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GLOBAL EXISTENCE OF SOLUTIONS TO A WAVE EQUATION WITH DAMPING AND SOURCE TERMS

Mohammed Aassila
Institut de Recherche Mathématique Avancée, Université Louis Pasteur et C.N.R.S., 7 rue René Descartes , 67084 Strasbourg cedex, France

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Abstract. This paper is concerned with investigating the global existence of a solution to a nonlinear wave equation with damping and source terms.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega := \Gamma$. We are concerned with the following mixed problem:

$$\begin{cases} u'' - \Delta u + g(u') = |u|^{q-1}u \quad \text{in} \quad \Omega \times \mathbb{R}_+, \\ u = 0 \quad \text{on} \quad \Gamma \times \mathbb{R}_+, \\ u(x,0) = u_0(x), \;\; u'(x,0) = u_1(x) \quad \text{in} \quad \Omega; \end{cases} \tag{P}$$

here g is a real-valued function, and Δ the Laplacian in $\mathbb{R}^n$.

Many authors have investigated the dynamics of the problem (P). In particular, since Sattinger [12] constructed the so-called stable set in 1968, the method of stable set (potential well) was used in order to construct global solutions; see for instance [2, 5, 7, 11, 14] among others.

Indeed, for the case when $g(x) = \delta x$ ($\delta > 0$), Ikehata and Suzuki [6] investigated the dynamics; they have shown that for sufficiently small initial data (u_0, u_1), the trajectory $(u(t), u'(t))$ tends to $(0,0)$ in $H_0^1(\Omega) \times L^2(\Omega)$ as $t \to +\infty$. When $g(x) = \delta|x|^{m-1}x$ ($m \geq 1$), Georgiev and Todorova [3] have shown that if the damping term dominates over the source, then a global solution exists for any initial data. Quite recently, Ikehata [4] proved that a global solution exists with no relation between q and m, and Todorova [13] proved that the energy decay rate is $E(t) \leq (1+t)^{-2/(m-1)}$ for $t \geq 0$.

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Our purpose in this paper is to study the global existence and the asymptotic behavior of a solution u to (P) when $g(x)$ does not necessarily have polynomial growth near zero, as for example $g(x) = e^{-1/x}$ for $x \in (0,1]$. The decay rate of the global solution depends on the polynomial growth near zero of $g(x)$, as was proved in [1, 8, 13]. In our problem, we cannot use the classical methods, but we shall introduce some new techniques to derive a decay rate of the solution. Our plan in this paper is as follows. In Section 2, we shall prepare some lemmas which will be needed later. Section 3 is devoted to the statement and proof of the main theorem.

2. Preliminaries

Throughout the paper, we assume that g is a C^1, odd, increasing function and

$$|g(x)| \leq c_1|x| \quad \text{if} \quad |x| \leq 1,$$
$$c_2|x| \leq |g(x)| \leq c_3|x| \quad \text{if} \quad |x| \geq 1,$$

where c_1, c_2 and c_3 are positive constants. We first state two well-known lemmas, and then we state and prove two other lemmas that will be needed later.

Lemma 2.1 (Sobolev-Poincaré). *If either $1 \leq q < +\infty$ $(n = 1, 2)$ or $1 \leq q \leq (n+2)/(n-2)$ $(n \geq 3)$, then there is a constant $c(\Omega, q+1)$ such that*

$$\|u\|_{q+1} \leq c(\Omega, q+1)\|\nabla u\|_2 \quad \text{for} \quad u \in H_0^1(\Omega).$$

Lemma 2.2 ([8, Theorem 9.1]). *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a nonincreasing function and assume that there are two constants $p \geq 1$ and $A > 0$ such that*

$$\int_S^{+\infty} E^{\frac{p+1}{2}}(t)\, dt \leq AE(S), \quad 0 \leq S < +\infty;$$

then we have

$$E(t) \leq cE(0)(1+t)^{\frac{-2}{p-1}} \ \ \forall t \geq 0, \quad \text{if } p > 1$$

and

$$E(t) \leq cE(0)e^{-\omega t} \ \ \forall t \geq 0, \quad \text{if } p = 1,$$

where c and ω are positive constants.

Lemma 2.3. *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a nonincreasing function and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ an increasing C^2 function such that $\phi(0) = 0$ and $\phi(t) \to +\infty$ as $t \to$*

$+\infty$. *Assume that there exist* $p \geq 1$ *and* $A > 0$ *such that*

$$\int_S^{+\infty} E(t)^{\frac{p+1}{2}}(t)\phi'(t)\,dt \leq AE(S), \quad 0 \leq S < +\infty;$$

then we have

$$E(t) \leq cE(0)(1+\phi(t))^{\frac{-2}{p-1}} \ \ \forall t \geq 0, \quad \textit{if } p > 1$$

and

$$E(t) \leq cE(0)e^{-\omega\phi(t)} \ \ \forall t \geq 0, \quad \textit{if } p = 1,$$

where c *and* ω *are positive constants.*

Proof of Lemma 2.3. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $f(x) := E(\phi^{-1}(x))$ (we remark that ϕ^{-1} makes sense by the hypotheses assumed on ϕ). f is nonincreasing and $f(0) = E(0)$, and if we set $x := \phi(t)$, we obtain

$$\int_{\phi(S)}^{\phi(T)} f(x)^{\frac{p+1}{2}}\,dx = \int_{\phi(S)}^{\phi(T)} E\left(\phi^{-1}(x)\right)^{\frac{p+1}{2}}\,dx$$
$$= \int_S^T E(t)^{\frac{p+1}{2}}\phi'(t)\,dt \leq AE(S) = Af(\phi(S)) \quad 0 \leq S < T < +\infty.$$

Setting $s := \phi(S)$ and letting $T \to +\infty$, we deduce that

$$\int_s^{+\infty} f(x)^{\frac{p+1}{2}}\,dx \leq Af(s), \quad 0 \leq s < +\infty.$$

Thanks to Lemma 2.2, we deduce the desired results.

Remark 2.4. The use of a "weight function" $\phi(t)$ to establish the decay rate of solutions to hyperbolic PDE's was successfully done by Aassila [1], Mochizuki and Motai [10], and Martinez [9].

Lemma 2.5. *There exists an increasing function* $\phi : \mathbb{R}_+ \to \mathbb{R}$ *such that* ϕ *is concave and* $\phi(t) \to +\infty$ *as* $t \to +\infty$, $\phi'(t) \to 0$ *as* $t \to +\infty$, *and*

$$\int_1^{+\infty} \phi'(t)\left(g^{-1}(\phi'(t))\right)^2\,dt < +\infty.$$

Proof of Lemma 2.5. If such a function exists, we can assume that $\phi(1) = 1$. Setting $s := \phi(t)$, we obtain

$$\int_1^{+\infty} \phi'(t)(g^{-1}(\phi'(t)))\,dt = \int_1^{+\infty} (g^{-1}(\phi'(\phi^{-1}(s))))^2 ds = \int_1^{+\infty} g^{-1}\big(\frac{1}{(\phi^{-1})'(s)}\big)^2 ds.$$

Let us define ψ by

$$\psi(t) := 1 + \int_1^t \frac{1}{g(\frac{1}{s})}\, ds, \quad t \geq 1.$$

ψ is increasing, of class C^2 and

$$\psi'(t) = \frac{1}{g(\frac{1}{t})} \to +\infty \quad \text{as} \quad t \to +\infty.$$

Hence, $\psi(t) \to +\infty$ as $t \to +\infty$, and

$$\int_1^{+\infty} \Big(g^{-1}\Big(\frac{1}{\psi'(s)}\Big)\Big)^2\, ds = \int_1^{+\infty} \frac{1}{s^2}\, ds < +\infty.$$

Furthermore, ψ' is nondecreasing, and hence ψ is convexe. Let us verify that ψ^{-1} is concave. From the fact that $\psi(\psi^{-1}(s)) = s$, we have

$$(\psi^{-1})''(s) = -\frac{\psi''(\psi^{-1}(s))\left((\psi^{-1})'(s)\right)^2}{\psi'\left(\psi^{-1}(s)\right)} = -\frac{\psi''\left(\psi^{-1}(s)\right)}{\left(\psi'(\psi^{-1}(s))\right)^3} \leq 0.$$

In conclusion, if we set $\phi(t) := \psi^{-1}(t)\ \forall t \geq 1$, we see that ϕ satisfies all the hypotheses of Lemma 2.5.

3. Global existence and asymptotic behavior

Let us define the potential well as $W := \{u \in H_0^1(\Omega) : \|\nabla u\|_2^2 - \|u\|_{q+1}^{q+1} > 0\} \cup \{0\}$, and the energy by

$$E(t) := \frac{1}{2}\|u'\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{q+1}\|u\|_{q+1}^{q+1}.$$

A simple computation shows that

$$E'(t) = -\int_\Omega u' g(u')\, dx \leq 0;$$

hence, the energy is nonincreasing, and we have in particular $E(t) \leq E(0)$ for all $t \geq 0$.

We employ the Faedo–Galerkin method to construct a global solution to problem (P). Let $(\lambda_j)_{j=1}^\infty$ be a sequence of eigenvalues of $-\Delta u = \lambda u$ in Ω, $u = 0$ on Γ, $w_j \in H^2(\Omega) \cap H_0^1(\Omega)$ be the eigenfunction corresponding to λ_j and take $(w_j)_{j=1}^\infty$ as a completely orthonormal system in $L^2(\Omega)$.

We construct approximate solutions u_m $(m = 1, 2, \ldots)$ in the form

$$u_m(t) = \sum_{j=1}^m g_{jm}(t) w_j,$$

where the g_{jm} $(j = 1, 2, \dots, m)$ are determined by the following ordinary differential equations, for $j = 1, 2, \dots, m$:

$$(u''_m(t), w_j)+(\nabla u_m(t), \nabla w_j)+(g(u'_m(t)), w_j)=(|u_m(t)|^{q-1}u_m(t), w_j) \quad (3.1)$$

$$u_m(0) = u_{0m} := \sum_{j=1}^{m}(u_0, w_j)w_j \to u_0 \text{ in } H^2 \cap H_0^1 \text{ as } m \to +\infty, \quad (3.2)$$

$$u'_m(0) = u_{1m} := \sum_{j=1}^{m}(u_1, w_j)w_j \to u_1 \text{ in } H_0^1 \text{ as } m \to +\infty. \quad (3.3)$$

The system (3.1)–(3.3) can be easily solved by Picard's iteration method; hence, it admits a unique solution on some interval $[0, T_m)$ with $0 < T_m \leq +\infty$. We shall see that $u_m(t)$, which is of class C^2, can be extended to $[0, +\infty)$. Our main tool is the use of the energy method to derive some a priori estimates for an assumed smooth solution $u(t)$ to the problem (P) (the results should be in fact applied to approximated solutions).

Lemma 3.1. *Assume that*

$$1 < q < +\infty \quad \textit{for} \quad n = 1, 2;$$

$$1 < q \leq \frac{n+2}{n-2} \quad \textit{for} \quad n \geq 3;$$

$$c(\Omega, q+1)^{q+1}\Big(\frac{2(q+1)}{q-1}\Big)^{\frac{q-1}{2}} E(0)^{\frac{q-1}{2}} < 1;$$

then, $u(t) \in W$ *for all* $t \in [0, +\infty)$*, and there exists a constant* $M = M(\|\nabla u_0\|_2, \|u_1\|_2) > 0$ *such that*

$$\|u'(t)\|_2^2 + \|\nabla u(t)\|_2^2 \leq M \quad \textit{for} \quad t \geq 0.$$

Proof. From the continuity of $u(t)$ and the fact that $u_0 \in W$, it follows that

$$\|\nabla u(t)\|_2^2 - \|u(t)\|_{q+1}^{q+1} \geq 0 \quad \text{for} \quad t \in [0, t_{\max}),$$

where $t_{\max}$ is a maximal time (possibly $t_{\max} = T_m$). On the other hand

$$\begin{aligned}&\frac{1}{2}\|\nabla u(t)\|_2^2 - \frac{1}{q+1}\|u(t)\|_{q+1}^{q+1}\\ &= \frac{q-1}{2(q+1)}\|\nabla u(t)\|_2^2 + \frac{1}{q+1}(\|\nabla u(t)\|_2^2 - \|u(t)\|_{q+1}^{q+1}),\end{aligned}$$

and then

$$\frac{1}{2}\|\nabla u(t)\|_2^2 - \frac{1}{q+1}\|u(t)\|_{q+1}^{q+1} \geq \frac{q-1}{2(q+1)}\|\nabla u(t)\|_2^2 \quad \text{on} \quad [0, t_{\max}).$$

Hence,

$$\begin{aligned}\|\nabla u(t)\|_2^2 &\le \frac{2(q+1)}{q-1}\Big(\frac{1}{2}\|\nabla u(t)\|_2^2 - \frac{1}{q+1}\|u(t)\|_{q+1}^{q+1}\Big)\\ &\le \frac{2(q+1)}{q-1}E(t) \quad \text{on} \quad [0,t_{\max})\\ &\le \frac{2(q+1)}{q-1}E(0) \quad \text{on} \quad [0,t_{\max}).\end{aligned}$$

By the Sobolev–Poincaré inequality, we deduce that

$$\begin{aligned}&\|u(t)\|_{q+1}^{q+1} \le c(\Omega,q+1)^{q+1}\|\nabla u(t)\|_2^{q+1} \le c(\Omega,q+1)^{q+1}\|\nabla u(t)\|_2^{q-1}\|\nabla u(t)\|_2^2\\ &\le c(\Omega,q+1)^{q+1}\Big(\frac{2(q+1)}{q-1}\Big)^{\frac{q-1}{2}}E(0)^{\frac{q-1}{2}}\|\nabla u(t)\|_2^2 < \|\nabla u(t)\|_2^2 \ \text{ on } \ [0,t_{\max}).\end{aligned}$$

Therefore, we get

$$\|\nabla u(t)\|_2^2 - \|u(t)\|_{q+1}^{q+1} > 0 \quad \text{on} \quad [0,t_{\max});$$

this implies that we can take $t_{\max} = T_m$. Furthermore, it follows from the nonincreasingness of the energy that

$$\begin{aligned}\|u'(t)\|_2^2 \le 2E(t) \le 2E(0) \quad \text{on} \quad [0,T_m),\\ \frac{1}{2}\|\nabla u(t)\|_2^2 - \frac{1}{q+1}\|u(t)\|_{q+1}^{q+1} \le E(t) \le E(0) \quad \text{on} \quad [0,T_m),\end{aligned}$$

and hence

$$\|\nabla u(t)\|_2^2 \le \frac{2(q+1)}{q-1}E(0) \quad \text{on} \quad [0,T_m).$$

Lemma 3.2. *Let $u(t)$ and q be as in Lemma 3.1. Then there exists $0 < \alpha_0 < 1$ such that*

$$\|u(t)\|_{q+1}^{q+1} \le (1-\alpha_0)\|\nabla u(t)\|_2^2 \quad \textit{on} \quad [0,+\infty).$$

Proof. From Lemma 3.1, we have

$$\begin{aligned}\|u(t)\|_{q+1}^{q+1} &\le c(\Omega,q+1)^{q+1}\Big(\frac{2(q+1)}{q-1}\Big)^{\frac{q-1}{2}}E(0)^{\frac{q-1}{2}}\|\nabla u(t)\|_2^2\\ &= (1-\alpha_0)\|\nabla u(t)\|_2^2 \quad \text{on} \quad [0,+\infty),\end{aligned}$$

where

$$\alpha_0 = 1 - c(\Omega,q+1)^{q+1}\Big(\frac{2(q+1)}{q-1}\Big)^{\frac{q-1}{2}}E(0)^{\frac{q-1}{2}} > 0.$$

Lemma 3.3. *Let $u(t)$ and q be as in Lemmas 3.1–3.2; then we have*

$$E(t) \le c\Big(G^{-1}\big(\frac{1}{t}v\big)\Big)^2 \quad \forall t > 0,$$

where c is a positive constant and $G(x) = xg(x)$. Furthermore, if $x \mapsto g(x)/x$ is nonincreasing on $[0, \eta]$ for some $\eta > 0$, then we have

$$E(t) \le c\Big(g^{-1}\big(\frac{1}{t}\big)\Big)^2 \quad \forall t > 0,$$

where c is a positive constant.

Examples. 1) If $g(x) = e^{-1/x^p}$ for $0 < x < 1$ and $p > 0$, then we have

$$E(t) \le \frac{c}{(\ln t)^{2/p}}.$$

2) If $g(x) = e^{e^{-1/x}}$ for $0 < x < 1$, then we have

$$E(t) \le \frac{c}{(\ln(\ln t))^2}.$$

Proof of Lemma 3.3. From now on, we denote by c various positive constants which may be different at different occurrences. We multiply the first equation of (P) by $E\phi' u$, where ϕ is a function satisfying all the hypotheses of Lemma 2.5; we obtain

$$\begin{aligned}
0 &= \int_S^T E\phi' \int_\Omega u(u'' - \Delta u + g(u') - |u|^{q-1}u)\,dx\,dt \\
&= \Big[E\phi' \int_\Omega uu'\,dx\Big]_S^T - \int_S^T (E'\phi' + E\phi'')\int_\Omega uu'\,dx\,dt - 2\int_S^T E\phi' \int_\Omega u'^2\,dx\,dt \\
&+ \int_S^T E\phi' \int_\Omega \Big(u'^2 + |\nabla u|^2 - \frac{2}{q+1}|u|^{q+1}\Big)\,dx\,dt + \int_S^T E\phi' \int_\Omega ug(u')\,dx\,dt \\
&+ \int_S^T E\phi' \int_\Omega \Big(\frac{2}{q+1} - 1\Big)|u|^{q+1}\,dx\,dt.
\end{aligned}$$

Since

$$\begin{aligned}
\Big(\frac{2}{q+1} - 1\Big)\int_\Omega |u|^{q+1}\,dx &\le (1-\alpha_0)\frac{q-1}{q+1}\int_\Omega |\nabla u|^2\,dx \\
&\le (1-\alpha_0)\frac{q-1}{q+1}\frac{2(q+1)}{q-1}E(t) = 2(1-\alpha_0)E(t),
\end{aligned}$$

we deduce that

$$2\alpha_0 \int_S^T E^2\phi'\,dt \le -\Big[E\phi' \int_\Omega uu'\,dx\Big]_S^T + \int_S^T (E'\phi' + E\phi'')\int_\Omega uu'\,dx\,dt$$

$$+ 2\int_S^T E\phi' \int_\Omega u'^2\,dx\,dt - \int_S^T E\phi' \int_\Omega ug(u')\,dx\,dt$$
$$\leq -\Big[E\phi' \int_\Omega uu'\,dx\Big]_S^T + \int_S^T (E'\phi' + E\phi'') \int_\Omega uu'\,dx\,dt$$
$$+ 2\int_S^T E\phi' \int_\Omega u'^2\,dx\,dt + c(\varepsilon)\int_S^T E\phi' \int_\Omega g(u')^2\,dx\,dt + \varepsilon \int_S^T E\phi' \int_\Omega u^2\,dx\,dt$$

for every $\varepsilon > 0$. Choosing ε small enough, we deduce that

$$\int_S^T E^2\phi'\,dt \leq -\Big[E\phi' \int_\Omega uu'\,dx\Big]_S^T + \int_S^T (E'\phi' + E\phi'') \int_\Omega uu'\,dx\,dt$$
$$+ c\int_S^T E\phi' \int_\Omega u'^2\,dx\,dt \leq cE(S) + c\int_S^T E\phi' \int_\Omega u'^2\,dx\,dt.$$

We want to majorize the last term of the above inequality; we have

$$\int_S^T E\phi' \int_\Omega u'^2 dx\,dt = \int_S^T E\phi' \int_{\Omega_1} u'^2 dx\,dt + \int_S^T E\phi' \int_{\Omega_2} u'^2 dx\,dt + \int_S^T E\phi' \int_{\Omega_3} u'^2 dx\,dt,$$

where, for $t \geq 1$, $\Omega_1 := \{x \in \Omega : |u'| \leq h(t)\}$, $\Omega_2 := \{x \in \Omega : h(t) < |u'| \leq h(1)\}$, $\Omega_3 := \{x \in \Omega : |u'| > h(1)\}$, and $h(t) := g^{-1}(\phi'(t))$, which is a positive nonincreasing function and satisfies $h(t) \to 0$ as $t \to +\infty$. Since

•

$$\int_S^T E\phi' \int_{\Omega_1} u'^2\,dx\,dt \leq c\int_S^T E(t)\phi'(t) \left(\int_{\Omega_1} h(t)^2\,ds\right) dt$$
$$\leq cE(S)\int_S^T \phi'(t)(g^{-1}(\phi'(t)))^2\,dt \leq cE(S);$$

• as g is nondecreasing, then for $x \in \Omega_2$ we have $\phi'(t) = g(h(t)) \leq |g(u')|$, and hence

$$\int_S^T E\phi' \int_{\Omega_2} u'^2\,dx\,dt \leq \int_S^T E \int_{\Omega_2} |g(u')|u'^2\,dx\,dt$$
$$\leq h(1)\int_S^T E \int_{\Omega_2} u'g(u')\,dx\,dt \leq \frac{h(1)}{2}E(S)^2;$$

• as $g(x) \geq cx$ for $x \geq h(1)$, we have

$$\int_S^T E\phi' \int_{\Omega_3} u'^2\,dx\,dt \leq c\int_S^T E\phi' \int_\Omega u'g(u')\,dx\,dt$$

$$\leq c\int_S^T E(-E')\,dx\,dt \leq cE(S)^2.$$

Then we deduce that

$$\int_S^T E^2\phi'\,dt \leq cE(S),$$

and thanks to Lemma 2.5 we obtain $E(t) \leq \frac{c}{\phi(t)}, \forall t \geq 1$. Let s_0 be such that $g(1/s_0) \leq 1$; since g is nondecreasing we have

$$\psi(s) \leq 1 + (s-1)\frac{1}{g(\frac{1}{s})} \leq s\frac{1}{g(\frac{1}{s})} = \frac{1}{G(\frac{1}{s})} \quad \forall s \geq s_0,$$

hence, $s \leq \phi(\frac{1}{G(\frac{1}{s})})$ and $\frac{1}{\phi(t)} \leq \frac{1}{s}$ with $t := \frac{1}{G(\frac{1}{s})}$. Thus, $\frac{1}{\phi(t)} \leq G^{-1}(\frac{1}{t})$. Now define $H(x) := \frac{g(x)}{x}$, H is nondecreasing, $H(0) = 0$, then we use the function $h(t) := H^{-1}(\phi'(t))$. On Ω_2 it holds that

$$\phi'(t)u'^2 \leq |H(u')|u'^2 = u'g(u'),$$

and the same calculations as above with

$$\phi^{-1}(t) = 1 + \int_1^t \frac{1}{H(\frac{1}{s})}\,ds$$

yield $E(t) \leq c(g^{-1}(\frac{1}{t}))^2$.

Lemma 3.4. *Let u, q and g be as in Lemmas 3.1–3.3, and assume that*

$$\int_0^{+\infty} \left(g^{-1}(\frac{1}{t})\right)^2 dt < +\infty.$$

Suppose that $u(t)$ is a local solution on $[0,T)$ such that

$$\sup\{\|\nabla u'(t)\|_2,\ \|\Delta u(t)\|_2\} \leq k \quad on \quad [0,T)$$

for some $k > 0$ and $T > 0$. If

$$1 < q < +\infty \quad for \quad n = 1,\, 2, \tag{3.4}$$

$$1 < q \leq 3 \quad for \quad n = 3, \tag{3.5}$$

then we have

$$\|\nabla u'(t)\|_2^2 + \|\Delta u(t)\|_2^2 \leq G_1(E_0, E_1, k),$$

and if

$$\frac{n}{n-2} \leq q \leq \min\left\{\frac{n+2}{n-2}, \frac{n-2}{[n-4]^+}\right\} \quad for \quad n \geq 3, \tag{3.6}$$

then we have

$$\|\nabla u'(t)\|_2^2 + \|\Delta u(t)\|_2^2 \leq G_2(E_0, E_1, k)$$

with

$$\lim_{E_0\to 0} G_j(E_0, E_1, k) = E_1 \quad (j = 1, 2)$$

and where we set $E_0 = E(0),\ E_1 = \|\nabla u_1\|_2^2 + \|\Delta u_0\|_2^2$.

Proof. Multiplying the first equation of (P) by $-\Delta u'(t)$ and integrating over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\Big[\|\nabla u'(t)\|_2^2 + \|\Delta u(t)\|_2^2\Big] + \Big(\nabla g(u'(t)), \nabla u'(t)\Big) = \Big(\nabla(|u(t)|^{q-1}u(t)), \nabla u'(t)\Big).$$

In the case (3.4)–(3.5), it follows from Hölder's inequality that

$$\begin{aligned}\Big|\Big(\nabla(|u(t)|^{q-1}u(t)), \nabla u'(t)\Big)\Big| &\le q\||u(t)|^{q-1}\nabla u(t)\|_2\|\nabla u'(t)\|_2\\ &\le q\|u(t)\|_{(q-1)a}^{q-1}\|\nabla u(t)\|_b\|\nabla u'(t)\|_2\end{aligned}$$

with $\frac{2}{b} + \frac{2}{a} = 1$. We take a and b such that

$$\frac{1}{(q-1)a} \ge \frac{1}{2} - \frac{1}{n} \quad \text{and} \quad \frac{1}{b} \ge \frac{1}{2} - \frac{1}{n}.$$

By Sobolev's inequality and regularity for elliptic equations, we have

$$\|u(t)\|_{(q-1)a} \le c\|\nabla u(t)\|_2 \quad \text{and} \quad \|\nabla u(t)\|_b \le c\|\Delta u(t)\|_2,$$

whence

$$\begin{aligned}\Big|\Big(\nabla(|u(t)|^{q-1}u(t)), \nabla u'(t)\Big)\Big| &\le c\|\nabla u(t)\|_2^{q-1}\|\Delta u(t)\|_2\|\nabla u'(t)\|_2\\ &\le c\,k^2\,E(t)^{\frac{q-1}{2}},\end{aligned}$$

and therefore

$$\frac{d}{dt}\Big[\|\nabla u'(t)\|_2^2 + \|\Delta u(t)\|_2^2\Big] + g'(u'(t))\|\nabla u'(t)\|_2^2 \le c\,k^2 E(t)^{\frac{q-1}{2}}.$$

That is,

$$\frac{d}{dt}\Big[\|\nabla u'(t)\|_2^2 + \|\Delta u(t)\|_2^2\Big] \le c\,k^2 E(t)^{\frac{q-1}{2}}.$$

Integrating this inequality over $[0, t]$, we get

$$\|\nabla u'(t)\|_2^2 + \|\Delta u(t)\|_2^2 \le E_1 + c\,k^2\int_0^t E(s)^{\frac{q-1}{2}}\,ds.$$

We deduce from Lemma 3.3 that

$$\|\nabla u'(t)\|_2^2 + \|\Delta u(t)\|_2^2 \le G_1(E_0, E_1, k).$$

In the case (3.6), we have

$$\Big|\Big(\nabla(|u(t)|^{q-1}u(t)), \nabla u'(t)\Big)\Big| \le q\||u(t)|^{q-1}\nabla u(t)\|_2\|\nabla u'(t)\|_2$$

$$\leq q\|u(t)\|_{(q-1)n}^{q-1}\|\nabla u(t)\|_{\frac{2n}{n-2}}\|\nabla u'(t)\|_2$$
$$\leq c\|u(t)\|_{(q-1)n}^{q-1}\|\Delta u(t)\|_2\|\nabla u'(t)\|_2.$$

From the Gagliardo–Nirenberg inequality we deduce that

$$\|u(t)\|_{(q-1)n}^{q-1} \leq c\|u(t)\|_{\frac{2n}{n-2}}^{(q-1)(1-\theta)}\|\Delta u(t)\|_2^{(q-1)\theta}$$
$$\leq c\|\nabla u(t)\|_2^{(q-1)(1-\theta)}\|\Delta u(t)\|_2^{(q-1)\theta} \leq c\, k^{(q-1)\theta}E(t)^{\frac{(q-1)(1-\theta)}{2}},$$

and hence

$$\left|\Big(\nabla(|u(t)|^{q-1}u(t)), \nabla u'(t)\Big)\right| \leq c\, k^{(q-1)\theta+2}E(t)^{\frac{(q-1)(1-\theta)}{2}}.$$

By the same arguments as the previous case, we obtain

$$\|\nabla u'(t)\|_2^2 + \|\Delta u(t)\|_2^2 \leq G_2(E_0, E_1, k).$$

Note that

$$\lim_{E_0\to 0} G_j(E_0, E_1, k) = E_1 \quad (j = 1, 2).$$

Remark 3.5. If $g(x) = |x|^{p-1}x$, $p \geq 1$, then $E(t) \leq cE(0)e^{-\omega t}$ $\forall t \geq 0,\ c > 0,\ \omega > 0$, if $p = 1$, or $E(t) \leq \frac{cE(0)}{(1+t)^{\frac{2}{p-1}}}$ $\forall t \geq 0$, $c > 0$ if $p > 1$. And $G_1(E_0, E_1, k) = E_1 + ck^2E_0^{\frac{q-1}{2}}$, $G_2(E_0, E_1, k) = E_1 + ck^{(q-1)\theta+2}E_0^{\frac{(q-1)(1-\theta)}{2}}$.

Main Theorem. *Under the hypotheses of Lemmas* 3.1–3.4, *there exists an open set* $S \subset (W \cap H^2(\Omega)) \times H_0^1(\Omega)$ *such that if* $(u_0, u_1) \in S$, *the problem (P) has a unique global solution* u *satisfying*

$$u \in L^\infty(\mathbb{R}_+, H^2(\Omega) \cap H_0^1(\Omega)) \cap W^{1,\infty}(\mathbb{R}_+, H_0^1(\Omega)) \cap W^{2,\infty}(\mathbb{R}_+, L^2(\Omega));$$

furthermore, we have the decay estimate

$$E(t) \leq c\Big(g^{-1}\big(\frac{1}{t}\big)\Big)^2 \quad \forall t > 0.$$

Proof. Let $k > 0$. Put $G(E_0, E_1, k) = G_1(E_0, E_1, k)$ or $G_2(E_0, E_1, k)$, $S_k := \{(u_0, u_1) \in (W \cap H^2(\Omega)) \times H_0^1(\Omega),\ G^{\frac{1}{2}}(E_0, E_1, k) < k\}$ and

$$S = \bigcup_{k>0} S_k.$$

Since $\lim_{E_0\to 0} G(E_0, E_1, k) = E_1$, the set S_k is not empty if E_0 is sufficiently small and E_1 satisfies $E_1 < k^2$. This means that S_k contains a small ball with the center at the origin with respect to the norm of (u_0, u_1) in $(W \cap H^2(\Omega)) \times H_0^1(\Omega)$. Moreover, if $(u_0, u_1) \in S_k$, then we have $\|\nabla u_1\|_2 < k$ and

$\|\Delta u_0\|_2 < k$; it is easy to verify that S is open in $(W \cap H^2(\Omega)) \times H_0^1(\Omega)$ and is unbounded in $(W \cap H^2(\Omega)) \times H_0^1(\Omega)$.

If $(u_0, u_1) \in S_k$ for some $k > 0$, then the local solution $u(t)$ exists on some interval $[0,T)$ and satisfies the estimates

$$\|\Delta u(t)\|_2 < k \quad \text{and} \quad \|\nabla u'(t)\|_2 < k \quad \text{on} \quad [0,T).$$

Hence, it follows from Lemma 3.4 that if $(u_0, u_1) \in S_k$ then

$$\begin{aligned}\sup\{\|\nabla u'(t)\|_2,\ \|\Delta u(t)\|_2\} &\leq \left(\|\nabla u'(t)\|_2^2 + \|\Delta u(t)\|_2^2\right)^{\frac{1}{2}} \\ &\leq G^{\frac{1}{2}}(E_0, E_1, k) < k \quad \text{on} \quad [0,T).\end{aligned}$$

Furthermore, we have $(u(t), u'(t)) \in S_k$ on $[0,T)$. Indeed, suppose that there is a number $t^* \in [0,T)$ such that $(u(t), u'(t)) \in S_k$ on $[0, t^*)$ and $(u(t^*), u'(t^*)) \notin S_k$; then it follows that

$$G(E(t^*), \|\nabla u'(t^*)\|_2^2, k) \geq k^2, \tag{3.7}$$

and hence, in the case $q = 3$ for example, we see from (3.7) and Lemma 3.4 that

$$E_1 + c\,k^2 \int_0^t E^{\frac{q-1}{2}}(s)\,ds \geq k^2. \tag{3.8}$$

But since $\lim_{E_0 \to 0} G(E_0, E_1, k) = E_1$, we may take E_1 so that $E_1 < k^2$ with sufficiently small E_0. This contradicts (3.8). Hence, we obtain $(u(t), u'(t)) \in S_k$ on $[0,T)$. If $(u_0, u_1) \in S_k$ for some $k > 0$, then the local solution $u(t)$ satisfies

$$\sup\{\|\nabla u'(t)\|_2, \|\Delta u(t)\|_2\} \leq k \quad \text{on} \quad [0,T),$$

and as a result $(u(t), u'(t))$ remains in S_k. Consequently, since we can repeat the continuation procedure indefinitely, we conclude that if $(u_0, u_1) \in S$, the solution $u(t)$ can be continued globally on $[0, +\infty)$ and $(u(t), u'(t)) \in S$ for all $t \geq 0$.

Lemma 3.6. (uniqueness) *The global solution $u(t)$ to (P) is unique.*

Proof. Let $u(t)$ and $v(t)$ be two solutions; then $w(t) = u(t) - v(t)$ satisfies

$$w'' - \Delta w + g(u') - g(v') = \left(|u|^{q-1}u - |v|^{q-1}v\right) \quad \text{in} \quad \Omega \times (0, +\infty), \tag{3.9}$$

$$w = 0 \quad \text{on} \quad \Gamma \times (0, +\infty), \tag{3.10}$$

$$w(0) = w'(0) = 0 \quad \text{in} \quad \Omega. \tag{3.11}$$

Taking the L^2 inner product of (3.9) with w', we find that

$$\frac{1}{2}\frac{d}{dt}\left[\|w'(t)\|_2^2 + \|\nabla w(t)\|_2^2\right] + \int_\Omega (g(u') - g(v'))(u' - v')\,dx$$

$$= \big(|u|^{q-1}u - |v|^{q-1}v, w'\big). \tag{3.12}$$

Note that

$$\int_\Omega (g(u') - g(v'))(u' - v')\,dx \geq 0$$

and

$$\begin{aligned}\||u|^{q-1}u - |v|^{q-1}v\|_2 &\leq c(\|u\|_{n(q-1)} + \|v\|_{n(q-1)})^{q-1}\|u - v\|_{\frac{2n}{n-2}}\\ &\leq c(\|\Delta u\|_2 + \|\Delta v\|_2)^{q-1}\|\nabla w\|_2.\end{aligned}$$

Hence,

$$\big(|u|^{q-1}u - |v|^{q-1}v, w'\big) \leq c\|\nabla w(t)\|_2\|w'(t)\|_2 \leq c\big(\|\nabla w(t)\|_2^2 + \|w'(t)\|_2^2\big).$$

Integrating (3.12) over $[0, t]$ we get

$$\|w'(t)\|_2^2 + \|\nabla w(t)\|_2^2 \leq c\int_0^t (\|w'(t)\|_2^2 + \|\nabla w(t)\|_2^2)\,ds,$$

which by Gronwall's inequality implies $w = 0$. This completes the proof of the main theorem.

Remark 3.7. If $g(x) = |x|^{q-1}x$ with $1 < q \leq \frac{n}{n-2}$ if $n > 2$ and $1 < q < +\infty$ if $n = 2$, then we have the relation $S = (W \cap H^2(\Omega)) \times H_0^1(\Omega)$ as in Georgiev–Todorova [3].

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