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SINGULARLY PERTURBED ELLIPTIC PROBLEMS IN EXTERIOR DOMAINS*

E.N. Dancer and Shusen Yan
School of Mathematics and Statistics, University of Sydney
NSW 2006, Australia

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1. Introduction. In this paper, we consider the following problem:

$$\begin{cases} -\varepsilon^2\Delta u + u = u^{p-1}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases} \tag{1.1}$$

where Ω is a domain in R^N satisfying that $R^N \setminus \Omega$ is a bounded open set, $2 < p < \frac{2N}{N-2}$ if $N > 2$ and $2 < p < +\infty$ if $N = 2$. The functional, or the energy, corresponding to (1.1) is

$$I(u) = \frac{1}{2}\int_\Omega \varepsilon^2|Du|^2 + u^2 - \frac{1}{p}\int_\Omega |u|^p. \tag{1.2}$$

Let $U(y)$ be the unique solution of

$$\begin{cases} -\Delta u + u = u^{p-1}, & \text{in } R^N, \\ u > 0, & \text{in } R^N, \\ u \in H^1(R^N), \quad u(0) = \max_{y\in R^N} u(y). \end{cases} \tag{1.3}$$

We define

$$A = \frac{1}{2}\int_{R^N} |DU|^2 + U^2 - \frac{1}{p}\int_{R^N} U^p. \tag{1.4}$$

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In [5], Cerami and Passaseo prove that (1.1) has at least $\mathrm{Cat}_\Omega(\Omega, R^N\setminus B_R(0))$ distinct solutions, whose energy is very close to $\varepsilon^N A$, where $R>0$ is a large number such that $R\setminus\Omega\subset B_R(0)$. Later they prove in [6] that (1.1) always has a solution for any Ω and the energy of this solution is strictly less than $2\varepsilon^N A$ but greater than that of the previous $\mathrm{Cat}_\Omega(\Omega, R^N\setminus B_R(0))$ solutions if ε is small enough. Because $U(y)$ is exponentially small in the infinity, using the blow-up argument, we know that for any solution u_ε of (1.1) satisfying $I(u_\varepsilon)\le C\varepsilon^N$, $I(u_\varepsilon)=(kA+o(1))\varepsilon^N$ for some positive integer k and thus this solution has exactly k local maximum points. So we see that the previous $\mathrm{Cat}_\Omega(\Omega, R^N\setminus B_R(0))$ solutions are all of single peak. As for the latter solution, we can only say it has at most two peaks. On the other hand, we can not get any information on the location of the peaks of the latter solution either. Although (1.1) has a single peak solution in some domain Ω with $\mathrm{Cat}_\Omega(\Omega, R^N\setminus B_R(0))=0$ (see the example in section 4), the result of Cerami and Passaseo at least suggests that (1.1) have no single peak solution in some domains Ω.

In this paper, we will prove that (1.1) has no single peak solution if $R^N\setminus\Omega$ is convex. We also prove that for any Ω, (1.1) always has at least one two-peak solution and if $R^N\setminus\Omega$ is convex, then both of the peaks of any two-peak solution of (1.1) must tend to infinity as $\varepsilon\to 0$. Before we state our results, we give some notation.

For any $u\in H^1(\Omega)$, let $P_{\varepsilon,\Omega}u$ be the unique solution of the following problem:

$$\begin{cases}-\varepsilon^2\Delta v+v=|u|^{p-2}u, & \text{in } \Omega\\ v\in H_0^1(\Omega).\end{cases} \tag{1.5}$$

We denote

$$\begin{aligned} E_{\varepsilon,x,2}=\Big\{v\in H_0^1(\Omega):\big\langle P_{\varepsilon,\Omega}U_{\varepsilon,x_j},v\big\rangle_\varepsilon=0,\\ \Big\langle\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_j}}{\partial x_{ji}},v\Big\rangle_\varepsilon=0,\ j=1,2,\ i=1,\dots,N\Big\},\end{aligned} \tag{1.6}$$

$$E_{\varepsilon,x}=\Big\{v\in H_0^1(\Omega):\big\langle P_{\varepsilon,\Omega}U_{\varepsilon,x},v\big\rangle_\varepsilon=\Big\langle\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x}}{\partial x_i},v\Big\rangle_\varepsilon=0,\ i=1,\dots,N\Big\}, \tag{1.7}$$

where $U_{\varepsilon,x}(y)=U\big(\frac{y-x}{\varepsilon}\big)$, $\langle u,v\rangle_\varepsilon=\int_\Omega\varepsilon^2DuDv+uv$, $x_j=(x_{j1},\cdots,x_{jN})$. We also denote $\|v\|_\varepsilon^2=\langle v,v\rangle_\varepsilon$.

The main results of this paper are the following:

Theorem 1.1. *There is an* $\varepsilon_0 > 0$*, such that for each* $\varepsilon \in (0, \varepsilon_0]$*,* (1.1) *has a solution of the form*

$$u_\varepsilon = \sum_{j=1}^{2} \alpha_{\varepsilon,j} P_{\varepsilon,\Omega} U_{\varepsilon, x_{\varepsilon,j}} + v_\varepsilon, \tag{1.8}$$

where $v_\varepsilon \in E_{\varepsilon,x,2}$ *and as* $\varepsilon \to 0$*,*

$$\alpha_{\varepsilon,j} \to 1, \ \frac{|x_{\varepsilon,1} - x_{\varepsilon,2}|}{\varepsilon} \to \infty, \ \frac{d(x_{\varepsilon,j}, \partial\Omega)}{\varepsilon} \to \infty, \ \|v_\varepsilon\|_\varepsilon^2 = o(\varepsilon^N). \tag{1.9}$$

Theorem 1.2. *Suppose that* $R^N \setminus \Omega$ *is convex, then*

(i) (1.1) *has no single peak solution, that is,* (1.1) *does not have solution of the form*

$$u_\varepsilon = \alpha_\varepsilon P_{\varepsilon,\Omega} U_{\varepsilon,x_\varepsilon} + v_\varepsilon, \tag{1.10}$$

where $v_\varepsilon \in E_{\varepsilon,x_\varepsilon}$*, and*

$$\alpha_\varepsilon \to 1, \ \frac{d(x_\varepsilon, \partial\Omega)}{\varepsilon} \to \infty, \ \|v_\varepsilon\|_\varepsilon^2 = o(\varepsilon^N), \quad as\ \varepsilon \to 0. \tag{1.11}$$

(ii) *For any solution of* (1.1) *of the form* (1.8) *satisfying* (1.9)*,* $x_{\varepsilon,j} \to \infty$ *and* $2d(x_{\varepsilon,j}, \partial\Omega) = (1 + o(1))|x_{\varepsilon,1} - x_{\varepsilon,2}|$*, where* $o(1) \to 0$ *as* $\varepsilon \to 0$*,* $j = 1, 2$*.*

Our calculations in the proof of Theorem 1.2 also lead to the following result.

Theorem 1.3. *Suppose that* (1.1) *has a single peak solution of the form* (1.10) *satisfying* (1.11)*. Then*

(i) x_ε *can neither move to infinity, nor to the boundary of* Ω*.*

(ii) *Assume that* $x_\varepsilon \to x_0 \in \Omega$*. Then the set* $\partial B_{d(x_0,\partial\Omega)}(x_0) \cap \partial\Omega$ *contains at least two points.*

(iii) *If* $\partial B_{d(x_0,\partial\Omega)}(x_0) \cap \partial\Omega = \{q_1, q_2\}$*, then* $q_1 - x_0 = -(q_2 - x_0)$*.*

Remark 1.4. Suppose that $R^N \setminus \Omega$ is convex and $x_{\varepsilon,1}$ and $x_{\varepsilon,2}$ are the peaks of a two peak solution. Let $z_{\varepsilon,j} = \frac{x_{\varepsilon,j}}{|x_{\varepsilon,1}|}$, $j = 1, 2$. Assume $z_{\varepsilon,j} \to z_j$ as $\varepsilon \to 0$. By Theorem 1.2, we see that $z_j \in S^{N-1}$, $j = 1, 2$ and $|z_1 - z_2| = 2$. Thus $z_1 = -z_2$. This implies that one of the peaks of the solution of the form (1.8) is roughly in the opposite direction of the other.

Remark 1.5. Let $\Omega = R^N \setminus \big(B_1(-x_1) \cup B_1(x_1)\big)$, $|x_1| > 1$. In this case, according to Theorem 1.3, single peak solution of (1.1) must concentrate at $x_0 = 0$.

There are several works considering the singularly perturbed elliptic problems on unbounded domain, see for example [3, 4, 8, 11, 18, 19] . But in these previous works, the effect of the coefficient or the boundary prevents the peaks of the solutions from moving to infinity. This means that the contribution from infinity is negligible. For the exterior domain problem, our arguments show that it is the contribution from infinity that makes (1.1) have a two-peak solution. As far as the authors know, this is the first time that we can find a problem which possesses a solution with the peaks moving to infinity.

If Ω is a bounded domain in R^N, we know (1.1) always has a least energy solution which is of single peak, see [13]. But the existence of two-peak solution depends on the domain shape [2, 9, 12], or the domain topology [7]. For the exterior domain problem, our results here show that (1.1) always has a two-peak solution and whether it has single peak solution depends on the domain topology or the domain geometry.

In order to characterize the locations of the peaks of the solutions for (1.1), it is essential to estimate certain quantities involving $\varphi_{\varepsilon,x} =: U_{\varepsilon,x} - P_{\varepsilon,\Omega} U_{\varepsilon,x}$. The method used in [13] to estimate these quantities in bounded domain problem depends heavily on the comparison theorem. This method is not applied in the exterior domain problem since it is very hard to determine the growth rate at infinity of the function $\psi_{\varepsilon,x} = -\varepsilon \ln \varphi_{\varepsilon,x}$. In section 2 of this paper, we give a direct and simple approach to estimate all the quantities needed in our argument.

It is also worth pointing out that the results in section 2 are still true if Ω is a bounded domain in R^N. Although these estimates are not as accurate as those in [13], they are good enough to get the main results in [13, 2, 15]. Also the estimate in Lemma 2.4 enables us to define a measure as in [16], so that we can get some information on the location of the peaks of the solution as in [16, 17].

This paper is organized as follows. In section 2, some basic estimates are presented. Section 3 is mainly devoted to the construction of a two-peak solution for (1.1). Also in section 3, we give another proof of the existence result in [5]. In section 4, we prove Theorems 1.2 and 1.3 and present an example which shows that if $R^N \setminus \Omega$ is not convex, the conclusions of Theorem 1.2 may not be true and (1.1) has single peak solutions even if $\mathrm{Cat}_\Omega(\Omega, R^N \setminus B_R(0)) = 0$.

2. Basic estimates. In this section, we present some basic estimates needed in the proof of the main results. First, let us recall the well known fact on the asymptotic behaviours of $U(y)$:

$$\lim_{|y|\to+\infty} |y|^{\frac{N-1}{2}} e^{|y|} U(y) = c_0 > 0.$$

From now on, we always assume that $\frac{d(x,\partial\Omega)}{\varepsilon} \geq M$ for some large constant $M > 0$. Let $\varphi_{\varepsilon,x} = U_{\varepsilon,x} - P_{\varepsilon,\Omega} U_{\varepsilon,x}$. Then $\varphi_{\varepsilon,x}$ satisfies

$$\begin{cases} -\varepsilon^2 \Delta \varphi_{\varepsilon,x} + \varphi_{\varepsilon,x} = 0, & y \in \Omega, \\ \varphi_{\varepsilon,x} = U_{\varepsilon,x}, & y \in \partial\Omega. \end{cases} \tag{2.1}$$

We denote $\tau_{\varepsilon,x} = \int_\Omega U_{\varepsilon,x}^{p-1} \varphi_{\varepsilon,x}$. We have the following estimate for $\tau_{\varepsilon,x}$.

Lemma 2.1. *For any $\theta > 0$, there are $c_0' > c_0 > 0$, such that*

$$\begin{aligned} \varepsilon^N c_0 \Big[\frac{\varepsilon}{d(x,\partial\Omega)+\varepsilon\theta}\Big]^{N-1} e^{-2\frac{d(x,\partial\Omega)+\varepsilon\theta}{\varepsilon}} &\leq \tau_{\varepsilon,x} \\ \leq \varepsilon c_0' \Big[\frac{\varepsilon}{d(x,\partial\Omega)-\varepsilon\theta}\Big]^{N-1} e^{-2\frac{d(x,\partial\Omega)-\varepsilon\theta}{\varepsilon}}&. \end{aligned} \tag{2.2}$$

Moreover, there are $C_2 > C_1 > 0$, such that

$$C_1 \varepsilon^N e^{-(2+\theta)\frac{d(x,\partial\Omega)}{\varepsilon}} \leq \tau_{\varepsilon,x} \leq C_2 \varepsilon^N e^{-(2-\theta)\frac{d(x,\partial\Omega)}{\varepsilon}}. \tag{2.3}$$

Proof. Multiplying (2.1) by $U_{\varepsilon,x}$ and integrating by parts, we get

$$\tau_{\varepsilon,x} = \varepsilon^2 \int_{\partial\Omega} \frac{\partial \varphi_{\varepsilon,x}}{\partial n} U_{\varepsilon,x} - \varepsilon^2 \int_{\partial\Omega} \frac{\partial U_{\varepsilon,x}}{\partial n} U_{\varepsilon,x}. \tag{2.4}$$

Besides, multiplying (2.1) by $\varphi_{\varepsilon,x}$ and integrating by parts, we obtain

$$\varepsilon^2 \int_{\partial\Omega} \frac{\partial \varphi_{\varepsilon,x}}{\partial n} U_{\varepsilon,x} = \|\varphi_{\varepsilon,x}\|_\varepsilon^2 > 0. \tag{2.5}$$

Combining (2.4) and (2.5), we obtain

$$\tau_{\varepsilon,x} > -\varepsilon^2 \int_{\partial\Omega} \frac{\partial U_{\varepsilon,x}}{\partial n} U_{\varepsilon,x} = -\varepsilon^2 \int_S \frac{\partial U_{\varepsilon,x}}{\partial n} U_{\varepsilon,x} - \varepsilon^2 \int_{\partial\Omega\setminus S} \frac{\partial U_{\varepsilon,x}}{\partial n} U_{\varepsilon,x}, \quad (2.6)$$

where $S = \{q : q \in \partial\Omega, |x-q| \le (1+\sigma_1)d(x,\partial\Omega)\}$.

Choose $\sigma_1 > 0$ small enough such that $\langle y-x, n\rangle \ge c_0 > 0$, $\forall y \in S$, where n is the outward unit normal of $\partial\Omega$ at y. Because $\frac{d(x,\partial\Omega)}{\varepsilon} \ge M$, we have $\partial\Omega \cap B_{\varepsilon\theta}(q) \subset S$ if $\theta > 0$ is small, where $q \in \partial\Omega$ is a point satisfying $|x-q| = d(x,\partial\Omega)$. Since there is a $\sigma > 0$ such that $U_{\varepsilon,x} \le Ce^{-(1+\sigma)\frac{d(x,\partial\Omega)}{\varepsilon}}$ for $x \in \partial\Omega \setminus S$, (2.6) gives

$$\begin{aligned}
\tau_{\varepsilon,x} &> -\varepsilon^2 \int_S \frac{\partial U_{\varepsilon,x}}{\partial n} U_{\varepsilon,x} + O\Big(\varepsilon^N e^{-(2+\sigma)\frac{d(x,\partial\Omega)}{\varepsilon}}\Big) \\
&\ge c'\varepsilon \int_S \Big(\frac{\varepsilon}{|y-x|}\Big)^{N-1} e^{-2\frac{|y-x|}{\varepsilon}} + O\Big(\varepsilon^N e^{-(2+\sigma)\frac{d(x,\partial\Omega)}{\varepsilon}}\Big) \\
&\ge c'\varepsilon \int_{\partial\Omega\cap B_{\varepsilon\theta}(q)} \Big(\frac{\varepsilon}{|y-x|}\Big)^{N-1} e^{-2\frac{|y-x|}{\varepsilon}} + O\Big(\varepsilon^N e^{-(2+\sigma)\frac{d(x,\partial\Omega)}{\varepsilon}}\Big) \\
&\ge \varepsilon^N c_0 \Big(\frac{\varepsilon}{d(x,\partial\Omega)+\varepsilon\theta}\Big)^{N-1} e^{-2\frac{d(x,\partial\Omega)+\varepsilon\theta}{\varepsilon}},
\end{aligned}$$

Thus we get the left hand side of (2.2). Since $\frac{d(x,\partial\Omega)}{\varepsilon} > M$, we see that (2.2) implies the left hand side of (2.3).

Next we claim

$$|D\varphi_{\varepsilon,x}(y)| \le \frac{C}{\varepsilon}\Big[\frac{\varepsilon}{|y-x|-\varepsilon\theta}\Big]^{\frac{N-1}{2}} e^{-\frac{|y-x|-\varepsilon\theta}{\varepsilon}}, \qquad \forall\, y \in \partial\Omega. \quad (2.7)$$

In fact, let $z = \varepsilon y + x$, $\Omega_{\varepsilon,x} = \{z : \frac{z-x}{\varepsilon} \in \Omega\}$. Then

$$\begin{cases} -\Delta\varphi_{\varepsilon,x} + \varphi_{\varepsilon,x} = 0, & z \in \Omega_{\varepsilon,x}, \\ \varphi_{\varepsilon,x} = U(z), & z \in \partial\Omega_{\varepsilon,x}. \end{cases} \quad (2.8)$$

Denote $\phi_{\varepsilon,x} = \varphi_{\varepsilon,x} - \xi U(z)$, where ξ is a smooth function satisfying $\xi = 1$ if $z \in \Omega_{\varepsilon,x,\theta/2}$, $\xi = 0$ if $z \in \Omega_{\varepsilon,x,\theta}$, where $\Omega_{\varepsilon,x,\theta} = \{z : d(z,\partial\Omega_{\varepsilon,x}) \le \theta\}$. Then,

$$\begin{cases} -\Delta\phi_{\varepsilon,x} + \phi_{\varepsilon,x} = -\Delta(\xi U) + \xi U, & z \in \Omega_{\varepsilon,x}, \\ \phi_{\varepsilon,x} = 0, & z \in \partial\Omega_{\varepsilon,x}. \end{cases} \quad (2.9)$$

It follows from the L^q-estimate in the elliptic theory that for $q > 1$,

$$\begin{aligned} &\|\phi_{\varepsilon,x}\|_{W^{2,q}(\Omega_{\varepsilon,x,\theta/2}\cap B_{\theta/2}(z))} \\ \le & C(\|\Delta(\xi U) + \xi U\|_{L^q(\Omega_{\varepsilon,x,\theta}\cap B_\theta(z))} + C\|\phi_{\varepsilon,x}\|_{L^q(\Omega_{\varepsilon,x,\theta}\cap B_\theta(z))} \\ \le & C\Big[\frac{1}{|z|-\theta}\Big]^{\frac{N-1}{2}} e^{-(|z|-\theta)}, \quad \forall\, z \in \partial\Omega_{\varepsilon,x}. \end{aligned} \tag{2.10}$$

Hence,

$$|D_y\varphi_{\varepsilon,x}(y)| = \frac{1}{\varepsilon}|D_z\varphi_{\varepsilon,x}(z)| \le \frac{C}{\varepsilon}\Big[\frac{\varepsilon}{|y-x|-\varepsilon\theta}\Big]^{\frac{N-1}{2}} e^{-\frac{|y-x|-\varepsilon\theta}{\varepsilon}}, \qquad \forall\, y \in \partial\Omega,$$

and (2.7) follows. It follows from (2.4) and (2.7) that

$$\begin{aligned} \tau_{\varepsilon,x} &\le C\varepsilon\int_{\partial\Omega}\Big[\frac{\varepsilon}{|y-x|-\varepsilon\theta}\Big]^{N-1} e^{-\frac{2|y-x|-\varepsilon\theta}{\varepsilon}} \\ &\le C\varepsilon\Big[\frac{\varepsilon}{d(x,\partial\Omega)-\varepsilon\theta}\Big]^{N-1} e^{-\frac{2d(x,\partial\Omega)-\varepsilon\theta}{\varepsilon}}, \quad \text{or} \end{aligned} \tag{2.11}$$

$$\begin{aligned} \tau_{\varepsilon,x} &\le C\varepsilon\int_{\partial\Omega} e^{-(2-2\theta)\frac{|y-x|}{\varepsilon}} \\ &\le C\varepsilon e^{-(2-3\theta)\frac{d(x,\partial\Omega)}{\varepsilon}}\int_{\partial\Omega} e^{-\theta\frac{|y-x|}{\varepsilon}} \le C\varepsilon^N e^{-(2-3\theta)\frac{d(x,\partial\Omega)}{\varepsilon}}, \end{aligned} \tag{2.12}$$

and hence the results.

Lemma 2.2. *There is a $\sigma > 0$, such that*

$$\int_\Omega \varphi_{\varepsilon,x}^2 U_{\varepsilon,x}^{p-2} = O\big(e^{-\sigma\frac{d(x,\partial\Omega)}{\varepsilon}}\tau_{\varepsilon,x}\big), \tag{2.13}$$

$$\int_\Omega \varphi_{\varepsilon,x}^p = O\big(e^{-\sigma\frac{d(x,\partial\Omega)}{\varepsilon}}\tau_{\varepsilon,x}\big). \tag{2.14}$$

Proof. Since $|\varphi_{\varepsilon,x}| \le U_{\varepsilon,x}$, we see that (2.14) follows from (2.13). Fix $\sigma > 0$ small. Because $U_{\varepsilon,x}(y) \le Ce^{-(1-\sigma)\frac{d(x,\partial\Omega)}{\varepsilon}}$ for any $y \in R^N \setminus B_{(1-\sigma)d(x,\partial\Omega)}(x)$, we have

$$\begin{aligned} &\int_{R^N\setminus B_{(1-\sigma)d(x,\partial\Omega)}(x)} \varphi_{\varepsilon,x}^2 U_{\varepsilon,x}^{p-2} \le \int_{R^N\setminus B_{(1-\sigma)d(x,\partial\Omega)}(x)} U_{\varepsilon,x}^p \\ &\le C\varepsilon^N e^{-p\frac{(1-2\sigma)d(x,\partial\Omega)}{\varepsilon}} = O\big(e^{-\sigma\frac{d(x,\partial\Omega)}{\varepsilon}}\tau_{\varepsilon,x}\big). \end{aligned} \tag{2.15}$$

On the other hand, for $y \in B_{(1-\sigma)d(x,\partial\Omega)}(x)$, we have

$$U_{\varepsilon,x}(y) \geq U\Big(\frac{(1-\sigma)d(x,\partial\Omega)}{\varepsilon}\Big).$$

Thus

$$\frac{\varphi_{\varepsilon,x}(y)}{U_{\varepsilon,x}(y)} \leq \frac{\max_{y\in\partial\Omega} U_{\varepsilon,x}(y)}{U_{\varepsilon,x}(y)} \leq Ce^{-\frac{d(x,\partial\Omega)}{\varepsilon}+\frac{(1-\sigma)d(x,\partial\Omega)}{\varepsilon}} = O\big(e^{-\sigma\frac{d(x,\partial\Omega)}{\varepsilon}}\big).$$

As a result,

$$\int_{B_{(1-\sigma)d(x,\partial\Omega)}(x)} \varphi_{\varepsilon,x}^2 U_{\varepsilon,x}^{p-2} = \int_{B_{(1-\sigma)d(x,\partial\Omega)}(x)} \varphi_{\varepsilon,x} U_{\varepsilon,x}^{p-1}\frac{\varphi_{\varepsilon,x}}{U_{\varepsilon,x}} = O\big(e^{-\sigma\frac{d(x,\partial\Omega)}{\varepsilon}}\tau_{\varepsilon,x}\big).$$

The result follows.

Remark 2.3. From the proof of Lemmas 2.1 and 2.2, we see the same conclusions still hold if Ω is a bounded domain in R^N. The estimates in these two lemmas are sufficient to prove that if Ω is a bounded domain in R^N, the maximum point of the least energy solution of (1.1) tends to a point which attains the maximum of the function $d(x,\partial\Omega)$. See the proof in [13].

Define $\tau'_{\varepsilon,x} = \varepsilon^2 \int_{\partial\Omega} \frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x}}{\partial n}\frac{\partial U_{\varepsilon,x}}{\partial r}$, where $r = |y-x|$ and n is the outward unit normal of $\partial\Omega$ at y.

Lemma 2.4. *For any $\theta > 0$, there are $c_1 > c_0 > 0$, such that*

$$c_0\varepsilon^{N-1}e^{-(2+\theta)\frac{d(x,\partial\Omega)}{\varepsilon}} \leq \tau'_{\varepsilon,x} \leq c_1\varepsilon^{N-1}e^{-(2-\theta)\frac{d(x,\partial\Omega)}{\varepsilon}}.$$

Proof. Similar to the proof of (2.7) in Lemma 2.1, we see that for any $\theta > 0$, there is a $C > 0$, such that

$$\Big|\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x}}{\partial n}\Big| \leq \frac{C}{\varepsilon}e^{-(1-\theta)\frac{|y-x|}{\varepsilon}}, \quad \forall y \in \partial\Omega.$$

Thus,

$$\begin{aligned}\tau'_{\varepsilon,x} &\leq C\int_{\partial\Omega} e^{-(2-2\theta)\frac{|y-x|}{\varepsilon}} \\ &\leq Ce^{-(2-3\theta)\frac{d(x,\partial\Omega)}{\varepsilon}}\int_{\partial\Omega} e^{-\theta\frac{|y-x|}{\varepsilon}} \leq c_1\varepsilon^{N-1}e^{-(2-3\theta)\frac{d(x,\partial\Omega)}{\varepsilon}}.\end{aligned} \tag{2.16}$$

Since $P_{\varepsilon,\Omega}U_{\varepsilon,x} \geq 0$, we have $\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x}}{\partial n} \leq 0$ on $\partial\Omega$. As a result,

$$\begin{aligned}\varepsilon^2 \int_{\partial\Omega} \frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x}}{\partial n}\frac{\partial U_{\varepsilon,x}}{\partial r} &\geq \varepsilon^2 \int_S \frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x}}{\partial n}\frac{\partial U_{\varepsilon,x}}{\partial r}\\ \geq -\, c_0\varepsilon \int_S \frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x}}{\partial n} e^{-\frac{(1+2\theta)d(x,\partial\Omega)}{\varepsilon}},&\end{aligned} \tag{2.17}$$

where $S = \{y :\ y \in \partial\Omega,\ |y-x| \leq d(x,\partial\Omega)(1+2\theta)\}$ and $c_0 > 0$. But

$$\begin{aligned}&\varepsilon\Big|\int_{\partial\Omega\setminus S} \frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x}}{\partial n}\Big| \leq C\int_{|y-x|\geq d(x,\partial\Omega)(1+2\theta)} e^{-(1-\theta)\frac{|y-x|}{\varepsilon}}\\ &\leq Ce^{-(1-\frac{3}{2}\theta)(1+2\theta)\frac{d(x,\partial\Omega)}{\varepsilon}}\int_{\partial\Omega} e^{-\frac{\theta}{2}\frac{|y-x|}{\varepsilon}} \leq C\varepsilon^{N-1}e^{-(1+\frac{1}{4}\theta)\frac{d(x,\partial\Omega)}{\varepsilon}}.\end{aligned} \tag{2.18}$$

Inserting (2.18) into (2.17), we obtain

$$\begin{aligned}\varepsilon^2 \int_{\partial\Omega} \frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x}}{\partial n}\frac{\partial U_{\varepsilon,x}}{\partial r} \geq -c_0\varepsilon \int_{\partial\Omega} \frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x}}{\partial n} e^{-\frac{(1+2\theta)d(x,\partial\Omega)}{\varepsilon}}&\\ + O\big(\varepsilon^{N-1}e^{-(2+\frac{9}{4}\theta)\frac{d(x,\partial\Omega)}{\varepsilon}}\big).&\end{aligned} \tag{2.19}$$

On the other hand,

$$\begin{aligned}&-\varepsilon^2 \int_{\partial\Omega} \frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x}}{\partial n} = \int_\Omega U_{\varepsilon,x}^{p-1} - \int_\Omega P_{\varepsilon,\Omega}U_{\varepsilon,x}\\ &= \int_{R^N} U_{\varepsilon,x}^{p-1} - \int_\Omega P_{\varepsilon,\Omega}U_{\varepsilon,x} + O\big(\varepsilon^N e^{-\frac{(p-1)d(x,\partial\Omega)}{\varepsilon}}\big)\\ &= \int_{R^N} U_{\varepsilon,x} - \int_\Omega P_{\varepsilon,\Omega}U_{\varepsilon,x} + O\big(\varepsilon^N e^{-\frac{(p-1)d(x,\partial\Omega)}{\varepsilon}}\big)\\ &\geq \int_{R^N\setminus\Omega} U_{\varepsilon,x} + O\big(\varepsilon^N e^{-\frac{(p-1)d(x,\partial\Omega)}{\varepsilon}}\big)\\ &\geq \int_{(R^N\setminus\Omega)\cap B_{d(x,\partial\Omega)\theta/8}(q)} U_{\varepsilon,x} + O\big(\varepsilon^N e^{-\frac{(p-1)d(x,\partial\Omega)}{\varepsilon}}\big)\\ &\geq c'(d(x,\partial\Omega))^N e^{-\frac{(1+\theta/8)d(x,\partial\Omega)}{\varepsilon}} \geq c_0\varepsilon^N e^{-\frac{(1+\theta/8)d(x,\partial\Omega)}{\varepsilon}},\end{aligned} \tag{2.20}$$

where $q \in \partial\Omega$, such that $|q-x| = d(x,\partial\Omega)$. In the last inequality, we have use the fact $\frac{d(x,\Omega)}{\varepsilon} \geq M$. Thus it follows from (2.19) and (2.20) that $\tau'_{\varepsilon,x} \geq c_0\varepsilon^{N-1}e^{-\frac{(2+17\theta/8)d(x,\partial\Omega)}{\varepsilon}}$. So the result follows.

Lemma 2.5. *We have*

$$\int_\Omega U_{\varepsilon,x}^{p-2}\frac{\partial U_{\varepsilon,x}}{\partial x_i}\varphi_{\varepsilon,x}=-\varepsilon^2\int_{\partial\Omega}\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x}}{\partial n}\frac{\partial U_{\varepsilon,x}}{\partial x_i}+O\big(\varepsilon^{N-1}e^{-\frac{pd(x,\partial\Omega)}{\varepsilon}}\big).$$

Moreover, if $\partial\Omega\cap\partial B_{d(x,\partial\Omega)}(x)$ *contains exactly one point* q*, then*

$$\sum_{i=1}^N\int_\Omega U_{\varepsilon,x}^{p-2}\frac{\partial U_{\varepsilon,x}}{\partial x_i}\varphi_{\varepsilon,x}\nu_i\ge c_0\varepsilon^{N-1}e^{-\frac{(2+\theta)d(x,\partial\Omega)}{\varepsilon}},$$

for any $\theta>0$*, where* ν *is the outward unit normal of* $\partial\Omega$ *at* q *and* $c_0>0$ *is a constant.*

Proof. Multiplying (2.1) by $\frac{\partial U_{\varepsilon,x}}{\partial x_i}$ and integrating by parts, we get

$$\begin{aligned}\int_\Omega U_{\varepsilon,x}^{p-2}\frac{\partial U_{\varepsilon,x}}{\partial x_i}\varphi_{\varepsilon,x}&=\varepsilon^2\int_{\partial\Omega}\frac{\partial\varphi_{\varepsilon,x}}{\partial n}\frac{\partial U_{\varepsilon,x}}{\partial x_i}-\varepsilon^2\int_{\partial\Omega}\varphi_{\varepsilon,x}\frac{\partial}{\partial n}\Big(\frac{\partial U_{\varepsilon,x}}{\partial x_i}\Big)\qquad(2.21)\\&=-\varepsilon^2\int_{\partial\Omega}\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x}}{\partial n}\frac{\partial U_{\varepsilon,x}}{\partial x_i}+\varepsilon^2\int_{\partial\Omega}\frac{\partial U_{\varepsilon,x}}{\partial n}\frac{\partial U_{\varepsilon,x}}{\partial x_i}-\varepsilon^2\int_{\partial\Omega}U_{\varepsilon,x}\frac{\partial}{\partial n}\Big(\frac{\partial U_{\varepsilon,x}}{\partial x_i}\Big).\end{aligned}$$

But

$$\begin{aligned}&\varepsilon^2\int_{\partial\Omega}\frac{\partial U_{\varepsilon,x}}{\partial n}\frac{\partial U_{\varepsilon,x}}{\partial x_i}-\varepsilon^2\int_{\partial\Omega}U_{\varepsilon,x}\frac{\partial}{\partial n}\Big(\frac{\partial U_{\varepsilon,x}}{\partial x_i}\Big)\qquad(2.22)\\&=\int_\Omega\varepsilon^2\Big[\Delta U_{\varepsilon,x}\frac{\partial U_{\varepsilon,x}}{\partial x_i}-U_{\varepsilon,x}\Delta\Big(\frac{\partial U_{\varepsilon,x}}{\partial x_i}\Big)\Big]\\&=-(p-2)\int_\Omega U_{\varepsilon,x}^{p-1}\frac{\partial U_{\varepsilon,x}}{\partial x_i}=(p-2)\int_{R^N\setminus\Omega}U_{\varepsilon,x}^{p-1}\frac{\partial U_{\varepsilon,x}}{\partial x_i}=O\big(\varepsilon^{N-1}e^{-\frac{pd(x,\partial\Omega)}{\varepsilon}}\big).\end{aligned}$$

Combining (2.21) and (2.22) yields

$$\int_\Omega U_{\varepsilon,x}^{p-2}\frac{\partial U_{\varepsilon,x}}{\partial x_i}\varphi_{\varepsilon,x}=-\varepsilon^2\int_{\partial\Omega}\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x}}{\partial n}\frac{\partial U_{\varepsilon,x}}{\partial x_i}+O\big(\varepsilon^{N-1}e^{-\frac{pd(x,\partial\Omega)}{\varepsilon}}\big).\qquad(2.23)$$

Suppose that $\partial\Omega\cap\partial B_{d(x,\partial\Omega)}(x)$ contains exactly one point q. Then $\nu=\frac{q-x}{|q-x|}$. Consequently,

$$\sum_{i=1}^N\frac{\partial U_{\varepsilon,x}}{\partial x_i}\nu_i=\frac{\partial U_{\varepsilon,x}}{\partial r}\sum_{i=1}^N\frac{x_i-y_i}{|x-y|}\nu_i\ge c_0\varepsilon^{N-1}e^{-\frac{(1+2\theta)d(x,\partial\Omega)}{\varepsilon}},$$

for all $y\in S$. Thus we can follow a similar procedure as in the proof of Lemma 2.4 to finish proving this lemma.

Lemma 2.6. *There is a $\sigma > 0$, such that*

$$\int_\Omega U_{\varepsilon,x_1} U^{p-2}_{\varepsilon,x_2} \varphi_{\varepsilon,x_2} = O\big(\varepsilon^N e^{-(1+\sigma)\frac{|x_1-x_2|}{\varepsilon}} + \varepsilon^N e^{-(2+\sigma)\frac{d(x_2,\partial\Omega)}{\varepsilon}}\big).$$

Proof. Fix $\beta \in [0,1)$ such that $p-3+\beta \geq 0$. Noting that $|\varphi_{\varepsilon,x_2}| \leq U_{\varepsilon,x_2}$ and using Lemma 2.2, we get

$$\begin{aligned} &\int_\Omega U_{\varepsilon,x_1} U^{p-2}_{\varepsilon,x_2} \varphi_{\varepsilon,x_2} \leq C\Big[\int_\Omega U^{1+\sigma}_{\varepsilon,x_1} U^{1+\sigma}_{\varepsilon,x_2} + \int_\Omega \big(U^{p-3}_{\varepsilon,x_2}\varphi_{\varepsilon,x_2}\big)^{\frac{1+\sigma}{\sigma}}\Big] \\ &= \int_\Omega \varphi_{\varepsilon,x_2}^{\frac{(1-\beta)(1+\sigma)}{\sigma}} + O\big(\varepsilon^N e^{-(1+\sigma)\frac{|x_1-x_2|}{\varepsilon}}\big) \qquad (2.24) \\ &= O\big(\varepsilon^N e^{-(1+\sigma)\frac{|x_1-x_2|}{\varepsilon}} + \varepsilon^N e^{-(2+\sigma)\frac{d(x_2,\partial\Omega)}{\varepsilon}}\big), \end{aligned}$$

if $\sigma > 0$ is small enough.

Lemma 2.7. *We have*

$$\int_\Omega U^{p-1}_{\varepsilon,x_1} \varphi_{\varepsilon,x_2} = O\big(\varepsilon^N \sum_{j=1}^2 e^{-(2-\theta)\frac{d(x_j,\partial\Omega)}{\varepsilon}}\big). \qquad (2.25)$$

where $\theta > 0$ is any small constant. Moreover, if $|x_j| \leq M$, then

$$\int_\Omega U^{p-1}_{\varepsilon,x_1} \varphi_{\varepsilon,x_2} = O\big(\varepsilon^N \sum_{j=1}^2 e^{-(2+\sigma)\frac{d(x_j,\partial\Omega)}{\varepsilon}} + \varepsilon^N e^{-(1+\sigma)\frac{|x_{\varepsilon,1}-x_{\varepsilon,2}|}{\varepsilon}}\big). \qquad (2.26)$$

Furthermore, if there are two sequences $x_{\varepsilon,j} \in \Omega$ such that at least one of the sequence $x_{\varepsilon,j}$ moves to infinity, then either

$$\int_\Omega U^{p-1}_{\varepsilon,x_{\varepsilon,1}} \varphi_{\varepsilon,x_{\varepsilon,2}} = O\big(\varepsilon^N \sum_{j=1}^2 e^{-(2+\sigma)\frac{d(x_{\varepsilon,j},\partial\Omega)}{\varepsilon}} + \varepsilon^N e^{-(1+\sigma)\frac{|x_{\varepsilon,1}-x_{\varepsilon,2}|}{\varepsilon}}\big) \text{ or} \qquad (2.27)$$

$$2d(x_{\varepsilon,j},\partial\Omega) = (1+o(1))|x_{\varepsilon,1}-x_{\varepsilon,2}|, \quad j=1,2. \qquad (2.28)$$

Proof. Similar to Lemma 2.1, we have

$$\begin{aligned} &\int_\Omega U^{p-1}_{\varepsilon,x_1}\varphi_{\varepsilon,x_2} = \varepsilon^2 \int_{\partial\Omega} \frac{\partial \varphi_{\varepsilon,x_2}}{\partial n} U_{\varepsilon,x_1} - \varepsilon^2 \int_{\partial\Omega} \frac{\partial U_{\varepsilon,x_1}}{\partial n} U_{\varepsilon,x_2} \qquad (2.29) \\ &= O\Big(\varepsilon \int_\Omega \big(\frac{\varepsilon}{|y-x_1|-\theta\varepsilon}\big)^{\frac{N-1}{2}} e^{-\frac{|y-x_1|-\theta\varepsilon}{\varepsilon}} \big(\frac{\varepsilon}{|y-x_2|-\theta\varepsilon}\big)^{\frac{N-1}{2}} e^{-\frac{|y-x_2|-\theta\varepsilon}{\varepsilon}}\Big). \end{aligned}$$

Thus (2.25) follows.

Suppose that (2.26) is not true. Then there are $x_{\varepsilon,j}\in\Omega$ with $|x_{\varepsilon,j}|\le M$, $j=1,2$, and $\sigma_\varepsilon\to 0$, such that

$$\int_\Omega U^{p-1}_{\varepsilon,x_{\varepsilon,1}}\varphi_{\varepsilon,x_{\varepsilon,2}}\ge \varepsilon^N\sum_{j=1}^2 e^{-(2+\sigma_\varepsilon)\frac{d(x_{\varepsilon,j},\partial\Omega)}{\varepsilon}}+\varepsilon^N e^{-(1+\sigma_\varepsilon)\frac{|x_{\varepsilon,1}-x_{\varepsilon,2}|}{\varepsilon}} \tag{2.30}$$

If $d(x_{\varepsilon,1},\partial\Omega)\ne(1+o(1))d(x_{\varepsilon,2},\partial\Omega)$, then (2.29) implies

$$\begin{aligned}\int_\Omega U^{p-1}_{\varepsilon,x_{\varepsilon,1}}\varphi_{\varepsilon,x_{\varepsilon,2}}&=O\big(\varepsilon^N e^{-(1-\theta)\frac{d(x_{\varepsilon,1},\partial\Omega)}{\varepsilon}}e^{-(1-\theta)\frac{d(x_{\varepsilon,2},\partial\Omega)}{\varepsilon}}\big)\\&=O\big(\varepsilon^N\sum_{j=1}^2 e^{-(2+\sigma)\frac{d(x_{\varepsilon,j},\partial\Omega)}{\varepsilon}}\big).\end{aligned} \tag{2.31}$$

Besides, if $2d(x_{\varepsilon,1},\partial\Omega)\le(1-\delta)|x_{\varepsilon,1}-x_{\varepsilon,2}|$, then

$$\begin{aligned}&\int_\Omega U^{p-1}_{\varepsilon,x_{\varepsilon,1}}\varphi_{\varepsilon,x_{\varepsilon,2}}\le\int_\Omega U^{p-1}_{\varepsilon,x_{\varepsilon,1}}U_{\varepsilon,x_{\varepsilon,2}}\\&=O\big(\varepsilon^N e^{-(1-\theta)\frac{|x_{\varepsilon,1}-x_{\varepsilon,2}|}{\varepsilon}}\big)\le O\big(\varepsilon^N\sum_{j=1}^2 e^{-(2+\sigma)\frac{d(x_{\varepsilon,j},\partial\Omega)}{\varepsilon}}\big).\end{aligned} \tag{2.32}$$

Furthermore, if $2d(x_{\varepsilon,1},\partial\Omega)\ge(1+\delta)|x_{\varepsilon,1}-x_{\varepsilon,2}|$, then from (2.29), we obtain

$$\int_\Omega U^{p-1}_{\varepsilon,x_{\varepsilon,1}}\varphi_{\varepsilon,x_{\varepsilon,2}}=O\big(\varepsilon^N\sum_{j=1}^2 e^{-(2-\theta)\frac{d(x_{\varepsilon,j},\partial\Omega)}{\varepsilon}}\big)=O\big(\varepsilon^N e^{-(1+\sigma)\frac{|x_{\varepsilon,1}-x_{\varepsilon,2}|}{\varepsilon}}\big). \tag{2.33}$$

Combining (2.30)–(2.33), we get

$$2d(x_{\varepsilon,j},\partial\Omega)=(1+o(1))|x_{\varepsilon,1}-x_{\varepsilon,2}|,\quad j=1,2. \tag{2.34}$$

Suppose that $x_{\varepsilon,j}\to x_j$, $j=1,2$. We distinguish two cases.

Suppose that $x_1\in\Omega$. Then

$$2d(x_2,\partial\Omega)=2d(x_1,\partial\Omega)=|x_1-x_2|>0. \tag{2.35}$$

Let $S_j=\{q,q\in\partial\Omega,|q-x_j|=d(x_j,\partial\Omega)\}$. Then $S_1\cap S_2=\emptyset$. Otherwise, let $q\in S_1\cap S_2$. We see that both $\frac{x_1-q}{|x_1-q|}$ and $\frac{x_2-q}{|x_2-q|}$ are the inward unit normal of $\partial\Omega$ at q. In view of the fact $|x_1 - q| = |x_2 - q|$, we get $x_1 = x_2$. This is a contradiction to (2.35).

So we see $|x_1 - y| > (1+\mu)d(x_2, \partial\Omega), y \in S_2$ for some $\mu > 0$. But

$$\begin{aligned}&\varepsilon^2\Big|\int_{\partial\Omega\setminus(S_{1,\sigma}\cup S_{2,\sigma})}\Big[\frac{\partial\varphi_{\varepsilon,x_{\varepsilon,2}}}{\partial n}U_{\varepsilon,x_{\varepsilon,1}} - \frac{\partial U_{\varepsilon,x_{\varepsilon,1}}}{\partial n}U_{\varepsilon,x_{\varepsilon,2}}\Big]\Big| \\ &\le \varepsilon\int_{\partial\Omega\setminus(S_{1,\sigma}\cup S_{2,\sigma})}\sum_{j=1}^{2}e^{-(2-\theta)\frac{|y-x_{\varepsilon,j}|}{\varepsilon}} \le \varepsilon^N\sum_{j=1}^{2}e^{-(2+\sigma)\frac{d(x_{\varepsilon,j},\partial\Omega)}{\varepsilon}},\end{aligned} \tag{2.36}$$

where $S_{j,\sigma} = \cup_{q\in S_j}\partial\Omega \cap B_{\varepsilon\sigma}(q)$. Thus,

$$\begin{aligned}&\varepsilon^2\Big|\int_{\partial\Omega}\Big[\frac{\partial\varphi_{\varepsilon,x_{\varepsilon,2}}}{\partial n}U_{\varepsilon,x_{\varepsilon,1}} - \frac{\partial U_{\varepsilon,x_{\varepsilon,1}}}{\partial n}U_{\varepsilon,x_{\varepsilon,2}}\Big]\Big| \\ &= \varepsilon^2\Big|\sum_{j=1}^{2}\int_{S_{j,\sigma}}\Big[\frac{\partial\varphi_{\varepsilon,x_{\varepsilon,2}}}{\partial n}U_{\varepsilon,x_{\varepsilon,1}} - \frac{\partial U_{\varepsilon,x_{\varepsilon,1}}}{\partial n}U_{\varepsilon,x_{\varepsilon,2}}\Big]\Big| + O\Big(\varepsilon^N\sum_{j=1}^{2}e^{-(2+\sigma)\frac{d(x_{\varepsilon,j},\partial\Omega)}{\varepsilon}}\Big) \\ &\le C\varepsilon\sum_{j=1}^{2}\int_{S_{j,\sigma}}e^{-(1-\theta)\frac{|y-x_{\varepsilon,j}|}{\varepsilon}}e^{-(1+\sigma)\frac{d(x_{\varepsilon,j},\partial\Omega)}{\varepsilon}} + O\Big(\varepsilon^N\sum_{j=1}^{2}e^{-(2+\sigma)\frac{d(x_{\varepsilon,j},\partial\Omega)}{\varepsilon}}\Big) \\ &= O\Big(\varepsilon^N\sum_{j=1}^{2}e^{-(2+\sigma)\frac{d(x_{\varepsilon,j},\partial\Omega)}{\varepsilon}}\Big).\end{aligned} \tag{2.37}$$

This is a contradiction to (2.30).

Suppose that $x_1 \in \partial\Omega$. Then by (2.34), $x_2 = x_1$ and

$$\partial\Omega \cap \partial B_{d(x_{\varepsilon,j},\partial\Omega)}(x_{\varepsilon,j}) = \{q_{\varepsilon,j}\}, \quad j = 1, 2.$$

In this case, we have $|x_{\varepsilon,1} - q_{\varepsilon,2}| \ge (1+o(1))|x_{\varepsilon,1} - x_{\varepsilon,2}| > \frac{3}{2}d(x_{\varepsilon,1}, \partial\Omega)$. So similar to (2.36) and (2.37), we can also get a contradiction.

To prove that either (2.27) or (2.28) holds, we just need to repeat the proof of (2.31)–(2.33) and thus we complete the proof of this lemma.

Finally, we give a proposition which will play an important role in the proof of Theorem 1.1.

Proposition 2.8. *We can choose $\rho = \rho(\varepsilon) > 0$ large enough, such that for any $x_1 \in \partial B_\rho(0)$ and $x_2 \in \partial B_{2\rho}(x_1)$, we have*

$$I(P_{\varepsilon,\Omega}U_{\varepsilon,x_1} + P_{\varepsilon,\Omega}U_{\varepsilon,x_2}) \le \varepsilon^N\big(2A - e^{-\frac{d(\varepsilon)}{\varepsilon}}\big),$$

where $d(\varepsilon)$ is a positive number.

Proof. First, we claim

$$I(P_{\varepsilon,\Omega}U_{\varepsilon,x_1}+P_{\varepsilon,\Omega}U_{\varepsilon,x_2})=\varepsilon^N 2A-\int_\Omega (P_{\varepsilon,\Omega}U_{\varepsilon,x_1})^{p-1}P_{\varepsilon,\Omega}U_{\varepsilon,x_2} \qquad (2.38)$$
$$+\frac{1}{2}(\tau_{\varepsilon,x_1}+\tau_{\varepsilon,x_2})+O\big(\varepsilon^N\sum_{j=1}^2 e^{-(2+\sigma)\frac{d(x_j,\partial\Omega)}{\varepsilon}}+\varepsilon^N e^{-(1+\sigma)\frac{|x_1-x_2|}{\varepsilon}}\big).$$

In fact, by Lemma 2.6, we have

$$\begin{aligned}&I(P_{\varepsilon,\Omega}U_{\varepsilon,x_1}+P_{\varepsilon,\Omega}U_{\varepsilon,x_2})\\&=\sum_{j=1}^2 I(P_{\varepsilon,\Omega}U_{\varepsilon,x_j})+\int_\Omega U_{\varepsilon,x_2}^{p-1}P_{\varepsilon,\Omega}U_{\varepsilon,x_1}-\int_\Omega(P_{\varepsilon,\Omega}U_{\varepsilon,x_2})^{p-1}P_{\varepsilon,\Omega}U_{\varepsilon,x_1}\\&-\int_\Omega(P_{\varepsilon,\Omega}U_{\varepsilon,x_1})^{p-1}P_{\varepsilon,\Omega}U_{\varepsilon,x_2}+O\big(\varepsilon^N U^{1+\sigma}(\frac{|x_1-x_2|}{\varepsilon})\big)\\&=\varepsilon^N 2A+\frac{1}{2}(\tau_{\varepsilon,x_1}+\tau_{\varepsilon,x_2})-\int_\Omega(P_{\varepsilon,\Omega}U_{\varepsilon,x_1})^{p-1}P_{\varepsilon,\Omega}U_{\varepsilon,x_2}\\&+O\big(\varepsilon^N\sum_{j=1}^2 e^{-(2+\sigma)\frac{d(x_j,\partial\Omega)}{\varepsilon}}+\varepsilon^N e^{-(1+\sigma)\frac{|x_1-x_2|}{\varepsilon}}\big).\end{aligned} \qquad (2.39)$$

On the other hand, it follows from Lemma 2.6 and (2.29) that

$$\begin{aligned}&\int_\Omega(P_{\varepsilon,\Omega}U_{\varepsilon,x_1})^{p-1}P_{\varepsilon,\Omega}U_{\varepsilon,x_2}=\int_\Omega U_{\varepsilon,x_1}^{p-1}U_{\varepsilon,x_2}-\int_\Omega U_{\varepsilon,x_1}^{p-1}\varphi_{\varepsilon,x_2}\\&\quad+O\big(\varepsilon^N\sum_{j=1}^2 e^{-(2+\sigma)\frac{d(x_j,\partial\Omega)}{\varepsilon}}+\varepsilon^N e^{-(1+\sigma)\frac{|x_1-x_2|}{\varepsilon}}\big)\\&=\int_\Omega U_{\varepsilon,x_1}^{p-1}U_{\varepsilon,x_2}+O\big(\varepsilon^N\sum_{j=1}^2\big(\frac{d(x_j,\partial\Omega)-\varepsilon\theta}{\varepsilon}\big)^{1-N}e^{-2\frac{d(x_j,\partial\Omega)-\varepsilon\theta}{\varepsilon}}\\&\quad+\varepsilon^N e^{-(1+\sigma)\frac{|x_1-x_2|}{\varepsilon}}\big)\\&=\int_{R^N}U_{\varepsilon,x_1}^{p-1}U_{\varepsilon,x_2}+O\big(\sum_{j=1}^2\big(\frac{d(x_j,\partial\Omega)-\varepsilon\theta}{\varepsilon}\big)^{1-N}e^{-2\frac{d(x_j,\partial\Omega)-\varepsilon\theta}{\varepsilon}}\\&+\varepsilon^N e^{-(1+\sigma)\frac{|x_1-x_2|}{\varepsilon}}\big)=(c+o(1))\varepsilon^N\Big(\frac{|x_1-x_2|}{\varepsilon}\Big)^{(1-N)/2}e^{-\frac{|x_1-x_2|}{\varepsilon}}\\&+O\big(\sum_{j=1}^2\big(\frac{d(x_j,\partial\Omega)-\varepsilon\theta}{\varepsilon}\big)^{1-N}e^{-2\frac{d(x_j,\partial\Omega)-\varepsilon\theta}{\varepsilon}}+\varepsilon^N e^{-(1+\sigma)\frac{|x_1-x_2|}{\varepsilon}}\big)\end{aligned} \qquad (2.40)$$

where $c > 0$. For any $x_1 \in \partial B_\rho(0)$ and $x_2 \in \partial B_{2\rho}(x_1)$, we have $d(x_j, \partial\Omega) \geq \rho - T$, $j = 1, 2$ for some $T > 0$. Thus, from Lemma 2.1, we see that for fixed $\varepsilon > 0$, if ρ is large enough,

$$\begin{aligned} &I(P_{\varepsilon,\Omega}U_{\varepsilon,x_1} + P_{\varepsilon,\Omega}U_{\varepsilon,x_2}) \leq \varepsilon^N \big[2A - c'\big(\frac{2\rho}{\varepsilon}\big)^{(1-N)/2} e^{-\frac{2\rho}{\varepsilon}}\big] + O(\tau_{\varepsilon,x_1} + \tau_{\varepsilon,x_2}) \\ &= \varepsilon^N \Big[2A - c'\big(\frac{2\rho}{\varepsilon}\big)^{(1-N)/2} e^{-\frac{2\rho}{\varepsilon}} + O\big[\big(\frac{\rho - T}{\varepsilon}\big)^{(1-N)} e^{-2\frac{\rho-T}{\varepsilon}}\big]\Big] \\ &\leq \varepsilon^N \big(2A - e^{-\frac{d(\varepsilon)}{\varepsilon}}\big), \end{aligned} \tag{2.41}$$

if we choose $\rho = \rho(\varepsilon)$ large enough and let $d(\varepsilon) = 3\rho$. Thus the proposition follows.

3. Existence results. First we define

$$D_\varepsilon = \{x = (x_1, x_2) : x_j \in \Omega,\ d(x_j, \partial\Omega) \geq \gamma_\varepsilon,\ j = 1, 2,\ \frac{|x_1 - x_2|}{\varepsilon} \geq R\},$$
$$M_\varepsilon = \{(\alpha, x, v) : |\alpha_j - 1| \leq \delta,\ j = 1, 2,\ x \in D_\varepsilon,\ v \in E_{\varepsilon,x,2},\ \|v\|_\varepsilon \leq \delta\varepsilon^{-N/2}\},$$

where $\delta > 0$ is a fixed small constant and $R > 0$ is a fixed large constant, and $\gamma_\varepsilon > 0$ will be suitably chosen later. We also define

$$J(\alpha, x, v) = I(\sum_{j=1}^{2} \alpha_j P_{\varepsilon,\Omega}U_{\varepsilon,x_j} + v), \quad \forall(\alpha, x, v) \in M_\varepsilon. \tag{3.1}$$

It is well known now (see [1, 14]) that if $\delta > 0$ is small enough and $R > 0$ is large enough,

$$u = \sum_{j=1}^{2} \alpha_j P_{\varepsilon,\Omega}U_{\varepsilon,x_j} + v,$$

is a positive critical point of $I(u)$ if and only if (α, x, v) is a critical point of $J(\alpha, x, v)$ in M_ε. So we need to solve the following system:

$$\frac{\partial J}{\partial \alpha_j} = 0, \quad j = 1, 2; \tag{3.2}$$

$$\frac{\partial J}{\partial x_{jl}} = \sum_{h=1}^{N} G_{hj}\big\langle \frac{\partial^2 P_{\varepsilon,\Omega}U_{\varepsilon,x_j}}{\partial x_{jh}\partial x_{jl}}, v\big\rangle, \quad j = 1, 2,\ l = 1, \ldots, N, \tag{3.3}$$

$$\frac{\partial J}{\partial v} = \sum_{j=1}^{2} B_j P_{\varepsilon,\Omega}U_{\varepsilon,x_j} + \sum_{j=1}^{2}\sum_{l=1}^{N} G_{jl}\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_j}}{\partial x_{jl}}, \tag{3.4}$$

for some constants $B_j, G_{jl} \in R,\ j=1,2,\ l=1,\dots,N$.

As usual, we solve (3.2) and (3.4) first. The procedure to achieve this is quite standard now. See for example [2] and also [14].

Proposition 3.1. *There is an $\varepsilon_0>0$, such that for each $\varepsilon \in (0,\varepsilon_0]$, there exists a C^1-map $(\alpha_\varepsilon(x), v_\varepsilon(x)) : D_\varepsilon \to R^2_+ \times H^1_0(\Omega)$ such that $v_\varepsilon(x) \in E_{\varepsilon,x,2}$, (3.2) and (3.4) hold for some constants B_j and G_{jl}. Moreover, we have*

$$|\alpha-1|\varepsilon^{N/2} + \|v\|_\varepsilon = O\big(\sum_{j=1}^2 \tau_{\varepsilon,x_j}^{1-\frac{1}{p}} + \varepsilon^{N/2} U^{\frac{1+\sigma}{2}}\big(\frac{|x_1-x_2|}{\varepsilon}\big)\big), \tag{3.5}$$

$$\varepsilon B_j,\ G_{jl} = O\big(\varepsilon e^{-(1-\theta)\frac{|x_1-x_2|}{\varepsilon}} + \varepsilon \sum_{j=1}^2 e^{-(2-\theta)\frac{d(x_j,\partial\Omega)}{\varepsilon}}\big). \tag{3.6}$$

Proof. The proof of the existence part of this proposition is similar to that of Proposition 4 in [14]. As for the estimates in (3.5) , we only need to use Lemma 2.2. To get (3.6), we only need to use the following estimates

$$\begin{aligned}\frac{\partial J}{\partial x_{1l}} &= \int_\Omega U^{p-1}_{\varepsilon,x_1} \frac{\partial U_{\varepsilon,1}}{\partial x_{1l}} \varphi_{\varepsilon,x_1} - (p-1)\int_\Omega U^{p-2}_{\varepsilon,x_1} U_{\varepsilon,x_2} \frac{\partial P_{\varepsilon,\Omega} U_{\varepsilon,1}}{\partial x_{1l}} \\ &\quad + O\big(\varepsilon^{N-1} e^{-(1+\sigma)\frac{|x_1-x_2|}{\varepsilon}} + \varepsilon^{N-1}\sum_{j=1}^2 e^{-(2+\sigma)\frac{d(x_j,\partial\Omega)}{\varepsilon}}\big) \qquad (3.7) \\ &= O\big(\varepsilon^{N-1} e^{-(1-\theta)\frac{|x_1-x_2|}{\varepsilon}} + \varepsilon^{N-1}\sum_{j=1}^2 e^{-(2-\theta)\frac{d(x_j,\partial\Omega)}{\varepsilon}}\big),\end{aligned}$$

$$\big\langle \frac{\partial J}{\partial v}, P_{\varepsilon,\Omega} U_{\varepsilon,1}\big\rangle = O\big(\varepsilon^N e^{-(1-\theta)\frac{|x_1-x_2|}{\varepsilon}} + \varepsilon^N \sum_{j=1}^2 e^{-(2-\theta)\frac{d(x_j,\partial\Omega)}{\varepsilon}}\big), \tag{3.8}$$

and solve a system as in [14], pp 22–23. We thus omit the details. We now define a new function: $K(x) = J(\alpha_\varepsilon(x), x, v_\varepsilon(x)),\ x \in D_\varepsilon$. In order to prove Theorem 1.1, we only need to prove that $K(x)$ has at least one critical point in D_ε. For each fixed $\varepsilon>0$, we choose $\rho(\varepsilon)$ large enough such that the estimates in Proposition 2.8 holds. Define $c_{\varepsilon,1} = \varepsilon^N(2A - \kappa U\big(\frac{d_1}{\varepsilon}\big))$, $c_{\varepsilon,2} = \varepsilon^N(2A - \frac{1}{2}e^{-\frac{d(\varepsilon)}{\varepsilon}})$, where $d_1>0$ and $\kappa>0$ are two fixed small constants and $d(\varepsilon)$ is the positive constant in Proposition 2.8. Define $K^c = \{x : x\in D_\varepsilon, K(x)\le c\}$. We are going to prove that for $\varepsilon>0$ small, $K(x)$ has a critical point in $K^{c_{\varepsilon,2}} \setminus K^{c_{\varepsilon,1}}$. First, we prove the following lemma:

Lemma 3.2. *For each $d_1 > 0$ small, we can find a γ_ε with $|\frac{1}{2}d_1 - \gamma_\varepsilon| \to 0$ as $\varepsilon \to 0$, such that $a_1 \le \frac{\tau_{\varepsilon,x}}{\varepsilon^N U(\frac{d_1}{\varepsilon})} \le a_2$, $\forall\, x \in \partial\Omega_{\gamma_\varepsilon}$, where $0 < a_1 < a_2 < 1$ are two fixed constants.*

Proof. We may assume that $x = d(x, \partial\Omega)(0, \dots, 0, -1)$. As in Lemma 2.1, we have

$$c_0\varepsilon \int_{\partial\Omega} \Big(\frac{\varepsilon}{|y-x|}\Big)^{N-1} e^{-\frac{2|y-x|}{\varepsilon}} \le \tau_{\varepsilon,x} \le c_1\varepsilon \int_{\partial\Omega} \Big(\frac{\varepsilon}{|y-x|}\Big)^{N-1} e^{-\frac{2|y-x|}{\varepsilon}}.$$

Besides,

$$C_1\varepsilon^N \int_{\partial\Omega} e^{-\frac{2|y-x|}{\varepsilon}} \le \varepsilon \int_{\partial\Omega} \Big(\frac{\varepsilon}{|y-x|}\Big)^{N-1} e^{-\frac{2|y-x|}{\varepsilon}} \le C_2\varepsilon^N \int_{\partial\Omega} e^{-\frac{2|y-x|}{\varepsilon}},$$

$$\int_{\partial\Omega} e^{-\frac{2|y-x|}{\varepsilon}} = \int_{\partial\Omega\cap B_\mu(0)} e^{-\frac{2|y-x|}{\varepsilon}} + O\big(e^{-\frac{2d(x,\partial\Omega)+\delta}{\varepsilon}}\big).$$

Thus, we obtain

$$\begin{aligned} &C_1\varepsilon^N\Big(\int_{\partial\Omega\cap B_\mu(0)} e^{-\frac{2|y-x|}{\varepsilon}} + O\big(e^{-\frac{2d(x,\partial\Omega)+\delta}{\varepsilon}}\big)\Big) \\ \le\, &\tau_{\varepsilon,x} \le C_2\varepsilon^N\Big(\int_{\partial\Omega\cap B_\mu(0)} e^{-\frac{2|y-x|}{\varepsilon}} + O\big(e^{-\frac{2d(x,\partial\Omega)+\delta}{\varepsilon}}\big)\Big). \end{aligned} \tag{3.9}$$

But

$$|y-x| = y_N + d(x,\partial\Omega) + \frac{1}{2}\frac{|y'|^2}{|y_N + d(x,\partial\Omega)|} + o\Big(\frac{|y'|^2}{|y_N + d(x,\partial\Omega)|}\Big),$$

and $y_N = O(|y'|^2)$, $\forall\, y \in \partial\Omega \cap B_\mu(0)$, where $y' = (y_1, \dots, y_{N-1})$. Hence we see that if $d(x, \partial\Omega)$ is small enough, then $\frac{1}{2}\frac{|y'|^2}{|y_N + d(x,\partial\Omega)|} = \frac{1}{2}(1 + o(1))\frac{|y'|^2}{d(x,\partial\Omega)}$ dominates the term $|y_N|$. But

$$\int_{\partial\Omega\cap B_\mu(0)} e^{-\frac{|y'|^2}{d(x,\partial\Omega)\varepsilon}} = O\big(\varepsilon^{(N-1)/2}\big).$$

As a result, we have

$$C_1\varepsilon^{\frac{N-1}{2}} e^{-\frac{2d(x,\partial\Omega)}{\varepsilon}} \le \int_{\partial\Omega\cap B_\mu(0)} e^{-\frac{2|y-x|}{\varepsilon}} \le C_2\varepsilon^{\frac{N-1}{2}} e^{-\frac{2d(x,\partial\Omega)}{\varepsilon}}. \tag{3.10}$$

Combining (3.9) and (3.10), we are led to

$$C_1\varepsilon^{\frac{3N-1}{2}}e^{-\frac{2d(x,\partial\Omega)}{\varepsilon}} \le \tau_{\varepsilon,x} \le C_2\varepsilon^{\frac{3N-1}{2}}e^{-\frac{2d(x,\partial\Omega)}{\varepsilon}}.$$

Let $\gamma_\varepsilon = \frac{1}{2}d_1 + t_\varepsilon$. Then for $x \in \partial\Omega_{\gamma_\varepsilon}$, we have

$$\frac{\varepsilon^{\frac{3N-1}{2}}e^{-\frac{2d(x,\partial\Omega)}{\varepsilon}}}{\varepsilon^N U(\frac{d_1}{\varepsilon})} = (c' + o(1))e^{-\frac{2t_\varepsilon}{\varepsilon}} = a,$$

if we let $t_\varepsilon = -\frac{1}{2}\varepsilon \ln \frac{a}{c'+o(1)}$. Thus we complete the proof of this lemma.

In order to apply the variational method to get a critical point for $K(x)$, we need the following lemma.

Lemma 3.3. *The following flow:*

$$\frac{dx(t)}{dt} = -DK(x(t)), \quad x(0) = x_0 \in K^{c_{\varepsilon,2}} \tag{3.11}$$

does not leave D_ε before it reaches $K^{c_{\varepsilon,1}}$.

Proof. We divide the proof of this lemma into two steps.

Step 1. Suppose that $\frac{|x_1-x_2|}{\varepsilon} = R$. We claim $x \in K^{c_{\varepsilon,1}}$. In fact, by using Proposition 3.1 and (2.39), we obtain

$$\begin{aligned} K(x) &= I(P_{\varepsilon,\Omega}U_{\varepsilon,x_1} + P_{\varepsilon,\Omega}U_{\varepsilon,x_2}) + O\big(\varepsilon^N|\alpha_\varepsilon - 1|^2 + \|v\|_\varepsilon^2\big) \\ &= \varepsilon^N 2A + \frac{1}{2}(\tau_{\varepsilon,x_1} + \tau_{\varepsilon,x_2}) - \int_\Omega (P_{\varepsilon,\Omega}U_{\varepsilon,x_1})^{p-1}P_{\varepsilon,\Omega}U_{\varepsilon,x_2} \\ &\quad + O\big(\sum_{j=1}^2 \tau_{\varepsilon,x_j}^{1+\sigma} + \varepsilon^N U^{1+\sigma}(\frac{|x_1-x_2|}{\varepsilon})\big) \end{aligned} \tag{3.12}$$

From (3.12) and Lemma 2.7, we have

$$\begin{aligned} K(x) &= \varepsilon^N 2A - \int_\Omega U_{\varepsilon,x_1}^{p-1}U_{\varepsilon,x_2} + O\big(\sum_{j=1}^2 e^{-(2-\theta)\frac{d(x_j,\partial\Omega)}{\varepsilon}} + \varepsilon^N U^{1+\sigma}(\frac{|x_1-x_2|}{\varepsilon})\big) \\ &\le \varepsilon^N(2A - cU(R)) + O\big(\sum_{j=1}^2 e^{-(2-\theta)\frac{d(x_j,\partial\Omega)}{\varepsilon}} + \varepsilon^N U^{1+\sigma}(\frac{|x_1-x_2|}{\varepsilon})\big) < c_{\varepsilon,1}, \end{aligned} \tag{3.13}$$

we know $x \in K^{c_{\varepsilon,1}}$.

Step 2. Suppose that $x_1 \in \partial\Omega_{\gamma_\varepsilon}$. We claim that either $K(x) < c_{\varepsilon,1}$, or $\frac{\partial K}{\partial n} > 0$, where n is the outward unit normal of $\partial\Omega_{\gamma_\varepsilon}$ at x_1. We have

$$
\begin{aligned}
&\frac{\partial K}{\partial x_{1l}} = \frac{\partial J}{\partial x_{1l}} + \Big\langle \frac{\partial J}{\partial v}, \frac{\partial v}{\partial x_{1l}} \Big\rangle \qquad (3.14)\\
&= \frac{\partial J}{\partial x_{1l}} + \sum_{h=1}^{N} G_{h1} \Big\langle \frac{\partial P_{\varepsilon,\Omega} U_{\varepsilon,x_1}}{\partial x_{1h}}, \frac{\partial v}{\partial x_{1l}} \Big\rangle_\varepsilon = \frac{\partial J}{\partial x_{1l}} - \sum_{h=1}^{N} G_{h1} \Big\langle \frac{\partial^2 P_{\varepsilon,\Omega} U_{\varepsilon,x_1}}{\partial x_{1h}\partial x_{1l}}, v \Big\rangle_\varepsilon \\
&= \frac{\partial J}{\partial x_{1l}} + O\big(\varepsilon^{N-1} e^{-(1+\sigma)\frac{|x_1-x_2|}{\varepsilon}} + \varepsilon^{N-1} \sum_{j=1}^{2} e^{-(2+\sigma)\frac{d(x_j,\partial\Omega)}{\varepsilon}}\big) \\
&= \int_\Omega \big(U_{\varepsilon,x_1}^{p-1} - (P_{\varepsilon,\Omega} U_{\varepsilon,x_1})^{p-1}\big) \frac{\partial U_{\varepsilon,x_1}}{\partial x_{1l}} - (p-1) \int_\Omega U_{\varepsilon,x_1}^{p-2} U_{\varepsilon,x_2} \frac{\partial U_{\varepsilon,x_1}}{\partial x_{1l}} \\
&+ O\big(\varepsilon^{N-1} e^{-(1+\sigma)\frac{|x_1-x_2|}{\varepsilon}} + \varepsilon^{N-1} \sum_{j=1}^{2} e^{-(2+\sigma)\frac{d(x_j,\partial\Omega)}{\varepsilon}}\big) \\
&= (p-1) \int_\Omega U_{\varepsilon,x_1}^{p-2} \frac{\partial U_{\varepsilon,x_1}}{\partial x_{1l}} \varphi_{\varepsilon,x_1} - (p-1) \int_\Omega U_{\varepsilon,x_1}^{p-2} U_{\varepsilon,x_2} \frac{\partial U_{\varepsilon,x_1}}{\partial x_{1l}} \\
&+ O\big(\varepsilon^{N-1} e^{-(1+\sigma)\frac{|x_1-x_2|}{\varepsilon}} + \varepsilon^{N-1} \sum_{j=1}^{2} e^{-(2+\sigma)\frac{d(x_j,\partial\Omega)}{\varepsilon}}\big).
\end{aligned}
$$

We distinguish two cases.

(i) Suppose that $\frac{\tau_{\varepsilon,x_1}}{\varepsilon^N U(\frac{|x_1-x_2|}{\varepsilon})} \le c'$, where $c' > 0$ is a small constant. In this case, we claim that $K(x) < c_{\varepsilon,1}$. In fact, we have

$$
\begin{aligned}
K(x) &= 2\varepsilon^N A + \frac{1}{2}(\tau_{\varepsilon,1} + \tau_{\varepsilon,2}) \\
&\quad - \int_\Omega U_{\varepsilon,x_1}^{p-1} U_{\varepsilon,x_2} + O\big(\sum_{j=1}^{2} \tau_{\varepsilon,x_j}^{1+\sigma} + \varepsilon^N U^{1+\sigma}(\frac{|x_1-x_2|}{\varepsilon})\big) \\
&\quad \le 2\varepsilon^N A - (c - c' - o(1)) \varepsilon^N U(\frac{|x_1-x_2|}{\varepsilon})\big). \qquad (3.15)
\end{aligned}
$$

On the other hand, according to Lemma 3.2, we have

$$
a_1 \frac{U(\frac{d_1}{\varepsilon})}{U(\frac{|x_1-x_2|}{\varepsilon})} \le \frac{\tau_{\varepsilon,x_1}}{\varepsilon^N U(\frac{|x_1-x_2|}{\varepsilon})} \le c',
$$

so we see $U(\frac{|x_1-x_2|}{\varepsilon}) \geq c''U(\frac{d_1}{\varepsilon})$. As a result,

$$K(x) \leq 2\varepsilon^N A - (c - c' - o(1))c''\varepsilon^N U(\frac{d_1}{\varepsilon}) < c_{\varepsilon,1},$$

if κ is small enough.

(ii) Suppose that $\frac{\tau_{\varepsilon,x_1}}{\varepsilon^N U(\frac{|x_1-x_2|}{\varepsilon})} \geq c'$. In this case, we prove that $\frac{\partial K(x)}{\partial n} > 0$, where n is the outward unit normal of $\partial\Omega_{\gamma_\varepsilon}$ at x_1. Since

$$\int_\Omega U^{p-2}_{\varepsilon,x_1} U_{\varepsilon,x_2} \frac{\partial U_{\varepsilon,x_1}}{\partial x_{1l}} = (c + o(1))\varepsilon^{N-1} U(\frac{|x_2 - x_1|}{\varepsilon}) \frac{x_{2l} - x_{1l}}{|x_2 - x_1|},$$

and $\big\langle \frac{x_2-x_1}{|x_2-x_1|}, n\big\rangle \leq \beta$, $\forall\, x_2 \in \Omega_{\gamma_\varepsilon} \cap B_\delta(x_1)$, we see that, $\forall\, x_2 \in \Omega_{\gamma_\varepsilon} \cap B_\delta(x_1)$,

$$\int_\Omega U^{p-2}_{\varepsilon,x_1} U_{\varepsilon,x_2} \frac{\partial U_{\varepsilon,x_1}}{\partial n} \leq \beta(c + o(1))\varepsilon^{N-1} U\big(\frac{|x_1 - x_2|}{\varepsilon}\big), \tag{3.16}$$

On the other hand, if $|x_2 - x_1| > \delta$, then

$$\int_\Omega U^{p-2}_{\varepsilon,x_1} U_{\varepsilon,x_2} \frac{\partial U_{\varepsilon,x_1}}{\partial n} = O\big(\varepsilon^{N-1} e^{-\frac{\delta}{\varepsilon}}\big) = O\big(\varepsilon^{N-1} e^{-\frac{2\gamma_\varepsilon+\sigma}{\varepsilon}}\big), \tag{3.17}$$

if d_1 is small. By Lemma 2.5, we get

$$\begin{aligned} &\frac{\varepsilon \int_\Omega U^{p-1}_{\varepsilon,x_1} \frac{\partial U_{\varepsilon,x_1}}{\partial n} \varphi_{\varepsilon,x_1}}{\tau_{\varepsilon,x_1}} \\ &= \frac{\varepsilon^2 \int_{\partial\Omega} \frac{\partial P_{\varepsilon,\Omega} U_{\varepsilon,x_1}}{\partial \nu} \frac{\partial U_{\varepsilon,x_1}}{\partial r} \big\langle \frac{y-x_1}{|y-x_1|}, n\big\rangle + O\big(\varepsilon^N e^{-\frac{pd(x_1,\partial\Omega)}{\varepsilon}}\big)}{-\varepsilon^2 \int_{\partial\Omega} \frac{\partial P_{\varepsilon,\Omega} U_{\varepsilon,x_1}}{\partial \nu} U_{\varepsilon,x_1}} \\ &\to \big\langle \frac{q - x_1}{|q - x_1|}, n\big\rangle = 1, \end{aligned} \tag{3.18}$$

where v is the outward unit normal of $\partial\Omega$ at y, $q \in \partial\Omega$ such that $|q - x_1| = d(x_1, \partial\Omega)$. Combining (3.14), (3.16)–(3.18), we obtain

$$\begin{aligned} \frac{\partial K(x)}{\partial n} &\geq (p-1)\big((1 + o(1))\varepsilon^{-1}\tau_{\varepsilon,x_1} - \beta(c + o(1))\varepsilon^{N-1} U(\frac{|x_1 - x_2|}{\varepsilon})\big) \\ &\quad + O\big(\varepsilon^{N-1} e^{-\frac{(2+\sigma)d(x_1,\partial\Omega)}{\varepsilon}} + \varepsilon^{N-1} U^{1+\sigma}(\frac{|x_1 - x_2|}{\varepsilon})\big) \\ &\geq (p-1)\varepsilon^{N-1} U(\frac{|x_1 - x_2|}{\varepsilon})\big((1 + o(1))c' - \beta(c + o(1))\big) > 0, \end{aligned} \tag{3.19}$$

if β is small.

Combining Steps 1 and 2, we complete the proof of this lemma.

Lemma 3.4. *For each fixed $\varepsilon > 0$ small, the function $K(x)$ satisfies the (PS) condition on $K^{c_{\varepsilon,2}} \setminus K^{c_{\varepsilon,1}}$.*

Proof. Suppose that there are $x_m \in K^{c_{\varepsilon,2}} \setminus K^{c_{\varepsilon,1}}$, such that

$$DK(x_m) \to 0, \quad \text{as } m \to +\infty. \tag{3.20}$$

We only need to prove that x_m is bounded in R^{2N}.

First, we prove that $x \in K^{c_{\varepsilon,2}}$ implies $|x_1 - x_2| \le 2d(\varepsilon)$. In fact, by using Proposition 3.1 and (2.39), we obtain

$$\begin{aligned} K(x) &= I(P_{\varepsilon,\Omega}U_{\varepsilon,x_1} + P_{\varepsilon,\Omega}U_{\varepsilon,x_2}) + O\big(\varepsilon^N|\alpha_\varepsilon - 1|^2 + \|v\|_\varepsilon^2\big) \\ &= \varepsilon^N 2A + \frac{1}{2}(\tau_{\varepsilon,x_1} + \tau_{\varepsilon,2}) - \int_\Omega (P_{\varepsilon,\Omega}U_{\varepsilon,x_1})^{p-1}P_{\varepsilon,\Omega}U_{\varepsilon,x_2} \\ &+ O\big(\sum_{j=1}^{2}\tau_{\varepsilon,x_j}^{1+\sigma} + \varepsilon^N U^{1+\sigma}(\frac{|x_1 - x_2|}{\varepsilon})\big) \ge \varepsilon^N 2A - c\int_{R^N} U_{\varepsilon,x_1}^{p-1}U_{\varepsilon,x_2}. \end{aligned} \tag{3.21}$$

Hence $x \in K^{c_{\varepsilon,2}}$ implies

$$-\int_{R^N} U_{\varepsilon,x_1}^{p-1}U_{\varepsilon,x_2} \le -C\varepsilon^N e^{-\frac{d(\varepsilon)}{\varepsilon}},$$

and the result follows.

Next, we prove that if $|x_j| \ge L =: 5d(\varepsilon)$ for some j, then $|DK(x)| \ge c_\varepsilon > 0$. In fact, we assume $j = 1$. Since $d(x_1, \partial\Omega) \ge L - T$ for some fixed $T > 0$, and $d(x_2, \partial\Omega) \ge d(x_1, \partial\Omega) - |x_1 - x_2| \ge L - T - 2d(\varepsilon) \ge \frac{5}{2}d(\varepsilon) > |x_1 - x_2|$, by (3.14), we have

$$\begin{aligned} \frac{\partial K}{\partial x_{1l}} &= (p-1)\int_\Omega U_{\varepsilon,x_1}^{p-1}\frac{\partial U_{\varepsilon,x_1}}{\partial x_{1l}}\varphi_{\varepsilon,x_1} - (p-1)\int_\Omega U_{\varepsilon,x_1}^{p-2}U_{\varepsilon,x_2}\frac{\partial U_{\varepsilon,x_1}}{\partial x_{1l}} \\ &\quad + O\big(\varepsilon^{N-1}e^{-(1+\sigma)\frac{|x_1-x_2|}{\varepsilon}} + \varepsilon^{N-1}\sum_{j=1}^{2} e^{-(2+\sigma)\frac{d(x_j,\partial\Omega)}{\varepsilon}}\big) \\ &\quad = -(p-1)\int_{R^N} U_{\varepsilon,x_1}^{p-2}U_{\varepsilon,x_2}\frac{\partial U_{\varepsilon,x_1}}{\partial x_{1l}} + O\big(\varepsilon^{N-1}e^{-(1+\sigma)\frac{|x_1-x_2|}{\varepsilon}}\big) \\ &\quad = (c + o(1))\varepsilon^{N-1}U\big(\frac{|x_1 - x_2|}{\varepsilon}\big)\frac{x_{2l} - x_{1l}}{|x_{2l} - x_{1l}|} + O\big(\varepsilon^{N-1}e^{-(1+\sigma)\frac{|x_1-x_2|}{\varepsilon}}\big), \end{aligned} \tag{3.22}$$

and hence the result.

So from (3.20), we obtain $|x_m| \le L$.

Before we give the proof of Theorem 1.1, we need the following lemma:

Lemma 3.5. *We have*

$$\{\,(x_1,x_2): x_1\in\partial B_{\rho(\varepsilon)}(0),\ x_2\in\partial B_{2\rho(\varepsilon)}(x_1)\,\}\subset K^{c_{2,\varepsilon}}.$$

Proof. It follows from Proposition 3.1 that

$$\begin{aligned}K(x) =&J(1,x,0)+O\big(|\alpha-1|^2\varepsilon^N+\|v_\varepsilon\|_\varepsilon^2\big)\\ =&J(1,x,0)+O\big(\sum_{j=1}^{2}\tau_{\varepsilon,x_j}^{1+\sigma}+U^{1+\sigma}\big(\frac{2\rho(\varepsilon)}{\varepsilon}\big)\big).\end{aligned}\tag{3.23}$$

Thus Proposition 2.8 and (3.23) give the result.

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. We claim that $K(x)$ has a critical point in $K^{c_{2,\varepsilon}}\setminus K^{c_{1,\varepsilon}}$. Suppose this is not true, then in view of Lemmas 3.3 and 3.4, we know that $K^{c_{2,\varepsilon}}$ can be deformed into $K^{c_{1,\varepsilon}}$. Using Lemma 3.5, we have $\{(x_1,x_2): x_1\in\partial B_{\rho(\varepsilon)}(0),\ x_2\in\partial B_{2\rho(\varepsilon)}(x_1)\}\subset K^{c_{2,\varepsilon}}$. On the other hand, it is not difficult to check that $K^{c_{1,\varepsilon}}\subset\{(x_1,x_2):|x_1-x_2|\le 2d_1\}$. As a result, we deduce that $\{\,(x_1,x_2): x_1\in\partial B_{\rho(\varepsilon)}(0),\ x_2\in\partial B_{2\rho(\varepsilon)}(x_1)\,\}$ can be deformed into a subset of $\{\,(x_1,x_2):|x_1-x_2|\le 2d_1\,\}$. But $\{\,(x_1,x_2): x_1\in\partial B_{\rho(\varepsilon)}(0), x_2\in\partial B_{2\rho(\varepsilon)}(x_1)\,\}$ is homotopically equivalent to $\partial B_{\rho(\varepsilon)}(0)\times\partial B_{\rho(\varepsilon)}(0)$, and thus $\partial B_{\rho(\varepsilon)}(0)\times\partial B_{\rho(\varepsilon)}(0)$ can be deformed into a subset of $\{\,(x_1,x_2):|x_1-x_2|\le 2d_1\,\}$. This is a contradiction to the following proposition.

Proposition 3.6. *The set $S=:\partial B_{\rho(\varepsilon)}(0)\times\partial B_{\rho(\varepsilon)}(0)$, or any set homotopically equivalent to S within $\Omega\times\Omega$, can not be deformed into a subset of $L=:\{\,(x_1,x_2):|x_1-x_2|\le 2d_1\,\}$ within $\Omega\times\Omega$.*

Proof. First, we mention that all the homologies here are with Z_2 coefficient. Let i be the inclusion map from S to $\Omega\times\Omega$. Since the inclusion i_1: $\partial B_{\rho(\varepsilon)}(0)\to\Omega$ induces a nontrivial map on the $N-1$ dimensional homology, by the Künneth formula for the homology of a product space as in [10], we see the induced map i_* is nontrivial in the $2N-2$ dimensional homology.

Let i_2 be the inclusion map from L to $\Omega\times\Omega$. Suppose that S can be deformed into a subset of L. Then there is a $H(x,t): S\times[0,1]\to\Omega\times\Omega$ satisfying $H(x,0)=x$, $H(x,1)\subset L$. Let $j(x)=H(x,1)$. Then by homotopy invariance, we have $i_*=j_*$. Write $j=i_2\circ j_1$, where $j_1: S\to L$ is defined as $j_1(x)=j(x)$. Then $j_*=i_{2*}\circ j_{1*}$. As a result, $i_*=i_{2*}\circ j_{1*}$. Since we can deform L into the diagonal $\{(x,x):x\in\Omega\}$ which is homeomorphic to Ω, L

only has nontrivial homology in dimensions less than or equal to $N-1$, and thus i_* is a trivial map in $2N-2$ homology. This is a contradiction.

Before we close this section, we give a new proof of the result in [5].

Theorem 3.7. *There is an $\varepsilon_0 > 0$, such that for each $\varepsilon \in (0, \varepsilon_0]$, (1.1) has at least $Cat_\Omega(\Omega, R^N \setminus B_R(0))$ solutions of the form*

$$u_\varepsilon = \alpha_\varepsilon P_{\varepsilon,\Omega} U_{\varepsilon,x_\varepsilon} + v_\varepsilon, \tag{3.24}$$

where $v_\varepsilon \in E_{\varepsilon,x_\varepsilon}$ and as $\varepsilon \to 0$, $\alpha_\varepsilon \to 1$, $\|v_\varepsilon\|_\varepsilon = o(\varepsilon^{N/2})$.

Proof. We just sketch the proof. Let $M > 2M_1 > 0$ be two large constants. We define $D_M = B_M(0) \cap \Omega_\delta$, $\Omega_\delta = \{x : x \in \Omega, d(x, \partial\Omega) \geq \delta\}$. Let $c'_{\varepsilon,2} = \varepsilon^N(A+\eta)$, $c'_{\varepsilon,1} = \varepsilon^N(A + e^{-\frac{M_1}{\varepsilon}})$, where $\eta > 0$ is a small constant. Let $J_1(\alpha, x, v) =: I(\alpha P_{\varepsilon,\Omega} U_{\varepsilon,x} + v)$.

Step 1. It is standard to prove that there is an $\varepsilon_0 > 0$, such that for each $\varepsilon \in (0, \varepsilon_0]$, there is a C^1-map $(\alpha_\varepsilon(x), v_\varepsilon(x)) : D_M \to R^+ \times H_0^1(\Omega)$, such that $v_\varepsilon(x) \in E_{\varepsilon,x}$ and

$$\frac{\partial J_1}{\partial \alpha} = 0 \tag{3.25}$$

$$\langle \frac{\partial J_1}{\partial v}, \omega \rangle = 0, \quad \forall\, \omega \in E_{\varepsilon,x}. \tag{3.26}$$

Moreover, $|\alpha_\varepsilon - 1|\varepsilon^{N/2} + \|v_\varepsilon\|_\varepsilon = O\big(\tau_{\varepsilon,x}^{1-\frac{1}{p}}\big) = O\big(\varepsilon^{N/2} e^{-(1+\sigma)\frac{d(x,\partial\Omega)}{\varepsilon}}\big)$, for some $\sigma > 0$.

Step 2. Define $K_1(x) = I(\alpha_\varepsilon(x) P_{\varepsilon,\Omega} U_{\varepsilon,x} + v_\varepsilon(x))$, $x \in D_M$. Then it follows from Lemma 2.5 that $\frac{\partial K_1}{\partial n} > 0$ for $x \in \partial\Omega_\delta$, where n is the outward unit normal of $\partial\Omega_\delta$ at x.

Step 3. We claim that $K_1(x) < c'_{\varepsilon,1}$ if $|x| = M$. In fact, from Lemma 2.1, we have

$$K_1(x) = \varepsilon^N\big(A + \tau_{\varepsilon,x} + o(\tau_{\varepsilon,x})\big) \leq \varepsilon^N\big(A + Ce^{-(2-\sigma)\frac{M-T}{\varepsilon}}\big) < c'_{\varepsilon,1}. \tag{3.27}$$

Combining Steps 1–3, we conclude

$$\#\{x :\ DK_1(x) = 0, x \in D_M,\ c'_{\varepsilon,1} < K(x) \leq c'_{\varepsilon,2}\} \geq \mathrm{Cat}_{D_M}(K_1^{c_{\varepsilon,2}}, K_1^{c_{\varepsilon,1}}).$$

But it is easy to check that $K_1^{c_{\varepsilon,2}} \subset D_M$, and $K_1^{c_{\varepsilon,1}} \subset \{x : \frac{M_1}{2} \leq |x| \leq M\}$. As a result, $\#\{x : DK_1(x) = 0, x \in D_M,\ c'_{\varepsilon,1} < K(x) \leq c'_{\varepsilon,2}\} \geq \mathrm{Cat}_{D_M}(D_M, \{x : \frac{M_1}{2} \leq |x| \leq M\})$ and the result follows.

4. Proof of Theorems 1.2 and 1.3. In this section, we will prove Theorems 1.2 and 1.3. First, we need the following lemmas.

Lemma 4.1. *Suppose that* $u_\varepsilon = \alpha_\varepsilon P_{\varepsilon,\Omega} U_{\varepsilon,x_\varepsilon} + v_\varepsilon$ *is a solution of* (1.1) *satisfying* $\alpha_\varepsilon \to 1$, $v_\varepsilon \in E_{\varepsilon,x_\varepsilon}$, $\frac{d(x_\varepsilon,\Omega)}{\varepsilon} \to +\infty$ *and* $\|v_\varepsilon\|_\varepsilon = o(\varepsilon^{N/2})$. *Then*

$$|\alpha_\varepsilon - 1|\varepsilon^{N/2} + \|v_\varepsilon\|_\varepsilon = O\big(\tau_{\varepsilon,x}^{1-\frac{1}{p}}\big) = O\big(\varepsilon^{N/2} e^{-(1+\sigma)\frac{d(x_\varepsilon,\partial\Omega)}{\varepsilon}}\big),$$

for some $\sigma > 0$.

Proof. We know that $u_\varepsilon = \alpha_\varepsilon P_{\varepsilon,\Omega} U_{\varepsilon,x_\varepsilon} + v_\varepsilon$ is a solution of (1.1) implies that $(\alpha_\varepsilon, x_\varepsilon, v_\varepsilon)$ is a critical point of $J_1(\alpha, x, v) =: I(\alpha P U_{\varepsilon,x} + v)$. Thus, Lemma 4.1 can be proved by following the same procedure as in [14], pp 15–17, and using Lemma 2.1. We omit the details here.

Lemma 4.2. *Suppose that* $u_\varepsilon = \sum_{j=1}^2 \alpha_{\varepsilon,j} P_{\varepsilon,\Omega} U_{\varepsilon,x_{\varepsilon,j}} + v_\varepsilon$ *is a solution of* (1.1) *satisfying* $\alpha_{\varepsilon,j} \to 1$, $\frac{d(x_{\varepsilon,j},\Omega)}{\varepsilon} \to +\infty$, $\frac{|x_{\varepsilon,1}-x_{\varepsilon,2}|}{\varepsilon} \to +\infty$, $v_\varepsilon \in E_{\varepsilon,x_\varepsilon,2}$ *and* $\|v_\varepsilon\|_\varepsilon = o(\varepsilon^{N/2})$. *Then*

$$\sum_{j=1}^2 |\alpha_{\varepsilon,j} - 1|\varepsilon^{\frac{N}{2}} + \|v_\varepsilon\|_\varepsilon = O\Big(\varepsilon^{\frac{N}{2}} e^{-(1+\sigma)\frac{d(x_\varepsilon,\partial\Omega)}{\varepsilon}} + \varepsilon^{N/2} U^{1+\sigma}\Big(\frac{|x_{\varepsilon,1} - x_{\varepsilon,2}|}{\varepsilon}\Big)\Big),$$

for some $\sigma > 0$.

Proof. The proof of this lemma is similar to that of Lemma 4.1.

Proof of Theorems 1.2. (i) Suppose that (1.1) has a single peak solution, that is, solution of the form $u_\varepsilon = \alpha_\varepsilon P_{\varepsilon,\Omega} U_{\varepsilon,x_\varepsilon} + v_\varepsilon$, $v_\varepsilon \in E_{\varepsilon,x_\varepsilon}$. Multiplying (1.1) by $\frac{\partial P_{\varepsilon,\Omega} U_{\varepsilon,x_\varepsilon}}{\partial x_j}$ and integrating by parts, we have

$$\int_\Omega \Big[\alpha_\varepsilon U_{\varepsilon,x_\varepsilon}^{p-1} - (\alpha_\varepsilon P_{\varepsilon,\Omega} U_{\varepsilon,x_\varepsilon} + v_\varepsilon)^{p-1}\Big] \frac{\partial P_{\varepsilon,\Omega} U_{\varepsilon,x_\varepsilon}}{\partial x_j} = 0. \tag{4.1}$$

Using Lemma 4.1, we get

$$\begin{aligned}
&\int_\Omega (\alpha_\varepsilon P_{\varepsilon,\Omega} U_{\varepsilon,x_\varepsilon} + v_\varepsilon)^{p-1} \frac{\partial P_{\varepsilon,\Omega} U_{\varepsilon,x_\varepsilon}}{\partial x_j} = \alpha_\varepsilon^{p-1} \int_\Omega (P_{\varepsilon,\Omega} U_{\varepsilon,x_\varepsilon})^{p-1} \frac{\partial P_{\varepsilon,\Omega} U_{\varepsilon,x_\varepsilon}}{\partial x_j} \\
&+ \alpha_\varepsilon^{p-2} \int_\Omega (P_{\varepsilon,\Omega} U_{\varepsilon,x_\varepsilon})^{p-2} \frac{\partial P_{\varepsilon,\Omega} U_{\varepsilon,x_\varepsilon}}{\partial x_j} v_\varepsilon + O\big(\varepsilon^{-1}\|v_\varepsilon\|_\varepsilon^2\big) \qquad (4.2) \\
&= \alpha_\varepsilon^{p-1} \int_\Omega (P_{\varepsilon,\Omega} U_{\varepsilon,x_\varepsilon})^{p-1} \frac{\partial P_{\varepsilon,\Omega} U_{\varepsilon,x_\varepsilon}}{\partial x_j} + O\big(\varepsilon^{-1}\|v_\varepsilon\|_\varepsilon \tau_{\varepsilon,x_\varepsilon}^{1-\frac{1}{p}} + \varepsilon^{-1}\|v_\varepsilon\|_\varepsilon^2\big) \\
&= \alpha_\varepsilon^{p-1} \int_\Omega (P_{\varepsilon,\Omega} U_{\varepsilon,x_\varepsilon})^{p-1} \frac{\partial P_{\varepsilon,\Omega} U_{\varepsilon,x_\varepsilon}}{\partial x_j} + O\big(\varepsilon^{N-1} e^{-(2+\sigma)\frac{d(x_\varepsilon,\partial\Omega)}{\varepsilon}}\big).
\end{aligned}$$

On the other hand, it follows from Lemma 2.2 that

$$\begin{aligned}\int_\Omega (P_{\varepsilon,\Omega}U_{\varepsilon,x_\varepsilon})^{p-1}\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_\varepsilon}}{\partial x_j} &= \int_\Omega U^{p-1}_{\varepsilon,x_\varepsilon}\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_\varepsilon}}{\partial x_j} \\ &- (p-1)\int_\Omega U^{p-2}_{\varepsilon,x_\varepsilon}\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_\varepsilon}}{\partial x_j}\varphi_{\varepsilon,x_\varepsilon} + O\big(\varepsilon^{-1}\int_\Omega U^{p-2}_{\varepsilon,x_\varepsilon}\varphi^2_{\varepsilon,x_\varepsilon}\big) \\ &= \int_\Omega U^{p-1}_{\varepsilon,x_\varepsilon}\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_\varepsilon}}{\partial x_j} - (p-1)\int_\Omega U^{p-2}_{\varepsilon,x_\varepsilon}\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_\varepsilon}}{\partial x_j}\varphi_{\varepsilon,x_\varepsilon} \\ &+ O\big(\varepsilon^{N-1}e^{-(2+\sigma)\frac{d(x_\varepsilon,\partial\Omega)}{\varepsilon}}\big).\end{aligned} \tag{4.3}$$

Inserting (4.2) and (4.3) into (4.1), we get

$$\begin{aligned}(\alpha_\varepsilon - \alpha_\varepsilon^{p-1})\int_\Omega U^{p-1}_{\varepsilon,x_\varepsilon}\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_\varepsilon}}{\partial x_j} + \alpha_\varepsilon^{p-1}(p-1)\int_\Omega U^{p-2}_{\varepsilon,x_\varepsilon}\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_\varepsilon}}{\partial x_j}\varphi_{\varepsilon,x_\varepsilon} \\ = O\big(\varepsilon^{N-1}e^{-(2+\sigma)\frac{d(x_\varepsilon,\partial\Omega)}{\varepsilon}}\big).\end{aligned} \tag{4.4}$$

It is easy to see

$$\begin{aligned}\int_\Omega U^{p-1}_{\varepsilon,x_\varepsilon}\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_\varepsilon}}{\partial x_j} &= \int_\Omega U^{p-1}_{\varepsilon,x_\varepsilon}\frac{\partial U_{\varepsilon,x_\varepsilon}}{\partial x_j} - \int_\Omega U^{p-1}_{\varepsilon,x_\varepsilon}\frac{\partial \varphi_{\varepsilon,x_\varepsilon}}{\partial x_j} \\ &= O\big(\varepsilon^{N-1}e^{-(1-\theta)\frac{d(x_\varepsilon,\partial\Omega)}{\varepsilon}}\big).\end{aligned} \tag{4.5}$$

Using Lemma 2.2 again, we get

$$\begin{aligned}\int_\Omega U^{p-2}_{\varepsilon,x_\varepsilon}\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_\varepsilon}}{\partial x_j}\varphi_{\varepsilon,x_\varepsilon} &= \int_\Omega U^{p-2}_{\varepsilon,x_\varepsilon}\frac{\partial U_{\varepsilon,x_\varepsilon}}{\partial x_j}\varphi_{\varepsilon,x_\varepsilon} - \int_\Omega U^{p-2}_{\varepsilon,x_\varepsilon}\frac{\partial \varphi_{\varepsilon,x_\varepsilon}}{\partial x_j}\varphi_{\varepsilon,x_\varepsilon} \\ = \int_\Omega U^{p-2}_{\varepsilon,x_\varepsilon}\frac{\partial U_{\varepsilon,x_\varepsilon}}{\partial x_j}\varphi_{\varepsilon,x_\varepsilon} &+ O\big(\varepsilon^{N-1}e^{-(2+\sigma)\frac{d(x_\varepsilon,\partial\Omega)}{\varepsilon}}\big).\end{aligned} \tag{4.6}$$

Combining (4.4), (4.5) and (4.6), noting Lemma 4.1, we conclude

$$\int_\Omega U^{p-2}_{\varepsilon,x_\varepsilon}\frac{\partial U_{\varepsilon,x_\varepsilon}}{\partial x_j}\varphi_{\varepsilon,x_\varepsilon} = O\big(\varepsilon^{N-1}e^{-(2+\sigma)\frac{d(x_\varepsilon,\partial\Omega)}{\varepsilon}}\big). \tag{4.7}$$

This is a contradiction to Lemma 2.5.

(ii) Suppose that (1.1) has a two peak solution, that is, solution of the form $u_\varepsilon = \sum_{j=1}^2 \alpha_{\varepsilon,j}P_{\varepsilon,\Omega}U_{\varepsilon,x_{\varepsilon,j}} + v_\varepsilon$, $v_\varepsilon \in E_{\varepsilon,x_\varepsilon,2}$. Multiplying (1.1) with

$\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_{\varepsilon,1}}}{\partial x_i}$ and integrating by parts, we have

$$\begin{aligned}&\int_\Omega \big(\alpha_{\varepsilon,1}U^{p-1}_{\varepsilon,x_{\varepsilon,1}}+\alpha_{\varepsilon,2}U^{p-1}_{\varepsilon,x_{\varepsilon,2}}\big)\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_{\varepsilon,1}}}{\partial x_i}\\ &-\int_\Omega \big(\alpha_{\varepsilon,1}P_{\varepsilon,\Omega}U_{\varepsilon,x_{\varepsilon,1}}+\alpha_{\varepsilon,2}P_{\varepsilon,\Omega}U_{\varepsilon,x_{\varepsilon,2}}+v_\varepsilon\big)^{p-1}\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_{\varepsilon,1}}}{\partial x_i}=0.\end{aligned}\tag{4.8}$$

By using Lemma 4.2, it is not difficult to deduce that (4.8) can be written as

$$\begin{aligned}&(p-1)\int_\Omega U^{p-2}_{\varepsilon,x_{\varepsilon,1}}\frac{\partial U_{\varepsilon,x_{\varepsilon,1}}}{\partial x_i}\varphi_{\varepsilon,x_{\varepsilon,1}}-(p-1)\int_\Omega U^{p-2}_{\varepsilon,x_{\varepsilon,1}}\frac{\partial U_{\varepsilon,x_{\varepsilon,1}}}{\partial x_i}P_{\varepsilon,\Omega}U_{\varepsilon,x_{\varepsilon,2}}\\ &=O\big(\varepsilon^{N-1}\sum_{j=1}^2 e^{-\frac{(2+\sigma)d(x_{\varepsilon,j},\partial\Omega)}{\varepsilon}}+\varepsilon^{N-1}U^{1+\sigma}\big(\frac{|x_{\varepsilon,1}-x_{\varepsilon,2}|}{\varepsilon}\big)\big).\end{aligned}\tag{4.9}$$

We argue by contradiction. Suppose that $d(x_{\varepsilon,1},\partial\Omega)=\min_{j=1,2}d(x_{\varepsilon,j},\partial\Omega)$ $\le C$. Then we can assume $x_{\varepsilon,1}\to x_1$. It follows from (4.9) that

$$|x_{\varepsilon,2}-x_{\varepsilon,1}|\le(2+\sigma)d(x_1,\partial\Omega)<+\infty.$$

Thus we assume $x_{\varepsilon,2}\to x_2$. So by Lemma 2.7, (4.9) is equivalent to

$$\begin{aligned}&(p-1)\int_\Omega U^{p-2}_{\varepsilon,x_{\varepsilon,1}}\frac{\partial U_{\varepsilon,x_{\varepsilon,1}}}{\partial x_i}\varphi_{\varepsilon,x_{\varepsilon,1}}-(p-1)\int_\Omega U^{p-2}_{\varepsilon,x_{\varepsilon,1}}\frac{\partial U_{\varepsilon,x_{\varepsilon,1}}}{\partial x_i}U_{\varepsilon,x_{\varepsilon,2}}\\ &=O\big(\varepsilon^{N-1}\sum_{j=1}^2 e^{-\frac{(2+\sigma)d(x_{\varepsilon,j},\partial\Omega)}{\varepsilon}}+\varepsilon^{N-1}U^{1+\sigma}\big(\frac{|x_{\varepsilon,1}-x_{\varepsilon,2}|}{\varepsilon}\big)\big).\end{aligned}\tag{4.10}$$

But

$$\begin{aligned}&\int_\Omega U^{p-2}_{\varepsilon,x_{\varepsilon,1}}\frac{\partial U_{\varepsilon,x_{\varepsilon,1}}}{\partial x_i}U_{\varepsilon,x_{\varepsilon,2}}\\ &=\int_{R^N} U^{p-2}_{\varepsilon,x_{\varepsilon,1}}\frac{\partial U_{\varepsilon,x_{\varepsilon,1}}}{\partial x_i}U_{\varepsilon,x_{\varepsilon,2}}+O\big(\sum_{j=1}^2\varepsilon^{N-1}e^{-\frac{(2+\sigma)d(x_{\varepsilon,j},\partial\Omega)}{\varepsilon}}\big).\end{aligned}\tag{4.11}$$

Inserting (4.11) into (4.10) yields

$$\begin{aligned}&\int_\Omega U^{p-2}_{\varepsilon,x_{\varepsilon,1}}\frac{\partial U_{\varepsilon,x_{\varepsilon,1}}}{\partial x_i}\varphi_{\varepsilon,x_{\varepsilon,1}}-\int_{R^N} U^{p-2}_{\varepsilon,x_{\varepsilon,1}}\frac{\partial U_{\varepsilon,x_{\varepsilon,1}}}{\partial x_i}U_{\varepsilon,x_{\varepsilon,2}}\\ &=O\big(\varepsilon^{N-1}\sum_{j=1}^2 e^{-\frac{(2+\sigma)d(x_{\varepsilon,j},\partial\Omega)}{\varepsilon}}+\varepsilon^{N-1}U^{1+\sigma}\big(\frac{|x_{\varepsilon,1}-x_{\varepsilon,2}|}{\varepsilon}\big)\big).\end{aligned}\tag{4.12}$$

Similarly, we have

$$\begin{aligned}&\int_\Omega U^{p-2}_{\varepsilon,x_{\varepsilon,2}}\frac{\partial U_{\varepsilon,x_{\varepsilon,2}}}{\partial x_i}\varphi_{\varepsilon,x_{\varepsilon,2}}-\int_{R^N}U^{p-2}_{\varepsilon,x_{\varepsilon,2}}\frac{\partial U_{\varepsilon,x_{\varepsilon,2}}}{\partial x_i}U_{\varepsilon,x_{\varepsilon,1}}\\&=O\big(\varepsilon^{N-1}\sum_{j=1}^{2}e^{-\frac{(2+\sigma)d(x_{\varepsilon,j},\partial\Omega)}{\varepsilon}}+\varepsilon^{N-1}U^{1+\sigma}\big(\frac{|x_{\varepsilon,1}-x_{\varepsilon,2}|}{\varepsilon}\big)\big).\end{aligned}\tag{4.13}$$

On the other hand, we have

$$\begin{aligned}&\frac{\int_{R^N}U^{p-2}_{\varepsilon,x_{\varepsilon,2}}\frac{\partial U_{\varepsilon,x_{\varepsilon,2}}}{\partial x_i}U_{\varepsilon,x_{\varepsilon,1}}}{U\big(\frac{|x_{\varepsilon,1}-x_{\varepsilon,2}|}{\varepsilon}\big)}\\&=-\varepsilon^{N-1}\Big(1+O\big(\frac{\varepsilon}{|x_{\varepsilon,1}-x_{\varepsilon,2}|}\big)\Big)\int_{R^N}U^{p-2}(|y|)\frac{\partial U(|y|)}{\partial|y|}\frac{y_i}{|y|}e^{-\langle y,\frac{x_{\varepsilon,1}-x_{\varepsilon,2}}{|x_{\varepsilon,1}-x_{\varepsilon,2}|}\rangle}\\&=-\varepsilon^{N-1}\Big(1+O\big(\frac{\varepsilon}{|x_{\varepsilon,1}-x_{\varepsilon,2}|}\big)\Big)\frac{(x_{\varepsilon,1}-x_{\varepsilon,2})_i}{|x_{\varepsilon,1}-x_{\varepsilon,2}|}\int_{R^N}U^{p-2}(|y|)\frac{\partial U(|y|)}{\partial|y|}\frac{y_1}{|y|}e^{-y_1}.\end{aligned}\tag{4.14}$$

Clearly, taking Lemma 2.5 into account, we deduce from (4.12)–(4.14) that

$$|x_{\varepsilon,1}-x_{\varepsilon,2}|=2d(x_{\varepsilon,j},\partial\Omega)+o(1),\quad j=1,2,\tag{4.15}$$

where $o(1)\to 0$ as $\varepsilon\to 0$. Let

$$\phi_{\varepsilon,x_{\varepsilon,j}}=\frac{\frac{\partial PU_{\varepsilon,x_{\varepsilon,j}}}{\partial n}\frac{\partial U_{\varepsilon,x_{\varepsilon,j}}}{\partial r}}{\int_{\partial\Omega}\frac{\partial PU_{\varepsilon,x_{\varepsilon,j}}}{\partial n}\frac{\partial U_{\varepsilon,x_{\varepsilon,j}}}{\partial r}}.$$

Then, in view of Lemma 2.4, we see $\phi_{\varepsilon,x_{\varepsilon,j}}$ is well defined and $\phi_{\varepsilon,x_{\varepsilon,j}}\in L^1(\partial\Omega)$. Since $\partial\Omega\cap\partial B_{d(x_{\varepsilon,j},\partial\Omega)}(x_{\varepsilon,j})=:\{q_{\varepsilon,j}\}$ and outside any neighbourhood of $q_j=\lim_{\varepsilon\to 0}q_{\varepsilon,j}$, $\frac{\partial PU_{\varepsilon,x_{\varepsilon,j}}}{\partial n}\frac{\partial U_{\varepsilon,x_{\varepsilon,j}}}{\partial r}$ is a higher order term of $e^{-(2+\theta)\frac{d(x,\partial\Omega)}{\varepsilon}}$, it is easy to prove that $\phi_{\varepsilon,x_{\varepsilon,j}}\rightharpoonup\delta_{q_j}$, Thus by Lemma 2.5, (4.12) and (4.13) become

$$\begin{aligned}&-\varepsilon^2\int_{\partial\Omega}\frac{\partial PU_{\varepsilon,x_{\varepsilon,1}}}{\partial n}\frac{\partial U_{\varepsilon,x_{\varepsilon,1}}}{\partial r}b_{i,1}-\int_{R^N}U^{p-2}_{\varepsilon,x_{\varepsilon,1}}\frac{\partial U_{\varepsilon,x_{\varepsilon,1}}}{\partial x_i}U_{\varepsilon,x_{\varepsilon,2}}\\&=O\big(\varepsilon^{N-1}\sum_{j=1}^{2}e^{-\frac{(2+\sigma)d(x_{\varepsilon,j},\partial\Omega)}{\varepsilon}}+\varepsilon^{N-1}U^{1+\sigma}\big(\frac{|x_{\varepsilon,1}-x_{\varepsilon,2}|}{\varepsilon}\big)\big),\end{aligned}\tag{4.16}$$

$$\begin{aligned} & - \varepsilon^2 \int_{\partial\Omega} \frac{\partial PU_{\varepsilon,x_{\varepsilon,2}}}{\partial n} \frac{\partial U_{\varepsilon,x_{\varepsilon,2}}}{\partial r} b_{i,2} - \int_{R^N} U^{p-2}_{\varepsilon,x_{\varepsilon,2}} \frac{\partial U_{\varepsilon,x_{\varepsilon,2}}}{\partial x_i} U_{\varepsilon,x_{\varepsilon,1}} \\ & = O\big(\varepsilon^{N-1} \sum_{j=1}^{2} e^{-\frac{(2+\sigma)d(x_{\varepsilon,j},\partial\Omega)}{\varepsilon}}\big) + \varepsilon^{N-1} U^{1+\sigma}\big(\frac{|x_{\varepsilon,1}-x_{\varepsilon,2}|}{\varepsilon}\big)\big), \end{aligned} \tag{4.17}$$

where $b_j = \lim_{\varepsilon\to 0} \frac{x_{\varepsilon,j}-q_{\varepsilon,j}}{|x_{\varepsilon,j}-q_{\varepsilon,j}|}$. It follows from (4.16) and (4.17) that as $\varepsilon \to 0$,

$$\frac{\varepsilon^2 \int_{\partial\Omega} \frac{\partial PU_{\varepsilon,x_{\varepsilon,j}}}{\partial n} \frac{\partial U_{\varepsilon,x_{\varepsilon,j}}}{\partial r}}{\varepsilon^{N-1} U\big(\frac{|x_{\varepsilon,1}-x_{\varepsilon,2}|}{\varepsilon}\big)} \to a_j > 0, \quad j = 1, 2. \tag{4.18}$$

We first show $x_1 \notin \partial\Omega$. Suppose that $x_1 \in \partial\Omega$. Then it follows from (4.15) that $x_2 = x_1$. In this case, we have $b_{i,1} = b_{i,2} = n_i$, where $n =: (n_1, \dots, n_N)$ is the outward unit normal of $\partial\Omega$ at x_1. Then it follows from (4.14) that

$$\begin{aligned} & \frac{(p-1)\varepsilon^2 \int_{\partial\Omega} \frac{\partial PU_{\varepsilon,x_{\varepsilon,2}}}{\partial n} \frac{\partial U_{\varepsilon,x_{\varepsilon,2}}}{\partial r} + (p-1)\varepsilon^2 \int_{\partial\Omega} \frac{\partial PU_{\varepsilon,x_{\varepsilon,1}}}{\partial n} \frac{\partial U_{\varepsilon,x_{\varepsilon,1}}}{\partial r}}{\varepsilon^{N-1} U\big(\frac{|x_{\varepsilon,1}-x_{\varepsilon,2}|}{\varepsilon}\big)} n_i \\ & = O\Big(\frac{\varepsilon}{|x_{\varepsilon,1}-x_{\varepsilon,2}|}\Big) \to 0. \end{aligned} \tag{4.19}$$

This is a contradiction to (4.18). As a result, we have

$$2d(x_1, \partial\Omega) = 2d(x_2, \partial\Omega) = |x_1 - x_2| > 0. \tag{4.20}$$

Combining (4.14), (4.16) and (4.17), we obtain

$$a_1 \frac{q_1 - x_1}{|q_1 - x_1|} + \gamma \frac{x_1 - x_2}{|x_1 - x_2|} = 0, \qquad a_2 \frac{q_2 - x_2}{|q_2 - x_2|} + \gamma \frac{x_2 - x_1}{|x_2 - x_1|} = 0,$$

where q_j is the unique point in $\partial\Omega$ such that $|q_j - x_j| = d(x_j, \partial\Omega)$ and γ is some constant. Hence, $\{x_1, x_2, q_1, q_2\} \subset \{x_1 + t(x_1 - q_1), \forall t \in R^1\}$. Without loss of generality, we assume $q_1 = 0$, $x_1 = (d, 0, \dots, 0)$, where $d = d(x_1, \partial\Omega) = d(x_2, \partial\Omega)$. Because $x_1 \neq x_2$ and $d(x_1, \partial\Omega) = d(x_2, \partial\Omega)$, we know $q_1 \neq q_2$. Since Ω is convex, we have $q_2 = (-T, 0, \dots, 0)$ for some $T > 0$. So $x_2 = (-d - T, 0, \dots, 0)$, and hence $|x_1 - x_2| > 2d$. This is impossible in view of (4.20). Thus we have proved that $x_{\varepsilon,j} \to \infty$, $j = 1, 2$. It follows from Lemma 2.7 that either

$$2d(x_{\varepsilon,j}, \partial\Omega) = (1 + o(1))|x_{\varepsilon,1} - x_{\varepsilon,2}|, \tag{4.21}$$

or (4.10) and hence (4.12) and (4.13) hold. But as above, (4.12), (4.13) and (4.14) imply (4.21).

Proof of Theorem 1.3. Suppose that $u_\varepsilon = \alpha_\varepsilon P_{\varepsilon,\Omega} U_{\varepsilon,x_\varepsilon} + v_\varepsilon$ is a single peak solution of (1.1). Then we have the following relation (see (4.7)):

$$\varepsilon^2 \int_{\partial\Omega} \frac{\partial P_{\varepsilon,\Omega} U_{\varepsilon,x_\varepsilon}}{\partial n} \frac{\partial U_{\varepsilon,x_\varepsilon}}{\partial x_i} = O\big(\varepsilon^{N-1} e^{-(2+\sigma)\frac{d(x_\varepsilon,\partial\Omega)}{\varepsilon}}\big). \tag{4.22}$$

(i) Suppose that $x_\varepsilon \to \infty$. Then we have

$$\Big\langle \frac{x_\varepsilon - q}{|x_\varepsilon - q|}, \nu \Big\rangle \geq c' > 0, \quad \forall\, q \in \partial\Omega \cap \partial B_{d(x_\varepsilon,\partial\Omega)}(x_\varepsilon),$$

where ν is the outward unit normal of $\partial\Omega$ at one of the fixed point in $\partial\Omega \cap \partial B_{d(x_\varepsilon,\partial\Omega)}(x_\varepsilon)$. Then as in Lemma 2.5, we have

$$-\varepsilon^2 \sum_{i=1}^N \int_{\partial\Omega} \frac{\partial P_{\varepsilon,\Omega} U_{\varepsilon,x_\varepsilon}}{\partial n} \frac{\partial U_{\varepsilon,x_\varepsilon}}{\partial x_i} \nu_i \geq c_0 \varepsilon^{N-1} e^{-(2+\theta)\frac{d(x_\varepsilon,\partial\Omega)}{\varepsilon}}.$$

Thus we get a contradiction. Suppose that x_ε tends to a point in the boundary. Then $\partial\Omega \cap \partial B_{d(x_\varepsilon,\partial\Omega)}(x_\varepsilon)$ contains exactly one point. Thus we can also get a contradiction from (4.22) and Lemma 2.5.

(ii) Let

$$\phi_{\varepsilon,x_\varepsilon} = \frac{\frac{\partial P_{\varepsilon,\Omega} U_{\varepsilon,x_\varepsilon}}{\partial n} \frac{\partial U_{\varepsilon,x_\varepsilon}}{\partial r}}{\int_{\partial\Omega} \frac{\partial P_{\varepsilon,\Omega} U_{\varepsilon,x_\varepsilon}}{\partial n} \frac{\partial U_{\varepsilon,x_\varepsilon}}{\partial r}}.$$

Then we may assume that as $\varepsilon \to 0$, $\phi_{\varepsilon,x_\varepsilon} \rightharpoonup \mu$, where μ is a measure in $\partial\Omega$. It is easy to check that $\mathrm{spt}(\mu) \subset \partial\Omega \cap \partial B_{d(x_0,\partial\Omega)}(x_0)$.

Suppose that $\partial\Omega \cap \partial B_{d(x_0,\partial\Omega)}(x_0) = \{q\}$. Then we have $\mu = \delta_q$ and we get from (4.22) $\frac{x_0 - q}{|x_0 - q|} = 0$. This is a contradiction.

(iii) Suppose that $\partial\Omega \cap \partial B_{d(x_0,\partial\Omega)}(x_0) = \{q_1, q_2\}$. Then we see $\mathrm{spt}(\mu) = \{q_1, q_2\}$. Otherwise we would have $\mu = \delta_{q_j}$ for some j. So we will get a contradiction as in (ii).

Thus we have $\mu = \beta_1 \delta_{q_1} + \beta_2 \delta_{q_2}$ for some $\beta_j > 0$, $j = 1, 2$ and $\beta_1 + \beta_2 = 1$. Hence (4.22) becomes

$$\beta_1 \frac{x_0 - q_1}{|x_0 - q_1|} + \beta_2 \frac{x_0 - q_2}{|x_0 - q_2|} = 0.$$

Hence $\beta_1 = \beta_2 = \frac{1}{2}$. But $|x_0 - q_1| = |x_0 - q_2| = d(x_0, \partial\Omega)$, so we have $x_0 - q_1 = -(x_0 - q_2)$.

Finally, let us look at the following example.

Example 4.3. Let $\Omega = R^N \setminus \big(B_1(0) \setminus \cup_{j=1}^2 B_\delta(x_j)\big)$, where $x_1 = -1 + \frac{3}{4}\delta$, $x_2 = 1 - \frac{3}{4}\delta$. Then

- (i) for each j and ε small, (1.1) has a single peak solution with its peak tending to x_j;
- (ii) (1.1) has a solution which has exactly two peaks $x_{\varepsilon,1}$ and $x_{\varepsilon,2}$ satisfying $x_{\varepsilon,j} \to x_j, j = 1, 2$.

In order to prove that above claim, let us consider

$$K_1(x) = I(\alpha_\varepsilon(x), x, v_\varepsilon(x)), \quad x \in \overline{B_\delta(x_j)},$$

where $(\alpha_\varepsilon, v_\varepsilon)$ is the map attained in Step 1 in Theorem 3.7. Then we have the following expansion: $K_1(x) = \varepsilon^N(A + \frac{1}{2}\tau_{\varepsilon,x} + o(\tau_{\varepsilon,x}))$. In view of Lemma 2.1, we can prove from the above expansion that the minimum point x_ε of $K_1(x)$ in $\overline{B_\delta(x_j)}$ is an interior point of $B_\delta(x_j)$, hence a critical point of $K(x)$. The existence of a two-peak solution can be proved similarly.

In example 4.3, (1.1) still has single peak solution even if $\mathrm{Cat}_\Omega(\Omega, R^N \setminus R_R(0)) = 0$. This example also shows that if $R^N \setminus \Omega$ is not convex, (1.1) may have a two-peak solution whose peaks do not move to infinity.

Remark 4.4. The two-peak solution u_ε constructed in example 4.3 satisfies the following estimate:

$$I(u_\varepsilon) = \varepsilon^N 2A + \frac{1}{2}\tau_{\varepsilon,x_{\varepsilon,1}} + \frac{1}{2}\tau_{\varepsilon,x_{\varepsilon,2}} + o(\tau_{\varepsilon,x_{\varepsilon,1}} + \tau_{\varepsilon,x_{\varepsilon,2}}) > 2\varepsilon^N A.$$

Thus this two-peak solution is different from that constructed in Theorem 1.1. So the problem whether (1.1) always has a two peak solution with both of its peaks moving to infinity is still open.

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