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EXISTENCE RESULTS ON PRESCRIBING ZERO SCALAR CURVATURE*

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Abstract. We study some time independent Schrödinger equations on complete noncompact manifolds. By showing that these equations have global positive solutions bounded away from zero, we prove that a large class of manifolds are conformal to complete manifolds with zero scalar curvature. They include those with Ricci curvature being nonnegative and the scalar curvature satisfying $V(x) \le C/d^2(x, x_0)$ for an arbitrary $C > 0$. As we have shown that there are noncompact manifolds with positive scalar curvature, which are not conformal to manifolds with positive constant scalar curvature, the issue of prescribing zero scalar curvature becomes interesting. The current result gives a partial answer to the Yamabe problem on noncompact manifolds with nonnegative Ricci curvatures.

1. Introduction. We shall study the global behavior of positive solutions to the following equation on manifolds.

$$\Delta u(x) - V(x)\ u(x) = 0, \qquad x \in \mathbf{M}^n \tag{1.1}$$

Here Δ is the Laplace-Beltrami operator on a complete noncompact manifold $\mathbf{M}^n$. $V = V(x)$ is a bounded nonnegative function. We will introduce the precise conditions on $\mathbf{M}^n$ and V in the next section.

Of course equation (1.1) is just a time independent Schrödinger equation that has been studied extensively. Unlike many other works on the equation, the motivation of the current paper comes from differential geometry.

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More specifically, we are motivated by the so called Yamabe problem on non-compact manifolds. i.e. whether the following elliptic problem has positive solutions on $\mathbf{M}^n$.

$$\Delta u - \frac{n-2}{4(n-1)} R_1 u + R_2 u^{(n+2)/(n-2)} = 0. \tag{1.2}$$

Here R_1 is the scalar curvature of the manifold and $R_2 = 1, 0, -1$. This problem has been asked by J. Kazdan [8] and S.T. Yau [19]. The compact version of the problem was proposed by Yamabe [18], proved by Trudinger [17] and Aubin [2] in some cases and eventually proved by R. Schoen [16] completely. In the non-compact case, when R_1 is negative somewhere there are both existence (Aviles and McOwen [5]) and nonexistence results ([7]) to the problem. However until recently , there had been no result on the noncompact Yamabe problem when $R_1 > 0$. In the papers [21, 22], one of us showed that in general equation (1.2) with $R_2 = 1$ can not be solved for manifolds with nonnegative Ricci curvature. In other words there exist complete noncompact manifolds with strictly positive scalar curvature, which are not conformal to manifolds with positive constant scalar curvature. In light of this development, equation (1.1), which relates to the problem of deforming to zero scalar curvature ($R_2 = 0$ in (1.2)), becomes more important than before from the geometric point of view. The noncompact Yamabe problem should now include the question: "What kind of complete noncompact manifolds with positive Ricci curvature are conformal to complete manifolds with zero scalar curvature?" In this paper we show that for a large class of manifolds, the answer to the above question is yes. We also mention that some existence results of (1.2) with $R_1 > 0$ and $R_2 = 1$ were obtained in [9] and [22] recently.

2. Statement of the result. Let p be a fixed point in $\mathbf{M}$. We shall introduce some classes of manifolds that include those with nonnegative Ricci or sectional curvature. More examples will be given in remarks below.

Definition 1.1. A manifold $\mathbf{M}$ is called type A^α, $0 \le \alpha \le 1$, if for large $R > 0$ and any fixed $\lambda > 1$, $\partial B_{\lambda R}(p)$ can be covered by a fixed number of geodesic balls of radius R^α.

Throughout the paper we make the following assumptions, unless stated otherwise.

(i) For some $\alpha \in [0,1]$, $\mathbf{M}$ is type A^α.

(ii) $\mathbf{M}$ has finitely many ends. For each end E and large $R > 0$, $E \cap \partial B(p, R)$ is connected.

(iii) $Ricci(x) \geq -k(x)$. Here $0 \leq k(x) \leq C/d(x,p)^2$.

(iv) The scalar curvature V is nonnegative and C^2. Let $R = d(x,p)$. When $\mathbf{M}$ is type A^α, for large R and some positive constant C (not necessarily small), $0 \leq V(x) \leq C/R^{2\alpha}$.

Theorem A. *Suppose* $\mathbf{M}$ *is a complete noncompact manifold satisfying* (i)-(iii) *and* V *is any function satisfying* (iv), *then* (1.1) *has global positive solutions bounded away from zero.*

Corollary A1. *Any complete noncompact manifold of dimension three and higher satisfying* (i)-(iv) *is conformal to a complete noncompact manifold of zero scalar curvature. These include* (a) *all complete manifolds with Ricci curvature being nonnegative and the scalar curvature satisfying* $V(x) \leq C/d^2(x,p)$ *for an arbitrary* $C > 0$; (b) *all asymptotically flat manifolds with nonnegative scalar curvature.*

Corollary A2. *Assuming* (i)-(iii) *and that* $0 \leq V(x) \leq C/d(x,p)^2$, *then there exist a positive solution of* (1.1) *and constants* $a, b, c > 0$ *such that* $c \leq u(x) \leq a[1 + d(x,p)^b]\ u(p)$.

The next theorem deals with the case when the scalar curvature is bounded away from zero. Assumption (iv) is no longer needed now.

Theorem B. *Suppose* $\mathbf{M}$ *is a complete noncompact manifold with nonnegative Ricci curvature and* $dim(\mathbf{M}) \geq 3$. *Let* V *be the scalar curvature of* $\mathbf{M}$ *and* C_1, C_2 *be any positive constants.*

(a) *Assuming that* $\mathbf{M}$ *is of type* A^α *with* $0 \leq \alpha < 1$, $C_2 > V \geq C_1 > 0$ *and* $|\nabla V|$, $|\Delta V|$ *are bounded, then* $\mathbf{M}$ *is conformal to a complete noncompact manifold of zero scalar curvature.*

(b) *Assuming that* $\mathbf{M}$ *is of type* A^α *with* $0 \leq \alpha < \frac{1}{2}$, $C_2 > V \geq C_1 > 0$, *then* $\mathbf{M}$ *is conformal to a complete noncompact manifold of zero scalar curvature.*

Remark 2.1. The following are well-known types of manifolds with finitely many ends and satisfying the assumptions (i), (iii).

(a) Manifolds with nonnegative Ricci curvature outside a compact set. $\alpha = 1$. (See [10] and [13]).

(b) Manifolds with nonnegative sectional curvature outside a compact set. $\alpha = 1$. See [12].

Remark 2.2. We mention that when the manifold has positive Ricci curvature, the decay rate $V \leq C/R^2$ is optimal in general. This is so because a variation of the Meyer theorem shows that if $Ricci(x) \geq C/R^{2-\epsilon}$, then $\mathbf{M}$ is compact.

Remark 2.3. By direct calculation, it is easy to see that the manifold given by $z = x_1^2 + x_2^2 + x_3^2$ endowed with the Euclidean metric is type $A^{1/2}$.

Remark 2.4. If the ends of $\mathbf{M}$ have uniformly bounded diameter then we can take $\alpha = 0$ in the assumptions. Then no decay of the scalar curvature is needed. An example is $\mathbf{M} = \mathbf{N} \times R^1$ with the product metric where $\mathbf{N}$ is a compact manifold with nonnegative Ricci curvature.

Remark 2.5. The notion given in Definition 1.1 is related to the concept of *diameter growth* introduced in [4]. However we do not know whether they are equivalent.

Remark 2.6. Example (b) in Corollary A1 is already given in [6], which uses a different method.

To end the section we mention that it is well known that (1.1) has global positive solutions. However these solutions in general do not generate complete metrics. The key is to construct solutions of (1.1), which either do not decay too fast or are bounded away from zero. To achieve this we will construct subsolutions and use the idea of Harnack inequality on spheres as in [12].

3. Proof of Theorem A.

Lemma 3.1. *Under the assumptions of Theorem A, let* Γ *be a connected component of* $\partial B(p, R)$ *with* R *sufficiently large. Suppose* u *is a nonnegative solution of* (1.1), *then there exists* $C_0 > 0$, *which is independent of* R, *such that* $\sup_\Gamma u \leq C_0 \inf_\Gamma u$.

Proof. Let $x_1, x_2 \in \Gamma$ be such that $u(x_1) = \inf_\Gamma u$ and $u(x_2) = \sup_\Gamma u$. By our assumption on the manifold, there exists a fixed number N independent of R and geodesic balls $B(p_j, R^\alpha/16)$, $j = 1, ..., N$ such that, $\Gamma \cap B(p_j, R^\alpha/16)$ is not empty and $\Gamma \subset \cup_{j=1}^N B(p_j, R^\alpha/16)$. Since Γ is connected, by rearranging the order of the balls if necessary, we claim that one can find a $N_1 \leq N$ and a sequence of points y_j, $j = 1, ..., N_1$, such that $y_1 = x_1$, $y_{N_1} = x_2$, $d(y_j, y_{j+1}) \leq R^\alpha/8$ and $y_j \in B(p_j, R^\alpha/16)$. The proof of the claim is as follows. With out loss of generality we assume that $x_1 \in B(p_1, R^\alpha/16)$ and $x_2 \in B(p_n, R^\alpha/16)$. It is apparent that one can find a chain of the balls starting from $B(p_1, R^\alpha/16)$ and ending at $B(p_n, R^\alpha/16)$ such that one ball overlaps the next. Otherwise the above balls would form two subgroups whose unions are disjoint. Hence Γ would not be connected, a contradiction. Now let $\{B(p_{j_k}, R^\alpha/16) | k = 1, ..., N_1\}$ be such a chain of balls. Noticing that $d(p_{j_k}, p_{j_{k+1}}) \le R^\alpha/8$, the claim is proved by picking y_k from $B(p_{j_k}, R^\alpha/16) \cap B(p_{j_{k+1}}, R^\alpha/16)$ where $k = 2, ..., N_1 - 1$.

Since $\Gamma \cap B(p_j, R^\alpha/16)$ is not empty we know that

$$d(y_j, p) \ge d(\Gamma, p) - \mathrm{diam} B(p_j, R^\alpha/16) = R - R^\alpha/8 \ge 7R/8.$$

Here the last inequality comes from the assumption that $R \ge 1$ and $0 \le \alpha \le 1$. Let γ_j be a shortest geodesic connecting y_j and y_{j+1}. Suppose $y \in \gamma_j$, then, by the last inequality,

$$d(y, p) \ge d(y_j, p) - d(y_j, y) \ge 7R/8 - R^\alpha/8 \ge 3R/4.$$

Let z_j be the midpoint of γ_j; then $\gamma_j \subset B(z_j, R^\alpha/16) \subset B(z_j, R^\alpha/8)$. This is because $d(x, z_j) \le R^\alpha/16$ for all $x \in \gamma_j$, $j = 1, ...N_1$. Moreover,

$$d(p, B(z_j, R^\alpha/8)) \ge d(p, y_j) - \mathrm{diam} B(z_j, R^\alpha/8) = 7R/8 - R^\alpha/8 \ge 3R/4.$$

Therefore, we have

$$\sup_{B(z_j, R^\alpha/8)} V \le C/R^{2\alpha}, \qquad \sup_{B(z_j, R^\alpha/8)} k(x) \le C/R^{2\alpha}.$$

Note that every solution of (1.1) in $B(z_j, R^\alpha/8)$ is also a solution of the corresponding parabolic equation in $B(z_j, R^\alpha/8) \times [R^{2\alpha}, 2R^{2\alpha}]$. We can use Theorem 5.3 of [15] to conclude,

$$\sup_{B(z_j, R^\alpha/16)} u \le e^{C(1+\beta R^{2\alpha} + K R^{2\alpha})} \inf_{B(z_j, R^\alpha/16)} u.$$

Here $\beta \equiv \sup_{B(z_j, R^\alpha/8)} V \le C/R^{2\alpha}$ and $K \equiv \sup_{B(z_j, R^\alpha/8)} k(x) \le C/R^{2\alpha}$. We remark that in Theorem 5.3 of [15], β and K were taken as the global bounds of $V = V(x)$ and $k = k(x)$. However since all the arguments in [15] leading to the theorem are local in nature, it is easy to check that the above choices of β and V are valid. By our assumptions on the curvature and since $y_{j+1}, y_j \in \bar{B}(z_j, R^\alpha/16)$, we have $u(y_{j+1}) \le C u(y_j)$. Now we conclude that $u(x_2) \le C^N u(x_1)$. $\square$

Lemma 3.2. *Suppose $V \ge 0$ and E is an end of $\mathbf{M}$, which satisfies* (i)-(iii). *There exist a positive solution of* (1.1) *$u = u(x)$ and $r_0 > 0$ such that for $R > r_0$, $\sup_{x \in \partial B(p,R) \cap E} u(x) \ge 1$ and $\sup_{x \in B(p,R)} u(x) = \sup_{x \in \partial B(p,R) \cap E} u(x)$.*

Proof. Let $E = E_1, E_2, ..., E_m$ be all the ends of $\mathbf{M}$. Let r_0 be such that for large $R > r_0$, $\partial B(p,R) = \cup_{i=1}^m E_i \cap \partial B(p,R)$ and $F_{i,R} \equiv E_i \cap \partial B(p,R)$ is connected.

For each R, r as above, let $D_{R,r}$ be the domain enclosed by $F_{1,r}, F_{i,R}$ with $i = 2, ..., m$ and $D_{R,\infty} = \cup_{r>r_0} D_{R,r}$. Obviously $D_{R,R} = B(p,R)$ and if $r > R$ then $D_{R,r} = B(p,R) \cup (E_1 \cap (B(p,r) - \bar{B}(p,R)))$. Let $w_{R,r}$ be a nonnegative solution of $\Delta u - Vu = 0$ in $D_{R,r}$, satisfying $w_{R,r}|_{F_{1,r}} = 1$ and $w_{R,r}|_{F_{i,R}} = 0$, for $i = 2, ..., m$. By the maximum principle we know that $w_{R,r}(p)$ is positive. Let $f_{R,r} \equiv w_{R,r}/w_{R,r}(p)$. Since $f_{R,r}(p) = 1$, by the maximum principle, $\sup_{x \in F_{1,R}} f_{R,r}(x) \geq 1$. Next suppose $R_1 < r$ and F_{1,R_1} is not empty. Consider $f_{R,r}$ in the domain enclosed by F_{1,R_1}, $F_{i,R}$ with $i = 2, ..m$. Since $f_{R,r}|_{F_{i,R}} = 0$ for $i = 2, ..., m$, we have by the maximum principle, again

$$\sup_{x \in F_{1,R_1}} f_{R,r}(x) \geq 1. \tag{3.1}$$

Another immediate consequence is

$$\sup_{x \in D_{R,R_1}} f_{R,r}(x) = \sup_{x \in F_{1,R_1}} f_{R,r}(x). \tag{3.2}$$

Now let r_1 be any number satisfying $r_0 < r_1 \leq \min\{R, r\}$. Then, since $B(p, r_1) \subset D_{R,r_1}$,

$$\sup_{x \in B(p,r_1)} f_{R,r}(x) \leq \sup_{x \in D_{R,r_1}} f_{R,r}(x) = \sup_{x \in F_{1,r_1}} f_{R,r}(x).$$

Here the last equality is by (3.2). Hence

$$\sup_{x \in B(p,r_1)} f_{R,r}(x) = \sup_{x \in \partial B(p,r_1) \cap E} f_{R,r}(x) \tag{3.3}$$

for all r_1 satisfying $r_0 < r_1 \leq \min\{R, r\}$.

Observe that $f_{R,r}(p) = 1$. By the Harnack inequality , on any compact subdomains of $D_{R,\infty}$ and for r sufficiently large, $f_{R,r}$ is uniformly bounded and equicontinuous. Here R is fixed for the moment. By the standard argument we can extract a subsequence of $\{f_{R,r}\}$, which converges, on any subdomain of $D_{R,\infty}$, to a nonnegative function u_R, which is a solution of equation (1.1) in $D_{R,\infty}$. By last paragraph $\sup_{x \in F_{1,R_1}} u_R(x) \geq 1$, $R_1 > r_0$. Moreover, by (3.3) we have

$$\sup_{x \in B(p,r_1)} u_R(x) = \sup_{x \in \partial B(p,r_1) \cap E} u_R(x), \quad r_1 < R. \tag{3.4}$$

Recall that $u_R(p) = 1$. Let D be any compact region in $\mathbf{M}$, which contains p. The Harnack inequality again implies that $\{u_R\}$ is uniformly bounded and equicontinuous in D. Therefore we can extract a subsequence $\{u_{R_k}\}$, which converges, on any subdomain of $\mathbf{M}$, to a nonnegative function u, which is a solution of equation (1.1) in $\mathbf{M}$. The Harnack inequality shows that u is positive. Clearly $\sup_{x\in F_{1,r}} u(x) \geq 1$, $r > r_0$. From (3.4) we also have $\sup_{x\in B(p,r)} u(x) = \sup_{x\in\partial B(p,r)\cap E} u(x)$, $r > r_0$.

Remark 3.1. We emphasize that in Lemma 3.2 one only needs V to be nonnegative. This fact will be useful in the proof of Theorem B to be given in the next section.

Proof of Theorem A. Let E_i, $i = 1, ..m$, be all the ends of $\mathbf{M}$. By Lemma 3.2, there exists a positive solution u_i of (1.1) such that $\sup_{x\in E_i\cap\partial B(p,R)} u_i(x) \geq 1$, when $R > r_0$. Here $i = 1, ..., m$. Let $u = \Sigma_{i=1}^m u_i$, then u is a positive solution of (1.1) and $\sup_{x\in E_i\cap\partial B(p,R)} u(x) \geq 1$, when $R > r_0$ for each $i = 1, ..., m$. By Lemma 3.1, there is a constant $C_0 > 0$ such that $\inf_{x\in E_i\cap\partial B(p,R)} u(x) \geq C_0$, when $R > r_0$ for each $i = 1, ..., m$, i.e., $\inf_{x\in\partial B(p,R)} u(x) \geq C_0$, $R > r_0$. The Harnack inequality shows that $u(x) > 0$ when $d(x, p) \leq r_0$. Hence $u(x) \geq C_1$ for all x in $\mathbf{M}$ and some $C_1 > 0$.

Proof of the Corollary A1. Let g_0 be the given metric of $\mathbf{M}$ and suppose $\mathbf{M}$ and the scalar curvature R_0 satisfy (i)-(iv). By Theorem A, the equation $\Delta u - \frac{n-2}{4(n-1)} R_0 u = 0$ has a positive solution u which is bounded away from zero. Let g be the pointwise conformal metric $g = u^{4/(n-2)} g_0$. Then it is standard to show that the scalar curvature corresponding to g is zero. Since $u \geq C > 0$, this new metric g is also complete.

If $\mathbf{M}$ is a complete manifold with Ricci curvature being nonnegative and the scalar curvature satisfying $V(x) \leq C/d^2(x, p)$ for an arbitrary $C > 0$, then clearly assumptions (i), (iii) and (iv) are satisfied. By Proposition 4.3 of [4], (ii) is also satisfied. Therefore, $\mathbf{M}$ is conformal to a manifold with zero scalar curvature.

Finally let $\mathbf{M}$ be an asymptotically flat manifold with nonnegative scalar curvature. By definition (see Definition 6.3 in [11]) there is a $\tau > 0$ and a decomposition $\mathbf{M} = \mathbf{M}_0 \cup \mathbf{M}_\infty$ (with $\mathbf{M}_0$ compact) and a diffeomorphism $\mathbf{M}_\infty$ to $\mathbf{R}^n - B_R$ for some $R > 0$, satisfying $g_{ij} = \delta_{ij} + O(\rho^{-\tau})$, $\partial_k g_{ij} = O(\rho^{-\tau-1})$, $\partial_k\partial_l g_{ij} = O(\rho^{-\tau-2})$, as $\rho = |z| \to \infty$ in the coordinates $\{z^i\}$ induced on $\mathbf{M}_\infty$.

By direct calculation, $Ricc(x) \geq -Cd(p, x)^{-2}$ and $V \leq C/d(p, x)^2$ for some $C > 0$. By our assumption, $V \geq 0$. It is also easy to see that $\mathbf{M}$ has only one end and is of type A^1. By Theorem A, $\mathbf{M}$ is conformal to a manifold with zero scalar curvature.

Proof of the Corollary A2. Let u be a solution as obtained in the end of the proof of Theorem A; we need to show that $u(x) \le a[1+d(x,p)^b]u(p)$ for some $a, b>0$. Without loss of generality we assume that $d(x,p)\ge 1$. Let $M(r)=\sup_{d(x,p)=r} u(x)$ and assume that $u(x_r)=M(r)$, where $d(x_r,p)=r$. Let γ be a shortest geodesic connecting p and x_r, which is parameterized by length. For $i=0,1,2,...,k$, we write $y_i=\gamma(2^i)$, where k is the greatest integer smaller than or equal to $\log_2 r$. Consider the chain of balls $B_i = B(y_i, 2^{i-1})$, $i=0,1,...,k$ and $B(x_r, 2^{k-1})$. Clearly $B_i\cap B_{i+1}$, $i=0,...,k-1$ and $B_k \cap B(x_r,2^{k-1})$ are nonempty and $dist(p,B_i)\ge 2^{i-1}$.

As shown in the proof of Lemma 3.1, there is a constant C independent of k such that $\sup_{y\in B_i} u(y)\le C\inf_{y\in B_i} u(y)$. Applying the above inequality for each of the balls above we have

$$u(x_r)\le C^{k+1}u(y_0)\le C^{k+2}u(p)\le CC^{\log_2 r}u(p).$$

This proves Corollary A2.

4. Proof of Theorem B.

Lemma 4.1. *Suppose* **M** *is a complete manifold with nonnegative Ricci curvature and V is bounded away from zero. Let u be any positive solution of* (1.1) *such that $u(p)=1$, then there exist positive numbers c_0 and c_1 such that, for large R,* $\sup_{x\in\partial B(p,R)} u(x)\ge c_1 e^{c_0 R}$.

Proof. We assume that $V\ge\lambda>0$. For $R>0$ let u_R be the solution of

$$\begin{cases} \Delta u_R - \lambda u_R = 0, \quad \text{in} \quad B(p,R) \\ u_R|_{\partial B(p,R)} = 1. \end{cases} \tag{4.1}$$

We are going to show that $u_R(0)\le ce^{-c_0R}$, where c, c_0 are suitable positive numbers. To this end let

$$u_0(x)=e^{c_0 r}+e^{-c_0 r}, \tag{4.2}$$

where $r=d(x,p)$ and $c_0>0$ will be chosen later. For u_0 we have, in the weak sense,

$$\Delta u_0 = u_0'' + \frac{n-1}{r}u_0' + u_0'\frac{\partial \log g^{1/2}}{\partial r},$$

where $u_0' = c_0(e^{c_0 r}-e^{-c_0 r}) = c_0 e^{c_0 r}(1-e^{-2c_0 r})\ge 0$, $u_0''=c_0^2(e^{c_0 r}+e^{-c_0 r})$. Since **M** has nonnegative Ricci curvature, it is well known that $\frac{\partial \log g^{1/2}}{\partial r}\le 0$ in the weak sense. As $u_0'\ge 0$, we have

$$\begin{aligned} \Delta u_0 &\le u_0'' + \frac{n-1}{r}u_0' = c_0^2(e^{c_0 r}+e^{-c_0 r}) + (n-1)c_0 e^{c_0 r}\frac{1-e^{-2c_0 r}}{r} \\ &\le c_0^2(e^{c_0 r}+e^{-c_0 r}) + 2(n-1)c_0^2 e^{c_0 r}, \end{aligned}$$

where we have used the elementary inequality $1 - e^{-2c_0 r} \leq 2c_0 r$. Now we know that

$$\begin{aligned}\Delta u_0 - \lambda u_0 &\leq c_0^2(e^{c_0 r} + e^{-c_0 r}) + 2(n-1)c_0^2 e^{c_0 r} - \lambda(e^{c_0 r} + e^{-c_0 r}) \\ &\leq (e^{c_0 r} + e^{-c_0 r})(c_0^2 + 2(n-1)c_0^2 - \lambda).\end{aligned}$$

Hence, when $0 < c_0 \leq [\lambda/(2n-1)]^{1/2}$,

$$\Delta u_0 - \lambda u_0 \leq 0. \tag{4.3}$$

Let u_R be the solution of (4.1) and $w_R \equiv u_R(x)(e^{c_0 R} + e^{-c_0 R})$, then

$$\Delta w_R - \lambda w_R = 0, \quad \text{in} \quad B(p,R)$$

and $w_R|_{\partial B(p,R)} = u_0|_{\partial B(p,R)}$. By (4.3) and the maximum principle we have

$$w_R(x) = u_R(x)(e^{c_0 R} + e^{-c_0 R}) \leq u_0(x), \quad \text{in} \quad B(p,R). \tag{4.4}$$

When $x = p$, $u_0(p) = 2$. Hence

$$u_R(p) \leq 2e^{-c_0 R}. \tag{4.5}$$

Now let u be a solution of $\Delta u - Vu = 0$ and $u(p) = 1$. For any $R > 0$ define $\bar{u} = \frac{u(x)}{\max_{\partial B(p,R)} u}$. Then $\bar{u}$ satisfies

$$\begin{cases} \Delta\bar{u} - V\bar{u} = 0, \quad \text{in} \quad B(p,R) \\ \bar{u}|_{\partial B(p,R)} \leq 1. \end{cases} \tag{4.6}$$

Since $V \geq \lambda$, we have, for $x \in B(p,R)$,

$$\Delta(\bar{u} - u_R) - \lambda(\bar{u} - u_R) = (V - \lambda)\bar{u} \geq 0.$$

Applying the maximum principle, we obtain that

$$\frac{u(x)}{\max_{\partial B(p,R)} u} = \bar{u}(x) \leq u_R(x), \quad \text{in} \quad B(p,R).$$

Recalling that $u(p) = 1$, we have, from (4.5),

$$\max_{\partial B(p,R)} u \geq \frac{1}{2} e^{c_0 R}. \quad \textit{q.e.d.} \tag{4.7}$$

Proof of Theorem B. (a) Since **M** has finitely many ends, as in the proof of Theorem A, it is enough to prove the following claim.

Claim. For each end E, there exist a positive solution of (1.1), $u = u(x)$, and $r_0 > 0$ such that, for $R > r_0$, $\inf_{x \in \partial B(p,R) \cap E} u(x) \geq C > 0$.

By Lemma 3.2, which does not need V to decay near infinity, for each end E, there exist a positive solution of (1.1) $u = u(x)$ and $r_0 > 0$ such that, for $R > r_0$,

$$\sup_{x \in \partial B(p,R) \cap E} u(x) = \sup_{x \in \partial B(p,R)} u(x). \tag{4.8}$$

For each $R > r_0$, let $\Gamma_R = E \cap \partial B(p, R)$. By Proposition 4.3 of [4], we know that Γ_R is connected.

Suppose $x_1, x_2 \in \Gamma_R$ be such that $u(x_1) = \inf_{\Gamma_R} u$ and $u(x_2) = \sup_{\Gamma_R} u$. Following the proof of Lemma 3.1, we can find a chain of balls $B(z_j, R^\alpha/16)$ and $y_i \in B(z_j, R^\alpha/16)$, $j = 1, ..., N$, such that (a) $y_1 = x_1, y_N = x_2$; (b) $\gamma_j \subset B(z_j, R^{\alpha/8})$, where γ_j is the shortest geodesic connecting y_j and y_{j+1}; (c) $d(p, B(z_j, R^{\alpha/8}) \geq 3R/4$.

Note that every solution of (1.1) in $B(z_j, R^\alpha/8)$ is also a solution of the corresponding parabolic equation in $B(z_j, R^\alpha/8) \times [R^{2\alpha}, 2R^{2\alpha}]$. We can use the gradient estimates in Theorem 1.2 of [14] to conclude, for any $\beta > 1$, there exist positive constants $C_1, ..., C_4$ depending on β, n, such that

$$\frac{|\nabla u(x)|^2}{u^2(x)} \leq \beta V + C_1 \beta^2 [R^\alpha]^{-2} (1 + R^\alpha K^{1/2}) + [C_2 \gamma^{4/3} + C_3 K^2 + C_4 \theta]^{1/2},$$

where $x \in B(z_j, R^\alpha/16)$, $\gamma = \sup_{B(z_j, R^\alpha/8)} |\nabla V|$ and $\theta = \sup_{B(z_j, R^\alpha/8)} |\Delta V|$; $-K$ is the lower bound of the Ricci curvature.

By our assumption on the curvature, we have

$$\frac{|\nabla u(x)|}{u(x)} \leq C_5,$$

when $x \in B(z_j, R^\alpha/16)$ and R is large. Recall that γ_j is a shortest geodesic connecting y_j and y_{j+1}. Hence

$$\log u(y_{j+1}) - \log u(y_j) = \int_{t_j}^{t_{j+1}} \frac{du(\gamma_j(t))}{dt} dt \leq \int_{t_j}^{t_{j+1}} \frac{|\nabla u(\gamma_j(t))|}{u(\gamma_j(t))} dt.$$

Since $\gamma_j \subset B(z_j, R^\alpha/16)$ and the length of γ_j is bounded from above by $R^\alpha/8$, the last inequality implies $\log u(y_{j+1}) - \log u(y_j) \leq C_5 R^\alpha$, or $u(y_{j+1}) \leq u(y_j) e^{C_5 R^\alpha}$. It follows that $u(x_2) \leq e^{N C_5 R^\alpha} u(x_1)$, i.e.,

$$\sup_{x \in \partial B_{p,R} \cap E} u \leq e^{N C_5 R^\alpha} \inf_{x \in \partial B_{p,R} \cap E} u.$$

By Lemma 4.1 $\sup_{x\in\partial B(p,R)} u(x) \ge c_1 e^{c_0 R}$, hence

$$\inf_{x\in\partial B_{p,R}\cap E} u \ge c_1 e^{c_0 R - NC_5 R^\alpha}.$$

Since $\alpha < 1$ and N is independent of R, we have, for R sufficiently large, $\inf_{x\in\partial B_{p,R}\cap E} u \ge C > 0$. This proves the claim and part (a) of Theorem B.

Proof of part (b). This is similar to that of part (a). The difference is that we will use the Harnack inequality in [15] instead of the gradient estimate in [14]. Retaining the notations from the proof of part (a) and using Theorem 5.3 in [15], we have $u(y_{j+1}) \le u(y_j) e^{CC_6 R^{2\alpha}}$, where C_6 is the upper bound of V. Therefore

$$\sup_{x\in\partial B_{p,R}\cap E} u \le e^{NCC_6 R^{2\alpha}} \inf_{x\in\partial B_{p,R}\cap E} u.$$

By Lemma 4.1 $\sup_{x\in\partial B(p,R)} u(x) \ge c_1 e^{c_0 R}$, hence

$$\inf_{x\in\partial B_{p,R}\cap E} u \ge c_1 e^{c_0 R - NCC_6 R^{2\alpha}}.$$

Since $\alpha < 1/2$ and N is independent of R, we have, for R sufficiently large,

$$\inf_{x\in\partial B_{p,R}\cap E} u \ge C > 0.$$

This proves the claim and part (b) of Theorem B. q.e.d.

Remark 4.1. In Theorem B, we can replace the assumption $Ricc(x) \ge 0$ by assumptions (ii) and (iii) in Section 2, which imply that $\frac{\partial \log g^{1/2}}{\partial_r} \le C$ when r is smooth. Only small adjustments are needed in constructing the super solution u_0 in Lemma 4.1.

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