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SOME SPECIAL SOLUTIONS OF THE THIN-FILM EQUATION

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1. Introduction

The one-dimensional spreading of a thin liquid film over a horizontal solid surface is often modelled by the following equation for the film height h:

$$\partial_t h + \partial_x \left(h^3 \partial^3_{xxx} h\right) = 0. \tag{1.1}$$

Here $h(x,t)$ is a nonnegative function which typically has compact support. The derivation of equation (1.1) is based on the following basic assumptions: the lubrication approximation with the no-slip condition for the fluid at the solid surface and the fact that the pressure is entirely created by surface tension. We observe that, if the lubrication approximation is assumed to be valid up to the edge of the support of the film (the contact lines), then the height h will satisfy a zero-flux condition at the contact lines. The main difficulty of the lubrication model is that the zero-flux condition implies the nonexistence of solutions of (1.1) with *advancing* contact lines. In this context in [3], Bernis, Peletier and Williams prove the nonexistence of compactly supported source-type solutions of self-similar form, and in [4], Boatto, Kadanoff and Olla prove the nonexistence of advancing travelling waves.

In the paper [1] we introduced a new mathematical model based on the lubrication approximation in the major part of the film where the film is not too thin (basic region), and in the remaining small regions near the contact

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angles (contour regions) it is assumed that the autonomy principle holds; i.e., the shape of the film in the contour region is completely determined by the velocity V of the contact line. This leads to the following free-boundary-value problem for $h(x,t)$: equation (1.1) holds in the basic region, and at the interface $x = x_0(t)$ separating the basic from the contour region the following three free-boundary conditions hold:

$$\begin{aligned} &h = h_0(V) \\ &h\partial^2_{xx}h = G(V) \\ &h^3\partial^3_{xxx}h - hV = C(V), \end{aligned} \tag{1.2}$$

where $V = x_0'(t)$ is the speed of propagation and h_0, G, C are given universal functions of V which depend on the type of liquid and solid in consideration. In [1] we considered a class of self-similar solutions to the free boundary problem assuming that $G = C = 0$ and $h_0 = KV^{1/6}$ is a positive monomial. In the first part of this paper we show the existence of such solutions. More precisely we look for positive solutions of problem (1.1)–(1.2) of the form

$$h(x,t) = \frac{A}{t^\alpha} f(\frac{x}{Bt^\beta})$$

where A, B, α, and β are positive constants to be chosen in such a way that (1.1) and (1.2) are satisfied for $G = C = 0$ and $h_0 = KV^k$. After some calculations (see [1] for the details) we find that $\alpha = \beta = \frac{1}{7}$ $B = \frac{Q^{6/7}(7)^{1/7}}{2^{6/7}K^{6/7}(\int_0^1 f(\eta)\,d\eta)^{6/7}}$ and $A = K(\frac{1}{7})^{1/6}B^{1/6}$, where, because of conservation of mass,

$$\int_{-x_0}^{x_0} h(x,t)\,dx = Q = const.$$

and f satisfies the following problem:

$$\begin{aligned} &f^2 f''' = \frac{\lambda\eta}{(\int_0^1 f(t)\,dt)^3} \qquad \text{if } 0 < \eta < 1 \text{ and } \lambda > 0 \\ &f'(0) = f''(1) = 0, \quad f(1) = 1, \end{aligned} \tag{1.3}$$

where $\lambda := \frac{Q^3}{8K^6}$. In the first part of this paper we prove the existence of a solution f_λ for any $\lambda > 0$. As we explained before, there is no hope to construct solutions of (1.2) with $V > 0$ and $h_0(V) = C(V) = 0$. In order to simplify the free-boundary problem (1.2) it is natural to ask the following question: is it possible to construct physically relevant solutions of (1.2) with $V > 0$, $h_0(V) = 0$ and $C(V) \neq 0$? In [1] we have given a formal argument which suggests a negative answer to this question. In the second part of this paper we provide a rigorous negative answer for a special class of solutions. More precisely, we study the problem

$$\text{(S)}\qquad \begin{cases} \partial_t h + \partial_x \left(h^3 \partial^3_{xxx} h\right) = 0 & \text{if } 0 < x < Vt,\ t > 0 \\ \partial_x h(0,t) = 0 & \text{if } t > 0 \\ h^3 \partial^3_{xxx} h(0,t) = 0 & \text{if } t > 0, \end{cases}$$

with the conditions at the interface

$$(\text{S}_0)\qquad \begin{cases} h(x,t) > 0 \quad \text{if } 0 \le x < Vt, t > 0 \\ h(Vt,t) = 0, \quad h^3 \partial^3_{xxx} h(Vt,t) = -\epsilon t \end{cases}$$

and look for positive solutions of the form

$$h(x,t) = t f(\eta), \qquad \eta = x/t \tag{1.4}$$

for any choice of ϵ and V; here V is the velocity of the contact line $x = Vt$ and $-\epsilon t$ is the flux at the contact line.

Substituting in (S) and in (S_0) we derive the following problem for f :

$$\begin{cases} (f^3 f''')' = \eta f' - f \quad \text{if } 0 < \eta < V \\ f(0) > 0, \quad f'(0) = 0, \quad f'''(0) = 0 \\ f(V) = 0 \quad \text{and} \quad 2\int_0^V f(\eta)\, d\eta = \epsilon. \end{cases} \tag{1.5}$$

We have chosen to look for self-similar solutions of the form (1.4) in order to get solutions h with constant velocity V of the contact line and with constant pressure at the origin (in fact by (1.2) $h\partial_{xx}h(0,t) = f(0)f''(0)$).

In Theorem 3.1 we prove existence of solutions of problem (1.5) for any positive V and ϵ. In Theorem 3.3 we establish an asymptotic formula for the solution f of problem 1.5, in a neighborhood of the interface. More precisely, we prove that as $\eta \to V$

$$f(\eta) = C(V-\eta)^{3/4} + o((V-\eta)^{3/4}) \tag{1.6}$$

where $C > 0$. This result proves that it is not possible to simplify the free-boundary problem (1.2) setting $h_0(V) = 0$. In fact the asymptotic formula (1.6) leads to the unphysical situation that $h\partial^2_{xx}$, which represents, up to a constant, the force acting on the contour region, is infinite at the edge of the fluid.

In order to prove Theorem 3.1 we investigate the initial value problem

$$\begin{cases} (f^3 f''')' = \eta f' - f \quad \text{for } \eta > 0 \\ f(0) = 1, \quad f'(0) = 0, \quad f''(0) = \beta, \quad f'''(0) = 0, \end{cases} \tag{1.7}$$

where $\beta \in \mathbf{R}$ is a shooting parameter. Problem (1.5) has been investigated by Peletier and Hastings in [2]. They give some very precise results about the behavior of solutions of (1.7) for different values of the parameter β.

Finally we observe that there exists an extensive literature on the thin film equations. For the modelling aspects we refer to [1] and the references therein; for the PDE results we refer to [5].

2. A class of self-similar solutions for a free-boundary problem

We consider the problem

$$\begin{aligned} &f^2 f''' = \frac{\lambda\eta}{(\int_0^1 f(t)\,dt)^3} \qquad \text{if } 0<\eta<1 \text{ and } \lambda>0 \\ &f'(0)=f''(1)=0, \quad f(1)=1. \end{aligned} \tag{2.1}$$

Let f_λ denote a solution of problem (2.1). It follows at once that f_λ is decreasing and concave. We start our analysis by studying the asymptotic behavior of solutions of (2.1) as $\lambda \to +\infty$.

In the sequel c will indicate a positive constant.

Proposition 2.1. *There exists a positive constant c such that for λ large enough*

$$f_\lambda(0) \geq c\lambda^{1/5}. \tag{2.2}$$

Furthermore,

$$v_\lambda(\eta) := \frac{f_\lambda(\eta)}{f_\lambda(0)} \to 1-\eta^2 \text{ in } C([0,1]) \cap C^2_{loc}([0,1)) \text{ as } \lambda \to +\infty \tag{2.3}$$

and v''_λ is uniformly bounded in $[0,1]$.

As a consequence we derive the following asymptotic formula for $f_\lambda(0)$.

Corollary 2.2. *As $\lambda \to +\infty$*

$$f_\lambda(0) = \frac{(27)^{1/5}}{2}\lambda^{1/5} + o(\lambda^{1/5}). \tag{2.4}$$

Proof. Consider the function v_λ defined above. Then v_λ satisfies

$$\begin{aligned} &v_\lambda^2 v_\lambda''' = \frac{\lambda\eta}{(f_\lambda(0))^6(\int_0^1 v_\lambda(t)\,dt)^3} \qquad \text{if } 0<\eta<1 \text{ and } \lambda>0 \\ &v'_\lambda(0)=v''_\lambda(1)=0, \quad v_\lambda(0)=1. \end{aligned}$$

It follows from Proposition 2.1 that

$$\frac{\lambda}{(f_\lambda(0))^6} \to 0 \text{ and } \int_0^1 v_\lambda(t)\,dt \to \frac{2}{3} \text{ as } \lambda \to +\infty \tag{2.5}$$

and

$$\int_0^1 (v_\lambda''(t))^2\,dt \to 4 \text{ as } \lambda \to +\infty. \tag{2.6}$$

Multiplying the equation for v_λ by v_λ' and integrating by parts over the interval $[0,\eta]$ we derive

$$v_\lambda' v_\lambda''(\eta) - \int_0^\eta (v_\lambda''(t))^2\,dt = \frac{-\lambda}{(f_\lambda(0))^5(\int_0^1 v_\lambda(t)\,dt)^3}\Big(\frac{t}{f_\lambda(t)}\Big|_0^\eta - \int_0^\eta \frac{1}{f_\lambda(t)}\,dt\Big)$$

which, for $\eta = 1$, becomes

$$\int_0^1 (v_\lambda''(t))^2\,dt = \frac{\lambda}{(f_\lambda(0))^5(\int_0^1 v_\lambda(t)\,dt)^3}\Big(1 - \int_0^1 \frac{1}{f_\lambda(t)}\,dt\Big).$$

Using (2.2), (2.3) and (2.6) and letting $\lambda \to +\infty$ in the above relation we get

$$\frac{\lambda}{(f_\lambda(0))^5} \to \frac{32}{27} \text{ as } \lambda \to +\infty$$

from which (2.4) follows.

Proof of Proposition 2.1. First of all let us prove that $\lim_{\lambda\to+\infty} f_\lambda(0) = +\infty$. Assume to the contrary that there exists a subsequence $\lambda_n \to +\infty$ and a constant B such that $f_{\lambda_n}(0) \le B$. Since f_λ is concave, f_λ'' is increasing and since $f_\lambda''(1) = 0$, one has

$$f_{\lambda_n}''(\eta) \le f_{\lambda_n}''(1/2) = -\int_{1/2}^1 f_{\lambda_n}'''(t)\,dt \le -\frac{\lambda_n}{B^5}\int_{1/2}^1 \eta\,d\eta = -\frac{3\lambda_n}{8B^5}, \quad \forall \eta \le 1/2.$$

Hence, integrating this relation twice in $(0,\eta)$ and setting $\eta = 1/2$, we obtain

$$f_{\lambda_n}(1/2) \le B - \frac{3\lambda_n}{32B^5},$$

which gives a contradiction for n large enough.

Let v_λ be defined by (2.3). We claim that

$$v_\lambda''(1/2) \ge -c \tag{2.7}$$

for some positive constant c which does not depend on λ. Arguing by contradiction we suppose that $v_{\lambda_n}''(1/2) \to -\infty$ as $\lambda_n \to +\infty$. Since $v_\lambda''' > 0$, this implies that $v''_{\lambda_n} \to -\infty$ uniformly on $[0,1/2]$ as $\lambda_n \to +\infty$, which contradicts the boundedness of v_λ:

$$0 < \frac{1}{f_\lambda(0)} = v_\lambda(1) < v_\lambda(\eta) < v_\lambda(0) = 1 \text{ if } 0 < \eta < 1.$$

Using the concavity of v_λ we have that

$$v'_\lambda(1/2) \geq -2(v_\lambda(1/2) - v_\lambda(1)) \geq -2v_\lambda(1/2) > -2. \tag{2.8}$$

By (2.7) and since $v'''_\lambda > 0$,

$$v_\lambda(\eta) \geq -c \text{ for } 1/2 \leq \eta \leq 1;$$

integrating this inequality in $(1/2,1)$ we obtain from (2.8) that $v'_\lambda(1) \geq -K$ for some $K > 0$ which does not depend on λ. This implies that

$$v_\lambda(\eta) \leq \beta_\lambda + K(1-\eta) \text{ for } 0 \leq \eta \leq 1$$

where we have set

$$\beta_\lambda = v_\lambda(1) = \frac{1}{f_\lambda(0)}.$$

In particular, observing that $\int_0^1 v_\lambda(t)\,dt > 1/2$, we find that

$$v'''_\lambda = \frac{\lambda\eta}{f_\lambda^6(0)(\int_0^1 v_\lambda(t)\,dt)^3 v_\lambda^2} \geq 8\lambda\beta^6 \frac{\eta}{(\beta_\lambda + K(1-\eta))^2} \text{ for } 0 \leq \eta \leq 1. \tag{2.9}$$

Integrating in $(1/2,1)$ and using that

$$\int_{1/2}^1 \frac{\eta}{(\beta_\lambda + K(1-\eta))^2}\,d\eta = \frac{1}{\beta_\lambda}(1 + o(\lambda)) \text{ as } \lambda \to +\infty$$

we conclude that

$$v''_\lambda(1/2) \leq -8\lambda\beta_\lambda^5(1 + o(\lambda)) \text{ as } \lambda \to +\infty.$$

Hence it follows from (2.7) that $\lambda\beta_\lambda^5$ is uniformly bounded, which implies (2.2). Let us now complete the proof of Proposition 2.1. By (2.2),

$$\frac{\lambda}{f_\lambda^6(0)} \to 0 \text{ as } \lambda \to +\infty.$$

Since $\int_0^1 v_\lambda(t)\,dt \geq 1/2$ this implies that

$$v_\lambda^2 v'''_\lambda \to 0 \text{ uniformly in } [0,1].$$

By the concavity of v_λ, $v_\lambda(\eta) \geq 1 - \eta$ for $\eta \in [0,1]$, and hence

$$v'''_\lambda \to 0 \text{ in } C_{loc}([0,1]) \text{ as } \lambda \to +\infty.$$

Since $v_\lambda(1) \to 0$ as $\lambda \to +\infty$, necessarily $v_\lambda \to 1-\eta^2$ uniformly in $[0,1]$ as $\lambda \to +\infty$ and $v''_\lambda \to -2$ in $C^2_{loc}([0,1])$ as $\lambda \to +\infty$. Since $v'''_\lambda > 0$ and $v''_\lambda(1) = 0$, v''_λ is uniformly bounded in $[0,1]$.

We now proceed to prove the existence of solutions to problem (2.1).

Theorem 2.3. *For all $\lambda > 0$, there exists a solution f_λ of problem* (2.1).

To prove Theorem 2.3 we introduce the auxiliary problem

$$\begin{aligned} &f^2 f''' = \frac{\lambda\eta}{\mu} \quad \text{if } 0<\eta<1 \text{ and } \mu>0 \\ &f'(0) = f''(1) = 0, \quad f(1) = 1. \end{aligned} \tag{2.10}$$

By changing the variable η into $-\eta$ and indicating the new variable again with η we obtain the equivalent problem

$$\begin{aligned} &f^2 f''' = \frac{\lambda\eta}{\mu} \quad \text{if } -1<\eta<0 \text{ and } \mu>0 \\ &f'(0) = f''(-1) = 0, \quad f(-1) = 1. \end{aligned} \tag{2.11}$$

We shall prove that problem (2.11) has a unique solution f_μ by using a shooting technique with $f'(-1)$ as shooting parameter. More precisely, let $f_{\mu,\gamma}$ be the solution of the problem

$$\begin{aligned} &f^2 f''' = \frac{\lambda\eta}{\mu} \quad \text{if } -1<\eta<0 \text{ and } \mu>0 \\ &f'(-1) = \gamma > 0,\ f''(-1) = 0, \quad f(-1) = 1 \end{aligned} \tag{2.12}$$

and let us indicate with S_1, S_2 the sets

$$\begin{aligned} S_1 &= \{\gamma > 0 : f'_{\mu,\gamma} > 0 \text{ in } [-1,0]\} \\ S_2 &= \{\gamma > 0 : \exists \eta_0 < 0 : f'_{\mu,\gamma}(\eta_0) = 0\}. \end{aligned}$$

If we show that S_1 and S_2 are nonempty open sets then, since $S_1 \cap S_2 = \emptyset$, it follows that $\mathbf{R}^+ \backslash S_1 \cup S_2$ is nonempty and there exists $\gamma_0 = \gamma_0(\mu,\lambda)$ such that

$$f'_{\mu,\gamma_0}(0) = 0$$

which proves the existence of a solution of problem (2.11).

Lemma 2.4. *S_1 is nonempty and open.*

Proof. Since $\gamma > 0$, $f_{\mu,\gamma}$ is initially increasing, and as long as $f_{\mu,\gamma} > 1$ we have that

$$f'''_{\mu,\gamma}(\eta) \geq \frac{\lambda\eta}{\mu},$$

which implies that

$$f'_{\mu,\gamma}(\eta) \geq \gamma + \frac{\lambda}{6\mu}(\eta^3+1) - \frac{\lambda}{2\mu}(\eta+1) \geq \gamma - \frac{\lambda}{2\mu}.$$

Hence, if $\gamma \geq \frac{\lambda}{2\mu}$, it follows that $f'_{\mu,\gamma}(\eta) > 0$ and $f_{\mu,\gamma}(\eta) > 1$ for all $\eta \in (-1,0]$. Hence S_1 is nonempty. S_1 is open by continuous dependence of the solution on the data.

Lemma 2.5. *S_2 is nonempty and open.*

Proof. Since $f_{\mu,\gamma}$ is concave and increasing one has that $f_{\mu,\gamma} < \gamma+1$ in $[-1,0]$. By equation (2.12) we get

$$f'_{\mu,\lambda,\gamma}(0) < \gamma - \frac{\lambda}{6\mu(\gamma+1)^2}$$

so that it follows that $f'_{\mu,\gamma}(0) < 0$ for γ small enough. Finally S_2 is open by continuous dependence.

Hence, we have proved the existence of a solution f_{μ,γ_0} to problem (2.12) such that

$$f'_{\mu,\gamma_0}(0) = 0. \tag{2.13}$$

Let us now show that such a solution is unique.

Lemma 2.6. *The solution f_{μ,γ_0} is unique.*

Proof. Assume that there exists some $\tilde{\gamma} \neq \gamma_0$ for which the corresponding solution $f_{\mu,\tilde{\gamma}}$ satisfies $f'_{\mu,\tilde{\gamma}}(0) = 0$. Then, if $\tilde{\gamma} < \gamma_0$, we will prove that

$$f_{\mu,\tilde{\gamma}} < f_{\mu,\gamma_0} \quad \text{in } (-1,0] \tag{2.14}$$

which, using equation (2.11), implies

$$f'_{\mu,\tilde{\gamma}} < f'_{\mu,\gamma_0} \quad \text{in } (-1,0],$$

which leads to a contradiction. Since $\tilde{\gamma} < \gamma_0$ it follows that (2.14) holds in some interval $(-1,\eta_0)$ where $\eta_0 = \sup\{\eta > -1 : f_{\mu,\tilde{\gamma}}(\eta) < f_{\mu,\gamma_0}(\eta)\}$. Assume that $\eta_0 < 0$. Then

$$f'_{\mu,\tilde{\gamma}}(\eta_0) \geq f'_{\mu,\gamma_0}(\eta_0).$$

On the other hand, by (2.11) and (2.14), we have

$$f'''_{\mu,\tilde{\gamma}} < f'''_{\mu,\gamma_0} \quad \text{in } (-1,\eta_0)$$

which gives a contradiction,

$$f'_{\mu,\tilde{\gamma}} < f'_{\mu,\gamma_0} \quad \text{in } (-1,\eta_0).$$

Hence, there exists a unique solution to problem (2.10) which we denote by f_μ. We finally prove Theorem 2.3 showing that $\int_0^1 f_\mu(t)\,dt$ is a continuous function of $\mu > 0$ satisfying that $\lim_{\mu\to 0}\int_0^1 f_\mu(t)\,dt = +\infty$ and $\lim_{\mu\to+\infty}\int_0^1 f_\mu(t)dt = 1$

Lemma 2.7. *Let f_μ denote the unique solution of problem* (2.12)–(2.13). *Then $\int_0^1 f_\mu(t)dt$ is a continuous function of $\mu > 0$ such that*

$$\lim_{\mu\to 0}\int_0^1 f_\mu(t)\,dt = +\infty \tag{2.15}$$

and

$$\lim_{\mu\to+\infty}\int_0^1 f_\mu(t)\,dt = 1 \tag{2.16}$$

Proof. The continuity follows from continuous dependence. To prove (2.15) we argue by contradiction. Assume that there exists a subsequence $\mu_n \to 0$ such that

$$\int_0^1 f_{\mu_n}(t)\,dt \le A$$

for some positive constant A. Since f_{μ_n} is a concave function, there exists some constant B such that $f_{\mu_n}(0) \le B$ so that

$$f'''_{\mu_n}(\eta) \ge \frac{\lambda}{\mu_n B^2}\eta;$$

arguing as in the proof of Proposition 2.1 we get that

$$f''_{\mu_n}(\eta) \le \frac{-3\lambda}{8\mu_n B^2} \quad \text{for} \quad \eta \in [0, 1/2].$$

Integrating twice this relation in $[0, \eta]$ and setting $\eta = 1/2$ we get

$$f_{\mu_n}(1/2) \le B - \frac{3\lambda}{64\mu_n B^2},$$

which gives that $f_{\mu_n}(1/2) < 1$ for n sufficiently large, a contradiction.

To prove (2.16) observe that

$$f'''_\mu(\eta) \le \frac{\lambda}{\mu} \to 0$$

as $\mu \to +\infty$. Hence, integrating in $[\eta, 1]$ we get

$$f''_\mu(1) - f''_\mu(\eta) \le \frac{\lambda}{\mu},$$

and since $f''_\mu(1) = f'_\mu(0) = 0$ an integration over $[0, \eta]$ gives

$$-f'_\mu(\eta) \le \frac{\lambda}{\mu},$$

and finally integrating over $[\eta, 1]$ gives $f_\mu(\eta) \le 1 + \frac{\lambda}{\mu}$, which proves (2.16).

3. A class of self-similar solutions

In this section we consider positive solutions of the form

$$h(x,t) = tf(\eta), \qquad \eta = x/t \tag{3.1}$$

of the problem

$$(\mathrm{S}) \quad \begin{cases} \partial_t h + \partial_x \left(h^3 \partial^3_{xxx} h\right) = 0 & \text{if } 0 < x < Vt,\ t > 0 \\ \partial_x h(0,t) = 0 & \text{if } t > 0 \\ h^3 \partial^3_{xxx} h(0,t) = 0 & \text{if } t > 0, \end{cases}$$

with the conditions

$$(\mathrm{S}_0) \quad \begin{cases} h(x,t) > 0 \quad \text{if } 0 \le x < Vt, t > 0 \\ h(Vt,t) = 0, \quad h^3 \partial^3_{xxx} h(Vt,t) = -\epsilon t. \end{cases}$$

Here $h(x,t)$ is a smooth function for $t > 0$ and $0 \le x < Vt$, and the conditions in (S_0) are to be considered in the sense of limits as $x \to (Vt)^-$.

We shall prove the following result:

Theorem 3.1. For any positive numbers V and ϵ there exists a solution h of problem (S)–(S_0) of the self-similar form (3.1).

Since we are looking for solutions of the form (3.1) a simple computation shows that f has to satisfy

$$\begin{cases} (f^3 f''')' = \eta f' - f & \text{if } 0 < \eta < V \\ f'(0) = 0, \quad f'''(0) = 0. \end{cases} \tag{3.2}$$

Integrating the differential equation in (3.2) over $(0, V)$ we obtain that

$$h^3 \frac{\partial^3 h}{\partial x^3}(Vt,t) = t(f^3 f''')(V) = -2t \int_0^V f(\eta)\, d\eta.$$

Hence, Theorem 3.1 is equivalent to the first part of the following result:

Theorem 3.2. *For any positive numbers V and ϵ there exists a solution f of (3.2), smooth in the interval $[0, V)$ and continuous in $[0, V]$, such that*

$$f(\eta) > 0 \quad \text{if } 0 \le \eta < V, \quad f(V) = 0 \quad \text{and} \quad \epsilon = 2 \int_0^V f(\eta)\, d\eta. \tag{3.3}$$

Furthermore, there exists a positive constant C which does not depend on V and ϵ such that

$$\frac{f_{\max}}{f(0)} \le C(V^7 \epsilon^{-3} + 1), \tag{3.4}$$

where f_{max} *indicates the maximum value of* f *in the interval* $[0, V]$.

We conjecture that the estimate (3.4) is optimal as $\epsilon \to 0^+$ for fixed V (see also Remark 3.5).

Theorem 3.3. *As* $\eta \to V$

$$f(\eta) = \frac{2\sqrt{2}}{15^{1/4}}\epsilon^{1/4}(V-\eta)^{3/4} + o((V-\eta)^{3/4}). \tag{3.5}$$

We omit the proof of this theorem, observing that the arguments used to prove (3.5) are similar to those used in [1] where we analyzed the asymptotic behavior of local traveling-wave solutions of equation (1.1) with nonvanishing flux at the contact angle.

A natural procedure to prove Theorem 3.2 is to use a shooting technique, i.e., to consider problem (3.2) as an initial value problem with the two shooting parameters $u(0)$ and $u''(0)$. Problem (3.2) however satisfies the following scaling property, which gives us the possibility of eliminating one of the shooting parameters: if u is a nontrivial solution of problem (3.2), for any $\gamma > 0$ the function

$$f_\gamma(\eta) = \gamma^{-4/3}u(\gamma\eta) \tag{3.6}$$

is a solution of (3.2) with V replaced by γV, and

$$\int_0^V f_\gamma(\eta)\,d\eta = \gamma^{-7/3}\int_0^{\gamma V} u(\eta)\,d\eta. \tag{3.7}$$

Therefore we shall consider the problem

$$\begin{cases} (u^3u''')' = \eta u' - u \quad \text{for } \eta > 0 \\ u(0) = 1, \quad u'(0) = 0, \quad u''(0) = \beta, \quad u'''(0) = 0, \end{cases} \tag{3.8}$$

where $\beta \in \mathbf{R}$ is the shooting parameter.

For every value β problem (3.8) has a unique local solution $u = u(\cdot, \beta)$, which can be continued as long as it remains bounded and positive. Let $[0, \eta_0(\beta))$ be the maximal interval in which $u(\cdot, \beta)$ can be continued; i.e., $\eta_0(\beta) \in (0, +\infty]$ and

$$\eta_0(\beta) = \sup\{t \geq 0 : u(\cdot, \beta) \text{ is positive and bounded in } [0, t)\}. \tag{3.9}$$

The following lemma collects some basic results about problem (3.8).

Lemma 3.1. *Let* $u(\cdot, \beta)$ *be the solution of problem* (3.8) *in the maximal interval* $[0, \eta_0(\beta))$.

(i) If $\eta_0(\beta) < +\infty$*, then*

$$u(\eta_0(\beta), \beta) \equiv \lim_{\eta \to \eta_0(\beta)-} u(\eta, \beta) = 0.$$

(ii) If $\beta \leq 0$*,* $u(\cdot, \beta)$ *is concave and decreasing in* $[0, \eta_0(\beta))$*,* $u'''(\cdot, \beta) < 0$ *in* $[0, \eta_0(\beta))$*, and* $\eta_0(\beta) < +\infty$*.*
(iii) If $u'(\eta_1, \beta) \leq 0$ *for some* $0 < \eta_1 < \eta_0(\beta)$*, then* $\eta_0(\beta) < +\infty$ *and*

$$u'(\cdot, \beta) < 0, \quad u''(\cdot, \beta) < 0 \quad \text{and} \quad u'''(\cdot, \beta) < 0 \quad \text{in } (\eta_1, \eta_0(\beta)).$$

(iv) If $\beta \geq \sqrt{2}$*,* $\eta_0(\beta) = +\infty$ *and* $u(\cdot, \beta)$ *is convex and increasing in* $[0, +\infty)$*.*
(v) Let β^* *be defined by*

$$\beta^* := \inf\{\beta \in \mathbf{R} : \eta_0(\beta) = +\infty\}. \tag{3.10}$$

Then $0 < \beta^* < \sqrt{2}$*,* $\eta_0(\beta^*) = +\infty$*, and* $u(\eta, \beta^*) \to +\infty$ *as* $\eta \to +\infty$*.*
(vi) η_0 *is continuous in* $(-\infty, \beta^*)$ *and*

$$\eta_0(\beta) \to \begin{cases} +\infty & \text{as } \beta \to \beta^{*-} \\ 0 & \text{as } \beta \to -\infty. \end{cases} \tag{3.11}$$

Remark 3.1. Numerical computations indicate that

$$0.73196 < \beta^* < 0.73197,$$

and that for all $\beta \geq \beta^*$ $\eta_0(\beta) = +\infty$ and $u(\eta, \beta) \to +\infty$ as $\eta \to +\infty$.

Remark 3.2. The continuity of η_0, announced in Lemma 3.1(vi) in the interval $(-\infty, \beta^*)$, actually holds in $\mathbf{R}$, considering η as a function from $\mathbf{R}$ to $(0, +\infty]$.

The proof of Lemma 3.1 is elementary, and we postpone it until the end of this section.

Our main estimate concerning problem (3.8) is the following:

Proposition 3.1. Let $u(\cdot, \beta)$ be the solution of problem (3.8) in the maximal interval $[0, \eta_0(\beta))$, and let β^* be defined by (3.10). Then there exists a positive constant C such that for any $\beta \leq \beta^*$

$$u(\eta, \beta) \leq C\eta + 1 \quad \text{for all } 0 < \eta < \eta_0(\beta). \tag{3.12}$$

Remark 3.3. As we shall see below it is possible to obtain more detailed and precise results about the behavior of $u(\cdot, \beta^*)$ (see Proposition 3.2 and Remark 3.6), some of which were obtained by Hastings and Peletier ([2]).

Before proving Proposition 3.1, we use it to prove Theorem 3.2.

Proof of Theorem 3.2. Let us fix V and ϵ. It follows from Lemma 3.1(vi) that for any positive γ there exists $\beta(\gamma) < \beta^*$ such that

$$u(\gamma V, \beta(\gamma)) = 0,$$

and the function f_γ, defined by (3.5), is a solution of problem (3.2) satisfying

$$f_\gamma(\eta) > 0 \quad \text{if } 0 \le \eta < V \quad \text{and} \quad f_\gamma(V) = 0.$$

Thus, in order to prove (3.3) we have to show that we can choose γ such that

$$\int_0^V f_\gamma(\eta)\, d\eta = \frac{\epsilon}{2},$$

i.e., in view of (3.7), such that

$$(\gamma V)^{-\frac{7}{3}} \int_0^{\gamma V} u(\eta, \beta(\gamma))\, d\eta = \frac{\epsilon}{2} V^{-\frac{7}{3}}. \tag{3.13}$$

By Proposition 3.1, for any $\beta < \beta^*$

$$\int_0^{\eta_0(\beta)} u(\eta, \beta)\, d\eta \le \frac{C}{2}\eta_0^2(\beta) + \eta_0(\beta),$$

and it follows from (3.11) that

$$\eta_0^{-\frac{7}{3}}(\beta) \int_0^{\eta_0(\beta)} u(\eta, \beta)\, d\eta \to 0 \quad \text{as } \beta \to \beta^{*-}. \tag{3.14}$$

On the other hand, by Lemma 3.1(ii), $u(\cdot, \beta)$ is concave if $\beta \le 0$, and thus

$$\int_0^{\eta_0(\beta)} u(\eta, \beta)\, d\eta \ge \frac{\eta_0(\beta)}{2},$$

which, by (3.11), implies that

$$\eta_0^{-\frac{7}{3}}(\beta) \int_0^{\eta_0(\beta)} u(\eta, \beta)\, d\eta \to +\infty \quad \text{as } \beta \to -\infty. \tag{3.15}$$

By Lemma 3.1(vi) and the continuous dependence of the solution of problem (3.2) on β, the expression in (3.14) depends continuously on $\beta \in (-\infty, \beta^*)$, and it follows from (3.14) and (3.15) that there exists γ for which (3.13) holds.

Finally, we prove the estimate (3.4). By (3.13) and Proposition 3.1

$$\epsilon = 2\gamma^{-\frac{7}{3}} \int_0^{\gamma V} u(\eta, \beta(\gamma))\, d\eta \le 2\gamma^{-7/3}\gamma V \max u(\gamma\eta, \beta(\gamma)) \le$$

$$\leq 2(C\gamma V+1)\gamma^{-\frac{7}{3}}(\gamma V) = \frac{2V^2}{\gamma^{\frac{1}{3}}}\Big(C+\frac{1}{\gamma V}\Big) \leq \frac{2V^2}{\gamma^{\frac{1}{3}}}(C+1) \quad \text{if } \gamma V \geq 1.$$

Hence,

$$\gamma \leq 8(C+1)^3\frac{V^6}{\epsilon^3} \quad \text{if } \gamma V \geq 1,$$

which, combined with Proposition 3.1, implies that

$$\frac{f_{\max}}{f(0)} = \max u(\gamma\eta, \beta(\gamma)) \leq C\gamma V + 1 \leq \begin{cases} C+1 & \text{if } \gamma V < 1 \\ 8C(C+1)^3\dfrac{V^7}{\epsilon^3}+1 & \text{if } \gamma V \geq 1. \end{cases}$$

Redefining the constant C we obtain (3.4).

Remark 3.4. Numerical evidence shows that the quantity

$$\eta_0^{-\frac{7}{3}}(\beta)\int_0^{\eta_0(\beta)} u(\eta,\beta)\,d\eta$$

is strictly decreasing with respect to β; obviously, the proof of such monotonicity would imply the uniqueness of the solutions h and f in Theorems 3.1 and 3.2.

The proof of Proposition 3.1 is based on the following differential inequality.

Lemma 3.2. *Let β^* be defined by* (3.10) *and let $u(\cdot,\beta)$ be the solution of problem* (3.8) *in the maximal interval $(0,\eta_0(\beta))$. If $\beta \in (0,\beta^*]$, then*

$$u'(\eta,\beta) \leq \frac{u(\eta,\beta)+\sqrt{2u(\eta,\beta)-1}}{\eta} \quad \textit{if } 0<\eta<\eta_0(\beta). \tag{3.16}$$

Proof. We shall denote $u(\eta,\beta)$ by $u(\eta)$. In view of Lemma 3.1(vi) and the continuous dependence of u of β, we may assume without loss of generality that $\beta < \beta^*$.

Let $\eta \in (0,\eta_0(\beta))$ be such that $u'(\eta) > 0$. Then, by Lemma 3.1(iii),

$$u' > 0 \quad \text{in } (0,\eta]. \tag{3.17}$$

Multiplying the equation for u by u'' and integrating over $(0,\eta)$ we obtain

$$u^3u''u'''(\eta) - \int_0^\eta u^3(u''')^2\,dt = \frac{1}{2}\eta(u'(\eta))^2 - uu'(\eta) + \frac{1}{2}\int_0^\eta (u')^2\,dt. \tag{3.18}$$

By Hölder's inequality

$$(u(\eta)-1)^2 = \Big(\int_0^\eta u'(t)\,dt\Big)^2 \leq \eta\int_0^\eta (u'(t))^2\,dt, \tag{3.19}$$

and it follows from (3.18) and (3.19) that

$$u^3 u'' u'''(\eta) + uu'(\eta) \geq \frac{1}{2}\eta(u'(\eta))^2 + \frac{1}{2\eta}(u(\eta)-1)^2. \tag{3.20}$$

If $(u''u''')(\eta) \leq 0$, (3.20) leads to

$$\eta(u'(\eta))^2 - 2u(\eta)u'(\eta) + \frac{1}{\eta}(u(\eta)-1)^2 \leq 0,$$

which implies that

$$u'(\eta) \leq \frac{u(\eta) + \sqrt{2u(\eta)-1}}{\eta} \quad \text{if } (u''u''')(\eta) \leq 0. \tag{3.21}$$

If $(u''u''')(\eta) > 0$ we distinguish two cases. If

$$u''(\eta) > 0 \quad \text{and} \quad u'''(\eta) > 0,$$

we claim that

$$u'(\eta) \leq \frac{u(\eta)}{\eta}, \tag{3.22}$$

which immediately implies (3.16).

To prove (3.22) we argue by contradiction and assume that $\eta u'(\eta) > u(\eta)$. Then we obtain from the equation for u that

$$\begin{cases} (tu'(t) - u(t))' = tu'' & \text{if } \eta < t < \eta_0(\beta) \\ (u^3 u''')'(t) = tu'(t) - u(t) & \text{if } \eta < t < \eta_0(\beta) \\ u'''(\eta) > 0, \quad u''(\eta) > 0, & \eta u'(\eta) - u(\eta) > 0, \end{cases}$$

and it follows at once that the subset of the phase space in which $tu' - u$, u'' and u''' are positive is invariant. Hence $u'(t) > 0$ if $\eta < t < \eta_0(\beta)$ and $\eta_0(\beta) = +\infty$, which leads to a contradiction since we have assumed that $\beta < \beta^*$.

It remains to consider the case in which

$$u''(\eta) < 0 \quad \text{and} \quad u'''(\eta) < 0.$$

Let η_1 indicate the last inflection point before η (such a point exists because the solution $u(\cdot, \beta)$ is initially convex if $\beta > 0$). By (3.21)

$$u'(\eta_1) \leq \frac{u(\eta_1) + \sqrt{2u(\eta_1)-1}}{\eta_1}. \tag{3.23}$$

Since $(tu' - u)' = tu'' < 0$ if $\eta_1 < t < \eta$, the function $tu' - u$ is decreasing in (η_1, η), and it follows from (3.22) that for $\eta_1 \leq t \leq \eta$

$$tu'(t) - u(t) \leq \eta_1 u'(\eta_1) - u(\eta_1) \leq \sqrt{2u(\eta_1)-1}.$$

Since u is increasing in (η_1,η) this implies that

$$tu'(t)-u(t)\le\sqrt{2u(\eta)-1}\quad\text{if } \eta_1\le t\le\eta,$$

and choosing $t=\eta$ we have found (3.16).

Proof of Proposition 3.1. In view of Lemma 3.1(ii) we may assume without loss of generality that $\beta>0$. Integrating the differential inequality (3.16) we find that

$$2\ln\frac{\sqrt{2u(\eta)-1}+1}{\sqrt{2u(1)-1}+1}\le\ln\eta+\frac{2}{\sqrt{2u(1)-1}+1}-\frac{2}{\sqrt{2u(\eta)-1}+1}$$

if $1\le\eta<\eta_0(\beta)$, and hence

$$u(\eta)\le C\eta+1\quad\text{if } 1\le\eta<\eta_0(\beta) \tag{3.24}$$

for some C which does not depend on β. Finally, we can redefine C such that (3.24) holds for all $0\le\eta<\eta_0(\beta)$, which completes the proof.

As we have seen, the information contained in Proposition 3.1 is sufficient to prove Theorem 3.2. It turns out that the upper bound in Proposition 3.1 for the growth of u_*, defined by

$$u_*(\eta)\equiv u(\eta,\beta^*)\quad\text{if } \eta\ge 0, \tag{3.25}$$

is sharp:

Proposition 3.2. *Let u_* be defined by* (3.25). *Then there exists $C>0$ such that*

$$\frac{u_*(\eta)}{\eta}\to C\qquad\text{as } \eta\to+\infty. \tag{3.26}$$

Proof. By Proposition 3.1 there exists a positive constant C such that

$$\int_1^\eta\Big(\frac{u_*(t)}{t}\Big)'dt\le C\quad\text{if } \eta\ge 1. \tag{3.27}$$

By Lemma 3.2 and Proposition 3.1

$$\Big(\frac{u_*(\eta)}{\eta}\Big)'=\frac{u_*'(\eta)}{\eta}-\frac{u_*(\eta)}{\eta^2}\le\frac{\sqrt{2u_*(\eta)-1}}{\eta^2}\le\frac{C}{\eta^{3/2}}\in L^1(1,+\infty), \tag{3.28}$$

and it follows from (3.27) and (3.28) that

$$\int_1^{+\infty}\Big|\Big(\frac{u_*(t)}{t}\Big)'\Big|dt<+\infty.$$

Hence, there exists a constant $C\ge 0$ such that

$$\lim_{\eta\to+\infty}\frac{u_*(\eta)}{\eta}=C. \tag{3.29}$$

It remains to show that the constant C in (3.29) is positive. By (3.18) and (3.19)

$$\begin{aligned} -\int_0^\eta u_*^3({u_*}''')^2\,d\eta + u_*^3{u_*}'''{u_*}''(\eta) \geq \\ \geq \frac{1}{2}\eta({u_*}'(\eta))^2 - u_*{u_*}'(\eta) + \frac{(u_*(\eta)-1)^2}{2\eta} \geq \\ \geq \min_{p\in\mathbf{R}}\left(\frac{1}{2}\eta p^2 - u_* p + \frac{(u_*(\eta)-1)^2}{2\eta}\right) = \frac{1}{2\eta} - \frac{u_*}{\eta}. \end{aligned} \tag{3.30}$$

We claim that there exists a sequence of points $\eta_k \to +\infty$ as $k \to +\infty$ such that

$$({u_*}''{u_*}''')(\eta_k) \leq 0. \tag{3.31}$$

Indeed, arguing by contradiction we suppose that both u'' and u''' are either positive or negative for sufficiently large η, but both possibilities lead at once to a contradiction (u_* grows at least quadratically in the first case, while $\eta_0(\beta^*) < +\infty$ in the second case).

From (3.30), (3.31) and Proposition 3.1 we obtain that

$$\int_0^{\eta_k} u_*^3({u_*}''')^2\,d\eta \leq \frac{2u_*(\eta_k)-1}{2\eta_k} \leq C \quad \text{for any } k.$$

Therefore,

$$\int_0^{+\infty} u_*^3({u_*}''')^2\,d\eta < \infty,$$

and it follows from (3.29) and (3.30) that

$$\lim_{\eta\to+\infty}\frac{u_*(\eta)}{\eta} = \lim_{\eta_k\to+\infty}\frac{u_*(\eta_k)}{\eta_k} \geq 2\int_0^{+\infty} u_*^3({u_*}''')^2\,d\eta > 0,$$

which concludes the proof.

Remark 3.5. As we have observed before, we conjecture that the estimate (3.4) is sharp as $\epsilon \to 0^+$ for fixed $V > 0$. It is not difficult to see that in terms of $u(\eta,\beta)$ this conjecture could be formulated as follows:

$$\frac{1}{\eta_0^2(\beta)}\int_0^{\eta_0(\beta)} u(s,\beta)\,ds \to C \quad \text{as } \beta \to \beta^{*-}$$

for some $C > 0$. In this context we observe that, by Proposition 3.1,

$$\int_0^{\eta_0(\beta)} u(s,\beta)\,ds \leq C\eta_0^2(\beta) \quad \text{for } 0 \leq \beta < \beta^*$$

for some $C > 0$, while, on the other hand, Proposition 3.2 implies that

$$\frac{1}{\eta^2}\int_0^\eta u(s,\beta^*)\,ds \to \frac{C}{2} \quad \text{as } \eta \to +\infty,$$

where C is the constant in (3.26).

Remark 3.6. Hastings and Peletier ([2]) have obtained some very precise results about the behavior of the solutions of problem (3.7). Among other things they have proved that $u_*{}''(\eta)$ changes sign infinitely many times and vanishes as $\eta \to +\infty$. Combining their results with ours, they have improved Proposition 3.2, showing that $u_*'(\eta)$ converges to a positive constant.

It remains to prove Lemma 3.1.

Proof of Lemma 3.1. We denote $u(\cdot,\beta)$ by u.

(i) The local solution u can be continued as long as it remains positive and bounded; hence, it is enough to show that u is a priori bounded in bounded intervals. Integrating the equation for u we obtain that

$$u^3u''' = \eta u - 2\int_0^\eta u(s)\,ds;$$

i.e.,

$$u'''(\eta) \le \frac{\eta}{u^2} \le \eta \quad \text{if } u(\eta) \ge 1,$$

from which the result follows at once.

(ii) By the equation for u we have that $(u^3u''')'(0) = -1$, and since $u''(0) = \beta \le 0$ and $u'(0) = 0$, this implies that u''', u'' and u' are negative for small positive values of η. It follows from the equation for u that $u''' < 0$, $u'' < 0$ and $u' < 0$ as long as $u > 0$.

(iii) If $\beta \le 0$ the result follows from (i). If $\beta > 0$ then u is increasing in a neighborhood of $\eta = 0$ and, since $u'(\eta_1) \le 0$, there exists $\eta_2 \in (0,\eta_1]$ at which u attains a maximum; i.e., $u'(\eta_2) = 0$ and $u''(\eta_2) \le 0$. Arguing as in the proof of (ii) it follows that u', u'' and u''' are negative in $(\eta_1, \eta_0(\beta))$ and that $\eta_0(\beta) < \infty$.

(iv) We define the function

$$\mathcal{E}_\beta(\eta) = \frac{1}{2}(u''(\eta))^2 - \frac{1}{u(\eta)} \quad \text{for } 0 \le \eta < \eta_0(\beta). \tag{3.32}$$

Dividing (3.17) by u^3 we obtain that

$$\mathcal{E}_\beta'(\eta) > 0 \quad \text{for } 0 < \eta < \eta_0(\beta). \tag{3.33}$$

If $\beta \geq \sqrt{2}$, then $\mathcal{E}_\beta(0) = \frac{1}{2}\beta^2 - 1 \geq 0$, and it follows from (3.33) that $\mathcal{E}_\beta(\eta) > 0$ for $0 < \eta < \eta_0(\beta)$, which implies that $u'' > 0$ in $(0, \eta_0(\beta))$. Hence, $u' > 0$ in $(0, \eta_0(\beta))$ and $\eta_0(\beta) = +\infty$.

(v) It follows from (ii) and (iv) and that $0 \leq \beta^* \leq \sqrt{2}$. Actually, since u depends continuously on β, it follows easily that $0 < \beta^* < \sqrt{2}$.

We claim that

$$u'(\eta, \beta^*) > 0 \quad \text{if } 0 < \eta < \eta_0(\beta^*); \tag{3.34}$$

indeed, arguing by contradiction it follows from (iii) that $\eta_0(\beta^*) < \infty$ and u is concave and decreasing in $(\eta_3, \eta_0(\beta^*))$ for some $\eta_3 \in (0, \eta_0(\beta^*))$. The continuous dependence of u on β implies that if $|\beta - \beta^*|$ is sufficiently small, u is concave and decreasing in a compact subset of $(\eta_3, \eta_0(\beta^*))$. Hence, by (iii), $\eta_0(\beta) < \infty$ if $|\beta - \beta^*|$ is sufficiently small, and we have found a contradiction with the definition of β^*.

It follows from (3.34) that $\eta_0(\beta^*) = \infty$. The equation for u and (3.33) imply that $u(\eta, \beta^*)$ can not remain bounded in $(0, \infty)$.

(vi) Arguing as in the proof of (v) it follows that $\eta_0(\beta) \to +\infty$ as $\beta \to \beta^{*-}$.

Also the continuous dependence of $\eta_0(\beta)$ follows easily. Fixing $\beta_0 < \beta^*$ and choosing $|\beta - \beta_0|$ sufficiently small, there exists $\delta > 0$ such that $u(\cdot, \beta) > 0$ in $[0, \eta_0(\beta_0) - \delta]$ and decreasing in $[\eta_0(\beta_0) - 2\delta, \eta_0(\beta_0) - \delta]$. Hence $u(\cdot, \beta)$ is concave in $[\eta_0(\beta_0) - \delta, \eta_0(\beta)]$ and $u(\eta_0(\beta_0) - \delta, \beta)$ and its derivatives at $\eta_0(\beta_0) - \delta$ depend continuously on β, which yield at once the result.

To prove that $\eta_0(\beta) \to 0$ as $\beta \to -\infty$, we use that, by (ii), for any $\beta < 0$ $u''(\eta) < u''(0) = \beta$ if $0 < \eta < \eta_0(\beta)$. Hence $u(\eta) < 1 + \frac{1}{2}\beta\eta^2$ if $0 < \eta < \eta_0(\beta)$, which implies that

$$\eta_0(\beta) < \sqrt{\frac{2}{-\beta}} \to 0 \quad \text{as } \beta \to -\infty.$$

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