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ESTIMATES FOR P-POISSON EQUATIONS*

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Abstract. We derive estimates for solutions to the equations like

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f\,,$$

where f belongs to weak L^q spaces. As applications of our results we show that the entropy solutions of

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = |u|^{a-1}u$$

are regular provided that $0 \le a < n(p-1)/(n-p)$.

1. Introduction. Throughout this paper we let Ω stand for a bounded open set in $\mathbf{R}^n$ and $1 < p \le n$ (we usually have $p < n$, for the case $p = n$ is rather simple concerning the questions we consider). We shall consider quasilinear operators

$$-\operatorname{div}\mathcal{A}(x,\nabla u)\,,$$

where $\mathcal{A}\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ is a mapping that satisfies the following assumptions for some constants $0 < \alpha \le \beta < \infty$:

$$\begin{aligned}&\text{the function } x \mapsto \mathcal{A}(x,\xi) \text{ is measurable for all } \xi \in \mathbf{R}^n, \text{ and}\\ &\text{the function } \xi \mapsto \mathcal{A}(x,\xi) \text{ is continuous for a.e. } x \in \mathbf{R}^n;\end{aligned} \tag{1.1}$$

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for all $\xi \in \mathbf{R}^n$ and a.e. $x \in \mathbf{R}^n$

$$\mathcal{A}(x,\xi)\cdot\xi \geq \alpha|\xi|^p\,, \tag{1.2}$$

$$|\mathcal{A}(x,\xi)| \leq \beta|\xi|^{p-1}\,. \tag{1.3}$$

A principal example is the p-Laplacian

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)\,.$$

Let $f \in L^1(\Omega)$. We consider the problem

$$\begin{cases} -\operatorname{div}\mathcal{A}(x,\nabla u(x)) = f \text{ in } \Omega \\ \qquad u = 0 \text{ on } \partial\Omega\,, \end{cases} \tag{1.4}$$

and our goal is to find estimates for u and ∇u in terms of the weak L^q norm of f. Since f is only an L^1 function, some remarks concerning the concept of the solution are needed; however, since we are going to prove estimates that are independent of the a priori regularity of the solution, one may proceed by proving such estimates for honest distributional solutions in $W_0^{1,p}(\Omega)$ and reach the desired estimates by an approximation. We shall use the concept of an entropy solution that was introduced by Benilan *et. al.* in [2], where the existence and uniqueness of such a solution was also established. A function u is called an *entropy solution* of the problem 1.4 if the truncations $T_k(u)$ belong to $W_0^{1,p}(\Omega)$ for each $k > 0$, and

$$\int_\Omega \mathcal{A}(x,\nabla u(x))\cdot\nabla T_k(u-\varphi)\,dx = \int_\Omega T_k(u-\varphi)f\,dx$$

for each $\varphi \in C_0^\infty(\Omega)$. Here and in what follows T_k is the truncation operator at level k, $T_k(s) = \min(1, k/|s|)s$. Note that an entropy solution is always a solution in the sense of distributions; here we use the definition

$$\nabla u(x) = \lim_{k\to\infty}\nabla T_k(u)(x)\,,$$

which is a.e. well defined. This gradient ∇u need not be distributional, however if u has a distributional gradient in L^1 (and for $p > 2 - 1/n$ u will have), then this new gradient is distributional. This all is quite simple; the reader may consult [2] or [9] for more details. What we really need is to be able to use truncations of u as test functions for equation (1.4). Some people prefer using a slightly different notion of a solution, called *renormalized* solution; see e.g. [12], [3]. In our case, where $f \in L^1$ renormalized and entropy solutions coincide.

We work in *weak* L^q spaces, known also as Marcinkiewicz spaces or Lorentz spaces $L^{(q,\infty)}$: if $q > 1$, then the space weak $-L^q(\Omega)$ consists of measurable functions g on Ω such that

$$\sup_{t>0} t|\{x \in \Omega \colon |g(x)| > t\}|^{1/q} < \infty \,. \tag{1.5}$$

This condition is equivalently stated as

$$|||g|||_q = \sup_{\substack{E \subset \Omega \\ |E|>0}} \frac{1}{|E|^{1/q'}} \int_E |g| \, dx < \infty \,, \tag{1.6}$$

where q' is the conjugated exponent of q, $1/q + 1/q' = 1$. It is a rather easy exercise to prove that weak $-L^q(\Omega)$ is a Banach space under $||| \cdot |||_q$ and, moreover, if the supremum in (1.5) is denoted by A, then $A \le |||g|||_q \le q'A$. For a detailed analysis of weak L^q spaces we refer to [16]. Note that both (1.5) and (1.6) make sense also for $q = 1$; however then the latter is strictly stronger condition and coincides with the definition of L^1.

The following is our main result in this paper.

Theorem 1.7. *Suppose that* $f \in$ weak $-L^q(\Omega) \cap L^1(\Omega)$ *and that* u *is the entropy solution of (1.4).*

i) *If* $1 \le q < n/p$ *and* $\gamma = \frac{nq(p-1)}{n-pq}$, *then* $u \in$ weak $-L^\gamma(\Omega)$ *and*

$$|||u|||_\gamma \le c|||f|||_q^{1/(p-1)} \,,$$

where $c = c(\alpha, n, p, q) > 0$.

ii) *If* $1 \le q < p^{*\prime}$ *and* $s = q^*(p-1) = \frac{nq(p-1)}{n-q}$, *then* $\nabla u \in$ weak $-L^s(\Omega)$ *and*

$$|||\nabla u|||_s \le c|||f|||_q^{1/(p-1)} \,;$$

here $c = c(\alpha, n, p, q) > 0$.

iii) *If* $q > p^{*\prime}$, *then* $u \in W_0^{1,p}(\Omega)$ *and*

$$||\nabla u||_{L^p(\Omega)} \le c|||f|||_q^{1/(p-1)} \,,$$

where $c = c(\alpha, n, p, q, |\Omega|) > 0$.

Here, as usually, $p^* = np/(n-p)$ is the Sobolev conjugate of p. An easy calculation shows that if we denote $\gamma = nq(p-1)/(n-pq)$ as above, then

$$q < p^{*\prime} \Leftrightarrow \gamma < q' \Leftrightarrow \gamma < p^*\,,$$

and this is further equivalent to $s = q^*(p-1) < p$. As to related results, Del Vecchio [4] proved that if f is in the Lorentz space $L(q, q^*)$, then the solution u is in $W^{1,q^*(p-1)}$.

If $q = n/p$, then it follows from iii) and [17, 4.2] that u is locally in BMO. The endpoint case $q = p^{*\prime}$ in ii) seems much harder. However, the methods of [7] might appear useful in trying to show that, then the gradient of solution lies in the weak-L^p.

If $q > n/p$, then the solution is bounded. Indeed, it follows from iii) that then an entropy solution is an ordinary $W_0^{1,p}(\Omega)$-solution. Hence the boundedness goes back to works of Ladyzhenskaya and Ural'tseva, and Serrin (see e.g. [11, Ch. 4 Thm 7.1] and [15]).

In the case iii) one cannot expect obtaining higher integrability exponent than p that wouldn't depend on the domain Ω, cf. [8, Remark 4.8.c]. Of course, in the setting of iii) one has that locally ∇u is s-integrable for some $s > p$ by [13].

In Section 2 we give examples showing that Theorem 1.7 is optimal. Our methods develop further ideas from [10].

We apply Theorem 1.7 to investigate the regularity of solutions to the problem

$$\begin{cases} -\operatorname{div}\mathcal{A}(x, \nabla u(x)) = |u|^{a-1}u \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \\ |u|^a \in L^1(\Omega)\,. \end{cases} \tag{1.8}$$

We show:

Theorem 1.9. *If u is the entropy solution of* (1.8) *and* $0 \le a < n(p-1)/(n-p)$*, then u is locally Hölder continuous in Ω. In the case of smooth $\mathcal{A}$, in particular for the p-Laplacian, u is in fact in $C^{1,\varepsilon}_{loc}(\Omega)$ for some $\varepsilon > 0$.*

The key point in Theorem 1.9 is to show that the solution u is bounded, whence $\operatorname{div}\mathcal{A}(x, \nabla u(x)) \in L^\infty(\Omega)$ by (1.8). After this, the Hölder regularity and the $C^{1,\varepsilon}$ regularity (which holds as soon as $\mathcal{A}$ is Hölder continuous in x) follow from known results [15] and [5], [6]. Also the corresponding global results could be stated in smooth domains, but we leave their formulation to the reader.

The critical exponent $a_c = n(p-1)/(n-p)$ in Theorem 1.9 is truly critical. We shall show in Example 3.1 below that in the supercritical case $a > a_c$ there are singular (unbounded) entropy solutions of

$$-\Delta_p(u) = u^a \, .$$

However, if we assume *a priori* that $u \in W^{1,p}(\Omega)$, then every solution of (1.8) is regular (i.e., Hölder continuous) if $a \leq p^* - 1 = a_c + \frac{p}{n-p}$; see [11, Ch. 4 Thm 7.1] for subcritical case and [18] for the critical case. Moreover, in the p-Laplacian case one can easily prove that there always exists a nontrivial regular solution of (1.8) if $0 \leq a < p^* - 1$, $a \neq p-1$, where the latter case corresponds to the eigenvalue problem. In the critical case $a = a_c$ we do not know whether there exists singular solutions if $p \neq 2$. For linear equations $p = 2$ such a solution was first found by Aviles [1]. Examples with large singular sets have been constructed by Pacard [14].

2. Weak L^q estimates. In this section we derive estimates that yield Theorem 1.7. Throughout the section we let u be an entropy solution of (1.4). First we prove a few lemmas.

Lemma 2.1. *Let $v \in L^1(\Omega)$ be such that*

$$k^b |\{|v| > 2k\}| \leq A |\{|v| > k\}|^a$$

for all $k > 0$, where $b > 0$ and $0 \leq a < 1$. Then $v \in \text{weak}-L^{b/(1-a)}(\Omega)$ and

$$\sup_{t>0} t^{b/(1-a)} |\{|v| > t\}| \leq A^{1/(1-a)} 2^{b(1-a)^{-2}} \, .$$

Proof. By induction, we have

$$\begin{aligned} |\{|v| > t\}| &= 2^b t^{-b} (\frac{t}{2})^b |\{|v| > t\}| \leq 2^b t^{-b} A |\{|v| > t/2\}|^a \\ &\leq A^{\sum_{j=1}^k a^{j-1}} 2^{b \sum_{j=1}^k j a^{j-1}} t^{-b \sum_{j=1}^k a^{j-1}} |\{|v| > 2^{-k} t\}|^{a^k} \, , \end{aligned}$$

which tends to $A^{1/(1-a)} 2^{b(1-a)^{-2}} t^{-b/(1-a)}$, as $k \to \infty$. □

In what follows $1 < p < n$.

Lemma 2.2. *If $k > 0$, then*

$$\alpha \int_{\{k<|u|<2k\}} |\nabla u|^p \, dx \le k |\{|u| > k\}|^{1/q'} |||f|||_q$$

$$k^{p^*/p'} |\{|u| > 2k\}| \le c |\{|u| > k\}|^{p^*/pq'} |||f|||_q^{p^*/p} \,,$$

here the constant c depends only on n, p, and $1/\alpha$.

Proof. We use $v = T_k(u - T_k(u))$, $k > 0$ as a test function. Then

$$\begin{aligned}\alpha \int_{\{k<|u|<2k\}} |\nabla u|^p \, dx &= \alpha \int_\Omega |\nabla v|^p \, dx \le \int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla v \, dx \\ &= \int_\Omega f v \, dx \le k \int_{\{|u|>k\}} |f| \, dx \le k |\{|u| > k\}|^{1/q'} |||f|||_q \,.\end{aligned}$$

Hence by the Sobolev inequality

$$k |\{|u| > 2k\}|^{1/p^*} \le \Big(\int_\Omega |v|^{p^*} \, dx \Big)^{1/p^*} \le c k^{1/p} |\{|u| > k\}|^{1/pq'} |||f|||_q^{1/p} \,,$$

and the lemma follows. □

Lemma 2.3. *Suppose that $\sup_{t>0} t^\gamma |\{|u| > t\}| \le B$ for some $0 \le \gamma < q'$. Then*

$$\alpha \int_{\{|u|<k\}} |\nabla u|^p \, dx \le c B^{1/q'} k^{1-\gamma/q'} |||f|||_q$$

for all $k > 0$; here c depends only on γ and q.

Proof. Using Lemma 2.2 and the assumption we infer that

$$\begin{aligned}\alpha \int_{\{|u|<k\}} |\nabla u|^p \, dx &= \alpha \sum_{j=0}^\infty \int_{\{k2^{-j-1}<|u|<k2^{-j}\}} |\nabla u|^p \, dx \\ &\le \sum_{j=0}^\infty k 2^{-j-1} |\{|u| > k2^{-j-1}\}|^{1/q'} |||f|||_q \\ &\le B^{1/q'} \sum_{j=0}^\infty (k 2^{-j-1})^{1-\gamma/q'} |||f|||_q \le \frac{B^{1/q'}}{2^{1-\gamma/q'} - 1} k^{1-\gamma/q'} |||f|||_q \,,\end{aligned}$$

as desired. □

Proof of Theorem 1.7.
Proof of claim i): By Lemma 2.2 the assumption of Lemma 2.1 holds with $a = \frac{p^*}{pq'}$, $b = \frac{p^*}{p'}$, $A = c|||f|||_q^{p^*/p}$, here $a < 1$ since $q < n/p$. Then

$$\frac{b}{1-a} = \frac{n(p-1)q'}{q'(n-p)-n} = \gamma\,,$$

and Lemma 2.1 yields

$$\sup_{t>0} t|\{|u| > k\}|^{1/\gamma} \le c|||f|||_q^{(p^*/p)(1/b)} = c|||f|||_q^{1/(p-1)}\,.$$

The claim follows from the equivalence of (1.5) and (1.6).
Proof of claim ii): We obtain from Lemma 2.3 that

$$\begin{aligned}|\{|\nabla u| > t\}| &\le |\{|u| > k\}| + t^{-p}\int_{\{|u|<k\}} |\nabla u|^p\,dx \\ &\le |\{|u| > k\}| + ct^{-p}|||u|||_\gamma^{\gamma/q'} k^{1-\gamma/q'}|||f|||_q \\ &\le |||u|||_\gamma^\gamma\big(k^{-\gamma} + ct^{-p}|||u|||_\gamma^{-\gamma/q}|||f|||_q k^{1-\gamma/q'}\big)\,.\end{aligned}$$

Next we minimize this in k, i.e., choose

$$k = \Big(\frac{\gamma t^p|||u|||_\gamma^{\gamma/q}}{c|||f|||_q(1-\gamma/q')}\Big)^{q/(q+\gamma)}\,,$$

and arrive at

$$|\{|\nabla u| > t\}| \le |||u|||_\gamma^{\gamma(1-\gamma/(q+\gamma))} ct^{-p\gamma q/(q+\gamma)}|||f|||_q^{\gamma q/(q+\gamma)}\,,$$

where c is a constant depending on n, p, q and α. Now we observe that

$$\frac{p\gamma q}{q+\gamma} = \frac{nq(p-1)}{n-q} = q^*(p-1) = s\,,$$

and hence

$$\{|\nabla u| > t\}| \le ct^{-s}|||u|||_\gamma^{s/p}|||f|||_q^{s/p}.$$

The proof is complete since we have by i) that

$$|||u|||_\gamma \le c|||f|||_q^{1/(p-1)}\,.$$

Proof of claim iii): By multiplying u with a constant we are free to assume that $|||f|||_q = 1$. Indeed, for $\lambda = |||f|||_q^{1/(1-p)}$ the function λu is an entropy solution of (1.4) with the mapping $\mathcal{A}$ replaced by $\tilde{\mathcal{A}}(x,\xi) = \lambda^{p-1}\mathcal{A}(x,\lambda^{-1}\xi)$ and f replaced by $\frac{f}{|||f|||_q}$; note that $\tilde{\mathcal{A}}$ satisfies exactly the same structural assumptions as $\mathcal{A}$ does.

We also assume, as we may that $q < p/n$ (observe that

$$|\Omega|^{1/q'}|||f|||_q \le |\Omega|^{1/\bar{q}'}|||f|||_{\bar{q}}$$

if $q \le \bar{q}$). Now we have

$$\begin{aligned}\alpha\int_{\{|u|\le 1\}} |\nabla u|^p\,dx &\le \int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla T_1(u)\,dx\\ &= \int_\Omega T_1(u) f\,dx \le |\Omega|^{1/q'}|||f|||_q = |\Omega|^{1/q'}\end{aligned}$$

and by Lemma 2.2

$$\begin{aligned}\alpha\int_{\{|u|>1\}} |\nabla u|^p\,dx &\le \alpha\sum_{j=0}^{\infty}\int_{\{2^j<|u|<k2^{j+1}\}} |\nabla u|^p\,dx\\ &\le \sum_{j=0}^{\infty} 2^j|\{|u|>2^j\}|^{1/q'} \le c\sum_{j=0}^{\infty} 2^{j(1-\gamma/q')};\end{aligned}$$

here the constant c comes from the fact that by i) $u \in \text{weak}-L^\gamma(\Omega)$, where $\gamma = nq(p-1)/(n-qp) > q'$. The proof is complete. □

Example 2.4. Let $u(x) = |x|^{-a} - 1$, $a > 0$. Then the truncations of u belong to $W_0^{1,p}(B)$, where $B = B(0,1)$ is the unit ball of $\mathbf{R}^n$. A direct computation shows that

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u)(x) = a^{p-1}(a(1-p)+n-p)|x|^{a(1-p)-p} = f(x)$$

if $x \ne 0$. Now we observe that $f \in L^1(B)$ if and only if $a < \frac{n-p}{p-1}$. Then u is an entropy solution of

$$-\Delta_p u = f$$

in B. Moreover, it follows that if $a = \frac{n-pq}{q(p-1)}$, then $f \in \text{weak}-L^q(B)$ and $u \in \text{weak}-L^\gamma(B)$ if and only if $\gamma \le \frac{nq(p-1)}{n-qp}$. Furthermore, $\nabla u \in \text{weak}-L^s(B)$ if and only if $s \le q^*(p-1)$. This shows that Theorem 1.7 is sharp.

3. Regularity of entropy solutions to equations $-\operatorname{div}\mathcal{A}(x,\nabla u) = |u|^{a-1}u$. In this section we study the regularity of solutions to the entropy solutions of (1.8) and prove Theorem 1.9.

Proof of Theorem 1.9. We use a bootstrap argument. To start with we let $f = |u|^{a-1}u$. Since $f \in L^1(\Omega)$ we have by Theorem 1.7 that $u \in$ weak $-L^{\gamma_1}(\Omega)$, $\gamma_1 = \frac{n(p-1)}{n-p}$. Therefore, $f \in$ weak $-L^{q_1}(\Omega)$, $q_1 = \frac{\gamma_1}{\alpha}$. Now we repeatedly use Theorem 1.7. At the jth step we obtain $u \in$ weak $-L^{\gamma_j}(\Omega)$ where $\gamma_j = \frac{nq_{j-1}(p-1)}{n-pq_{j-1}}$, and $f \in$ weak $-L^{q_j}(\Omega)$, $q_j = \frac{\gamma_j}{\alpha}$ here we put $q_0 = 1$. By recursion

$$q_j = \frac{n}{n(\frac{a}{p-1})^j - p\sum_{k=1}^{j}(\frac{a}{p-1})^k}$$

provided that $q_{j-1} < n/p$. Since $0 \le a < n(p-1)/(n-p)$, it is immediate that q_j is an increasing sequence and moreover, there is an $\delta > 0$ such that

$$\begin{aligned}\sum_{k=0}^{j}(\frac{a}{p-1})^{k-j} = \sum_{k=0}^{j}(\frac{a}{p-1})^{-k} &\ge \delta + \sum_{k=0}^{j}(1-p/n)^k \\ &= \delta + \frac{n}{p} - \frac{n}{p}(1-p/n)^{j+1} > \frac{n}{p}\end{aligned}$$

if j is large enough. One easily checks that for such a j it holds that $q_j > n/p$. Therefore we conclude that $f \in$ weak $-L^q(\Omega)$ for some $q > n/p$ and hence u is bounded by the remark after Theorem 1.7. Moreover, as indicated in that remark it follows for instance from [15] that u is then locally Hölder continuous. In the case of the p-Laplacian it follows that u is in $C^{1,\varepsilon}$ for some $\varepsilon > 0$ by e.g. [5]. □

Example 3.1. Suppose that $a > a_c = \frac{n(p-1)}{n-p}$. Let $b = \frac{p}{a-p+1}$ and $u(x) = |x|^{-b}$. Since $a > a_c$ u is an entropy solution of

$$-\Delta_p(u) = f\,, \tag{3.2}$$

where $f(x) = b^{p-1}\big(b(1-p)+n-p\big)|x|^{-ab} \in L^1(B(0,1))$. Because of the nonhomogeneity of (3.2) there is a constant $\lambda > 0$ such that $v = \lambda u$ satisfies

$$-\Delta_p(v) = v^a\,,$$

but v is not regular (bounded).

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