0}(t) - \int_0^t S(t-\tau) \mathcal{P}_N f(u_N'(\tau))\, d\tau, \quad (3.6)$$

$$u_N'(t) = V_{N,0}(t) - \int_0^t C(t-\tau) \mathcal{P}_N f(u_N'(\tau))\, d\tau, \quad (3.7)$$

where

$$\begin{aligned} U_{N,0}(t) &= C(t)\mathcal{P}_N(u^0 - Rg(0)) + S(t)\mathcal{P}_N u^1 + \ \mathcal{P}_N Rg(t) \\ &\quad - \int_0^t C(t-\tau)\mathcal{P}_N Rg'(\tau)\, d\tau, \\ V_{N,0}(t) &= C(t)\mathcal{P}_N u^1 - AS(t)\mathcal{P}_N(u^0 - Rg(0)) + \int_0^t AS(t-\tau)\mathcal{P}_N Rg'(\tau)\, d\tau. \end{aligned}$$

A priori estimates: We shall show that $T_N = T$. If $T_N < T$, then it must be the case that $\|u_N(t)\|_{H^1_{0,\Gamma_0}(\Omega)} + \|u_N'(t)\|_{L^2(\Omega)} \to \infty$ as $t \to T_N$. However, in view of the a priori estimates for u_N obtained below, this behavior is impossible. Although these estimates can be obtained directly from the differential equation (3.3), we derive them easily from the integral equations (3.6) and (3.7).

In the remaining parts of the paper, we shall refer to the following Hilbert spaces repeatedly: $X = L^2(0, T, L^2(\Omega))$ and $Y = L^2(0, T, H^1_{0,\Gamma_0}(\Omega))$, where $T > 0$ is arbitrary. In what follows all generic constants will be denoted by C; they may depend on arguments like T, Ω, m as indicated, and they may change from line to line.

Lemma 3.1. *The sequence of approximate solutions u_N satisfies the following:*

(i) $\{u_N\}$ *is bounded in* Y.

(ii) $\{u_N'\}$ *and* $\{\mathcal{P}_N f(u_N')\}$ *are bounded in* X.

Proof. Thanks to Lemma 2.1 and assumptions (H2) and (H3), we have $U_{N,0}(\cdot) \in C([0,T], H^1_{0,\Gamma_0}(\Omega))$ and $V_{N,0}(\cdot) \in C([0,T], L^2(\Omega))$. Moreover, by

Hölder's inequality, we have

$$\left\|\mathcal{P}_N f(u_N')\right\|_X^2 \leq \int_0^T \int_\Omega |u_N'(t)|^{2m} dx\ dt \leq |\Omega|^{1-m} T^{1-m} \left\|u_N'\right\|_X^{2m}. \tag{3.8}$$

Now, it follows from (3.8), Lemma 2.1 and Hölder's inequality that

$$\int_0^* S(*-\tau)\mathcal{P}_N f(u_N'(\tau))d\tau \in C([0,T], H^1_{0,\Gamma_0}(\Omega)),$$

and

$$\begin{aligned}
&\left\|\int_0^* S(*-\tau)\mathcal{P}_N f(u_N'(\tau))d\tau\right\|_Y^2 = \int_0^T \left\|\int_0^t S(t-\tau)\mathcal{P}_N f(u_N'(\tau))d\tau\right\|_{H^1(\Omega)}^2 dt \\
&\leq \int_0^T t\int_0^t \left\|f(u_N'(\tau))\right\|_{L^2(\Omega)}^2 d\tau\, dt \leq |\Omega|^{1-m} \int_0^T t\int_0^t \left\|u_N'(\tau)\right\|_{L^2(\Omega)}^{2m} d\tau\, dt \\
&= \frac{1}{2}|\Omega|^{1-m}\int_0^T (T^2-\tau^2)\left\|u_N'(\tau)\right\|_{L^2(\Omega)}^{2m} d\tau \\
&\leq \frac{1}{2}|\Omega|^{1-m}T^2\int_0^T \left\|u_N'(\tau)\right\|_{L^2(\Omega)}^{2m} d\tau \leq \frac{1}{2}|\Omega|^{1-m}T^{3-m}\left\|u_N'\right\|_X^{2m},
\end{aligned} \tag{3.9}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. Similarly, one has

$$\int_0^* C(*-\tau)\mathcal{P}_N f(u_N'(\tau))\, d\tau \in C([0,T], L^2(\Omega))$$

and

$$\left\|\int_0^* C(*-\tau)\mathcal{P}_N f(u_N'(\tau))\, d\tau\right\|_X^2 \leq \frac{1}{2}|\Omega|^{1-m}T^{3-m}\left\|u_N'\right\|_X^{2m}. \tag{3.10}$$

It follows from (3.9), (3.10), and the integral equations (3.6) and (3.7) that

$$\|u_N\|_Y \leq \|U_{N,0}\|_Y + C(T,\Omega)\left\|u_N'\right\|_X^m, \tag{3.11}$$

and

$$\left\|u_N'\right\|_X \leq \|V_{N,0}\|_X + C(T,\Omega)\left\|u_N'\right\|_X^m. \tag{3.12}$$

Thanks to Lemma 2.1 and assumptions (H1) and (H2), it is easy to see that $U_{N,0} \to U_0$ strongly in Y, and $V_{N,0} \to V_0$ strongly in X, as $N \to \infty$. So, there exists an $M > 0$ such that

$$\|u_N\|_Y \leq \|U_0\|_Y + M + C(T,\Omega)\left\|u_N'\right\|_X^m, \tag{3.13}$$

and

$$\left\|u_N'\right\|_X \leq \|V_0\|_X + M + C(T,\Omega)\left\|u_N'\right\|_X^m, \tag{3.14}$$

for all $N \geq 1$. Let $b = M + \|V_0\|_X$; then (3.14) becomes

$$\left\|u'_N\right\|_X - C(T, \Omega)\left\|u'_N\right\|_X^m - b \leq 0. \tag{3.15}$$

Let $\alpha = \alpha(T, \Omega, m)$ be the only positive root of the function $h : [0, \infty) \to \mathbb{R}$ defined by $h(y) = y - C(T, \Omega)y^m - b$. Then, it follows from (3.15) that $\|u'_N\|_X \leq \alpha$ for all N; i.e., $\{u'_N\}$ is bounded in X. Finally, (3.13) and (3.8) yield that $\{u_N\}$ is bounded in Y, and $\{\mathcal{P}_N f(u'_N)\}$ is bounded in X, which completes the proof of the lemma.

The following compactness theorem is included only for the sake of completeness. For instance, see [14] for a more general theorem.

Compactness Theorem. *Let X and Y be the Hilbert spaces as described above. Let $\mathcal{Y}$ be the space of functions $\mathcal{Y} = \{u \in Y, u' \in X\}$ endowed with the natural norm $\|u\|^2_{\mathcal{Y}} = \|u\|^2_Y + \|u'\|^2_X$. Then the imbedding $\mathcal{Y} \overset{i}{\hookrightarrow} X$ is compact.*

Now, by using Lemma 3.1 and the compactness theorem above, we can extract a subsequence of $\{u_N\}$ (still denoted by $\{u_N\}$) and find functions $u \in Y$ and $w \in X$ such that

$$\begin{cases} u_N \to u \text{ strongly in } X, \\ u_N \to u \text{ weakly in } Y, \\ u'_N \to u' \text{ weakly in } X, \\ \mathcal{P}_N f(u'_N) \to w \text{ weakly in } X. \end{cases} \tag{3.16}$$

Moreover, since the mapping $H^1_{0,\Gamma_0}(\Omega) \overset{\gamma}{\to} L^2_{0,\Gamma_0}(\Gamma)$ is continuous, it follows that $\{\gamma_1 u_N\}$ is bounded in $L^2(0, T, L^2(\Gamma_1))$. Therefore, there exists a subsequence, still denoted by $\{u_N\}$, such that

$$\gamma_1 u_N \to \gamma_1 u \text{ weakly in } L^2(0, T, L^2(\Gamma_1)). \tag{3.17}$$

It follows from the integral equations (3.6), (3.7), and (3.16) that $u \in Y$, $u' \in X$, and that u satisfies the integral equations

$$u(t) = U_0(t) - \int_0^t S(t - \tau)w(\tau)\, d\tau, \tag{3.18}$$

$$u'(t) = V_0(t) - \int_0^t C(t - \tau)w(\tau)\, d\tau. \tag{3.19}$$

In view of Remark 2.1, we conclude that the limit function u is a regular solution on $[0, T]$ (in the sense of Definition 1) to the initial–boundary value problem

$$u_{tt} - \Delta u + w = 0, \quad \text{in } \Omega \times (0,T), \tag{3.20}$$

$$u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x), \quad \text{in } \Omega, \tag{3.21}$$

$$\mathcal{B}u(t) = \alpha g(t) \quad \text{on } \Gamma \times (0,T), \tag{3.22}$$

where w is the weak limit of $\mathcal{P}_N f(u_N')$ in X.

Remark 3.1. The following are immediate consequences of (3.16)–(3.19): (i) There exists a subsequence of $\{u_N\}$, which is still denoted by $\{u_N\}$, such that for almost all $t \in [0,T]$, we have

$$\begin{cases} u_N(t) \to u(t) \text{ strongly in } L^2(\Omega), \\ u_N(t) \to u(t) \text{ weakly in } H^1_{0,\Gamma_0}(\Omega), \\ u_N'(t) \to u'(t) \text{ weakly in } L^2(\Omega), \\ \gamma_1 u_N(t) \to \gamma_1 u(t) \text{ weakly in } L^2(\Gamma_1). \end{cases} \tag{3.23}$$

(ii) The pointwise convergence in (3.23) is indeed everywhere on $[0,T]$, and moreover

$$u \in C([0,T], H^1_{0,\Gamma_0}(\Omega)) \text{ and } u' \in C([0,T], L^2(\Omega)). \tag{3.24}$$

Moreover, we have the following lemma.

Lemma 3.2. *Let $u^0 \in H^1_{0,\Gamma_0}(\Omega)$, $u^1 \in L^2(\Omega)$, $w \in X$ and u be a regular solution to the initial–boundary value problem* (3.20)–(3.22). *Then $u'' \in L^2(0,T,(H^1_{0,\Gamma_0}(\Omega))')$ with*

$$\|u''\|^2_{L^2(0,T,(H^1_{0,\Gamma_0}(\Omega))')} \le C\big[\, \|u\|_Y^2 + \|g\|^2_{L^2(0,T,L^2(\Gamma_1))} + \|w\|_X^2 \,\big], \tag{3.25}$$

for some constant $C > 0$.

Proof. Recall that since u is a regular solution of (3.20)–(3.22),

$$\langle u'(t), \phi \rangle_{L^2(\Omega)} - \langle u^1, \phi \rangle_{L^2(\Omega)} + \int_0^t \left\langle A^{1/2} u(s), A^{1/2}\phi \right\rangle_{L^2(\Omega)} ds$$
$$- \int_0^t \Big[\langle g(s), \gamma_1 \phi \rangle_{L^2(\Gamma_1)} + \langle w(s), \phi \rangle_{L^2(\Omega)} \Big] ds \quad = \quad 0$$

for all $\phi \in H^1_{0,\Gamma_0}(\Omega)$ and for almost all $t \in [0,T]$. Let $< \cdot, \cdot >$ denote the standard pairing of $(H^1_{0,\Gamma_0}(\Omega))'$ and $H^1_{0,\Gamma_0}(\Omega)$. Then

$$\big| < u''(t), \phi > \big| = \left| \frac{d}{dt} < u'(t), \phi > \right| = \left| \frac{d}{dt} \langle u'(t), \phi \rangle_{L^2(\Omega)} \right|$$
$$\le \left| \left\langle A^{1/2} u(t), A^{1/2}\phi \right\rangle_{L^2(\Omega)} \right| + \left| \langle g(t), \gamma_1 \phi \rangle_{L^2(\Gamma_1)} \right| + \left| \langle w(t), \phi \rangle_{L^2(\Omega)} \right|.$$

Use of the Cauchy–Schwartz inequality shows that

$$| < u''(t), \phi > | \leq \left\| A^{1/2} u(t) \right\|_{L^2(\Omega)} \|\phi\|_{H^1_{0,\Gamma_0}(\Omega)} + \|g(t)\|_{L^2(\Gamma_1)} \|\gamma_1 \phi\|_{L^2(\Gamma_1)} + \|w(t)\|_{L^2(\Omega)} \|\phi\|_{H^1_{0,\Gamma_0}(\Omega)}$$

for all $\phi \in H^1_{0,\Gamma_0}(\Omega)$ and for almost all $t \in [0,T]$. Because the mapping $H^1_{0,\Gamma_0}(\Omega) \xrightarrow{\gamma_1} L^2(\Gamma_1)$ is continuous, one can conclude that for almost every $t \in [0,T]$

$$\left\| u''(t) \right\|_{(H^1_{0,\Gamma_0}(\Omega))'} \leq C\big(\|u(t)\|_{H^1_{0,\Gamma_0}(\Omega)} + \|g(t)\|_{L^2(\Gamma_1)} + \|w(t)\|_{L^2(\Omega)} \big), \tag{3.26}$$

for some constant C. Therefore,

$$\left\| u'' \right\|^2_{L^2(0,T,(H^1_{0,\Gamma_0}(\Omega))')} \leq C\big[\|u\|^2_Y + \|g\|^2_{L^2(0,T,L^2(\Gamma_1))} + \|w\|^2_X \big], \tag{3.27}$$

which completes the proof of the lemma.

4. Uniqueness and the fact $w = f(u')$

We first show that $w = f(u')$. We begin by deriving an energy identity for the approximate solutions $\{u_N\}$. By multiplying equation (3.3) by $u'_{N,k}(t)$, summing from 1 to N, and integrating from 0 to t, we obtain

$$E_N(t) = E_N(0) \text{ for } t \in [0,T], \tag{4.1}$$

where

$$\begin{aligned} E_N(t) = \frac{1}{2} \left(\left\| u'_N(t) \right\|^2_{L^2(\Omega)} + \left\| A^{1/2} u_N(t) \right\|^2_{L^2(\Omega)} \right) \\ - \ \langle g(t), \gamma_1 u_N(t) \rangle_{L^2(\Gamma_1)} + \int_0^t \left\langle g'(\tau), \gamma_1 u_N(\tau) \right\rangle_{L^2(\Gamma_1)} d\tau \\ + \int_0^t \left\langle f(u'_N(\tau)), u'_N(\tau) \right\rangle_{L^2(\Omega)} d\tau. \end{aligned} \tag{4.2}$$

Due to the fact that u, the solution of the initial–boundary value problem (3.20)–(3.22), is not sufficiently regular, obtaining the energy identity in Lemma 4.1 is not straightforward. However, by modifying the proof of Lemma 8.3 of Lions and Magenes [10], Lemma 4.1 follows. Thus, its proof is omitted.

Lemma 4.1. *Let $u \in C([0,T], H^1_{0,\Gamma_0}(\Omega))$ and $u' \in C([0,T], L^2(\Omega))$ such that u is a regular solution to the initial–boundary value problem* (3.20)–(3.22). *Then, u satisfies*

$$E(t) = E(0), \tag{4.3}$$

where

$$E(t) = \frac{1}{2}\left(\left\|u'(t)\right\|^2_{L^2(\Omega)} + \left\|A^{1/2}u(t)\right\|^2_{L^2(\Omega)}\right) - \langle g(t), \gamma_1 u(t)\rangle_{L^2(\Gamma_1)}$$
$$+ \int_0^t \left\langle g'(\tau), \gamma_1 u(\tau)\right\rangle_{L^2(\Gamma_1)} d\tau + \int_0^t \left\langle w(\tau), u'(\tau)\right\rangle_{L^2(\Omega)} d\tau. \qquad (4.4)$$

The following lemma is straightforward and follows from the continuity and monotonicity of the function $f(\phi) = |\phi|^m\, sgn(\phi)$.

Lemma 4.2. *Let $f(\phi) = |\phi|^m\, sgn(\phi)$. Then f generates a monotone operator from $L^2(\Omega)$ into $L^2(\Omega)$; i.e.,*

$$\langle f(\phi) - f(\psi), \phi - \psi\rangle_{L^2(\Omega)} \geq 0, \text{ for all } \phi, \psi \in L^2(\Omega).$$

Moreover, the mapping $\lambda \mapsto \langle f(\phi + \lambda\psi), \eta\rangle_{L^2(\Omega)}$ is continuous from $\mathbb{R}$ to $\mathbb{R}$ for every fixed $\phi, \psi, \eta \in L^2(\Omega)$.

Proof of $w = f(u')$. Since

$$E_N(0) = \frac{1}{2}\Big(\left\|\mathcal{P}_N u^1\right\|^2_{L^2(\Omega)} + \left\|A^{1/2}\mathcal{P}_N u^0\right\|^2_{L^2(\Omega)}\Big) - \left\langle g(t), \gamma_1 \mathcal{P}_N u^0\right\rangle_{L^2(\Gamma_1)},$$

it follows that

$$\lim_{N\to\infty} E_N(0) = E(0) = \frac{1}{2}\Big(\left\|u^1\right\|^2_{L^2(\Omega)} + \left\|A^{1/2}u^0\right\|^2_{L^2(\Omega)}\Big) - \left\langle g(t), \gamma_1 u^0\right\rangle_{L^2(\Gamma_1)}.$$

Therefore, for all $t \in [0, T]$, we have

$$\liminf_{N\to\infty} E_N(t) = \lim_{N\to\infty} E_N(0) = E(0) = E(t). \qquad (4.5)$$

It follows from (3.12), (3.23) and (4.5) that

$$E(t) = \liminf_{N\to\infty} E_N(t) \geq \frac{1}{2}\liminf_{N\to\infty}\left\|u_N'(t)\right\|^2_{L^2(\Omega)} + \frac{1}{2}\liminf_{N\to\infty}\left\|A^{1/2}u_N(t)\right\|^2_{L^2(\Omega)}$$
$$- \langle g(t), \gamma_1 u(t)\rangle_{L^2(\Gamma_1)} + \int_0^t \left\langle g'(\tau), \gamma_1 u(\tau)\right\rangle_{L^2(\Gamma_1)} d\tau$$
$$+ \liminf_{N\to\infty}\int_0^t \left\langle f(u_N'(\tau)), u_N'(\tau)\right\rangle_{L^2(\Omega)} d\tau \qquad (4.6)$$
$$\geq \frac{1}{2}\left\|u'(t)\right\|^2_{L^2(\Omega)} + \frac{1}{2}\left\|A^{1/2}u(t)\right\|^2_{L^2(\Omega)} - \langle g(t), \gamma_1 u(t)\rangle_{L^2(\Gamma_1)}$$
$$+ \int_0^t \left\langle g'(\tau), \gamma_1 u(\tau)\right\rangle_{L^2(\Gamma_1)} d\tau + \liminf_{N\to\infty}\int_0^t \left\langle f(u_N'(\tau)), u_N'(\tau)\right\rangle_{L^2(\Omega)} d\tau.$$

Therefore, (4.4) and (4.6) yield

$$\int_0^t \left\langle w(\tau), u'(\tau)\right\rangle_{L^2(\Omega)} d\tau \geq \liminf_{N\to\infty} \int_0^t \left\langle f(u_N'(\tau)), u_N'(\tau)\right\rangle_{L^2(\Omega)} d\tau. \tag{4.7}$$

Now, let $\phi \in L^2(0, T, L^2(\Omega))$ be arbitrary. Then it follows from (4.7) that

$$\begin{aligned}\int_0^t &\left\langle w(\tau) - f(\phi(\tau)), u'(\tau) - \phi(\tau)\right\rangle_{L^2(\Omega)} d\tau \\ &\geq \liminf_{N\to\infty} \int_0^t \left\langle f(u_N'(\tau)) - f(\phi(\tau)), u_N'(\tau) - \phi(\tau)\right\rangle_{L^2(\Omega)} d\tau. \end{aligned}\tag{4.8}$$

Therefore, Lemma 4.2 and (4.8) yield that

$$\int_0^t \left\langle w(\tau) - f(\phi(\tau)), u'(\tau) - \phi(\tau)\right\rangle_{L^2(\Omega)} d\tau \geq 0, \tag{4.9}$$

for all $\phi \in L^2(0, T, L^2(\Omega))$. By choosing $\phi(t) = u'(t) - \lambda\psi(t)$, where $\lambda \in \mathbb{R}$, (4.9) yields

$$\int_0^t \left\langle w(\tau) - f(u'(\tau) - \lambda\psi(\tau)), \psi(\tau)\right\rangle_{L^2(\Omega)} d\tau \geq 0, \tag{4.10}$$

for all $\lambda \geq 0$ and for all $\psi \in L^2(0, T, L^2(\Omega))$. By letting $\lambda \to 0$ and using Lemma 4.2, one has

$$\int_0^t \left\langle w(\tau) - f(u'(\tau)), \psi(\tau)\right\rangle_{L^2(\Omega)} d\tau \geq 0, \tag{4.11}$$

for all $\psi \in L^2(0, T, L^2(\Omega))$. In particular, by letting $\psi(t) = f(u'(t)) - w(t)$, (4.11) shows that

$$\left\| w(\cdot) - f(u'(\cdot))\right\|_{L^2(0,T,L^2(\Omega))} = 0. \tag{4.12}$$

Consequently, u is in fact a regular solution to the original initial–boundary value problem (1.1)–(1.3).

Proof of uniqueness. Let v_1 and v_2 be regular solutions of the initial–boundary value problem (1.1)–(1.3). Thus, for $i = 1, 2$, we have

$$\begin{aligned}\left\langle v_i'(t), \phi\right\rangle_{L^2(\Omega)} - \left\langle u^1, \phi\right\rangle_{L^2(\Omega)} + \int_0^t \Big[&\left\langle A^{1/2} v_i(s), A^{1/2}\phi\right\rangle_{L^2(\Omega)} \\ &- \langle g(s), \gamma_1\phi\rangle_{L^2(\Gamma_1)} + \left\langle f(v_i'(s)), \phi\right\rangle_{L^2(\Omega)}\Big] ds = 0, \end{aligned}\tag{4.13}$$

for all $\phi \in H^1_{0,\Gamma_0}(\Omega)$ and almost every $t \in [0,T]$. Let $v = v_1 - v_2$. Then, (4.13) yields

$$\left\langle v'(t), \phi \right\rangle_{L^2(\Omega)} + \int_0^t \left\langle A^{1/2} v(s), A^{1/2}\phi \right\rangle_{L^2(\Omega)} ds + \int_0^t \left\langle f(v_1'(s)) - f(v_2'(s)), \phi \right\rangle_{L^2(\Omega)} ds = 0, \tag{4.14}$$

for all $\phi \in H^1_{0,\Gamma_0}(\Omega)$ and almost every $t \in [0,T]$. Therefore, v is a regular solution to the initial–boundary value problem

$$v'' - \Delta v + f(v_1') - f(v_2') = 0, \quad \text{in } \Omega \times (0,T), \tag{4.15}$$

$$v(x,0) = 0, \quad v'(x,0) = 0, \quad \text{in } \Omega, \tag{4.16}$$

$$\mathcal{B}v = 0, \quad \text{on } \Gamma. \tag{4.17}$$

However, by applying Lemma 4.1, we have

$$\frac{1}{2}\Big(\|v'(t)\|^2_{L^2(\Omega)} + \left\|A^{1/2}v(t)\right\|^2_{L^2(\Omega)} \Big) = -\int_0^t \left\langle f(v_1'(\tau)) - f(v_2'(\tau)), v_1'(\tau) - v_2'(\tau) \right\rangle_{L^2(\Omega)} d\tau. \tag{4.18}$$

Therefore, by Lemma 4.2 it follows that

$$\left\|A^{1/2}v(t)\right\|^2_{L^2(\Omega)} = 0 \tag{4.19}$$

for almost every $t \in [0,T]$, which completes the proof of uniqueness.

5. Appendix A

In this appendix, we provide the proof of Remark 2.1, which states that the weak formulation in Definition 1 is equivalent to the integral equations (2.12) and (2.13). Assume $u \in L^2(0,T,H^1_{0,\Gamma_0}(\Omega))$, $u' \in L^2(0,T,L^2(\Omega))$ such that u and u' satisfy integral equations (2.12) and (2.13). To simplify notation, unless otherwise indicated all inner products are taken on $L^2(\Omega)$. In what follows, Lemma 2.1 will be used repeatedly. Recall the definitions of the operators K and L given below:

$$Kg(t) = \int_0^t C(t-\tau)Rg(\tau)\,d\tau \text{ and } Lg(t) = \int_0^t AS(t-\tau)Rg(\tau)\,d\tau.$$

In addition, set

$$Fu(t) = \int_0^t S(t-\tau)f(u'(\tau))\,d\tau \text{ and } Gu(t) = \int_0^t C(t-\tau)f(u'(\tau))\,d\tau.$$

We shall show that u satisfies (2.5). From (2.12) we see that for any $\phi \in H^1_{0,\Gamma_0}(\Omega)$,

$$\int_0^t \left\langle A^{1/2}u(s), A^{1/2}\phi \right\rangle ds = -\int_0^t \left\langle A^{1/2}Fu(s), A^{1/2}\phi \right\rangle ds$$
$$+ \int_0^t \left\langle A^{1/2}S(s)u^1, A^{1/2}\phi \right\rangle ds + \int_0^t \left\langle A^{1/2}C(s)u^0, A^{1/2}\phi \right\rangle ds \tag{5.1}$$
$$+ \int_0^t \left\langle A^{1/2}(Rg(s) - C(s)Rg(0) - Kg'(s)), A^{1/2}\phi \right\rangle ds.$$

Next we examine each of the above terms.

$$\begin{aligned}\int_0^t \left\langle A^{1/2}Fu(s), A^{1/2}\phi \right\rangle\, ds &= \left\langle \int_0^t \int_0^s A^{1/2}S(s-\tau)f(u'(\tau))\, d\tau\, ds, A^{1/2}\phi \right\rangle \\ &= \left\langle -\int_0^t \int_\tau^t \frac{d}{ds}\left[A^{-1/2}C(s-\tau)f(u'(\tau))\right]\, ds\, d\tau, A^{1/2}\phi \right\rangle \\ &= \left\langle -\int_0^t A^{-1/2}\left[C(t-\tau)f(u'(\tau)) - f(u'(\tau))\right]\, d\tau, A^{1/2}\phi \right\rangle \\ &= -\left\langle Gu(t), \phi \right\rangle + \left\langle \int_0^t f(u'(\tau))\, d\tau, \phi \right\rangle .\end{aligned} \tag{5.2}$$

Using an argument similar to (5.2),

$$\int_0^t \left\langle A^{1/2}Kg'(s), A^{1/2}\phi \right\rangle ds = \left\langle Lg'(t), \phi \right\rangle . \tag{5.3}$$

We can also easily see that

$$\begin{aligned}\int_0^t \left\langle A^{1/2}C(s)u^0, A^{1/2}\phi \right\rangle ds &= \left\langle A^{1/2}\int_0^t C(s)u^0 ds, A^{1/2}\phi \right\rangle \\ &= \left\langle A^{1/2}S(t)u^0, A^{1/2}\phi \right\rangle = \left\langle AS(t)u^0, \phi \right\rangle .\end{aligned} \tag{5.4}$$

Again, by arguments similar to (5.4),

$$\int_0^t \left\langle A^{1/2}S(s)u^1, A^{1/2}\phi \right\rangle ds = \left\langle u^1, \phi \right\rangle - \left\langle C(t)u^1, \phi \right\rangle \tag{5.5}$$

and

$$\int_0^t \left\langle A^{1/2}C(s)Rg(0), A^{1/2}\phi \right\rangle ds = \left\langle AS(t)Rg(0), \phi \right\rangle . \tag{5.6}$$

Hence, by combining (5.1)–(5.6), we have that

$$\begin{aligned}&\left\langle u^1, \phi \right\rangle - \int_0^t \langle A^{1/2}u(s), A^{1/2}\phi\rangle - \int_0^t \langle f(u'(s)), \phi\rangle\, ds + \int_0^t \langle A^{1/2}Rg(s), A^{1/2}\phi\rangle\, ds \\ &= \left\langle C(t)u^1, \phi \right\rangle - \left\langle As(t)u^0, \phi \right\rangle + \left\langle As(t)Rg(0), \phi \right\rangle + \left\langle Lg'(t), \phi \right\rangle - \left\langle Gu(t), \phi \right\rangle .\end{aligned} \tag{5.7}$$

In addition, notice that (2.13) yields

$$\langle u'(t),\phi\rangle = \left\langle C(t)u^1,\phi\right\rangle - \left\langle As(t)u^0,\phi\right\rangle + \langle As(t)Rg(0),\phi\rangle + \langle Lg'(t),\phi\rangle - \langle Gu(t),\phi\rangle . \tag{5.8}$$

Therefore, (5.7) and (5.8) show that

$$\begin{aligned}\langle u'(t),\phi\rangle \quad &= \quad \left\langle u^1,\phi\right\rangle - \int_0^t \left\langle A^{1/2}u(s), A^{1/2}\phi\right\rangle ds \\ &\quad - \int_0^t \langle f(u'(s)),\phi\rangle\, ds + \int_0^t \left\langle A^{1/2}Rg(s), A^{1/2}\phi\right\rangle ds. \end{aligned}\tag{5.9}$$

Finally, by the definition of the Dirichlét–Neumann map R given in Section 2, we have

$$\langle A^{1/2}Rg(s), A^{1/2}\phi\rangle = \langle g(s),\gamma_1\phi\rangle_{L^2(\Gamma_1)}\,, \tag{5.10}$$

and so (2.5) holds.

Now suppose that $u\in L^2(0,T,H^1_{0,\Gamma_0}(\Omega))$, $u'\in L^2(0,T,L^2(\Omega))$, and that (2.5) holds for any $\phi\in H^1_{0,\Gamma_0}(\Omega)$ and almost every $t\in[0,T]$. Since $e_n\in H^1_{0,\Gamma_0}(\Omega)$ for all n, (2.5) and (5.10) imply that for each $n\in\mathbb{N}$,

$$\frac{d}{dt}\langle u'(t),e_n\rangle + \langle A^{\frac12}u(t), A^{\frac12}e_n\rangle - \langle A^{\frac12}Rg(t), A^{\frac12}e_n\rangle + \langle f(u'(t)),e_n\rangle = 0. \tag{5.11}$$

Setting $u_n(t)=\langle u(t),e_n\rangle$, we have that $u_n(t)$ satisfies the following:

$$u_n''(t)+\lambda_n u_n(t) = \lambda_n^{1/2}\langle A^{1/2}Rg(t),e_n\rangle - \langle f(u'(t)),e_n\rangle \tag{5.12}$$

$$u_n(0)=u_n^0,\quad u_n'(0)=u_n^1, \tag{5.13}$$

where $u_n^0 = \left\langle u^0,e_n\right\rangle$ and $u_n^1 = \left\langle u^1,e_n\right\rangle$. Hence, by the variation-of-parameters formula,

$$\begin{aligned}u_n(t) &= \cos(\lambda_n^{1/2}t)u_n^0 + \lambda_n^{-1/2}\sin(\lambda_n^{1/2}t)u_n^1 + \int_0^t \sin(\lambda_n^{1/2}(t-\tau))\langle A^{1/2}Rg(\tau),e_n\rangle\, d\tau \\ &\quad - \int_0^t \lambda_n^{-1/2}\sin(\lambda_n^{1/2}(t-\tau))\,\langle f(u'(\tau)),e_n\rangle\, d\tau. \end{aligned}\tag{5.14}$$

Integration by parts yields that

$$\begin{aligned}u_n(t) &= \cos(\lambda_n^{1/2}t)(u_n^0 - \lambda_n^{-1/2}\langle A^{1/2}Rg(0),e_n\rangle) + \lambda_n^{-1/2}\sin(\lambda_n^{1/2}t)u_n^1 \\ &\quad + \lambda_n^{-1/2}\langle A^{1/2}Rg(t),e_n\rangle - \int_0^t \lambda_n^{-1/2}\cos(\lambda_n^{1/2}(t-\tau))\langle A^{1/2}Rg'(\tau),e_n\rangle\, d\tau \\ &\quad - \int_0^t \lambda_n^{-1/2}\sin(\lambda_n^{1/2}(t-\tau))\,\langle f(u'(\tau)),e_n\rangle\, d\tau. \end{aligned}\tag{5.15}$$

Therefore, one has

$$u_n(t) = \cos(\lambda_n^{1/2}t)(u_n^0 - \langle Rg(0),e_n\rangle) + \lambda_n^{-1/2}\sin(\lambda_n^{1/2}t)u_n^1 + \langle Rg(t),e_n\rangle$$

$$- \int_0^t \cos(\lambda_n^{1/2}(t-\tau)) \langle Rg'(\tau), e_n \rangle \, d\tau - \int_0^t \lambda_n^{-1/2} \sin(\lambda_n^{1/2}(t-\tau)) \langle f(u'(\tau)), e_n \rangle \, d\tau,$$

which shows that u satisfies equation (2.12). Equation (2.13) immediately follows.

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